APPROXIMATION OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS ON SPHERES

A Dissertation

by

QUOC THONG LE GIA

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2003

Major Subject: Mathematics

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ABSTRACT

Approximation of Linear Partial Differential Equations on Spheres. (August 2003)

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The theory of interpolation and approximation of solutions to differential and integral equations on spheres has attracted considerable interest in recent years; it has also been applied fruitfully in fields such as physical geodesy, potential theory, oceanography, and meteorology. In this dissertation we study the approximation of linear partial differential equations on spheres, namely a class of elliptic partial differential equations and the heat equation on the unit sphere. The shifts of a spherical basis function are used to construct the approximate solution. In the elliptic case, both the finite element method and the collocation method are discussed. In the heat equation, only the collocation method is considered. Error estimates in the supremum norms and the Sobolev norms are obtained when certain regularity conditions are imposed on the spherical basis functions.

To my parents, Mr. and Mrs. Le Gia Than

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CHAPTER I

INTRODUCTION

The theory of interpolation and approximation of solutions to differential and integral equations on spheres has attracted considerable interest in recent years; it has also been applied fruitfully in fields such as physical geodesy, potential theory, oceanography, and meteorology [11, 12, 23]. As more satellites are being launched into space, the acquisition of global data is becoming more important, and there is a growing demand for the processing and mathematical modeling of such data.

Differential or, more generally, pseudo-differential equations arise in many areas of earth sciences. Pseudo-differential operators of order t on spheres are operators that have eigenvalues $\Lambda(\ell) : \ell = 0, 1, \ldots,$ which are asymptotic to $(\ell + 1/2)^t$. A detailed discussion on pseudo-differential operators with their applications can be found in [5, 12, 16, 42].

Given a pseudo-differential operator $\mathcal{L}$ and a continuous function f defined on the unit sphere $S^n \subset \mathbb{R}^{n+1}$, we shall discuss the approximation of solutions of the equation

$$\mathcal{L}u = f \text{ on } S^n.$$

The approximate solution will be constructed as a linear combination of spherical basis functions that are derived from zonal kernels $\Phi : S^n \times S^n \to \mathbb{R}$ of the form

$$\Phi(x, y) = \phi(x \cdot y), \qquad x, y \in S^n,$$

where ϕ is a univariate function defined on $[-1, 1]$, and $x \cdot y$ is the Euclidean dot product of the position vectors of the points $x, y \in S^n$. For a fixed x the value of

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$\Phi(x, y)$ depends only on the geodesic distance from x to y, so the function $\Phi(x, \cdot)$ is radially symmetric with respect to the point x, and is called a spherical basis function (SBF).

The theory of interpolation of continuous functions by SBFs is well understood, see [9, 17, 29, 30], while the implications of the results to approximation for partial differential equations (PDEs) on spheres remain unexplored. In this dissertation we aim to explore the use of various types of SBFs in approximation of elliptic PDEs and parabolic PDEs on spheres.

A. Spherical harmonics

First, we need some background on the space of square integrable functions on the unit sphere, $L^2(S^n)$. An important orthonormal basis for $L^2(S^n)$ is constructed from the set of all spherical harmonics, which are polynomials $Y(x)$ satisfying $\Delta_x Y(x) = 0$ and are restricted to S^n, where Δ_x is the Laplacian operator in $\mathbb{R}^{n+1}$, i.e.,

$$\Delta_x = \left(\frac{\partial}{\partial x_1}\right)^2 + \ldots + \left(\frac{\partial}{\partial x_{n+1}}\right)^2.$$

A more detailed discussion on spherical harmonics can be found in [25, 26]. The space of all spherical harmonics of degree ℓ on S^n, denoted by V_ℓ, has an orthonormal basis

$$\{Y_{\ell k}(x) : k = 1, \ldots, N(n, \ell)\},$$

where

$$N(n, 0) = 1 \text{ and } N(n, \ell) = \frac{(2\ell + n - 1)\Gamma(\ell + n - 1)}{\Gamma(\ell + 1)\Gamma(n)} \text{ for } \ell \geq 1.$$

To describe spherical harmonics intrinsically, we first define the Laplace-Beltrami operator. If we introduce local coordinates $\{\theta_1, \ldots, \theta_n\}$ for a coordinate patch on S^n, then the corresponding patch embedded in $\mathbb{R}^{n+1}$ will have the parametrization

$x_i = f_i(\theta_1, \ldots, \theta_n)$, $i = 1, \ldots, n+1$. The metric g_{jk} on S^n is then induced via restricting $ds^2 = \sum_{i=1}^{n+1} dx_i^2$ to S^n; that is,

$$\sum_{j,k=1}^{n} g_{jk} d\theta_j d\theta_k \equiv \sum_{i=1}^{n+1} \left(\sum_{k=1}^{n} \frac{\partial f_i}{\partial \theta_k} d\theta_k \right)^2 .$$

We follow standard conventions in letting g^{jk} be the matrix inverse of g_{jk} and $g = \det g_{jk}$. The Laplace-Beltrami operator on S^n is given by

$$\Delta := g^{-1/2} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial}{\partial \theta_j} \left(g^{1/2} g^{jk} \frac{\partial}{\partial \theta_k} \right).$$

The eigenfunctions of the Laplace-Beltrami operator are the spherical harmonics Y_ℓ; more precisely,

$$-\Delta Y_\ell = \lambda_\ell Y_\ell, \qquad \lambda_\ell = \ell(\ell + n - 1).$$

Every function $f \in L^2(S^n)$ can be expanded in terms of spherical harmonics

$$f = \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(n,\ell)} \widehat{f}_{\ell k} Y_{\ell k}, \qquad \widehat{f}_{\ell k} = \langle f, Y_{\ell k} \rangle = \int_{S^n} f \overline{Y_{\ell k}} dS,$$

where dS is the surface measure of the unit sphere. The $L^2(S^n)$ norm of f, given by the familiar formula

$$\|f\|_2 = \left(\int_{S^n} |f|^2 dS \right)^{1/2},$$

and can also be expressed, via Parseval's identity, as follows:

$$\|f\|_2 = \left(\sum_{\ell=0}^{\infty} \sum_{k=1}^{N(n,\ell)} |\widehat{f}_{\ell k}|^2 \right)^{1/2}.$$

Given any nonnegative integer ℓ, we define

$$Pf_\ell := \sum_{k=1}^{N(n,\ell)} \widehat{f}_{\ell k} Y_{\ell k};$$

since the set $\{Y_{\ell k} : k = 1, \ldots, N(n,\ell)\}$ is orthonormal, we find that

$$\|Pf_\ell\|_2 = \left(\sum_{k=1}^{N(n,\ell)} |\widehat{f}_{\ell k}|^2 \right)^{1/2} .$$

The Sobolev space $H^s := H^s(S^n)$ on the unit sphere is defined as

$$H^s := \left\{ f : \|f\|_{H^s}^2 = \sum_{\ell=0}^{\infty} (1+\lambda_\ell)^s \|Pf_\ell\|_2^2 < \infty \right\}.$$

For more details on Sobolev space with real parameters s we refer to [20, §1.7].

B. Positive definite bizonal functions

Second, we introduce a class of *bizonal functions* on spheres used in approximation methods. Bizonal functions on S^n are bivariate functions $\Phi(x,y)$ that can be represented as $\phi(x \cdot y)$ for all $x, y \in S^n$ where $\phi(t)$ is a continuous function over $[-1,1]$. We shall be concerned exclusively with bizonal kernels of the type

$$\Phi(x,y) = \phi(x \cdot y) = \sum_{\ell=0}^{\infty} a_\ell P_\ell(n+1; x \cdot y), \quad a_\ell \geq 0, \quad \sum_{\ell=0}^{\infty} a_\ell < \infty, \tag{1.1}$$

where $\{P_\ell(n+1;t)\}_{\ell=0}^{\infty}$ is the sequence of $(n+1)$-dimensional Legendre polynomials. Recall from [25] that

$$\int_{-1}^{1} P_\ell(n+1;t) P_k(n+1;t)(1-t^2)^{(n-2)/2} dt = 0 \text{ for } \ell \neq k,$$

and

$$\int_{-1}^{1} [P_\ell(n+1;t)]^2 (1-t^2)^{(n-2)/2} dt = \frac{|S^n|}{|S^{n-1}| N(n,\ell)},$$

where $|S^n|$ is the surface area of S^n and $|S^{n-1}|$ is the surface area of S^{n-1}.

Thanks to the seminal work of Schoenberg [39], we know that such a Φ is positive definite on S^n, that is, the matrix $A := [\Phi(x_i, x_j)]_{i,j=1}^{m}$ is positive semidefinite for every set of distinct points $\{x_1, \ldots, x_m\}$ on S^n and every positive integer m. When the coefficients a_ℓ are positive for every ℓ, we say that Φ is strictly positive definite. In this case the matrix A becomes positive definite, hence invertible, for every set of distinct points $\{x_1, \ldots, x_m\}$ on S^n and every m (see [52]).

Using the addition theorem for spherical harmonics (see, for example, [25, page 18]), we can write

$$\Phi(x, y) = \phi(x \cdot y) = \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(n,\ell)} \widehat{\phi}(\ell) Y_{\ell k}(x) \overline{Y_{\ell k}(y)}, \text{ where } \widehat{\phi}(\ell) = \frac{|S^n|}{N(n, \ell)} a_\ell. \tag{1.2}$$

Spherical basis functions (SBFs) are constructed from the above class of strictly positive definite bizonal kernels. The smoothness of the kernels depends on how fast the coefficients $\widehat{\phi}(\ell)$ decay.

The *native space* induced by Φ is defined to be the closure of the set

$$N_\Phi := \left\{ f \in \mathcal{D}'(S^n) : \|f\|_\Phi^2 = \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(n,\ell)} |\widehat{f}_{\ell k}|^2 / \widehat{\phi}(\ell) < \infty \right\},$$

where $\mathcal{D}'(S^n)$ denotes the set of all tempered distributions defined on S^n. Note that Φ is the reproducing kernel in N_Φ in the sense that for every $f \in N_\Phi$ and for any fixed $x \in S^n$,

$$\langle \Phi(\cdot, x), f \rangle_\Phi = \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(n,\ell)} \frac{\widehat{\phi}(\ell) Y_{\ell k}(x) \widehat{f}_{\ell k}}{\widehat{\phi}(\ell)} = f(x).$$

C. Interpolation on spheres by SBFs

Let $X = \{x_1, \ldots, x_m\}$ be a set of scattered distinct points on S^n. The density of the scattered points is measured by the mesh norm

$$h_X = \sup_{y \in X} \operatorname{dist}(y, X),$$

where $\text{dist}(y, X) = \inf_{x \in X} \theta(y, x)$. Here θ is the geodesic distance on S^n which is defined as $\theta(x, y) = \cos^{-1}(x \cdot y)$, where x and y are represented as two unit vectors in $\mathbb{R}^{n+1}$. The separation radius is defined via

$$q_X = \frac{1}{2} \min_{j \neq k} \theta(x_j, x_k).$$

It is easy to see that $h_X \geq q_X$; equality can hold only for a uniform distribution of points on the circle S^1. The *mesh ratio*

$$\rho_X := h_X / q_X \geq 1$$

provides a measure of how uniformly points in X are distributed on S^n.

As pointed out in the previous section, with a class of strictly positive definite bizonal kernels, the matrix $A = [\Phi(x_i, x_j)]_{i,j=1\dots m}$ is positive definite, hence invertible, for any set X of distinct m points on S^n, where m is an arbitrary positive integer. This special property has been used in the interpolation problem on spheres.

Given a continuous function f on S^n, a positive integer m, and a set of distinct points $\{x_1, \dots, x_m\}$ on S^n, there uniquely exists a sequence of numbers $\{c_j\}_{j=1}^m$ such that the function

$$I_X f := \sum_{j=1}^{m} c_j \Phi(x, x_j) \tag{1.3}$$

satisfies the interpolating condition

$$I_X f(x_k) = f(x_k), \qquad 1 \leq k \leq m.$$

The functions $\phi_j(x) := \Phi(x, x_j)$ are called spherical basis functions (SBFs).

Let I_X be the interpolation operator $I_X : C(S^n) \to V_X$ such that $I_X f(x_j) = f(x_j)$ for all $x_j \in X$. The operator I_X is well-defined for every function $f \in C(S^n)$ since the matrix

$$[\Phi(x_i, x_j)]_{i,j=1\ldots m}$$

is positive definite, hence invertible, for every configuration of the set X.

If the sampling function f is in the native space N_Φ, we have the following error estimates, as in [17, 23]:

Theorem I.1 *Assume that $f \in N_\Phi$ and the set X has mesh norm h_X. Let L be a positive integer so that $1/(2L+2) < h_X \leq 1/(2L)$, then there exists a positive constant C so that*

$$\|f - I_X f\|_\infty \leq C \left(\sum_{\ell > L} \widehat{\phi}(\ell) N(n, \ell) \right)^{1/2} \|f\|_\Phi.$$

If the spherical harmonic coefficients $\widehat{\phi}(\ell)$ decay algebraically, i.e. $\widehat{\phi}(\ell) \sim (1 + \lambda_\ell)^{-\sigma}$ for some $\sigma > n/2$ then by the fact that $N(n, \ell) \sim \ell^{n-1}$, we have:

Corollary I.1 *Assume that $f \in N_\Phi$ and the set X has mesh norm h_X. Let ϕ be an SBF satisfying $\widehat{\phi}(\ell) \sim (1 + \lambda_\ell)^{-\sigma}$, then there exists a positive constant C so that*

$$\|f - I_X f\|_\infty \leq C h_X^{\sigma - n/2} \|f\|_\Phi.$$

In fact, f can be in a larger space than N_Φ, as pointed out in [30, Theorem 3.2]. We introduce the following norm in $C^{2k}(S^n)$:

$$\|f\|_{2k} := \max\{\|f\|_\infty, \|\Delta^k f\|_\infty\}, \qquad f \in C^{2k}(S^n).$$

Theorem I.2 *Let ϕ be an SBF satisfying $\widehat{\phi}(\ell) \sim (1 + \lambda_\ell)^{-\sigma}$ for $\sigma > 2k \geq n/2$. If $f \in C^{2k}(S^n)$ and $I_X f$ is defined as in (1.3) then there exists a positive constant C independent of f and X such that*

$$\|f - I_X f\|_\infty \leq C \rho_X^{\sigma - 2k} h_X^{2k - n/2} \|f\|_{2k}.$$

In the following we will outline an error estimate in the $\|\cdot\|_\Phi$ norm, which has proven to be useful in the error analysis of the parabolic differential equation on S^n.

Lemma I.1 *For every $f \in N_\Phi$, we have*

$$\|I_X f\|_\Phi^2 + \|f - I_X f\|_\Phi^2 = \|f\|_\Phi^2.$$

Proof. Since Φ is the reproducing kernel in the reproducing Hilbert space N_Φ, the interpolating condition $I_X f(x_j) = f(x_j)$ for all $x_j \in X$ is equivalent to

$$\langle I_X f - f, \Phi(x_j, \cdot)\rangle_\Phi = 0 \quad \forall x_j \in X.$$

Since $I_X f$ is a linear combination of $\Phi(x_j, \cdot)$'s, we have the orthogonal property

$$\langle I_X f - f, I_X f\rangle_\Phi = 0.$$

Hence the desired relation follows from the Pythagorean theorem. □

We define the convolution kernel of Φ by

$$\Phi * \Phi(x, y) := \int_{S^n} \Phi(x, z)\Phi(z, y) dS(z), \quad x, y \in S^n.$$

In terms of Fourier expansions we have

$$\Phi * \Phi(x, y) = \sum_{\ell=0}^{\infty} (\widehat{\phi}(\ell))^2 \sum_{k=1}^{N(n,\ell)} Y_{\ell k}(x)\overline{Y_{\ell k}(y)}.$$

This observation allows us to define a convolution native space by

$$N_{\Phi*\Phi} = \left\{ f \in L^2(S^n) : \|f\|_{\Phi*\Phi}^2 = \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(n,\ell)} \frac{|\widehat{f}_{\ell k}|^2}{(\widehat{\phi}(\ell))^2} < \infty \right\}.$$

If the kernel Φ satisfies the condition $\widehat{\phi}(\ell) \sim (1 + \lambda_\ell)^{-\sigma}$ then

$$N_{\Phi*\Phi} \cong H^{2\sigma}(S^n) \subset H^\sigma(S^n) \cong N_\Phi.$$

Based on [24, Theorem 4.7], we have the following theorem:

Theorem I.3 *Let* Φ *be a positive definite kernel with* $\widehat{\phi}(\ell) \sim (1+\lambda_\ell)^{-\sigma}$, *and* $f \in H^{2\sigma}(S^n)$. *Then there exists a constant* C, *independent of* h_X, *such that*

$$\|f - I_X f\|_\Phi \leq C h_X^\sigma \|f\|_{H^{2\sigma}}.$$

Proof.

$$\begin{aligned} \|f - I_X f\|_\Phi^2 &= \langle f, f - I_X f\rangle_\Phi \\ &= \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(n,\ell)} \frac{\widehat{f}_{\ell k}(\widehat{f}_{\ell k} - \widehat{I_X f}_{\ell k})}{\widehat{\phi}(\ell)} \\ &\leq \left(\sum_{\ell=0}^{\infty} \sum_{k=1}^{N(n,\ell)} \frac{|\widehat{f}_{\ell k}|^2}{\widehat{\phi}(\ell)^2} \right)^{1/2} \left(\sum_{\ell=0}^{\infty} \sum_{k=1}^{N(n,\ell)} (\widehat{f}_{\ell k} - \widehat{I_X f}_{\ell k})^2 \right)^{1/2} \\ &\leq \|f\|_{\Phi * \Phi} \|f - I_X f\|_2. \end{aligned} \tag{1.4}$$

Then by using [24, Theorem 4.4] with $p = 2$, it follows that

$$\|f - I_X f\|_2 \leq C h_X^\sigma \|f - I_X f\|_\Phi. \tag{1.5}$$

Combining inequalities (1.4) and (1.5) and noting that $N_{\Phi * \Phi} \cong H^{2\sigma}(S^n)$, we obtain

$$\|f - I_X f\|_\Phi \leq C h_X^\sigma \|f\|_{H^{2\sigma}}.$$

□

In [48] Wendland introduced a class of locally supported positive definite radial basis function defined on $\mathbb{R}^{n+1}$. These functions $\psi(x)$ are rotation invariant and thus are functions of $|x|$ only. So, the corresponding convolution kernel $\psi(x-y), x, y \in S^n$, is a function of $|x-y| = \sqrt{2 - 2x \cdot y}$. We may therefore define a function

$$\Phi(x, y) = \phi(x \cdot y) := \psi(x - y), \quad x, y \in S^n. \tag{1.6}$$

Note that $\Phi(x,y)$ inherits the property of positive definiteness from ψ, and $\widehat{\phi}(\ell) \sim (1+\lambda_\ell)^{-\sigma}$ for some $\sigma > 0$ (see Section 4 in [30]). Theorem I.1 and Theorem I.2 provide error estimates for SBFs as in (1.6) which have fixed support. In this dissertation we will investigate error estimates when these SBFs are scaled or dilated. The scaled SBFs enable the interpolation matrix $[\Phi(x_i,x_j)]_{i,j=1\ldots m}$ to be band-limited, facilitate the use of iterative algorithms in solving large linear systems.

D. Approximation of elliptic PDEs on spheres

When dealing with elliptic PDEs on spheres, we restrict our concern to a class of elliptic equations of the form:

$$-\Delta u(x) + \omega^2 u(x) = f(x), \qquad x \in S^n,$$

where Δ is the Laplace-Beltrami operator and ω is some real non-zero constant. Other classes of elliptic PDEs are deferred for future research. We investigate and analyze error estimates only for two common methods of approximation: the finite element method and the collocation method.

1. Finite element method

The mathematics of the finite element method on $\mathbb{R}^n$ can be found in [3]. Here we attempt to give a version similar to it for S^n. To begin, we set up the weak formulation for the PDE: find u such that

$$\left\langle -\Delta u + \omega^2 u, v \right\rangle = \langle f, v \rangle, \quad \forall v \in H^1(S^n),$$

where for any functions $u, v \in C(S^n)$,

$$\langle u, v \rangle := \int_{S^n} u\overline{v}\, dS.$$

The bilinear form $a(u,v) := \langle -\Delta u + \omega^2 u, v\rangle$ is bounded and coercive, so by the Lax-Milgram theorem, the weak formulation has a unique solution. Then we approximate the weak formulation by using a sequence of finite dimensional subspaces of $H^1(S^n)$. For a set of points $X = \{x_1, \dots, x_m\} \subset S^n$, a possible finite dimensional subspace of $H^1(S^n)$ can be defined as

$$V_X := \text{ span } \{\phi_i(x) : x_i \in X\},$$

where $\phi_i(x) := \Phi(x, x_i) = \phi(x \cdot x_i)$. Assume that the SBFs used in the construction of the approximate solution are required to have the Fourier coefficients algebraically decaying:

$$\widehat{\phi}(\ell) \sim (1 + \lambda_\ell)^{-\sigma}, \quad \sigma > n/2 + 2.$$

It is easy to see that $\phi_i(x) := \Phi(x, x_i)$ is in $H^1(S^n)$ since we require $\sigma > n/2 + 2$. The Ritz Galerkin approximation problem is:

$$\text{Find } u_h \in V_X \text{ such that } a(u_h, \chi) = \langle f, \chi\rangle, \quad \forall \chi \in V_X.$$

In this dissertation we will derive an error estimate in the Sobolev norm of the approximate solution. The strategy is that $\|u - u_h\|_{H^1}$ will be estimated via $\|u - I_X u\|_{H^1}$, where $I_X u$ is the SBF interpolant of u.

2. Collocation method

In a collocation method one seeks an approximate solution of a differential equation in a finite dimensional space of sufficient regular functions by requiring that the equation is satisfied exactly at a finite number of points. Such a procedure for a two-point boundary problem in one space variable was analyzed by de Boor and Swartz [6], and for parabolic equations in one space variable was studied by Douglas and

Dupont [8].

In this dissertation we discuss a collocation method for a more general class of elliptic differential operators, namely

$$\mathcal{L}u = f,$$

in which the differential operator $\mathcal{L}$ has eigenvalues asymptotic to $(1+\lambda_\ell)^{\beta/2}$. In other words, for spherical harmonics of order ℓ, where $\ell = 0, 1, \dots,$ there are numbers

$$a_\ell \sim (1+\lambda_\ell)^{\beta/2} \text{ such that } \mathcal{L}Y_\ell = a_\ell Y_\ell.$$

In the collocation method, we require that the differential equation to be exact on the set of points X. In effect, we would like to find u_X which lies in some finite dimensional space V_X such that

$$\mathcal{L}u_X|_{x=x_j} = f(x_j), \quad \forall x_j \in X.$$

We have to go back to the original framework set out by Golomb and Weinberger [13] to obtain error estimates.

E. Approximation of parabolic PDEs on spheres

Evolution equations on spherical geometry such as shallow water equations have been studied in weather forecasting services [14, 50]. Error estimates of pseudo-differential operator (which are time-independent) were studied in [12, 23] but error estimates for the evolution equations remain unexplored.

In this dissertation we only consider the following parabolic partial differential equation

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) - \Delta u(x,t) &= F(x,t), \\ u(x,0) &= f(x), \quad f \in H^s(S^n). \end{cases} \tag{1.7}$$

It is known that the PDE describes the heat diffusion process on the surface of the unit sphere with external heat source $F(x,t)$. We shall study methods of approximation in two steps: First, we shall approximate $u(x,t)$ by means of a function $u_X(x,t)$ which, for each fixed t, belongs to a finite dimensional linear space V_X spanned by the SBFs. This function will be a solution of a finite system of ordinary differential equations and is referred to as a semi-discrete solution. Second, we discretize (1.7) also in the time variable so as to produce a completely discrete scheme for the approximate solution of our problem.

1. Semi-discrete problem

We seek the approximate solution $u_X \in V_X$ such that

$$\begin{cases} \frac{\partial}{\partial t}u_X(x_j,t) - \Delta u_X(x_j,t) &= F(x_j,t), \quad \forall x_j \in X \\ u_X(x,0) &= I_X f(x). \end{cases} \tag{1.8}$$

We can express $u_X(x,t)$ as $u_X(x,t) = \sum_{i=1}^m c_i(t)\phi_i(x)$. In the homogeneous case, i.e. when $F = 0$, equation (1.8) is reduced to the following system of ordinary differential equations:

$$\frac{d}{dt}\mathbf{c}(t) = A^{-1}B\mathbf{c}(t),$$

subject to the initial condition

$$A\mathbf{c}(0) = [f(x_j)]_{j=1}^m,$$

where $\mathbf{c}(t) = [c_1(t) \ldots c_m(t)]^T$, $A = [\phi_i(x_j)]_{i,j=1\ldots m}$, and $B = [\Delta\phi_i(x)|_{x=x_j}]_{i,j=1\ldots m}$. The solution for the homogeneous semi-discrete problem is

$$u_X(x,t) = [\phi_1(x) \ldots \phi_m(x)] \exp(A^{-1}Bt)\mathbf{c}(0), \text{ where } \mathbf{c}(0) = A^{-1}f|_X.$$

In this dissertation we will investigate error estimates between u_X and the exact solution u.

2. Backward Euler method

Let us discretize the time derivative using backward Euler method with time-step τ as

$$\frac{u(x,t) - u(x,t-\tau)}{\tau} + o(1) - \Delta u(x,t) = F(x,t).$$

By neglecting the term $o(1)$, we seek the approximate solution $u_X \in V_X$ which satisfies the following collocation equation

$$u_X(x_j,t) - u_X(x_j,t-\tau) - \tau\Delta u_X(x_j,t) = \tau F(x_j,t), \quad \forall x_j \in X, \tag{1.9}$$

subject to the initial condition

$$u_X(x,0) = I_X f(x).$$

Let us define $t_N := N\tau$ and $U_N := u_X(x,t_N)$. The collocation equation (1.9) can be rewritten as

$$U_N - \tau\Delta U_N = U_{N-1} + \tau F(x_j,t_N), \quad \forall x_j \in X. \tag{1.10}$$

With $U_0 = I_X f$, equation (1.10) defines an iterative algorithm to obtain an approximate solution as any given time t_N. In this dissertation we will derive error estimates $u - u_X$ in the Sobolev norm. Some numerical experiments will be presented.

3. Crank-Nicolson method

In Crank-Nicolson method, the semi-discrete equation is discretized in a symmetric fashion around the point $t_{N-1/2} := (N - 1/2)\tau$, which will produce second accuracy in time. More precisely, U_N can be defined recursively by

$$\frac{U_N - U_{N-1}}{\tau} - \Delta(U_N(x_j) + U_{N-1}(x_j))/2 = F(x_j, t_{N-1/2}), \quad \forall x_j \in X.$$

Here, we also set $U_0 = I_X f$. In this dissertation we aim to obtain error estimates and to implement the algorithm.

CHAPTER II

INTERPOLATION ON SPHERES BY DILATED SBFs

The theory of interpolation on spheres using SBFs has been outlined in Section A of the introduction chapter. In effect, we have to invert the matrix $A = [\Phi(x_i, x_j)]_{i,j=1,\ldots,m}$ to solve the linear system $A\mathbf{c} = [f(x_j)]_{j=1}^m$. By choosing a strictly positive definite kernel Φ, the matrix A is positive definite, so we can employ iterative algorithms to invert A. It is more efficient for the algorithms if the matrix A is band-limited, which can be achieved by rescaling the support of the spherical basis functions. In this chapter we shall prove a new result on interpolation using locally supported SBFs with support scaled by a factor of α, for $\alpha > 0$.

A. Approximation theorems

We first state several results concerning approximation of functions on S^n by spherical harmonics in $\mathcal{P}_L$, where $\mathcal{P}_L$ denotes the space of all spherical harmonics of degree at most L. These results, obtained by Pawelke [34, 35], involve the notions of spherical mean and spherical modulus of continuity (see below). We shall use Pawelke's results later in the chapter.

Let $f \in C(S^n)$, $x \in S^n$, and $0 < h \leq \pi$. The spherical mean of f over the spherical cap of radius h centered at x is defined as follows:

$$T_h f(x) := \frac{1}{|S^{n-1}|(\sin\ h)^{n-1}} \int_{x\cdot y=\cos h} f(y) d\sigma_x(y),$$

where $d\sigma_x$ is the volume element corresponding to $x \cdot y = \cos(h)$ and $|S^{n-1}|$ is the surface area of S^{n-1}. The spherical modulus of continuity of f is defined to be

$$\omega(f;\epsilon) := \sup_{0<h\leq\epsilon} \|T_h f - f\|_\infty, \quad \epsilon > 0.$$

Given $f \in C(S^n)$, we define the distance from f to the polynomial space $\mathcal{P}_L$ in the usual manner:

$$\operatorname{dist}(f, \mathcal{P}_L) := \inf_{P \in \mathcal{P}_L} \|f - P\|_\infty.$$

Theorem II.1 *([34, Satz 5.1], [35]) Suppose $f \in C^{2k}(S^n)$ and $L \in \mathbb{Z}^+$. Then there is a positive constant M, independent of both f and L, for which*

$$dist(f, \mathcal{P}_L) \leq M\omega(f; 1/L),$$

and for which

$$dist(f, \mathcal{P}_L) \leq M^k L^{-2k} \|\Delta^k f\|_\infty, \ k \in \mathbb{Z}^+.$$

The remaining approximation theorems that we will use in the proofs are dealing with the norm of iterates of Δ applied to the best and near-best approximants from $\mathcal{P}_L$.

Proposition II.1 *[35, Satz 4.4] Suppose $f \in C^{2k}(S^n)$ and let P_L^* be the best approximation to f from $\mathcal{P}_L$, i.e., $\|f - P_L^*\|_\infty = dist(f, \mathcal{P}_L)$. Then there exists a positive constant C, independent of f and L, for which*

$$\|\Delta^k P_L^*\|_\infty \leq C\|\Delta^k f\|_\infty.$$

The preceding theorem has been extended in [30] to a class of near-best approximants from $\mathcal{P}_L$.

Corollary II.1 *[30, Corollary 2.5] Let $f \in C^{2k}(S^n)$ and let $P_L \in \mathcal{P}_L$, $L = 1, 2, \ldots$, be a sequence of polynomials satisfying $\|f - P_L\|_\infty \leq K\, dist(f, \mathcal{P}_L)$, with K independent of f and L. Then there is a constant C_1, independent of f and L, such that*

$$\|\Delta^k P_L\|_\infty \leq C_1 \|\Delta^k f\|_\infty.$$

In the proof, we need to construct, for every $f \in C(S^n)$, spherical harmonics that are both near-best approximants to f from $\mathcal{P}_L$ and also interpolate f on the point set X. This is precisely the content of the following theorem:

Theorem II.2 *[30, Theorem 3.1] Let $X \subset S^n$ be a finite set of distinct points and let $\beta > 1$. If we set $L = \lceil \frac{M(\beta+1)}{q_X(\beta-1)} \rceil$, where M as in Theorem II.1, then for any $f \in C(S^n)$ there exists a spherical harmonic $P_L \in \mathcal{P}_L$ which interpolates f on X and satisfies*

$$\|f - P_L\|_\infty \le (1+\beta) dist(f, \mathcal{P}_L).$$

B. Locally supported basis functions on $\mathbb{R}^{n+1}$ and S^n

We then review the locally supported spherical basis functions constructed via a class of compactly supported radial basis functions proposed in [48, 49]. The dilated strictly positive definite radial basis functions and the corresponding dilated SBF shall be analyzed. The Fourier transform in $\mathbb{R}^{n+1}$ and Bessel functions play a crucial role in the analysis.

1. Compactly supported strictly positive definite functions on $\mathbb{R}^{n+1}$

We investigate a class of radial basis functions $\Psi(x) = \psi(\|x\|)$, $x \in \mathbb{R}^{n+1}$, where $\psi(t)$ is of the following form:

$$\psi(t) = \begin{cases} p(t) & 0 \le t \le 1, \\ 0 & t > 1, \end{cases}$$

with a univariate polynomial $p(t) = \sum_{j=1}^{N} c_j t^j$, $c_N \neq 0$. The Fourier transform of $\Psi(x)$ in $\mathbb{R}^{n+1}$ is

$$\begin{aligned}
\widehat{\Psi}(x) \equiv \widehat{\psi}(r) &= (2\pi)^{-(n+1)/2} \int_{\mathbb{R}^{n+1}} \Psi(\omega) e^{x\cdot\omega} d\omega \\
&= r^{-(n-1)/2} \int_0^\infty \psi(t) t^{(n+1)/2} J_{(n-1)/2}(rt) dt \\
&= r^{-(n+1)} \int_0^r p(t/r) t^{(n+1)/2} J_{(n-1)/2}(t) dt.
\end{aligned}$$

Here J_ν denotes the Bessel function of the first kind. Bochner's theorem establishes the fact that Ψ, which is compactly supported, is strictly positive definite if and only if $\widehat{\Psi}$ is nonnegative and positive at least on an open set.

The α-dilation (for $\alpha > 0$) of ψ is defined as

$$\psi_\alpha(t) = \begin{cases} p(\alpha t) & 0 \le t \le 1/\alpha, \\ 0 & t > 1/\alpha. \end{cases}$$

The Fourier transform of the α-dilation of ψ is

$$\begin{aligned}
\widehat{\psi_\alpha}(x) &= r^{-(n-1)/2} \int_0^{1/\alpha} p(\alpha t) t^{(n+1)/2} J_{(n-1)/2}(rt) dt \\
&= r^{-(n+1)} \int_0^{r/\alpha} p(\alpha t/r) t^{(n+1)/2} J_{(n-1)/2}(t) dt \\
&= \alpha^{-(n+1)} \rho^{-(n+1)} \int_0^{\rho} p(t/\rho) t^{(n+1)/2} J_{(n-1)/2}(t) dt, \quad \rho = r/\alpha, \\
&= \alpha^{-(n+1)} \widehat{\psi}(\rho) = \alpha^{-(n+1)} \widehat{\psi}(r/\alpha).
\end{aligned}$$

Suppose that there are positive constants c and C such that

$$c(1+r^2)^{-s} \le \widehat{\psi}(r) \le C(1+r^2)^{-s}, \quad s > (n+1)/2,$$

then there are positive constants c_1 and C_1 such that

$$c_1 \alpha^{n+1} \left(1 + (r/\alpha)^2\right)^{-s} \le \widehat{\psi_\alpha}(r) \le C_1 \alpha^{n+1} \left(1 + (r/\alpha)^2\right)^{-s}.$$

2. Locally supported strictly positive definite functions on S^n

If the function $\Psi(x)$ is restricted on the unit sphere $S^n \subset \mathbb{R}^{n+1}$, then $\Psi(x-y)$ on S^n is a function of $|x-y| = \sqrt{2-2x\cdot y}$. Consequently, the restriction $\Psi(x-y)|_{x,y\in S^n}$ is a function of $x \cdot y$. We define the function

$$\Phi(x,y) := \Psi(x-y), \quad x, y \in S^n.$$

In the spherical harmonics expansion

$$\Phi(x,y) = \sum_{\ell=0}^{\infty}\sum_{k=1}^{N(n,\ell)} \widehat{\phi}(\ell) Y_{\ell k}(x)\overline{Y_{\ell k}(y)},$$

the coefficients $\widehat{\phi}(\ell)$, as in [30, Theorem 4.1], are given as

$$\widehat{\phi}(\ell) = \int_0^\infty r\widehat{\psi}(r) J^2_{\ell+(n-1)/2}(r)dr.$$

Now we will follow a framework set out in [30] in order to investigate the behavior of $\widehat{\phi_\alpha}(\ell)$. For $\widehat{\psi_\alpha} \sim \alpha^{n+1}(1+(r/\alpha)^2)^{-s}$, we need to consider the following integral

$$\widehat{\chi}(\ell) = \alpha^{n+1}\int_0^\infty \frac{rJ_\nu^2(r)}{(1+(r/\alpha)^2)^s}dr \text{ where } \nu := \ell + \frac{n-1}{2} \text{ and } s > \frac{n+1}{2}. \tag{2.1}$$

It is noted that $\widehat{\chi} \sim \widehat{\phi_\alpha}$. We need the hyper-geometric functions

$${}_pF_q(a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q; z) := \sum_{j=0}^{\infty} \frac{(a_1)_j(a_2)_j\ldots(a_p)_j z^j}{(b_1)_j(b_2)_j\ldots(b_q)_j j!},$$

where Pochammer's symbol $(\lambda)_j := \lambda(\lambda+1)\cdots(\lambda+j-1)$ when $j \geq 1$ and $(\lambda)_0 := 1$.

Lemma II.1

$$\widehat{\chi}(\ell) = A\ {}_1F_2(\nu+\frac{1}{2}; \nu+2-s, 2\nu+1; \alpha^2) + B\ {}_1F_2(s-\frac{1}{2}; s+\nu, s-\nu; \alpha^2),$$

where

$$A = \frac{\alpha^{n+3-2\nu-4s}\Gamma(s-1-\nu)}{2^{1+2\nu}\Gamma(\nu+1)\Gamma(s)}, \qquad B = \frac{\alpha^{n+1-2s}\Gamma(\nu+1-s)\Gamma(s-\frac{1}{2})}{2\sqrt{\pi}\Gamma(s)\Gamma(\nu+s)}.$$

Proof. As in [46, Eq. (1), §5.43], we express $J_\nu^2(r)$ as

$$J_\nu^2(r) = \frac{2}{\pi}\int_0^{\pi/2} J_{2\nu}(2r\cos\theta)d\theta,$$

then insert in (2.1) and use Fubini's theorem to interchange integrals to get

$$\frac{2\alpha^{n+1-2s}}{\pi}\int_0^{\pi/2}\int_0^{\infty}\frac{rJ_{2\nu}(2r\cos\theta)}{(\alpha^2+r^2)^s}drd\theta.$$

By [46, §13.6, Eq. (1)] with our parameters, it has the form

$$\begin{aligned}\int_0^{\infty}\frac{rJ_{2\nu}(2r\cos\theta)}{(\alpha^2+r^2)^s}dr &= \frac{\Gamma(\nu+1)\Gamma(s-1-\nu)\alpha^{2\nu-2s+2}}{2\Gamma(s)\Gamma(2\nu+1)}\cos^{2\nu}(\theta)\\ &\quad \times {}_1F_2(\nu+1;\nu+2-s,2\nu+1;\alpha^2\cos^2(\theta))\\ &\quad + \frac{\Gamma(\nu+1-s)}{2\Gamma(s+\nu)}\cos^{2s-2}(\theta){}_1F_2(s;s+\nu,s-\nu;\alpha^2\cos^2(\theta)).\end{aligned}$$

We now consider the integral of the following form

$$\int_0^{\pi/2}\cos^{\mu}(\theta){}_1F_2(a_1;b_1,b_2;\alpha^2\cos^2(\theta))d\theta.$$

Let $u=\cos^2(\theta)$ then such integrals transform to

$$\frac{1}{2}\int_0^1 u^{(\mu-1)/2}(1-u)^{-1/2}{}_1F_2(a_1;b_1,b_2;\alpha^2 u)du.$$

Using formula [15, §7.512, #12], we have

$$\begin{aligned}\int_0^1 u^{(\mu-1)/2}(1-u)^{-1/2} \quad & {}_1F_2(a_1;b_1,b_2;\alpha^2 u)du = \\ & \frac{\sqrt{\pi}\Gamma(\frac{\mu+1}{2})}{\Gamma(1+\frac{\mu}{2})}{}_2F_3\left(\frac{\mu+1}{2},a_1;1+\frac{\mu}{2},b_1,b_2;\alpha^2\right).\end{aligned}$$

Using this result, we have that

$$\begin{aligned}\widehat{\chi}(\ell) &= A\ {}_2F_3(\nu+\frac{1}{2},\nu+1;\nu+1,\nu+2-s,2\nu+1;\alpha^2)\\ &\quad +B\ {}_2F_3(s-\frac{1}{2},s;s,s+\nu,s-\nu;\alpha^2),\end{aligned}$$

where A and B are the accumulated factors that are given by

$$A=\frac{\alpha^{n+3-2\nu-4s}\Gamma(s-1-\nu)\Gamma(\nu+\frac{1}{2})}{2\sqrt{\pi}\Gamma(s)\Gamma(2\nu+1)}=\frac{\alpha^{n+3-2\nu-4s}\Gamma(s-1-\nu)}{2^{1+2\nu}\Gamma(\nu+1)\Gamma(s)},$$

and

$$B=\frac{\alpha^{n+1-2s}\Gamma(\nu+1-s)\Gamma(s-\frac{1}{2})}{2\sqrt{\pi}\Gamma(s)\Gamma(\nu+s)}.$$

Using the cancellation property for the hyper-geometric functions, we arrive at

$$\widehat{\chi}(\ell)=A\ {}_1F_2(\nu+\frac{1}{2};\nu+2-s,2\nu+1;\alpha^2)+B\ {}_1F_2(s-\frac{1}{2};s+\nu,s-\nu;\alpha^2).$$

□

For a class of compactly supported positive definite radial functions introduced by Wendland in [48, 49], we have $s=\frac{n+1}{2}+k+\frac{1}{2}$, where k is a positive integer, and so

$$\nu-s=\ell-k-\frac{3}{2}\text{ and }\nu+s=\ell+n+k+\frac{1}{2}.$$

We now investigate the behavior of $\widehat{\chi}(\ell)$ as $\ell\to\infty$.

Lemma II.2 *For fixed values of n and k, the asymptotic behavior of $\widehat{\chi}(\ell)$ is*

$$\widehat{\chi}(\ell)=\mathcal{O}\left(\ell^{-n-2k-1}\exp\left(\frac{\alpha^2}{\ell}\right)\right).$$

Proof.

$$\begin{aligned}{}_1F_2(s-\frac{1}{2};s+\nu,s-\nu;\alpha^2) &= {}_1F_2(\frac{n+1}{2}+k;\ell+k+n+\frac{1}{2},k-\ell+\frac{3}{2};\alpha^2)\\ &\le C\sum_{j=0}^{\infty}\frac{\alpha^{2j}}{\ell^j j!}\le C\exp\left(\frac{\alpha^2}{\ell}\right)\end{aligned}$$

and

$$
\begin{aligned}
{}_1F_2(\nu+\frac{1}{2};\nu+2-s,2\nu+1;\alpha^2) &= {}_1F_2(\ell+\frac{n}{2};\ell-k+\frac{1}{2},2\ell+n;\alpha^2) \\
&\leq C\sum_{j=0}^{\infty}\frac{\alpha^{2j}}{(2\ell)^j j!} \leq C\exp\left(\frac{\alpha^2}{2\ell}\right).
\end{aligned}
$$

The coefficients A and B become

$$
\begin{aligned}
A &= \frac{2^{-n-2\ell}\alpha^{-2\ell-2n}}{\Gamma(\frac{n}{2}+k+1)}\frac{\Gamma(k-\ell+\frac{1}{2})}{\Gamma(\ell+\frac{n+1}{2})} \\
&= \frac{2^{-n-2\ell}\alpha^{-2\ell-2n}}{\Gamma(\frac{n}{2}+k+1)}\frac{\pi\csc(\pi(\ell-k+\frac{1}{2}))}{\Gamma(\ell-k+\frac{1}{2})\Gamma(\ell+\frac{n+1}{2})} = \mathcal{O}(2^{-n-2\ell}\alpha^{-2\ell-2n}(\ell!)^{-2})
\end{aligned}
$$

In above equation, we use the relation $\Gamma(1-z)\Gamma(z)=\pi\csc(\pi z)$. For B,

$$
B = \frac{\alpha^{-2k-1}\Gamma(\frac{n+1}{2}+k)\Gamma(\ell-k-\frac{1}{2})}{2\sqrt{\pi}\Gamma(\frac{n+1}{2}+k+\frac{1}{2})\Gamma(\ell+n+k+\frac{1}{2})} = \mathcal{O}(\ell^{-n-2k-1}).
$$

□

C. Interpolation on spheres using dilated locally supported SBFs

As pointed out in the previous section, the dilated positive definite function Ψ_α on $\mathbb{R}^{n+1}$ induces the corresponding dilated Φ_α on the unit sphere S^n. The asymptotic behavior of the spherical harmonics coefficients $\widehat{\phi_\alpha}$ as $\ell\to\infty$ was given in Lemma II.2. To avoid singularity at $\ell=0$, we assume that

$$
\widehat{\phi_\alpha}(\ell) \sim (1+\lambda_\ell)^{-\sigma}\exp\left(\frac{\alpha^2}{1+\ell}\right). \tag{2.2}
$$

For a function $f\in C(S^n)$, the interpolant $I_X^{(\alpha)}f$ of f is defined as

$$
I_X^{(\alpha)}f(x) = \sum_{j=1}^{m} c_j\Phi_\alpha(x_j,x) \text{ such that } I_X^{(\alpha)}f(x_k)=f(x_k) \text{ for all } x_k\in X.
$$

The reproducing kernel Hilbert space induced by the kernel $\Phi_\alpha(x,y)$ is defined as

$$N_{\Phi_\alpha} = \left\{ f \in L^2(S^n) : \|f\|^2_{\Phi_\alpha} = \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(n,\ell)} \frac{|\widehat{f}_{\ell k}|^2}{\widehat{\phi_\alpha}(\ell)} < \infty \right\}. \tag{2.3}$$

Theorem II.3 *Assume that* $f \in N_{\Phi_\alpha}$ *and there is a positive integer* L *that satisfies* $1/(2L+2) < h_X \leq (1/2L)$. *Then there is a positive constant* C *such that the following holds:*

$$\|f - I_X^{(\alpha)} f\|_\infty \leq C h_X^{\sigma - n/2} \exp(\alpha^2 h_X) \|f\|_{\Phi_\alpha}.$$

Proof. By [17, Corollary 2] with an improvement pointed out in [23], we have

$$\|f - I_X^{(\alpha)} f\|_\infty \leq C \left(\sum_{\ell > L} \widehat{\phi_\alpha}(\ell) N(n,\ell) \right)^{1/2} \|f\|_{\Phi_\alpha}.$$

Since $N(n,\ell) \sim \ell^{n-1}$ and $\widehat{\phi_\alpha}(\ell) \sim (1+\ell)^{-2\sigma} e^{\alpha^2/(\ell+1)}$ we have

$$\sum_{\ell > L} \widehat{\phi_\alpha}(\ell) N(n,\ell) \leq \int_L^\infty (1+x)^{-2\sigma+n-1} e^{\alpha^2/(\ell+1)} dx \leq C L^{-2\sigma+n} e^{\alpha^2/(L+1)}.$$

Since h_X is of order $1/2L$ we obtain the result. □

Now by adopting the same strategy as in [30], we first construct a spherical harmonic P_L that satisfies:

(A) P_L interpolates f on X, where $L = \lceil 2M/q_X \rceil$ with M independent of X, as in Theorem II.1.

(B) $\|f - P_L\|_\infty \leq 4 \operatorname{dist}(f, \mathcal{P}_L)$.

The existence of P_L is guaranteed by Theorem II.2.

Theorem II.4 *Assume that* $f \in C^{2k}(S^n)$ *and the interpolant* $I_X^{(\alpha)} f$ *is constructed from the shifts of a positive definite kernel that satisfies condition (2.2). Then there* *are positive constants C_1 and C_2 so that the following holds:*

$$\|f - I_X^{(\alpha)} f\|_\infty \le \left(C_1 \rho_X^{n/2-2k} + C_2 \rho_X^{\sigma-2k} \exp\left(\alpha^2 (h_X - q_X/M)\right)\right) h_X^{2k-n/2} \|f\|_{2k}.$$

Proof. We start with the estimate

$$\|f - I_X^{(\alpha)} f\|_\infty \le \|f - P_L\|_\infty + \|P_L - I_X^{(\alpha)} P_L\|_\infty + \|I_X^{(\alpha)} P_L - I_X^{(\alpha)} f\|_\infty. \tag{2.4}$$

Since $I_X^{(\alpha)} P_L(x_j) = I_X^{(\alpha)} f(x_j) = f(x_j)$ for all $j = 1 \ldots m$ and both functions are in the finite dimensional space V_X of dimension m, $I_X^{(\alpha)} P_L \equiv I_X^{(\alpha)} f$. So we have to estimate only the first two terms on the right hand side of (2.4). The spherical harmonic polynomial P_L is in any reproducing kernel space N_{Φ_α}. So

$$\|P_L - I_X^{(\alpha)} P_L\|_\infty \le C h_X^{\sigma - n/2} \exp(\alpha^2 h_X) \|P_L\|_{\Phi_\alpha}. \tag{2.5}$$

From the definition (2.3),

$$\|P_L\|_{\Phi_\alpha} \le \exp\left(-\frac{\alpha^2}{2(L+1)}\right) \|P_L\|_{H^\sigma} \le \exp\left(-\frac{\alpha^2}{2(L+1)}\right) (1+\lambda_L)^{\sigma/2-k} \|P_L\|_{2k}. \tag{2.6}$$

From (2.5) and (2.6) we obtain

$$\begin{aligned} \|P_L - I_X^{(\alpha)} P_L\|_\infty &\le C \exp\left(\alpha^2 h_X - \frac{\alpha^2}{2L+2}\right) h_X^{\sigma-n/2} (1+\lambda_L)^{\sigma/2-k} \|P_L\|_{2k} \\ &\le C \exp\left(\alpha^2 (h_X - q_X/M)\right) h_X^{\sigma-n/2} (1+\lambda_L)^{\sigma/2-k} \|P_L\|_{2k}. \end{aligned}$$

From condition B) we can see that $\|P_L\|_\infty \le 5\|f\|_\infty$, and from Corollary II.1, we also have $\|\Delta^k P_L\|_\infty \le C_1 \|\Delta^k f\|_\infty$, so that $\|P_L\|_{2k} \le \max\{5, C_1\} \|f\|_{2k}$ and, consequently,

$$\|P_L - I_X^{(\alpha)} P_L\|_\infty \le C \exp\left(\alpha^2 (h_X - q_X/M)\right) h_X^{\sigma-n/2} (1+\lambda_L)^{\sigma/2-k} \|f\|_{2k}. \tag{2.7}$$

From (2.4) and (2.7) and $\lambda_L = L(L+n-1) \sim L^2$, we arrive at this bound on the interpolation error for f,

$$\begin{aligned}
\|f - I_X^{(\alpha)} f\|_\infty &\leq 4\mathrm{dist}(f, \mathcal{P}_L) + \|P_L - I_X^{(\alpha)} P_L\|_\infty \\
&\leq 4M^k L^{-2k} \|\Delta^k f\|_\infty + C \exp\left(\alpha^2 (h_X - q_X/M)\right) h_X^{\sigma - n/2} L^{\sigma - 2k} \|f\|_{2k} \\
&\leq \left(C_0 L^{n/2-2k} + C \exp\left(\alpha^2 (h_X - q_X/M)\right) h_X^{\sigma-n/2} L^{\sigma-2k}\right) \|f\|_{2k} \\
&\leq \left(C_0 (h_X L)^{n/2-2k} + C \exp\left(\alpha^2 (h_X - q_X/M)\right) (h_X L)^{\sigma-2k}\right) h_X^{2k-n/2} \|f\|_{2k}.
\end{aligned}$$

Since we use $L = \lceil 2M/q_X \rceil = \lceil 2M\rho_X/h_X \rceil$ from A), then we get

$$\|f - I_X^{(\alpha)} f\|_\infty \leq \left(C_1 \rho_X^{n/2-2k} + C_2 \rho_X^{\sigma-2k} \exp\left(\alpha^2 (h_X - q_X/M)\right)\right) h_X^{2k-n/2} \|f\|_{2k}. \quad (2.8)$$

□

We arrive at some simple corollaries:

Corollary II.2 *If the scaling constant α satisfies $\alpha^2(h_X - q_X/M) < \gamma$, where γ is a positive constant then there exists a positive constant C so that*

$$\|f - I_X^{(\alpha)} f\|_\infty \leq C \rho_X^{\sigma-2k} h_X^{2k-n/2} \|f\|_{2k}.$$

Proof. From inequality (2.8) with the new condition that $\alpha^2(h_X - q_X/M) < \gamma$, we obtain

$$\|f - I_X^{(\alpha)} f\|_\infty \leq \left(C_1 \rho_X^{n/2-2k} + C_3 \rho_X^{\sigma-2k}\right) h_X^{2k-n/2} \|f\|_{2k},$$

where $C_3 := C_2 e^\gamma$. Since $\rho_X \geq 1$ and $\sigma > n/2$, it follows that

$$\|f - I_X^{(\alpha)} f\|_\infty \leq C \rho_X^{\sigma-2k} h_X^{2k-n/2} \|f\|_{2k}.$$

□

The following corollary has a more practical meaning.

Corollary II.3 *If the set of points X is quasi-uniform, i.e. $\rho_X \leq C$ then for every*

$f \in C^{2k}(S^n)$,

$$\|f - I_X^{(\alpha)} f\|_\infty \leq C[1 + \exp(\alpha^2 h_X)] h_X^{2k-n/2} \|f\|_{2k}.$$

D. Open problems

As we increase α, the approximation rate will get worse, but the interpolation matrix is more sparse, and hence the inversion of the matrix is more stable numerically. What is the optimal α to balance the two effects ?

CHAPTER III

APPROXIMATION OF ELLIPTIC PDEs ON SPHERES

In [23] a collocation method based on spherical basis functions is used to approximate the solutions of a class of pseudo-differential equations $\mathcal{L}u = f$ on S^n. The collocation method requires the approximate solution to satisfy the differential equations at a certain given set of points on the unit sphere. In this chapter we use the Galerkin method, with the approximate solution being spanned by spherical basis functions. The operator $\mathcal{L}$ is restricted to a class of pseudo-differential operators of the form $-\Delta + \omega^2$, where Δ is the Laplace-Beltrami operator on the unit sphere and $\omega \neq 0$. We aim to make use of the recent results in [30] to derive error estimates for the Galerkin approximation on S^n of the following elliptic partial differential equation

$$-\Delta u(x) + \omega^2 u(x) = f(x), \qquad x \in S^n,$$

where $f \in C^{2k}(S^n)$ for some $k \geq 1$.

The finite dimensional subspace used to approximate the solution of the PDE will be the space spanned by shifts of a spherical basis function. Such spaces are used extensively in the interpolation problem on spheres in [9, 29, 30]. Assume that the exact solution u is in $C^{2k}(S^n)$, the main result of this chapter is the following Sobolev type error estimate

$$\|u - u_h\|_{H^1} \leq C \cdot h_X^{2k-n/2-1} \max\{\|u\|_\infty, \|\Delta^k u\|_\infty\},$$

where u_h is the finite element approximation of u, h_X is the mesh norm of the set of scattered points X used to define the space of SBFs. The SBFs used in the construction of the approximation space are required to have the Fourier coefficients algebraically decaying:

$$\widehat{\phi}(\ell) \sim (1+\lambda_\ell)^{-\sigma}, \quad \sigma > n/2+2.$$

For a more general class of elliptic PDE $\mathcal{L}u = f$, a classical theorem in [13] is revisited in the context of deriving error estimates for the collocation method. The pseudo-differential operator $\mathcal{L}$ is assumed to have eigenvalues asymptotic to $(1+\lambda_\ell)^{\beta/2}$.

A. Positive definite kernels and the power functions

A conjugate symmetric, complex-valued kernel $\Phi \in C(S^n \times S^n) \cap H^{2s}(S^n \times S^n)$ is said to be *positive definite* if for every finite subset $X = \{x_1, \ldots, x_m\} \subset S^n$ of m distinct points, the matrix A with entries $A_{i,j} = \Phi(x_i, x_j)$ is positive semidefinite. In terms of distribution, the positive definiteness of Φ is equivalent to the following [9, Theorem 2.1]: for every nonzero distribution w in the dual Sobolev space $H^{-s}(S^n)$,

$$(\overline{w} \otimes w, \Phi) := \int_{S^n} w(x) \left(\int_{S^n} w(y)\Phi(x,y)dS(y) \right) dS(x) \geq 0.$$

If $(\overline{w} \otimes w, \Phi) > 0$ for every $w \neq 0$, we will call Φ *strictly positive definite*. The kernel Φ is positive definite (or strictly positive definite) if and only if all of coefficients a_ℓ in the Legendre polynomial expansion (1.1) non-negative (or positive) [29]. We define

$$\Phi * w(x) := (\delta_x \otimes w, \Phi), \quad x \in S^n,$$

where δ_x is the Dirac point evaluation functional. Let $\mathcal{P}$ be a finite dimensional subspace of functions in $C^k(S^n)$, and let $\mathcal{P}^\perp$ be a space of all distributions over $C^k(S^n)$ such that $(\overline{w}, p) = 0$ for all $p \in \mathcal{P}$. Given a strictly positive definite kernel Φ, we can define an inner product on $\mathcal{P}^\perp$:

$$[v, w]_\Phi := (\overline{v} \otimes w, \Phi), \quad v, w \in \mathcal{P}^\perp.$$

The interpolation problem can be put into distributional framework in the following way. Let $W = \{w_1, \ldots, w_m\}$ be a linearly independent set of distributions defined on $C^k(S^n)$, and let f be a function in $C^k(S^n)$. Given the data $d_j = (\overline{w}_j, f), j = 1, \ldots, m$, we seek to find $w \in \mathrm{span}\{W\} \cap \mathcal{P}^\perp$ and $p \in \mathcal{P}$ such that $f_X = \Phi * w + p$ satisfies $(\overline{w}_j, f_X) = d_j$ for every $1 \leq j \leq m$, and if $f \in \mathcal{P}$, then $f_X = p = f$. The latter requirement that the interpolation process reproduces $\mathcal{P}$ implies that the set $\overline{W}|_{\mathcal{P}} = \{\overline{w}_1|_{\mathcal{P}}, \ldots, \overline{w}_m|_{\mathcal{P}}\}$ spans $\mathcal{P}^*$, the dual of $\mathcal{P}$.

Suppose that the function f generating the data has the form $f = \Phi * v + q$ with $q \in \mathcal{P}$ and $v \in \mathcal{P}^\perp$. Let η be a distribution defined on functions in $C^k(S^n)$, for example $\eta = \delta_x$. In order to estimate the error $f - f_X$, we need to estimate $|(\overline{\delta_x}, f - f_X)|$ for every value of x. For a general η, in order to estimate $|(\overline{\eta}, f - f_X)|$, we observe that, by construction, $(\overline{w}_j, f - f_X) = 0$ for $j = 1, \ldots, m$; and so if we can find c_j's such that $\eta - \sum_{j=1}^m c_j w_j$ is in $\mathcal{P}^\perp$, then

$$\begin{aligned} (\overline{\eta}, f - f_X) &= (\overline{\eta - \sum_j c_j w_j}, \Phi * (v - w) + q - p) \qquad (3.1) \\ &= (\overline{\eta - \sum_j c_j w_j}, \Phi * (v - w)) \\ &= [v - w, \eta - \sum_j c_j w_j]_\Phi . \end{aligned}$$

If we set $\eta = w \in \mathcal{P}^\perp \cap \mathrm{span}\{W\}$ in (3.1) then the left hand side of (3.1) is 0 and the right hand side is $[v - w, w]_\Phi = 0$, since we can take all c_j's to be 0. It then follows that $[\![v]\!]_\Phi^2 = [\![v - w]\!]_\Phi^2 + [\![w]\!]_\Phi^2$, which yields

$$[\![w]\!]_\Phi < [\![v]\!]_\Phi \text{ and } [\![v - w]\!]_\Phi < [\![v]\!]_\Phi . \qquad (3.2)$$

By applying Schwarz's inequality to the right-hand side of (3.1), and using (3.2), we obtain

$$|(\overline{\eta}, f - f_X)| \leq [\![v]\!]_\Phi [\![\eta - \sum_j c_j w_j]\!]_\Phi, \text{ where } \sum_j c_j w_j|_{\mathcal{P}} = \eta|_{\mathcal{P}}. \tag{3.3}$$

We define the *power function* [38] to be

$$P^\eta_{\Phi,W} := \min \left\{ [\![\eta - \sum_j c_j w_j]\!]_\Phi : \sum_j c_j w_j|_{\mathcal{P}} = \eta|_{\mathcal{P}} \right\}. \tag{3.4}$$

Let $\Phi_{\mathcal{P}} \in \mathcal{P} \otimes \overline{\mathcal{P}}$ be an appropriate conjugate symmetric kernel that approximates Φ. We define

$$\begin{aligned} \Delta_0 &:= |(\bar{\eta} \otimes \eta, \Phi - \Phi_{\mathcal{P}})|, \\ \Delta_1 &:= \max_j |(\bar{\eta} \otimes w_j, \Phi - \Phi_{\mathcal{P}})|, \end{aligned}$$

and

$$\Delta_2 := \max_{j,k} |(\bar{w}_k \otimes w_j, \Phi - \Phi_{\mathcal{P}})|.$$

Theorem III.1 *([32, §3]) For any set of coefficients satisfying the constraint*

$$\sum_j c_j w_j|_{\mathcal{P}} = \eta|_{\mathcal{P}},$$

we have the following bound on the power function:

$$(P^\eta_{\Phi,W})^2 \leq \Delta_0 + 2\|c\|_1 \Delta_1 + \|c\|_1^2 \Delta_2.$$

B. Finite element method

In this section, we set up the weak formulation for the PDE on the unit sphere and prove a version for Cea's lemma (see [4] for the version on $\mathbb{R}^n$) for our equation on spheres.

1. Weak formulation

Let ω be a non-zero real constant and consider the following differential equation

$$-\Delta u(x) + \omega^2 u(x) = f(x), \quad x \in S^n.$$

The weak formulation of this equation is

$$\left\langle -\Delta u + \omega^2 u, v \right\rangle = \langle f, v \rangle, \quad \forall v \in H^1,$$

where

$$\langle u, v \rangle = \int_{S^n} u\overline{v} dS.$$

Defining the bilinear form $a(u,v) := \langle -\Delta u + \omega^2 u, v \rangle$, we find that the weak formulation becomes:

$$a(u,v) = \langle f, v \rangle, \quad \forall v \in H^1.$$

Lemma III.1 *There exist positive constants C and α such that*

$$|a(u,v)| \leq C\|u\|_{H^1}\|v\|_{H^1} \text{ and } |a(u,u)| \geq \alpha\|u\|_{H^1}^2.$$

Proof.

$$\begin{aligned} a(u,v) &= \sum_{\ell=0}^{\infty}\sum_{k=1}^{N(n,\ell)} (\lambda_\ell + \omega^2)\widehat{u}_{\ell k}\widehat{v}_{\ell k} \\ &\leq \left(\sum_{\ell=0}^{\infty}\sum_{k=1}^{N(n,\ell)} (\lambda_\ell + \omega^2)|\widehat{u}_{\ell k}|^2\right)^{1/2} \left(\sum_{\ell=0}^{\infty}\sum_{k=1}^{N(n,\ell)} (\lambda_\ell + \omega^2)|\widehat{v}_{\ell k}|^2\right)^{1/2} \\ &\leq \max\{1, \omega^2\}\|u\|_{H^1}\|v\|_{H^1}. \end{aligned}$$

We also have

$$a(u,u) = \sum_{\ell=0}^{\infty}\sum_{k=1}^{N(n,\ell)} (\lambda_\ell + \omega^2)|\widehat{u}_{\ell k}|^2 \geq \min\{1, \omega^2\}\sum_{\ell=0}^{\infty}\sum_{k=1}^{N(n,\ell)} (\lambda_\ell + 1)|\widehat{u}_{\ell k}|^2.$$

□

The preceding lemma shows that the bilinear form $a(u, v)$ is bounded and coercive, so by the Lax-Milgram theorem (cf. [3]) the weak formulation has a unique solution. It is easy to see that $\phi_i(x) := \Phi(x, x_i) = \phi(x \cdot x_i)$ is in H^1 since we require $\sigma > n/2 + 2$. We now define a finite dimensional subspace of H^1:

$$V_X := \text{span}\{\Phi(x, x_i) : x_i \in X\}.$$

The Ritz Galerkin approximation problem is the following:

$$\text{find } u_h \in V_X \text{ such that } a(u_h, \chi) = \langle f, \chi \rangle, \quad \forall \chi \in V_X. \tag{3.5}$$

The following is a version of Cea's lemma for unit spheres.

Lemma III.2 *The following holds:*

$$\|u - u_h\|_{H^1} \le C \inf_{v \in V_X} \|u - v\|_{H^1}.$$

Proof. It is noted that $a(u-u_h, \chi) = 0$ for all $\chi \in V_X$. In particular, $a(u-u_h, v-u_h) = 0$ for any $v \in V_X$. Thus,

$$a(u - u_h, u - u_h) = a(u - u_h, u - v + v - u_h) = a(u - u_h, u - v).$$

By Lemma III.1, we have

$$\alpha\|u - u_h\|_{H^1}^2 \le a(u - u_h, u - u_h) = a(u - u_h, u - v) \le C\|u - u_h\|_{H^1}\|u - v\|_{H^1}.$$

Dividing $\|u - u_h\|_{H^1}$ on both sides and taking infimum over $v \in V_X$, we obtain the required result. □

Lemma III.3 *Assume that $u \in H^1$, the following inequality holds:*

$$\|u\|_{H^1} \le (\|\Delta u\|_2 + \|u\|_2)^{1/2}\|u\|_2^{1/2}.$$

Proof.

$$\begin{aligned}
\|u\|_{H^1}^2 &= \sum_{\ell=0}^{\infty}\sum_{k=1}^{N(n,\ell)} (\lambda_\ell+1)|\widehat{u}_{\ell k}|^2 \\
&\le \sum_{\ell=0}^{\infty}\sum_{k=1}^{N(n,\ell)} \lambda_\ell|\widehat{u}_{\ell k}|^2 + \sum_{\ell=0}^{\infty}\sum_{k=1}^{N(n,\ell)} |\widehat{u}_{\ell k}|^2 \\
&\le \left(\sum_{\ell=0}^{\infty}\sum_{k=1}^{N(n,\ell)} \lambda_\ell^2|\widehat{u}_{\ell k}|^2\right)^{1/2} \left(\sum_{\ell=0}^{\infty}\sum_{k=1}^{N(n,\ell)} |\widehat{u}_{\ell k}|^2\right)^{1/2} + \sum_{\ell=0}^{\infty}\sum_{k=1}^{N(n,\ell)} |\widehat{u}_{\ell k}|^2 \\
&= \|\Delta u\|_2\|u\|_2 + \|u\|_2^2.
\end{aligned}$$

□

The foregoing lemma enables us to use recent results in [30] to estimate $\|u - I_X u\|_{H^1}$, where $I_X u \in V_X$ is the interpolant of u on X, i.e. $u(x_j) = I_X u(x_j)$ for all $x_j \in V_X$.

2. The estimate of $\|\Delta^s u - \Delta^s I_X u\|_\infty$

We shall estimate the error in two steps: first, u is assumed to be in the native space N_Φ and the error will be bounded by a factor of $\|u\|_\Phi$; second, we let u escape to a larger space $C^{2k}(S^n)$ and estimate the error in terms of $\|u\|_{2k}$.

To estimate the error in terms of $\|u\|_\Phi$, we need to estimate the *power function*, introduced in part A). Bounding the power function is done via employing *norming sets*, the use of which in the context of scattered data interpolation was initiated in [17].

Let V be a finite dimensional vector space with norm $\|\cdot\|_V$ and let $Z \subset V^*$ be a finite set of cardinality m. We will say that Z is a norming set for V if the mapping $T : V \to T(V) \subset \mathbb{R}^m$ defined by $T(u) = (z(u))_{z\in Z}$ is injective. The operator T is called the *sampling operator*. The norm of its inverse is given by

$$\|T^{-1}\| = \sup_{v\in V}\{\|v\|_V : \max_{z\in Z}|z(v)| = 1\}.$$

Proposition III.1 *[22, Proposition 4.1] Let Z be a norming set for V with T being the corresponding sampling operator. If $\lambda \in V^*$ with $\|\lambda\|_{V^*} \leq A$, then there exist real numbers $\{a_z : z \in Z\}$ depending only on λ such that for every $v \in V$,*

$$\lambda(v) = \sum_{z \in Z} a_z z(v), \ \ \textit{and} \ \sum_{z \in Z} |a_z| \leq A\|T^{-1}\|.$$

The second result needed is the Markov-Bernstein inequality for spherical harmonics of order L. A proof of the inequality may be found in [35].

Theorem III.2 *If $P_L \in \mathcal{P}_L$, then*

$$\|\Delta P_L\|_\infty \leq D_n L^2 \|P_L\|_\infty,$$

where the constant D_n depends only on the dimension of the ambient space.

Remark. It is known that $D_2 = 4$ (see [32]).

Corollary III.1

$$\|\Delta^s P_L\|_\infty \leq D_n^s L^{2s} \|P_L\|_\infty.$$

Next we need to adapt [32, Theorem 6.4] to the case S^n.

Proposition III.2 *If the mesh norm of X satisfies $h_X < 1/(2L)$, then for any fixed x there exist numbers $\alpha_j(x)$, $1 \leq j \leq m$, such that*

$$\sum_{j=1}^m \alpha_j(x) Y(x_j) = \Delta^s Y(x) \ \textit{for all} \ Y \in \mathcal{P}_L,$$

and

$$\sum_{j=1}^m |\alpha_j(x)| \leq 2 D_n^s L^{2s}.$$

Proof. Let T be the point-sampling operator, namely, $T(Y) = (Y(x_1), \ldots, Y(x_m))$, and let $\lambda(Y) = \Delta^s Y(x)$. The upper bound for $\|\lambda\|$ is given by Theorem III.2. Moreover, if the mesh norm $h_X < 1/(2L)$ then $\|T^{-1}\| \leq 2$ (see [17]). The required result now follows via Proposition III.1. □

Defining the ordinary differential operator

$$\mathcal{L} := -(1-t^2)^{(2-n)/2}\frac{d}{dt}(1-t^2)^{n/2}\frac{d}{dt} = -(1-t^2)\left(\frac{d}{dt}\right)^2 + nt\frac{d}{dt},$$

we recall from [26, page 38] that the $(n+1)$-dimensional Legendre polynomials $P_\ell(n+1;t)$ satisfy the differential equation

$$\mathcal{L}P_\ell(n+1;t) = \lambda_\ell P_\ell(n+1;t).$$

We approximate the kernel ϕ by the truncated kernel ϕ_L:

$$\phi_L(x\cdot y) := \sum_{\ell=0}^{L} a_\ell P_\ell(n+1; x\cdot y),$$

which belongs to the space $\mathcal{P}_L \otimes \overline{\mathcal{P}}_L$.

Lemma III.4 *Let ϕ be a kernel as in (1.1). If $\phi(t) \in C^{(2k+2j)}[-1,1]$, then*

$$|\mathcal{L}^k[\phi-\phi_L](x\cdot y)| \leq \frac{\mathcal{L}^{k+j}[\phi-\phi_L](1)}{(L+n-1)^{2j}} \leq \frac{\mathcal{L}^{k+j}\phi(1)}{(L+n-1)^{2j}}.$$

Proof. We have

$$|\mathcal{L}^k\phi(x\cdot y) - \mathcal{L}^k\phi_L(x\cdot y)| \leq \sum_{\ell\geq L+1} \lambda_\ell^k a_\ell |P_\ell(n+1;x\cdot y)|.$$

Since the Legendre polynomials satisfy the inequality $|P_\ell(n+1;t)| \leq P_\ell(n+1;1) = 1$ for every t in $[-1,1]$, (see [26, page 15]), we have

$$\begin{aligned}\sum_{\ell>L} \lambda_\ell^k a_\ell P_\ell(n+1;t) &\leq \sum_{\ell>L} \lambda_\ell^k a_\ell P_\ell(n+1;1)\\ &\leq (L+n-1)^{-2j}\sum_{\ell>L} \lambda_\ell^{k+j} a_\ell P_\ell(n+1;1)\\ &\leq \frac{\mathcal{L}^{k+j}[\phi-\phi_L](1)}{(L+n-1)^{2j}}.\end{aligned}$$

The lemma follows by observing that $\mathcal{L}^{k+j}[\phi - \phi_L](1) \leq \mathcal{L}^{k+j}\phi(1)$. □

We are now in a position to obtain an error estimate for $\Delta^s(u - I_X u)$, where $I_X u$ is the SBF interpolant of u on the set X.

Proposition III.3 *Suppose that Φ is a positive definite function of the form (1.2), $\phi(t) \in C^{4s}[-1,1]$, and let X be a finite set of distinct points on S^n with mesh norm $h_X \leq 1/(2L)$. If u belongs to the native space N_Φ and $I_X u$ is an interpolant of the form $\sum_{j=1}^m c_j \Phi(x, x_j)$ which interpolates u on the set X, then*

$$\|\Delta^s u - \Delta^s I_X u\|_\infty \leq C \left(\sum_{\ell > L}^{\infty} \widehat{\phi}(\ell) N(n,\ell) \lambda_\ell^{2s} \right)^{1/2} \|u\|_\Phi,$$

where C is a constant depending only on n and s.

Proof. Recalling the distributional framework set out in part A), we consider the following particular linear functional:

$$\eta(u) = \Delta^s u(x).$$

For a given point $x \in S^n$, we shall use inequality (3.3) in Chapter II to estimate $|\Delta^s u(x) - \Delta^s I_X u(x)|$. Now Theorem III.1 and Proposition III.2 provides the following bound:

$$(P^\eta_{\Phi,W})^2 \leq \Delta_0 + 4D_n^s L^{2s} \Delta_1 + 4D_n^{2s} L^{4s} \Delta_2,$$

where the Δ_j's are given by

$$\Delta_0 = |\mathcal{L}^{2s}\phi(1) - \mathcal{L}^{2s}\phi_L(1)|,$$

$$\Delta_1 = \max_j |\mathcal{L}^s \phi(x \cdot x_j) - \mathcal{L}^s \phi_L(x \cdot x_j)|,$$

and

$$\Delta_2 = \max_{j,k} |\phi(x_k \cdot x_j) - \phi_L(x_k \cdot x_j)|.$$

Applying Lemma III.4 to bound these quantities and then using the resulting bounds in the power-function estimate above, we obtain

$$\begin{aligned}(P^{\eta}_{\Phi,W})^2 &\leq \left(1 + \frac{4D_n^s L^{2s}}{(L+n-1)^{2s}} + \frac{4D_n^{2s} L^{4s}}{(L+n-1)^{4s}}\right) \mathcal{L}^{2s}[\phi - \phi_L](1) \\ &\leq C\mathcal{L}^{2s}[\phi - \phi_L](1),\end{aligned}$$

where C is a constant that depends only on n and s. The required result follows by the following relation

$$\mathcal{L}^{2s}[\phi - \phi_L](1) = \frac{1}{|S^n|}\sum_{\ell > L} \lambda_\ell^{2s}\widehat{\phi}(\ell)N(n,\ell).$$

□

We now derive a simple consequence for our choice of kernels.

Corollary III.2 *Suppose that* $\phi(t) \in C^{2s}[-1,1]$, $\widehat{\phi}(\ell) \sim C(1+\lambda_\ell)^{-\sigma}$ *for some* $\sigma > n/2 + 2s$ *and the mesh norm* h_X *satisfying* $1/(2L+2) \leq h_X \leq 1/2L$. *Then*

$$\|\Delta^s u - \Delta^s I_X u\|_\infty \leq C h_X^{\sigma - n/2 - 2s}\|u\|_\Phi.$$

Proof. Since $(1+\lambda_\ell) \leq C\ell^2$ and $N(n,\ell) = \mathcal{O}(\ell^{n-1})$ we have

$$\sum_{\ell=L+1}^{\infty} \widehat{\phi}(\ell)N(n,\ell)\lambda_\ell^{2s} \leq C\int_L^\infty x^{n-1+4s-2\sigma}dx \leq CL^{n+4s-2\sigma}.$$

The result follows directly from Proposition III.3 and the condition

$$1/(2L+2) \leq h_X \leq 1/2L.$$

□

In the proof of the main result, we need to construct for every $u \in C(S^n)$, spherical harmonics that are both near-best approximants to u from $\mathcal{P}_L$ and also interpolate u on the point set X. This is precisely the content of the following theorem:

Theorem III.3 *[30, Theorem 3.1] Let $X \subset S^n$ be a finite set of distinct points and let $\beta > 1$. If we set $L = \lceil \frac{M(\beta+1)}{q_X(\beta-1)} \rceil$, with M as in Theorem II.1, then for $u \in C(S^n)$ there exists a spherical harmonic $P_L \in \mathcal{P}_L$ that interpolates u on X and also satisfies the estimate*

$$\|u - P_L\|_\infty \leq (1+\beta)\, dist\, (u, \mathcal{P}_L).$$

Lemma III.5 *Suppose $u \in C^{2s}(S^n)$, where s is a positive integer, and let P_L be the best approximation to u from $\mathcal{P}_L$, i.e., $dist\,(u, \mathcal{P}_L) = \|u - P_L\|_\infty$. Then there is a constant C, independent of u and L, such that*

$$\|\Delta^s u - \Delta^s P_L\|_\infty \leq C\, dist\, (\Delta^s u, \mathcal{P}_L).$$

Proof. We prove the lemma by induction on s. We consider the case $s = 1$. Note that if Q is a spherical harmonic of degree L, for $L > 0$, then so is ΔQ, because spherical harmonics are eigenfunctions of Δ. Therefore, the space of all spherical harmonics of degree $\leq L$ except constants, denote by $\mathcal{P}_L \setminus V_0$, is isomorphic to $\Delta(\mathcal{P}_L \setminus V_0)$. Let Q be a spherical harmonic without constant term in $\mathcal{P}_L$ so that ΔQ is the best approximation to Δu. So,

$$\|\Delta u - \Delta Q\|_\infty = \text{dist}\, (\Delta u, \mathcal{P}_L).$$

Let $R \in \mathcal{P}_L$ be the best approximation to $u - Q$, so that

$$\|R - (u - Q)\| = \text{dist}\, (u - Q, \mathcal{P}_L) = \text{dist}\, (u, \mathcal{P}_L).$$

Since P_L is unique, we obtain $P_L = R + Q$. By the estimate in Theorem II.1 in Chapter II,

$$\|\Delta R\|_\infty \leq C \|\Delta u - \Delta Q\|_\infty = C\, \text{dist}\, (\Delta u, \mathcal{P}_L).$$

Thus

$$\|\Delta u - \Delta P_L\|_\infty \le \|\Delta u - \Delta Q\|_\infty + \|\Delta R\|_\infty \le 2C\,\mathrm{dist}\,(\Delta u, \mathcal{P}_L).$$

Now let $s > 1$, and suppose that there is a constant C_0 so that

$$\|\Delta^{s-1} u - \Delta^{s-1} P_L\|_\infty \le C_0\,\mathrm{dist}\,(\Delta^{s-1} u, \mathcal{P}_L).$$

Using the induction hypothesis for Δu and ΔQ, we have

$$\|\Delta^{s-1}\Delta u - \Delta^{s-1}\Delta Q\|_\infty \le C_0\,\mathrm{dist}\,(\Delta^s u, \mathcal{P}_L).$$

Using Theorem II.1 in Chapter II once again, we have

$$\|\Delta^s R\|_\infty \le C_1 \|\Delta^s u - \Delta^s Q\|_\infty \le C_2\,\mathrm{dist}\,(\Delta^s u, \mathcal{P}_L),$$

where $C_2 = C_1 C_0$. Thus

$$\|\Delta^s u - \Delta^s P_L\|_\infty \le \|\Delta^s u - \Delta^s Q\|_\infty + \|\Delta^s R\|_\infty \le C_3\,\mathrm{dist}\,(\Delta^s u, \mathcal{P}_L),$$

with $C_3 = \max(C_0, C_2)$. □

We extend the result of the previous lemma to a broader class of near best approximants to u.

Lemma III.6 *Suppose that $u \in C^{2k}(S^n)$ and P is a near best approximation to u from $\mathcal{P}_L$ in the sense that there is a constant K, independent of L and u, so that*

$$\|u - P\|_\infty \le K\,dist\,(P, \mathcal{P}_L).$$

Then there exists a positive constant C_1 so that for any integer $s \le k$,

$$\|\Delta^s u - \Delta^s P\|_\infty \le C_1 L^{-2k+2s} \|\Delta^k u\|_\infty.$$

Proof. Let P_L be the best approximation to u from $\mathcal{P}_L$. The preceding lemma implies the estimate

$$\|\Delta^s u - \Delta^s P_L\|_\infty \le C \,\mathrm{dist}\,(\Delta^s u, \mathcal{P}_L).$$

By the Markov-Bernstein inequality (Theorem III.2),

$$\begin{aligned}\|\Delta^s P_L - \Delta^s P\|_\infty &\le D_n^s L^{2s}\|P_L - P\|_\infty \\ &\le D_n^s L^{2s}(\|P_L - u\|_\infty + \|u - P\|_\infty) \\ &\le D_n^s L^{2s}(K+1)\,\mathrm{dist}\,(u, \mathcal{P}_L).\end{aligned}$$

Combining the two estimates above, we obtain

$$\begin{aligned}\|\Delta^s u - \Delta^s P\|_\infty &\le \|\Delta^s u - \Delta^s P_L\|_\infty + \|\Delta^s P_L - \Delta^s P\|_\infty \\ &\le C_1 \,\mathrm{dist}\,(\Delta^s u, \mathcal{P}_L) + DL^{2s}\,\mathrm{dist}\,(u, \mathcal{P}_L),\end{aligned}$$

where $D := D_n^s(K+1)$. Now by the second part of Theorem II.1 in Chapter II,

$$\mathrm{dist}\,(\Delta^s u, \mathcal{P}_L) \le M_1 L^{-2k+2s}\|\Delta^k u\|_\infty$$

and

$$\mathrm{dist}\,(u, \mathcal{P}_L) \le M_2 L^{-2k}\|\Delta^k u\|_\infty,$$

so the required result follows by setting $C_1 = \max\{CM_1, DM_2\}$. □

Now we adapt the proof in [30] to estimate $\|u - I_X u\|_\infty$ for $u \in C^{2k}(S^n)$, which is in general a larger space of functions than the native space induced by the kernel Φ.

Theorem III.4 *Let Φ be an SBF satisfying $\widehat{\phi}(\ell) \sim (1+\lambda_\ell)^{-\sigma}$ and suppose that $\sigma > 2k \ge n/2 + 2s$. If $u \in C^{2k}(S^n)$ and $I_X u \in V_X$ interpolates u on X then for any integer $s < k - n/4$,*

$$\|\Delta^s u - \Delta^s I_X u\|_\infty \le C h_X^{2k-2s-n/2}\|\Delta^k u\|_\infty.$$

Proof. By Theorem II.2 in Chapter II with $\beta = 3$, there exists a $P_L \in \mathcal{P}_L$ that interpolates u on X, where $L = \lceil 2M/q_X \rceil$, where M is as in Theorem II.1 in Chapter II, and

$$\|u - P_L\|_\infty \leq 4\,\text{dist}\,(u, \mathcal{P}_L).$$

Let P_X be the interpolant of P_L in the space V_X, then

$$\|\Delta^s u - \Delta^s I_X u\|_\infty \leq \|\Delta^s u - \Delta^s P_L\|_\infty + \|\Delta^s P_L - \Delta^s P_X\|_\infty + \|\Delta^s (P_X - I_X u)\|_\infty. \quad (3.6)$$

Since $P_X(x_j) = P_L(x_j) = u(x_j) = I_X u(x_j)$ for all $x_j \in X$ and both P_X and $I_X u$ lie in the same finite dimensional space V_X, we have $P_X \equiv I_X u$ and the final term in the previous inequality vanishes. By Lemma III.6, we have the estimate

$$\|\Delta^s u - \Delta^s P_L\|_\infty \leq C_0 L^{-2k+2s} \|\Delta^k u\|_\infty.$$

By the assumption on $\widehat{\phi}(\ell)$, Corollary III.2 holds and, since the norms $\|\cdot\|_\Phi$ and $\|\cdot\|_{H^\sigma}$ are equivalent, we can estimate the second term in the right hand side of (3.6) as

$$\|\Delta^s P_L - \Delta^s P_X\|_\infty \leq C_1 h_X^{\sigma - n/2 - 2s} \|P_L\|_{H^\sigma}.$$

Using the definition of Sobolev norm and the fact that P_L is a polynomial,

$$\|P_L\|_{H^\sigma} \leq (1 + \lambda_L)^{\sigma/2 - k} \|P_L\|_{H^{2k}} \leq 2^k |S^n|^{1/2} (1 + \lambda_L)^{\sigma/2 - k} \|P_L\|_{2k}.$$

From the assumption, $\|P_L\|_\infty \leq 5\|u\|_\infty$, so by Theorem II.1 in Chapter II, we also have $\|\Delta^k P_L\|_\infty \leq C_1 \|\Delta^k u\|_\infty$, so that

$$\|P_L\|_{2k} \leq \max\{5, C_1\} \|u\|_{2k}.$$

So, if we set $C_2 = 2^k |S^n|^{1/2} \max\{5, R\}$ then

$$\|\Delta^s P_L - \Delta^s P_X\|_\infty \leq C_2 h_X^{\sigma - n/2 - 2s} (1 + \lambda_L)^{\sigma/2 - k} \|u\|_{2k}. \quad (3.7)$$

From (3.6), (3.7) and $\lambda_L = L(L+n-1) \le CL^2$,

$$\begin{aligned}\|\Delta^s u - \Delta^s I_X u\|_\infty &\le (C_1 L^{2s-2k} + C_2 L^{\sigma-2k} h_X^{\sigma-n/2-2s})\|u\|_{2k} \\ &\le (C_1 L^{n/2+2s-2k} + C_2 L^{\sigma-2k} h_X^{\sigma-n/2-2s})\|u\|_{2k} \\ &\le [C_1 (h_X L)^{n/2+2s-2k} + C_2 (h_X L)^{\sigma-2k}] h_X^{2k-n/2-2s}\|u\|_{2k}.\end{aligned}$$

If we use $L = \lceil 2M/q_X \rceil = \lceil 2M\rho_X/h_X \rceil$, then we get

$$\|\Delta^s u - \Delta^s I_X u\|_\infty \le (C_3 \rho_X^{n/2+2s-2k} + C_4 \rho_X^{\sigma-2k}) h_X^{2k-n/2-2s}\|u\|_{2k}.$$

Finally, since $\rho_X \ge 1$ and $\sigma > 2s + n/2$, it follows that

$$\|\Delta^s u - \Delta^s I_X u\|_\infty \le C\rho_X^{\sigma-2k} h_X^{2k-n/2-2s}\|u\|_{2k}.$$

□

So, we have all the results to estimate the H^1 error for the finite element solution.

Theorem III.5 *The finite element solution u_h satisfies the following error estimate*

$$\|u - u_h\|_{H^1} \le Ch_X^{2k-n/2-1}\|u\|_{2k}.$$

Proof. By Theorem III.4, we have

$$\|\Delta u - \Delta I_X u\|_2 \le \sqrt{|S^n|}\|\Delta u - \Delta I_X u\|_\infty \le C_1 h_X^{2k-n/2-2}\|u\|_{2k}.$$

By Theorem I.2 we also have

$$\|u - I_X u\|_2 \le \sqrt{|S^n|}\|u - I_X u\|_\infty \le C_2 h_X^{2k-n/2}\|u\|_{2k}.$$

So by Lemma III.3, we conclude

$$\|u - I_X u\|_{H^1} \le Ch_X^{2k-n/2}\sqrt{1 + h_X^{-2}}\|u\|_{2k}.$$

Now, using Cea's lemma (Lemma III.2), we finally have

$$\|u - u_h\|_{H^1} \leq C\|u - I_X u\|_{H^1} \leq C h_X^{2k-n/2}\sqrt{1 + h_X^{-2}}\|u\|_{2k}.$$

□

C. Collocation method

In this section, we discuss a collocation method for a more general class of elliptic differential operator, namely

$$\mathcal{L}u = f, \tag{3.8}$$

in which the differential operator $\mathcal{L}$ has eigenvalues asymptotic to $(1 + \lambda_\ell)^{\beta/2}$. In other words, for spherical harmonics of order ℓ, where $\ell = 0, 1, \ldots,$ there are numbers

$$a_\ell \sim (1 + \lambda_\ell)^{\beta/2} \text{ such that } \mathcal{L}Y_\ell = a_\ell Y_\ell. \tag{3.9}$$

In the collocation method, we require that the differential equation to be exact on the set of points X. In effect, we would like to find u_X which lies in some finite dimensional space V_X such that

$$\mathcal{L}u_X|_{x=x_j} = f(x_j), \quad \forall x_j \in X. \tag{3.10}$$

Before outlining the structure of the space V_X, we need to recall a classical framework set out in [13]:

Theorem III.6 *Let $F_i, 1 \leq i \leq m$, be m linearly independent continuous linear functionals of the native space N_Φ. We define the space*

$$\mathcal{S}_m^\perp := \{w \in N_\Phi : F_i(w) = 0 \textit{ for all } 1 \leq i \leq m\},$$

and for given data $d = (d_i : 1 \le i \le m) \in \mathbb{R}^m$ *let*

$$\mathcal{S}_d := \{w \in N_\Phi : F_i(w) = d_i \text{ for all } 1 \le i \le m\}.$$

Then there exists a unique interpolant $v \in \mathcal{S}_m \cap \mathcal{S}_d$ *so that*

(i) For any $w \in \mathcal{S}_d$,

$$\|w\|_\Phi^2 = \|v\|_\Phi^2 + \|w - v\|_\Phi^2.$$

(ii) For another continuous linear functional F *on* N_Φ *the value* Fv *is the best approximation to* $\{Fw : w \in \mathcal{S}_d, \|w\|_\Phi = r\}$, *meaning that*

$$\sup_{w\in C_r} |Fw - Fz| \ge \sup_{w\in C_r} |Fw - Fv|$$

holds for all $z \in N_\Phi$. *Here* $C_r := \{w \in \mathcal{S}_d : \|w\|_\Phi = r\}, r > 0$, *denotes a hypercircle.*

Furthermore, the hypercircle inequality is satisfied: If $y \in \mathcal{S}_m^\perp$ *denotes an element with unit norm for which* $F|_{\mathcal{S}_m^\perp}$ *attains its least upper bound, then*

$$|Fw - Fv|^2 \le |Fy|^2(r^2 - \|v\|_\Phi^2) \text{ for all } w \in C_r.$$

Let $\mathcal{L}^*$ be the dual operator of $\mathcal{L}$ which is defined as $(\mathcal{L}^*\delta_x)(w) = \delta_x(\mathcal{L}w)$. In our case $F = \mathcal{L}^*\delta_x$ and $F_i = \mathcal{L}^*\delta_{x_i}$, $i = 1, \ldots, m$. For the sake of simplicity, let us assume that all the linear functionals $\mathcal{L}^*\delta_{x_i}$ are linearly independent. Since Φ is the reproducing kernel in N_Φ, we have

$$F_i(w) = \mathcal{L}w(x)|_{x=x_i} = \langle \mathcal{L}w, \Phi(\cdot, x_i)\rangle_\Phi = \langle w, \mathcal{L}^*\Phi(\cdot, x_i)\rangle_\Phi .$$

Hence, the space $\mathcal{S}_m^\perp$ takes the following form

$$\mathcal{S}_m^\perp := \{w \in N_\Phi : \langle w, \mathcal{L}^*\Phi(\cdot, x_i)\rangle_\Phi = 0 \text{ for all } 1 \le i \le m\}.$$

Thus the approximation space is $\mathcal{S}_m := \text{span}\{\phi_i(x) : i = 1, \ldots, m\}$, where

$$\phi_i(\cdot) := \mathcal{L}^*\Phi(\cdot, x_i) = \mathcal{L}_y\Phi(\cdot, y)|_{y=x_i}, \quad i = 1, \ldots, m.$$

So by Theorem III.6, there exists a unique solution for our equation (3.10) provided that we restrict the problem to a subspace of N_Φ in which $F = \mathcal{L}^*\delta_x$ and $F_i = \mathcal{L}^*\delta_{x_i}$ are well-defined. We have to solve the linear system:

$$\mathcal{L}\left(\sum_{i=1}^{m} c_i\phi_i(x)\right)\Bigg|_{x=x_j} = f(x_j) \text{ or } A\mathbf{c} = \mathbf{f}, \tag{3.11}$$

where $A_{ij} = \mathcal{L}(\mathcal{L}^*\Phi(x_j, x_i))$, $\mathbf{c} = [c_1 \ldots c_m]^T$ and $\mathbf{f} = [f(x_1) \ldots f(x_m)]^T$. It is noted that A is a positive definite matrix since the kernel Φ is strictly positive definite and the operator $\mathcal{L}$ has positive eigenvalues with respect to spherical harmonics of order ℓ.

In order to estimate the term $|F(y)|$ in the hypercircle inequality we need to state a few preliminary results.

Markov's inequality. Let $P \in \mathcal{P}_L$, we have the following general Markov's inequality:

$$|D_T P(p)| \leq L\|P\|_\infty,$$

where D_T denotes any unit tangential derivative at p and the maximum norm is on S^n. This is a simple case of the general Markov inequality for polynomials on compact smooth algebraic sub-manifolds of $\mathbb{R}^{n+1}$ without boundary (see [2]). We will use it later in its integrated form

$$|P(p) - P(q)| \leq L\theta(p, q)\|P\|_\infty, \quad p, q \in S^n. \tag{3.12}$$

Lemma III.7 *Suppose that the set of scattered points X has mesh norm h_X that* *satisfies* $h_X \leq 1/2L$. *We assume further that there is a constant* Λ *such that*

$$\|\mathcal{L}P\|_\infty \geq \Lambda \|P\|_\infty \quad \forall P \in \mathcal{P}_L.$$

Then the following holds:

$$\|\mathcal{L}P|_X\|_\infty \geq \frac{\Lambda}{2}\|P\|_\infty, \quad \forall P \in \mathcal{P}_L.$$

Proof. We show that $\|Q|_X\|_\infty \geq \frac{1}{2}\|Q\|_\infty$ for all $Q \in \mathcal{P}_L$. Without loss of generality, we can assume that $\|Q\|_\infty = 1$ and $Q(x_0) = 1$ for some $x_0 \in S^n$. Then for any $x_i \in X$,

$$1 = |Q(x_0)| \leq |Q(x_0) - Q(x_i)| + |Q(x_i)| \leq L\|Q\|_\infty \theta(x_0, x_i) + |Q(x_i)|.$$

The second inequality follows from the intermediate value theorem and the Markov's inequality. Since $h_X \leq 1/2L$, we can choose x_i so that $\theta(x_0, x_i) \leq 1/2L$ and hence $|Q(x_i)| \geq 1/2$.

Since $\mathcal{L}$ is surjective, given any $Q \in \mathcal{P}_L$, there is a P such that $Q = \mathcal{L}P$ and hence

$$\|\mathcal{L}P|_X\|_\infty = \|Q|_X\|_\infty \geq \frac{1}{2}\|Q\|_\infty = \frac{1}{2}\|\mathcal{L}P\|_\infty \geq \frac{\Lambda}{2}\|P\|_\infty.$$

□

In our specific choice of differential operators which satisfy condition (3.9), we can work out the value of the constant Λ in terms of L as in the following lemma:

Lemma III.8 *For any spherical harmonic* $P \in \mathcal{P}_L$, *the following holds:*

$$\|\mathcal{L}P\|_\infty \geq CL^{1/2-n}\|P\|_\infty.$$

Proof. From [25], we recall that each spherical harmonic in the orthonormal basis can be bounded by the following inequality:

$$\|Y_{\ell k}\|_\infty \le \sqrt{\frac{N(n,\ell)}{|S^n|}}, \text{ for } k = 1, \dots, N(n,\ell). \tag{3.13}$$

We can express P as

$$P(x) = \sum_{\ell=0}^{L} \sum_{k=1}^{N(n,\ell)} \widehat{P}_{\ell k} Y_{\ell k}(x).$$

Using the Cauchy-Schwarz inequality and inequality (3.13) together with the fact that $N(n,\ell) = \mathcal{O}(\ell^{n-1})$, we have

$$\begin{aligned} |P(x)|^2 &\le \left(\sum_{\ell=0}^{L} \sum_{k=1}^{N(n,\ell)} |\widehat{P}_{\ell k}||Y_{\ell k}(x)| \right)^2 \\ &\le \left(\sum_{\ell=0}^{L} \sum_{k=1}^{N(n,\ell)} |\widehat{P}_{\ell k}|^2 \right) \left(\sum_{\ell=0}^{L} \sum_{k=1}^{N(n,\ell)} \frac{N(n,\ell)}{|S^n|} \right) \\ &\le C\|P\|_2^2 \sum_{\ell=0}^{L} \ell^{2n-2} \le C\|P\|_2^2 L^{2n-1}. \end{aligned}$$

Thus, we obtain $\|P\|_\infty \le C\|P\|_2 L^{n-1/2}$, where C is a constant depending only on n. On the other hand, $\|P\|_2 \le |S^n|^{1/2}\|P\|_\infty$. Since $\mathcal{L}Y_{\ell k} \sim (1+\lambda_\ell)^{\beta/2} Y_{\ell k}$ (condition (3.9)), we have

$$\|\mathcal{L}P\|_2 \ge \|P\|_2.$$

Combining all the inequalities above, the following holds:

$$\|\mathcal{L}P\|_\infty \ge \frac{1}{|S^n|^{1/2}}\|\mathcal{L}P\|_2 \ge \frac{1}{|S^n|^{1/2}}\|P\|_2 \ge \frac{CL^{1/2-n}}{|S^n|^{1/2}}\|P\|_\infty.$$

□

Remark The previous lemma gives a lower bound for $\|\mathcal{L}\|$. In [41], we can find many upper bounds for norms of operators of similar type.

We now prove a proposition, which is a generalized version of [17, Proposition 2].

Proposition III.4 *Assume that* $X = \{x_i : 1 \le i \le m\}$ *is a given set of points on* S^n *and its mesh norm* $h_X \le 1/2L$*. Let* $\mathcal{P}_L$ *be the space of spherical harmonics of degree less than or equal* L*. Then the dual space* $\mathcal{P}_L^*$ *can be identified with the space*

$$span\{\mathcal{L}^*\delta_x|_{\mathcal{P}_L} : x \in X\}.$$

Moreover, for any $w^* \in \mathcal{P}_L^*$ *with* $\|w^*\| = \|\mathcal{L}\|$*,* w^* *can be identified with some*

$$\sum_{x\in X} a_x \mathcal{L}^*\delta_x|_{\mathcal{P}_L} \text{ where } \sum_{x\in X} |a_x| \le 2/\Lambda.$$

Proof. Let $T : C^k(S^n) \to C(X)$ be defined as

$$T(f) = \mathcal{L}f|_X$$

Let T_0 be the restriction of T to $\mathcal{P}_L$, i.e.

$$\begin{aligned} T_0 = T|_{\mathcal{P}_L} : \quad \mathcal{P}_L &\to T(\mathcal{P}_L) \\ P &\mapsto \mathcal{L}P|_X \end{aligned}$$

By Lemma III.7, T_0 is a 1-1 isomorphism and $\|T_0^{-1}\| \le \frac{2}{\Lambda}$. Also

$$T_0^* : T(\mathcal{P}_L)^* \to \mathcal{P}_L^*$$

is an isomorphism with $\|(T_0^*)^{-1}\| \le \frac{2}{\Lambda}$. So for any $w^* \in \mathcal{P}_L^*$ with $\|w^*\| = \|\mathcal{L}\|$ there is $t^* \in T(\mathcal{P}_L)^*$ such that $T_0^*(t^*) = w^*$ and $\|t^*\| \le \frac{2}{\Lambda}\|\mathcal{L}\|$. Using the Hahn-Banach theorem, we can extend $t^* \in T(\mathcal{P}_L)^*$ to $l^* \in C(X)^*$,

$$l^* = \sum_{x\in X} a_x \mathcal{L}^*\delta_x|_{C(X)}$$

such that $t^* = l^*|_{T(\mathcal{P}_L)}$ and

$$\|t^*\| = \|l^*\| = \|\mathcal{L}\| \sum_{x \in X} |a_x| \leq \frac{2}{\Lambda} \|\mathcal{L}\|.$$

Thus $\sum_{x \in X} |a_x| \leq \frac{2}{\Lambda}$. For a fixed $w^* \in \mathcal{P}_L^*$, let $l^* = \sum_{x \in X} \mathcal{L}^* \delta_x|_{C(X)}$ be the extension of $(T_0^*)^{-1} w^*$. Then

$$\begin{aligned} (w^*, P) = (w^*, T_0^{-1} T_0 P) &= ((T_0^*)^{-1} w^*, T_0 P) \\ &= \left(\sum_x a_x \mathcal{L}^* \delta_x|_{C(X)}, T_0 P \right) \\ &= \left(\sum_x a_x \mathcal{L}^* \delta_x|_{\mathcal{P}_L}, P \right), \quad \forall P \in \mathcal{P}_L. \end{aligned}$$

□

We are ready to state a theorem.

Theorem III.7 *Assume that X has mesh norm $h_X \leq 1/(2L)$, $L \in \mathbb{Z}^+$, x is a given point on S^n, then there exist c_i's, $i = 1, \ldots, m$ such that $\sum_{i=1}^m |c_i| \leq 2/\Lambda$ and*

$$\left(\mathcal{L}^* \delta_x - \sum_{i=1}^m c_i \mathcal{L}^* \delta_{x_i} \right) P = 0 \quad \forall P \in \mathcal{P}_L.$$

The term $|Fy|^2$ in the hypercircle inequality can be bounded by

$$|\mathcal{L}^* \delta_x(y)| \leq \left(1 + \frac{2}{\Lambda} \right) \max_{t \in S^n} \left(\sum_{\ell > L} \sum_{k=1}^{N(n,\ell)} \widehat{\phi}(\ell) |\mathcal{L} Y_{\ell k}(t)|^2 \right)^{1/2}.$$

Proof. Let $c_0 = -1$. The existence of $\{c_i : 1 \le i \le m\}$ is given by Proposition III.4. By orthogonality $F_i(y) = \mathcal{L}y(x_i) = 0$ for $i = 1, \dots, m$, so we have

$$\begin{aligned}
|F(y)| &= \left| \sum_{\ell > L} \sum_{k=1}^{N(n,\ell)} \left(F - \sum_{i=1}^{m} c_i F_i \right) \widehat{y}_{\ell k} Y_{\ell k} \right| \\
&= \left| \sum_{\ell > L} \sum_{k=1}^{N(n,\ell)} \left(\mathcal{L}Y_{\ell k}(x) - \sum_{i=1}^{m} c_i \mathcal{L}Y_{\ell k}(x_i) \right) \widehat{y}_{\ell k} \right| \\
&\le \left(\sum_{i=0}^{m} |c_i| \left| \sum_{\ell > L} \sum_{k=1}^{N(n,\ell)} \widehat{y}_{\ell k} \mathcal{L}Y_{\ell k}(x_i) \right| \right) \\
&\le \left(1 + \frac{2}{\Lambda} \right) \max_{t \in \{x\} \cup X} \left| \sum_{\ell > L} \sum_{k=1}^{N(n,\ell)} \widehat{y}_{\ell k} \mathcal{L}Y_{\ell k}(t) \right|
\end{aligned}$$

We also have $\|y\|_\Phi = 1$ so by Cauchy-Schwarz's inequality

$$\begin{aligned}
\left(\sum_{\ell > L} \sum_{k=1}^{N(n,\ell)} \widehat{y}_{\ell k} \mathcal{L}Y_{\ell k}(t) \right)^2 &\le \left(\sum_{\ell > L} \sum_{k=1}^{N(n,\ell)} \frac{|\widehat{y}_{\ell k}|^2}{\widehat{\phi}(\ell)} \right) \left(\sum_{\ell > L} \sum_{k=1}^{N(n,\ell)} \widehat{\phi}(\ell) |\mathcal{L}Y_{\ell k}(t)|^2 \right) \\
&\le \sum_{\ell > L} \sum_{k=1}^{N(n,\ell)} \widehat{\phi}(\ell) |\mathcal{L}Y_{\ell k}(t)|^2.
\end{aligned}$$

Combine the two above inequalities we have the result. □

Now we are in a position to give an $L^2(S^n)$ error estimate between the exact solution u of (3.8) and the SBF approximate solution of (3.10).

Theorem III.8 *Suppose that Φ is an SBF satisfying $\widehat{\phi}(\ell) \sim (1 + \lambda_\ell)^{-\sigma}$ and the set X has mesh norm h_X. The approximate solution u_X is constructed from*

$$\textit{span}\ \{\phi_i(x) := \mathcal{L}\phi(x \cdot x_i), \quad x_i \in X\}.$$

Then there exists a positive constant C_3 such that:

$$\|u - u_X\|_2 \le C_3 h_X^{(\sigma - \beta) - n/2} \|u\|_\Phi.$$

Proof. Choosing an integer L so that $1/(2L+2) \leq h_X \leq 1/2L$, we can use $\Lambda = CL^{n-1/2}$ as in Lemma III.8 and the estimate in Theorem III.7 to obtain:

$$\begin{aligned}
\|u - u_X\|_2 &\leq \|\mathcal{L}(u - u_X)\|_2 \\
&\leq C_0 \|u\|_\Phi \left(1 + \frac{2}{\Lambda}\right) \max_{t \in S^n} \left(\sum_{\ell > L} \sum_{k=1}^{N(n,\ell)} \widehat{\phi}(\ell) |\mathcal{L} Y_{\ell k}(t)|^2 \right)^{1/2} \\
&\leq C_1 \|u\|_\Phi (1 + 2L^{1/2-n}) \left(\sum_{\ell > L} (1 + \lambda_\ell)^{\beta - \sigma} \frac{N(n,\ell)}{|S^n|} \right)^{1/2} \\
&\leq C_2 \|u\|_\Phi (1 + 2L^{1/2-n}) L^{(\beta-\sigma)+n/2} \\
&\leq C_3 \|u\|_\Phi h_X^{(\sigma-\beta)-n/2}.
\end{aligned}$$

□

D. Implementation of the two methods on S^2

In the implementation of the Galerkin method on $S^2 \subset \mathbb{R}^3$, there are two main issues to be addressed: the quadrature rule used in approximating the bilinear form $a(u,v)$ and the construction of spherical basis functions.

Since $\Phi(x,y)$ is a zonal function, we can reduce the surface integrals in the bilinear form $a(\Phi(x_i,\cdot), \Phi(x_j,\cdot))$ into one dimensional series of Legendre polynomials as discussed in Section D.1. For the surface integrals $\langle f, \Phi(x_i,\cdot)\rangle$'s, we have to derive a quadrature rule over the surface of the unit sphere as in Section D.2.

1. Inner product of two zonal functions

Let $\phi(t)$ and $\psi(t)$, for $t \in [-1,1]$, be two zonal functions on S^2. We can expand $\phi(t)$ and $\psi(t)$ in terms of series of Legendre polynomials

$$\phi(t) = \sum_{\ell=0}^{\infty} a_\ell P_\ell(t), \quad \psi(t) = \sum_{\ell=0}^{\infty} b_\ell P_\ell(t),$$

where

$$a_\ell = \frac{\int_{-1}^{+1} \phi(t) P_\ell(t) dt}{\int_{-1}^{+1} [P_\ell(t)]^2 dt} = \frac{2\ell+1}{2} \int_{-1}^{+1} \phi(t) P_\ell(t) dt \tag{3.14}$$

and

$$b_\ell = \frac{\int_{-1}^{+1} \psi(t) P_\ell(t) dt}{\int_{-1}^{+1} [P_\ell(t)]^2 dt} = \frac{2\ell+1}{2} \int_{-1}^{+1} \psi(t) P_\ell(t) dt. \tag{3.15}$$

In the approximation of the bilinear form $a(u, v) = \langle -\Delta u + \omega^2 u, v\rangle$, we need the following useful lemma:

Lemma III.9 *Let $\Psi(x, y) = \psi(x \cdot y)$ and $\Phi(x, y) = \phi(x \cdot y)$ be two zonal functions on S^2. For two distinct fixed points $p, q \in S^2$, the following relation holds:*

$$\int_{S^2} \phi(p \cdot x)\psi(q \cdot x) dS(x) = 4\pi \sum_{\ell=0}^{\infty} \frac{a_\ell b_\ell}{(2\ell+1)} P_\ell(p \cdot q).$$

Proof. We have

$$\phi(p \cdot x) = \sum_{\ell=0}^{\infty} a_\ell P_\ell(p \cdot x) = 4\pi \sum_{\ell=0}^{\infty} \frac{a_\ell}{(2\ell+1)} \sum_{k=-\ell}^{\ell} Y_{\ell,k}(p)\overline{Y_{\ell,k}(x)},$$

and

$$\psi(q \cdot x) = \sum_{\ell=0}^{\infty} b_\ell P_\ell(q \cdot x) = 4\pi \sum_{\ell=0}^{\infty} \frac{b_\ell}{(2\ell+1)} \sum_{k=-\ell}^{\ell} Y_{\ell,k}(q)\overline{Y_{\ell,k}(x)}.$$

Since $\{Y_{\ell,k} : \ell = 0, 1, 2, \ldots; k = -\ell \ldots \ell\}$ is an orthonormal set, we can use Parseval's identity to obtain

$$\begin{aligned} \int_{S^2} \phi(p \cdot x)\psi(q \cdot x) dS(x) &= 16\pi^2 \sum_{\ell=0}^{\infty} \frac{a_\ell b_\ell}{(2\ell+1)^2} \sum_{k=-\ell}^{\ell} Y_{\ell,k}(p)\overline{Y_{\ell,k}(q)} \\ &= 16\pi^2 \sum_{\ell=0}^{\infty} \frac{a_\ell b_\ell}{(2\ell+1)^2} \frac{(2\ell+1)}{4\pi} P_\ell(p \cdot q) \\ &= 4\pi \sum_{\ell=0}^{\infty} \frac{a_\ell b_\ell}{(2\ell+1)} P_\ell(p \cdot q). \end{aligned}$$

□

For numerical approximation the integration in (3.14) and (3.15) can be approximated by a Gaussian quadrature formula over the interval $[-1, +1]$.

2. Quadrature formula

We seek a spherical quadrature rule that integrates exactly all polynomials up to a certain degree L, i.e., we seek a set of points $\Xi := \{\eta_1, \dots, \eta_N\}$ and a set of positive weights $\{w_1, \dots, w_N\}$ such that

$$\int_{S^2} P(x)dS = \sum_{j=1}^{N} w_j P(\eta_j), \qquad \forall P \in \mathcal{P}_L.$$

If all the weights are equal, namely $w_j = 4\pi/N$ for all $j = 1 \dots N$, then the set Ξ are called spherical L-design, see [1, 7, 40]. It can be shown that a pair of antipodal points, the vertexes of a regular tetrahedron, the regular octahedron, and the regular icosahedron give 1-,2-, and 5-designs, respectively. The following existence theorem, proved in [22], provides a general quadrature formula for S^n.

Theorem III.9 *Let L be an integer with $L \leq \alpha/h_\Xi$, where h_Ξ is the mesh norm of the set Ξ and α is some real constant. Then there exist nonnegative weights $\{w_j : j = 1 \dots N\}$ such that*

$$\int_{S^n} P(x)dS = \sum_{j=1}^{N} w_j P(x_j), \quad \forall P \in \mathcal{P}_L,$$

and the cardinality of the set of weights, N, is comparable to the dimension of $\mathcal{P}_L$.

In principle, Ξ can be any set of scattered points on the unit sphere. However, if the points are uniformly distributed in some sense then the quadrature scheme achieves higher accuracy, see [5, 18, 37]. Here we shall use the set of points that is constructed by dividing the surface of the sphere into N cells of roughly equal area (see [18]).

Let $\mathbf{w} := \{w_1, \ldots, w_N\}$, the weights are computed by solving the following quadratic programming problem:

$$\min \mathbf{w} \cdot \mathbf{w}^T$$

subject to the following linear constraints:

$$\begin{aligned} \sum_{j=1}^{N} w_j Y_{\ell k}(\eta_j) &= 4\pi Y_{0,0}\delta_{\ell,0}, \quad \ell = 0, \ldots, L, \quad -\ell \leq k \leq \ell; \\ w_j &\geq 0, \quad j = 1, \ldots, N. \end{aligned}$$

This optimization program can be solved numerically using the subroutine **quadprog** in MATLAB 6.0. The strategy is to start with a high value of L, say $L = \lfloor \sqrt{N} - 1 \rfloor$, and step it down by 1 until we reach a value of L for which we obtain a solution. Figure 1 shows weights associated with 2500 points calculated according to the previous algorithm.

3. The spherical basis functions

In [48] Wendland introduced a class of locally supported positive definite radial basis function defined on $\mathbb{R}^{n+1}$. These functions $\psi(x)$ are rotation invariant and thus are functions of $|x|$ only. So the corresponding convolution kernel $\psi(x-y)$, $x, y \in S^n$, is a function of $|x-y| = \sqrt{2 - 2x \cdot y}$. We may therefore define a function

$$\Phi(x, y) = \phi(x \cdot y) := \psi(x - y), \quad x, y \in S^n. \tag{3.16}$$

Note that $\Phi(x, y)$ inherits the property of positive definiteness from ψ, and $\widehat{\phi}(\ell) \sim (1 + \lambda_\ell)^{-\sigma}$ for some $\sigma > 0$ (see Section 4 in [30]).

For our numerical study, we use the function $\psi(r) = (1 - r)_+^4(4r + 1)$, where

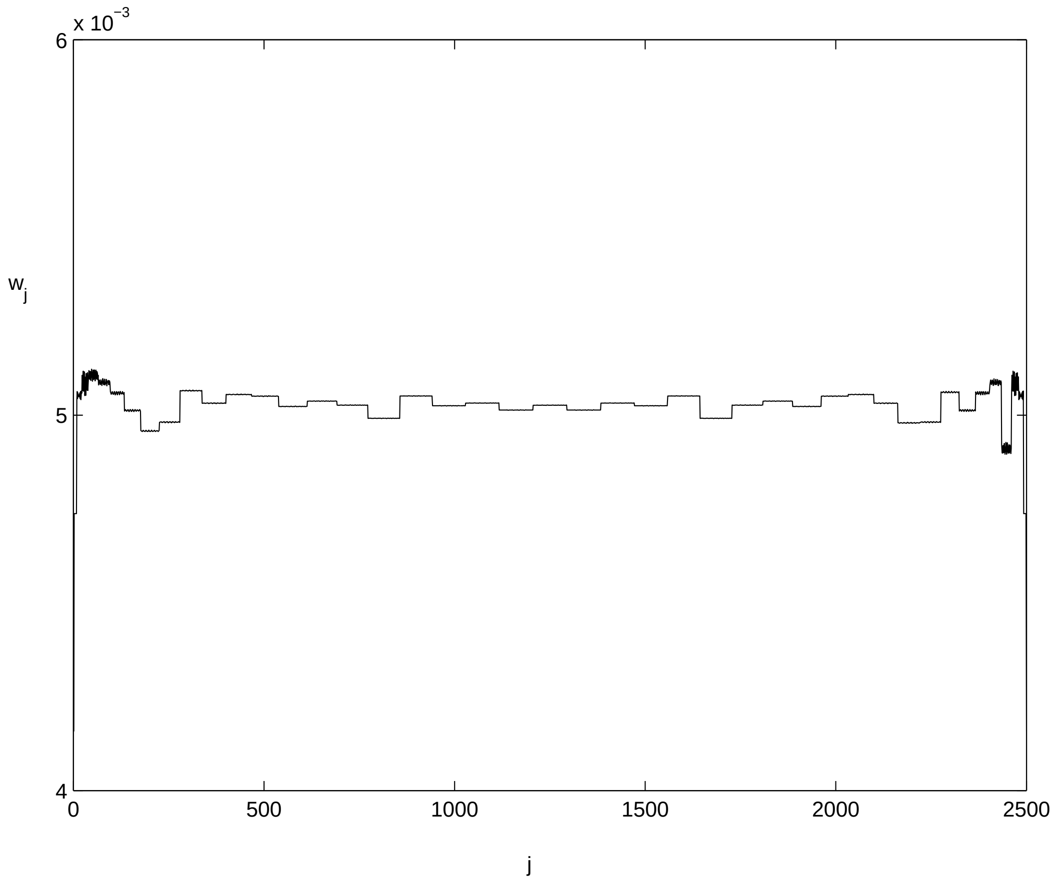

Fig. 1. Weights associated with 2500 quadrature points. The associated quadrature rule integrates exactly all polynomials up to degree 45.

$r = \sqrt{2 - 2x \cdot y}$.

The set of points that are used in constructing the SBFs is generated according to an algorithm in [37]. These points are generated uniformly, in the sense that each point is a center of a cell on the unit sphere of area $4\pi/N$.

4. Stability results for the two methods

The stability of the interpolation matrix $[\Phi(x_i, x_j)]_{i,j=1}^{m}$ has been studied in [33], which can be summarized in the following theorem:

Theorem III.10 *Let $\Phi(x, y) = \phi(x \cdot y)$ be a strictly positive definite function on S^2*

with $\widehat{\phi}(\ell) \sim \ell^{-2\sigma}$*. Then there is a positive constant* γ *depending on* Φ *but not on the data, such that the minimal eigenvalue of the interpolation matrix* $[\Phi(x_i, x_j)]_{i,j=1}^m$ *satisfying*

$$\lambda_{min} \geq \gamma q_X^{2\sigma-2},$$

holds for sufficiently dense data sets X *and this order is optimal.*

The same approach has been used in [19] in proving the stability results when Φ is a strictly positive definite function on S^n.

In our collocation method, the collocation matrix is $A := [\mathcal{L}\mathcal{L}^*\Phi(x_i, x_j)]_{i,j=1}^m$ can be understood as an interpolation matrix having positive definite kernel with $\widehat{\phi}(\ell) \sim (1 + \lambda_\ell)^{\beta-\sigma}$. Using Theorem III.10, we conclude that the minimal eigenvalue of the collocation matrix satisfies

$$\lambda_{min}(A) \geq \gamma q_X^{2\sigma-2\beta-2},$$

where γ is a constant depending on Φ but not on the data.

In the finite element method, we have to find the inverse of matrix B with entries

$$\begin{aligned} B_{i,j} &= \int_{S^2} (-\Delta + \omega^2)\phi(x_i \cdot x)\phi(x \cdot x_j) dS(x) \\ &= \sum_{\ell=0}^{\infty} (\lambda_\ell + \omega^2)(\widehat{\phi}(\ell))^2 \sum_{k=-\ell}^{\ell} Y_{\ell k}(x_i)\overline{Y_{\ell k}(x_j)} \text{ (Lemma III.9)} \end{aligned}$$

The matrix B can be viewed as interpolation matrix having positive definite kernel with $\widehat{\phi}(\ell) \sim (1 + \lambda_\ell)^{-2\sigma+1}$. Using Theorem III.10, we have the minimal eigenvalue of A is bounded below as

$$\lambda_{min}(B) \geq \gamma q_X^{4\sigma-4},$$

where γ is a constant independent from the data.

5. Numerical results

We aim to solve numerically the following differential equation:

$$-\Delta u + \omega^2 u = f,$$

where $f = e^z(z^2 + 2z + \omega^2 - 1)$ and (x, y, z) satisfying $x^2 + y^2 + z^2 = 1$. Note that the function f depends only on the geodesic distance from a point to the north pole. The exact solution of the differential equation is $u = e^z$. Table I shows the errors between the exact solution and the approximate solution obtained via the Galerkin method using the SBFs as in (3.16). The experiments use various values of ω. The errors are computed over a grid $\mathcal{C}$ of 10^4 points on the sphere. The ℓ_2 errors are computed as follows:

$$e_2 := \left(\frac{1}{|\mathcal{C}|} \sum_{\xi \in \mathcal{C}} |u(\xi) - u_X(\xi)|^2 \right)^{1/2}.$$

The supremum errors are computed as

$$e_\infty := \max_{\xi \in \mathcal{C}} |u(\xi) - u_X(\xi)|.$$

Table I. Numerical errors for different values of ω

ω	m	h_X	e_∞	e_2
0.01	100	0.267187	1.1327189	1.1186866
	400	0.128840	0.1239192	0.1233412
	800	0.095016	0.0020322	0.0013973
0.1	100	0.267187	0.3375578	0.3177237
	400	0.128840	0.0019762	0.0013812
	800	0.095016	0.0020748	3.0189262e-04
	1600	0.067870	4.2467432e-04	3.4931946e-04
1.0	100	0.267187	0.0258676	0.0074494
	400	0.128840	6.863848e-04	1.239210e-04
	1600	0.067870	4.596362e-05	1.325570e-05
10	100	0.267187	0.0213299	0.0058496
	400	0.128840	7.064579e-04	1.228608e-04
	1000	0.084946	6.519814e-04	1.275973e-04
	1600	0.067870	4.827265e-04	8.777684e-05

CHAPTER IV

APPROXIMATION OF PARABOLIC PDEs ON SPHERES

In this chapter we consider the following parabolic partial differential equation defined on the unit sphere $S^n \subset \mathbb{R}^{n+1}$:

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) - \Delta u(x,t) &= F(x,t) \\ u(x,0) &= f(x), \qquad f \in H^s(S^n), \end{cases} \tag{4.1}$$

where Δ is the Laplace-Beltrami operator on S^n. It is known that equation (4.1) describes the heat diffusion process on the surface of the sphere with external heat source $F(x,t)$.

In many applications in geophysics and global weather forecast, it is common that the functions f and F are not known analytically everywhere but only at a finite set of scattered points.

We propose a collocation method in which the spherical basis functions are used to construct the approximate solution. The approximate solution of the partial differential equation will be of the form

$$u_X(x,t) = \sum_{i=1}^{m} c_i(t)\phi_i(x),$$

subject to the initial condition

$$u_X(x,0) = I_X f(x),$$

where $\phi_i(x) = \phi(x_i \cdot x) = \Phi(x_i, x)$'s are the shifts of a spherical basis function (SBF) ϕ and $I_X f$ is the SBF interpolant of the function f. In case the basis function ϕ satisfies certain regularity conditions, we are able to obtain error estimates in certain Sobolev norms. Throughout the chapter we make further assumption that $\widehat{\phi}(\ell) \sim (1+\lambda_\ell)^{-\sigma}$, i.e. there are positive constants c and C and $\sigma > n/2$ such that

$$c(1+\lambda_\ell)^{-\sigma} \leq \widehat{\phi}(\ell) \leq C(1+\lambda_\ell)^{-\sigma}. \tag{4.2}$$

A. Semi-discrete problem

The numerical analysis here follows a framework set out in [45], which was used to analyze the approximation of solutions of the heat equation on a bounded domain $\Omega \subset \mathbb{R}^n$ for the finite element method. However, the framework of [45] is modified significantly with the structure of the reproducing kernel Hilbert space N_Φ for a collocation method on S^n.

1. The homogeneous problem

By the method of separation of variables, see [36, §5.7], the exact solution for the homogeneous problem:

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) &= \Delta u(x,t) \\ u(x,0) &= f(x), \quad f \in L^2(S^n), \end{cases}$$

is given as the infinite series

$$u(x,t) = \sum_{\ell=0}^{\infty} e^{-\lambda_\ell t} \sum_{k=1}^{N(n,\ell)} \widehat{f}_{\ell k} Y_{\ell k}(x).$$

Let the approximate solution be of the following form:

$$u_X(x,t) = \sum_{i=1}^{m} c_i(t)\phi_i(x),$$

where $\phi_i(x) := \Phi(x_i, x)$. The homogeneous semi-discrete problem is formulated as the following: we require the equation (4.1) to be exact on the set X, i.e.

$$\begin{cases} \frac{\partial}{\partial t} u_X(x_j, t) &= \Delta u_X(x_j, t), \quad \forall x_j \in X, \\ u_X(x, 0) &= I_X f(x), \end{cases} \tag{4.3}$$

where $I_X f$ is the interpolant of f in V_X. Equation (4.3) can be rewritten as the following:

$$\frac{d}{dt} \sum_{i=1}^{m} c_i(t)\phi_i(x_j) = \sum_{i=1}^{m} c_i(t)\Delta\phi_i(x_j), \quad \forall x_j \in X, \tag{4.4}$$

subject to the following initial condition:

$$\sum_{i=1}^{m} c_i(0)\phi_i(x_j) = f(x_j), \quad \forall x_j \in X.$$

If we set $A := [\phi_i(x_j)]_{i,j=1,\ldots,m}$ and $B := [\Delta\phi_i(x_j)]_{i,j=1,\ldots,m}$ then equation (4.4) can be written as the following system of ordinary differential equations in time:

$$\frac{d}{dt}\mathbf{c}(t) = A^{-1}B\mathbf{c}(t), \tag{4.5}$$

where $\mathbf{c}(t) = [c_1(t), \ldots, c_m(t)]^T$. It is known that (see, for example, [27]), in order to solve the system (4.5), we have to compute the distinct eigenvalues $r_1, \ldots, r_k$ of the matrix $A^{-1}B$ with multiplicities $n_1, \ldots, n_k$. For each eigenvalue r_i, we find n_i linearly independent generalized eigenvectors. Each independent solution of (4.5) is of the form

$$\exp(A^{-1}Bt)\mathbf{v} = e^{rt}\left(\mathbf{v} + t(A^{-1}B - rI)\mathbf{v} + \frac{t^2}{2}(A^{-1}B - rI)^2\mathbf{v} + \ldots\right),$$

where r is an eigenvalue and $\mathbf{v}$ is a corresponding generalized eigenvector. If r has multiplicity n_i, then the above series reduces to the first n_i terms. The linearly independent solutions form column vectors of a matrix $E(t)$, and then the fundamental matrix $\exp(A^{-1}Bt)$ is given as

$$\exp(A^{-1}Bt) = E(t)E^{-1}(0).$$

The solution of the homogeneous semi-discrete problem is

$$u_X(x,t) = [\phi_1(x) \dots \phi_m(x)] \exp(A^{-1}Bt)\mathbf{c}(0) \text{ where } \mathbf{c}(0) = A^{-1}f|_X. \tag{4.6}$$

We shall express the solution $u_X(x,t)$ in terms of some evolution operator. Let us consider the following operator:

$$\begin{aligned} \Delta I_X : C(S^n) &\rightarrow \text{span } \{\Delta\phi_i(x) : i = 1, \dots, m\} \\ f &\mapsto \Delta(I_X f). \end{aligned}$$

Lemma IV.1 *At the set of points X, we have*

$$A(A^{-1}B)^n A^{-1}[f(x_j)]_{j=1}^m = [(\Delta I_X)^n f|_{x=x_j}]_{j=1}^m.$$

Here and thereafter, the notation $[a_j]_{j=1}^m$ stands for $[a_1 \dots a_m]^T$ which is a vector in $\mathbb{R}^m$.

Proof. For $n = 1$, we have

$$AA^{-1}BA^{-1}[f(x_j)]_{j=1}^m = BA^{-1}[f(x_j)]_{j=1}^m = [\Delta I_X f|_{x=x_j}]_{j=1}^m.$$

Now assume that for $k > 1$

$$A(A^{-1}B)^k A^{-1}[f(x_j)]_{j=1}^m = [(\Delta I_X)^k f(x_j)]_{j=1}^m$$

Then

$$\begin{aligned} A(A^{-1}B)^{k+1}A^{-1}[f(x_j)]_{j=1}^m &= A(A^{-1}B)(A^{-1}B)^kA^{-1}[f(x_j)]_{j=1}^m \\ &= BA^{-1}A(A^{-1}B)^kA^{-1}[f(x_j)]_{j=1}^m \\ &= BA^{-1}[(\Delta I_X)^k f|_{x=x_j}]_{j=1}^m \\ &= [(\Delta I_X)(\Delta I_X)^k f|_{x=x_j}]_{j=1}^m \\ &= [(\Delta I_X)^{k+1} f|_{x=x_j}]_{j=1}^m. \end{aligned}$$

□

Lemma IV.2 *For small $t > 0$ we have*

$$u_X(x,t) = I_X f + tI_X\Delta I_X f + \frac{t^2}{2}I_X(\Delta I_X)^2 f + \ldots + \frac{t^n}{n!}I_X(\Delta I_X)^n f + \ldots$$

Proof. Using equation (4.6) and Lemma IV.1, we have

$$\begin{aligned} [u_X(x_j,t)]_{j=1}^m &= A\exp(A^{-1}Bt)A^{-1}[f(x_j)]_{j=1}^m \\ &= A\left(I + tA^{-1}B + \ldots \frac{t^n}{n!}(A^{-1}B)^n + \ldots\right)A^{-1}[f(x_j)]_{j=1}^m \\ &= [f(x_j)]_{j=1}^m + t[\Delta I_X f|_{x=x_j}]_{j=1}^m + \ldots + \frac{t^n}{n!}[(\Delta I_X)^n f|_{x=x_j}]_{j=1}^m + \ldots \end{aligned}$$

Since $u_X \in V_X$, this implies

$$u_X(x,t) = I_X f + tI_X\Delta I_X f + \ldots + \frac{t^n}{n!}I_X(\Delta I_X)^n f + \ldots$$

□

Let us define the following evolution operator

$$E_X(t) := I + tI_X\Delta + \ldots + \frac{t^n}{n!}(I_X\Delta)^n + \ldots$$

then $u_X(x,t) = E_X(t)I_X f(x)$. We can show that $E_X(t)$ is a stable operator in V_X in the $\|\cdot\|_\Phi$ norm by the following lemma:

Lemma IV.3 *For every $\psi \in V_X$,*

$$\|E_X(t)\psi(x)\|_\Phi \leq \|\psi(x)\|_\Phi.$$

Proof. Let $\theta(x,t)$ be defined as

$$\theta(x,t) = \sum_{i=1}^{m} c_i(t)\phi_i(x).$$

We wish to solve the following PDE by a collocation method

$$\frac{\partial}{\partial t}\theta(x,t) = \Delta\theta(x,t),$$

subject to the initial condition

$$\theta(x,0) = \psi(x).$$

In our collocation method, it is required that the PDE is exact on the set of given points X, i.e.

$$\frac{\partial}{\partial t}\theta(x_j,t) = \Delta\theta(x_j,t), \quad \forall x_j \in X,$$

subject to the initial condition

$$\theta(x_j,0) = \psi(x_j), \quad \forall x_j \in X.$$

Since Φ is the reproducing kernel in the Hilbert space N_Φ,

$$\left\langle \frac{\partial}{\partial t}\theta(\cdot,t), \Phi(\cdot,x_j) \right\rangle_\Phi = \langle \Delta\theta(\cdot,t), \Phi(\cdot,x_j)\rangle_\Phi, \quad \forall x_j \in X. \tag{4.7}$$

Since V_X is spanned by $\Phi(x,x_j)$, for $j = 1,\ldots,m$, equation (4.7) implies that for every function $v \in V_X$,

$$\left\langle \frac{\partial\theta}{\partial t}, v \right\rangle_\Phi = \langle \Delta\theta, v\rangle_\Phi .$$

Since $\theta \in V_X$, we can take $v = \theta$ to obtain

$$\frac{1}{2}\frac{\partial}{\partial t}\|\theta\|_\Phi^2 = \left\langle \frac{\partial \theta}{\partial t}, \theta \right\rangle_\Phi = \langle \Delta\theta, \theta \rangle_\Phi .$$

From the definition of $\langle \cdot, \cdot \rangle_\Phi$,

$$\langle \Delta\theta, \theta \rangle_\Phi = \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(n,\ell)} \frac{-\lambda_\ell |\widehat{\theta}_{\ell k}|^2}{\widehat{\phi}(\ell)} \leq 0, \quad \forall \theta \in V_X. \tag{4.8}$$

Thus, we obtain the result $\|\theta(x,t)\|_\Phi \leq \|\theta(x,0)\|_\Phi$ or in other words

$$\|E_X(t)\psi(x)\|_\Phi \leq \|\psi(x)\|_\Phi.$$

□

2. The non-homogeneous problem

The approximation of the non-homogeneous equation will be tackled via an elliptic projection from the space of the exact solution u to the finite dimensional space V_X, which is somehow similar to the Ritz projection in the finite element method. To begin, let us define the following operator:

$$\begin{aligned} P : H^{2\sigma+2}(S^n) &\rightarrow V_X \\ u &\mapsto u_P, \end{aligned}$$

where

$$\begin{cases} \Delta u_P(x_j) = \Delta u(x_j) \qquad \forall x_j \in X, \\ \int_{S^n} u_P dS = \int_{S^n} u dS. \end{cases} \tag{4.9}$$

It is noted that Δ has zero as an eigenvalue, thus the matrix $B = [\Delta\Phi(x_i, x_j)]_{i,j=1\ldots m}$ is not invertible. The null space of B has dimension 1. We fix the null space problem by finding $u_P = \sum_{j=1}^{m} \alpha_j \phi_j(x)$, where $\alpha := (\alpha_1, \ldots, \alpha_m)^T$ solves the following system of linear equations

$$\begin{cases} B\alpha & = & [\Delta u(x_j)]_{j=1}^m \\ \sum_{i=1}^m \alpha_i \int_{S^n} \phi_i dS & = & \int_{S^n} u dS. \end{cases}$$

We notice that u_P is well-defined since the solution α is unique. It is also from the definition that

$$I_X \Delta P = I_X \Delta. \tag{4.10}$$

Lemma IV.4 *Let $u \in H^{2\sigma+2}(S^n)$, and $u_P \in V_X$ be constructed from a linear combination of shifts of SBF Φ with $\widehat{\phi}(\ell) \sim (1+\lambda_\ell)^{-\sigma}$. Then there is a positive constant C, independent of h_X, so that*

$$\|u_P - u\|_\Phi \le C h_X^\sigma \|u\|_{H^{2\sigma+2}}.$$

Proof. Since Δu_P is the interpolation of Δu, by Theorem I.3, we have

$$\|\Delta u_P - \Delta u\|_\Phi \le C h_X^\sigma \|\Delta u\|_{H^{2\sigma}} \le C h_X^\sigma \|u\|_{H^{2\sigma+2}}. \tag{4.11}$$

Let $\psi = u_P - u$, then from the definition (4.9)

$$\widehat{\psi}_0 = \int_{S^n} (u_P - u) dS = 0.$$

Hence,

$$\begin{aligned} \|\psi\|_\Phi^2 = \sum_{\ell=0}^\infty \sum_{k=1}^{N(n,\ell)} |\widehat{\psi}_{\ell k}|^2 / \widehat{\phi}(\ell) \quad & \le \quad \left(|\widehat{\psi}_0|^2 + \sum_{\ell=1}^\infty \sum_{k=1}^{N(n,\ell)} \lambda_\ell^2 |\widehat{\psi}_{\ell k}|^2 / \widehat{\phi}(\ell) \right) \\ & \le \quad \|\Delta \psi\|_\Phi^2. \end{aligned}$$

Combining with (4.11), we have

$$\|u_P - u\|_\Phi \le \|\Delta u_P - \Delta u\|_\Phi \le C h_X^\sigma \|u\|_{H^{2\sigma+2}}.$$

□

The collocation semi-discrete equation (4.3) now takes the following form

$$\frac{\partial}{\partial t}u_X(x_j,t) - \Delta u_X(x_j,t) = F(x_j,t), \quad \forall x_j \in X, \tag{4.12}$$

subject to the initial condition

$$u_X(x,0) = I_X f(x).$$

Theorem IV.1 *Let $f, u_t \in H^{2\sigma+2}(S^n)$ and u, u_X be the solution for (4.1) and (4.12) respectively. The approximate solution u_X is constructed as a linear combination of shifts of a spherical basis function $\Phi(x,y) = \phi(x\cdot y)$ which satisfies $\widehat{\phi}(\ell) \sim (1+\lambda_\ell)^{-\sigma}$. Then there is a constant C, independent of h_X, so that the following error estimate holds:*

$$\|u(T) - u_X(T)\|_\Phi \le Ch_X^\sigma \left(\|f\|_{H^{2\sigma}} + \|f\|_{H^{2\sigma+2}} + \int_0^T \|u_t\|_{H^{2\sigma+2}} ds \right).$$

Proof. Let $\theta := u_X - u_P$, and let $\gamma := u_P - u$. Note that $\theta \in V_X$. When being restricted on the set X, using the relation $\Delta u_P|_X = \Delta u|_X$ we have the following equations:

$$\begin{aligned} \left(\frac{\partial}{\partial t}\theta - \Delta\theta\right)\Big|_X &= \left(\frac{\partial}{\partial t}u_X - \Delta u_X\right)\Big|_X - \left(\frac{\partial}{\partial t}u_P - \Delta u_P\right)\Big|_X \\ &= F|_X - \left(\frac{\partial u_P}{\partial t} - \Delta u\right)\Big|_X \\ &= F|_X - \left(\frac{\partial u}{\partial t} - \Delta u\right)\Big|_X + \left(\frac{\partial u}{\partial t} - \frac{\partial u_P}{\partial t}\right)\Big|_X \\ &= \frac{\partial}{\partial t}(u - u_P)\Big|_X, \end{aligned}$$

or in terms of a PDE in the finite dimensional space V_X,

$$\frac{\partial \theta}{\partial t} - \Delta\theta = -I_X\frac{\partial \gamma}{\partial t}. \tag{4.13}$$

By Duhamel's principle, see [36, §3.11], we have

$$\theta(T) = E_X(T)\theta(0) - \int_0^T E_X(T-s) I_X \frac{\partial \gamma}{\partial t} ds.$$

Since $\|E_X(T)v\|_\Phi \leq \|v\|_\Phi$ for all $v \in V_X$ (by Lemma IV.3), we have

$$\|\theta(T)\|_\Phi \leq \|\theta(0)\|_\Phi + \int_0^T \left\| I_X \frac{\partial \gamma}{\partial t}(s) \right\|_\Phi ds.$$

Here,

$$\begin{aligned} \|\theta(0)\|_\Phi = \|I_X f - Pf\|_\Phi &\leq \|I_X f - f\|_\Phi + \|Pf - f\|_\Phi \\ &\leq Ch^\sigma \left(\|f\|_{H^{2\sigma}} + \|f\|_{H^{2\sigma+2}} \right). \end{aligned}$$

We can use Lemma IV.4 to obtain

$$\|\gamma_t\|_\Phi = \left\| \frac{\partial}{\partial t}(u - u_P) \right\|_\Phi \leq Ch_X^\sigma \|u_t\|_{H^{2\sigma+2}}.$$

Using Lemma I.1, we obtain

$$\begin{aligned} \|I_X \gamma_t\|_\Phi &\leq \|\gamma_t\|_\Phi \\ &\leq Ch_X^\sigma \|u_t\|_{H^{2\sigma+2}}. \end{aligned}$$

We know from Lemma IV.4 that

$$\begin{aligned} \|\gamma(T)\|_\Phi &= \|u(T) - u_P(T)\|_\Phi \\ &\leq Ch_X^\sigma \|u(T)\|_{H^{2\sigma+2}} \\ &\leq Ch_X^\sigma \left(\left\| f + \int_0^T u_t(s) ds \right\|_{H^{2\sigma+2}} \right) \\ &\leq Ch_X^\sigma \left(\|f\|_{H^{2\sigma+2}} + \int_0^T \|u_t\|_{H^{2\sigma+2}} ds \right). \end{aligned}$$

Therefore, after adjusting the constant C, we obtain

$$\begin{aligned}\|u - u_X\|_\Phi &\leq \|\theta(T)\|_\Phi + \|\gamma(T)\|_\Phi \\ &\leq Ch_X^\sigma \left(\|f\|_{H^{2\sigma}} + \|f\|_{H^{2\sigma+2}} + \int_0^T \|u_t\|_{H^{2\sigma+2}} ds \right).\end{aligned}$$

□

B. Time discretization

1. Backward Euler method

Let us discretize the time derivative using backward Euler method as

$$\frac{u(x,t) - u(x,t-\tau)}{\tau} + o(1) - \Delta u(x,t) = F(x,t).$$

The collocation equation for u_X is

$$u_X(x_j,t) - u_X(x_j,t-\tau) - \tau\Delta u_X(x_j,t) = \tau F(x_j,t), \quad \forall x_j \in X. \tag{4.14}$$

Let us define $t_N := N\tau$, $U_N(x) := u_X(x,t_N)$ and introduce the notation

$$\overline{\partial}_t U_N := \frac{U_N - U_{N-1}}{\tau}.$$

The collocation equation (4.14) can be rewritten as

$$\overline{\partial}_t U_N(x_j) - \Delta U_N(x_j) = F(x_j,t_N), \quad \forall x_j \in X, \tag{4.15}$$

subject to the initial condition

$$U_0 = I_X f.$$

If we write $U_N = \sum_{i=1}^m c_{N,i}\phi_i(x)$ then in terms of matrices A and B, defined in Section 3, we have

$$(A - \tau B)\mathbf{c}_N = A\mathbf{c}_{N-1} + \tau[F(x_j,N\tau)]_{j=1}^m, \tag{4.16}$$

with the initial condition

$$A\mathbf{c}_0 = [f(x_j)]_{j=1}^m.$$

We now estimate the difference between U_N and the exact solution u at the time t_N.

Theorem IV.2 *Let us assume that $u_{tt} \in H^\sigma(S^n)$ and $u_t, f \in H^{2\sigma+2}(S^n)$ and let U_N be the solution of (4.15). The approximate solution U_N is constructed from shifts of a spherical basis function Φ with $\widehat{\phi}(\ell) \sim (1+\lambda_\ell)^{-\sigma}$. Then there are positive constants C_1 and C_2 so that we have the following error estimate:*

$$\|U_N - u(t_N)\|_\Phi \le C_1 h_X^\sigma \Gamma(f, u_t) + C_2 \tau \int_0^{t_N} \|u_{tt}\|_{H^\sigma} ds.$$

where

$$\Gamma(f, u_t) := \|f\|_{H^{2\sigma}} + \|f\|_{H^{2\sigma+2}} + \int_0^{t_N} \|u_t(s)\|_{H^{2\sigma+2}} ds + \tau \sum_{j=1}^N \|u_t(t_j)\|_{H^{2\sigma}}.$$

Proof.

$$U_N - u(t_N) = U_N - Pu(t_N) + Pu(t_N) - u(t_N) =: \theta_N + \gamma_N.$$

We already know

$$\begin{aligned} \|Pu(t_N) - u(t_N)\|_\Phi &= \|\gamma_N\|_\Phi \\ &\le Ch_X^\sigma \|u(t_N)\|_{H^{2\sigma+2}} \\ &\le Ch_X^\sigma \left(\|f\|_{H^{2\sigma+2}} + \int_0^{t_N} \|u_t(s)\|_{H^{2\sigma+2}} ds \right). \end{aligned}$$

Similar to (4.13), we have

$$\overline{\partial}_t \theta_N(x_j) - \Delta\theta_N(x_j) = -\omega_N(x_j), \quad \forall x_j \in X, \tag{4.17}$$

where

$$\omega_N = \overline{\partial}_t Pu(t_N) - I_X u_t(t_N).$$

We can rewrite equation (4.17) as

$$(1-\tau\Delta)\theta_N(x_j) = \theta_{N-1}(x_j) - \tau\omega_N(x_j), \quad \forall x_j \in X. \tag{4.18}$$

In terms of the inner-product $\langle \cdot, \cdot \rangle_\Phi$ in the reproducing kernel Hilbert space N_Φ,

$$\langle \theta_N - \tau\Delta\theta_N, \Phi(x_j, \cdot) \rangle_\Phi = \langle \theta_{N-1} - \tau\omega_N, \Phi(x_j, \cdot) \rangle_\Phi, \quad \forall x_j \in X. \tag{4.19}$$

Since V_X is spanned by $\Phi(x_j, \cdot)$'s, $j = 1, \ldots, m$, this means for every $v \in V_X$,

$$\langle \theta_N - \tau\Delta\theta_N, v \rangle_\Phi = \langle \theta_{N-1} - \tau\omega_N, v \rangle_\Phi. \tag{4.20}$$

By taking $v = \theta_N$, we have

$$\begin{aligned} \langle \theta_N - \tau\Delta\theta_N, \theta_N \rangle_\Phi &= \langle \theta_{N-1} - \tau\omega_N, \theta_N \rangle_\Phi \\ \|\theta_N\|_\Phi^2 - \tau \langle \Delta\theta_N, \theta_N \rangle_\Phi &= \langle \theta_{N-1}, \theta_N \rangle_\Phi - \tau \langle \omega_N, \theta_N \rangle_\Phi. \end{aligned}$$

Since $\langle \Delta\theta_N, \theta_N \rangle_\Phi \leq 0$ (cf. inequality (4.8)), we can conclude

$$\begin{aligned} \|\theta_N\|_\Phi^2 &\leq \langle \theta_{N-1}, \theta_N \rangle_\Phi + \tau |\langle \omega_N, \theta_N \rangle_\Phi| \\ &\leq \|\theta_{N-1}\|_\Phi \|\theta_N\|_\Phi + \tau \|\omega_N\|_\Phi \|\theta_N\|_\Phi. \end{aligned}$$

Simplifying $\|\theta_N\|_\Phi$ on both sides, we obtain

$$\|\theta_N\|_\Phi \leq \|\theta_{N-1}\|_\Phi + \tau\|\omega_N\|_\Phi.$$

By repeated application,

$$\|\theta_N\|_\Phi \leq \|\theta_0\|_\Phi + \tau \sum_{j=1}^{N} \|\omega_j\|_\Phi.$$

Here, as before,

$$\begin{aligned}\|\theta_0\|_\Phi &= \|I_X f - Pf\|_\Phi \\ &\leq Ch_X^\sigma(\|f\|_{H^{2\sigma}} + \|f\|_{H^{2\sigma+2}}).\end{aligned}$$

Now for every $1 \leq j \leq N$,

$$\begin{aligned}\omega_j &= \overline{\partial}_t Pu(t_j) - \overline{\partial}_t u(t_j) + (\overline{\partial}_t u(t_j) - I_X u_t(t_j)) \\ &=: \omega_{j,1} + \omega_{j,2}.\end{aligned}$$

We note that

$$\omega_{j,1} = (P-I)\tau^{-1}\int_{t_{j-1}}^{t_j} u_t ds = \tau^{-1}\int_{t_{j-1}}^{t_j}(P-I)u_t ds,$$

whence

$$\begin{aligned}\tau\sum_{j=1}^N \|\omega_{j,1}\|_\Phi &\leq \sum_{j=1}^N Ch_X^\sigma \int_{t_{j-1}}^{t_j} \|u_t(s)\|_{H^{2\sigma+2}} ds \\ &= Ch_X^\sigma \int_0^{t_N} \|u_t(s)\|_{H^{2\sigma+2}} ds.\end{aligned}$$

Further,

$$\begin{aligned}\omega_{j,2} &= \frac{u(t_j) - u(t_{j-1})}{\tau} - u_t(t_j) + u_t(t_j) - I_X u_t(t_j) \\ &= -\frac{1}{\tau}\int_{t_{j-1}}^{t_j}(s - t_{j-1})u_{tt}(s) ds + u_t(t_j) - I_X u_t(t_j),\end{aligned}$$

so that

$$\begin{aligned}\tau\sum_{j=1}^N \|\omega_{j,2}\|_\Phi &\leq \sum_{j=1}^N \left\|\int_{t_{j-1}}^{t_j}(s-t_{j-1})u_{tt}(s)ds\right\|_\Phi + \tau\sum_{j=1}^N \|u_t(t_j) - I_X u_t(t_j)\|_\Phi \\ &\leq \tau\int_0^{t_N}\|u_{tt}\|_\Phi ds + C\tau h_X^\sigma \sum_{j=1}^N \|u_t(t_j)\|_{H^{2\sigma}}.\end{aligned}$$

Therefore, by setting $C_1 := C$ and noting $\|\cdot\|_\Phi \sim \|\cdot\|_{H^\sigma}$, we obtain a constant C_2 so that

$$\begin{aligned}\tau\sum_{j=1}^N \|\omega_j\|_\Phi &\le \tau\sum_{j=1}^N \|\omega_{j,1}\|_\Phi + \tau\sum_{j=1}^N \|\omega_{j,2}\|_\Phi \\ &\le C_1 h_X^\sigma \left(\int_0^{t_N} \|u_t(s)\|_{H^{2\sigma+2}} ds + \tau\sum_{j=1}^N \|u_t(t_j)\|_{H^{2\sigma}}\right) \\ &\quad + C_2\tau\int_0^{t_N} \|u_{tt}(s)\|_{H^\sigma} ds.\end{aligned}$$

Thus

$$\begin{aligned}\|u(T)-U_N(T)\|_\Phi &\le \|\gamma_N\|_\Phi + \|\theta_N\|_\Phi \\ &\le \|\gamma_N\|_\Phi + \|\theta_0\|_\Phi + \tau\sum_{j=1}^N \|\omega_j\|_\Phi \\ &\le C_1 h_X^\sigma \Gamma(f,u_t) + C_2\tau\int_0^{t_N} \|u_{tt}(s)\|_{H^\sigma} ds,\end{aligned}$$

where

$$\Gamma(f,u_t) := \|f\|_{H^{2\sigma}} + \|f\|_{H^{2\sigma+2}} + \int_0^{t_N} \|u_t(s)\|_{H^{2\sigma+2}} ds + \tau\sum_{j=1}^N \|u_t(t_j)\|_{H^{2\sigma}}.$$

□

2. Crank-Nicolson method

We now turn to the Crank-Nicolson method in which the semi-discrete equation is discretized in a symmetric fashion around the point $t_{N-1/2} := (N-1/2)\tau$, which will produce a second order in time accurate method. More precisely, U_N in V_X can be defined recursively by

$$\overline{\partial}_t U_N(x_j) - \Delta(U_N(x_j) + U_{N-1}(x_j))/2 = F(x_j, t_{N-1/2}), \quad \forall x_j \in X, \tag{4.21}$$

given that

$$U_0 = I_X f.$$

In matrix form

$$(A - \frac{1}{2}\tau B)\mathbf{c}_N = (A + \frac{1}{2}\tau B)\mathbf{c}_{N-1} + \tau[F(x_j, t_{N-1/2})]_{j=1}^m,$$

given that

$$A\mathbf{c}_0 = [f(x_j)]_{j=1}^m.$$

Theorem IV.3 *Let U_N and u be the solutions of (4.21) and (4.1), respectively. We assume that $f, u_t \in H^{2\sigma+2}(S^n)$ and $u_{ttt}, \Delta u_{tt} \in H^\sigma(S^n)$. The approximate solution U_N is constructed from shifts of a spherical basis function Φ with $\widehat{\phi}(\ell) \sim (1+\lambda_\ell)^{-\sigma}$. Then there are positive constants C_1 and C_2, independent of h_X, so that the following holds:*

$$\|U_N - u(t_N)\|_\Phi \le C_1 h_X^\sigma \Gamma(f, u_t) + C_2\tau^2 \left(\int_0^{t_N} \|u_{ttt}\|_{H^\sigma} + \|\Delta u_{tt}\|_{H^\sigma} ds \right),$$

where

$$\Gamma(f, u_t) \;\; := \;\; \|f\|_{H^{2\sigma}} + \|f\|_{H^{2\sigma+2}} + \int_0^{t_N} \|u_t(s)\|_{H^{2\sigma+2}} ds + \tau \sum_{j=1}^N \|u_t(t_{j-1/2})\|_{H^{2\sigma}}.$$

Proof. Let

$$U_N - u(t_N) = U_N - Pu(t_N) + Pu(t_N) - u(t_N) =: \theta_N + \gamma_N.$$

With the above notation we have

$$\overline{\partial}_t \theta_N(x_j) - \Delta(\theta_N(x_j) + \theta_{N-1}(x_j))/2 = -\eta_N(x_j), \quad \forall x_j \in X,$$

where now

$$\begin{aligned}
\eta_N &= \overline{\partial}_t Pu(t_N) - \partial_t I_X u(t_{N-1/2}) + I_X \Delta \left(u(t_{N-1/2}) - \frac{u(t_N) + u(t_{N-1})}{2} \right) \\
&= (P - I)\overline{\partial}_t u(t_N) + (\overline{\partial}_t u(t_N) - I_X u_t(t_{N-1/2})) + \\
&\quad I_X \Delta \left(u(t_{N-1/2}) - \frac{u(t_N) + u(t_{N-1})}{2} \right) \\
&=: \eta_{N,1} + \eta_{N,2} + \eta_{N,3}
\end{aligned}$$

Applying arguments similar to (4.19) and (4.20) we arriving at

$$\left\langle \theta_N - \theta_{N-1} - \frac{\tau}{2}\Delta(\theta_N + \theta_{N-1}), \chi \right\rangle_\Phi = -\tau \left\langle \eta_N, \chi \right\rangle_\Phi, \quad \forall \chi \in V_X.$$

By taking $\chi = \theta_N + \theta_{N-1}$ and note that $\langle \Delta(\theta_N + \theta_{N-1}), \theta_N + \theta_{N-1} \rangle_\Phi \leq 0$ (cf. inequality (4.11)), we have

$$\|\theta_N\|_\Phi^2 - \|\theta_{N-1}\|_\Phi^2 \leq -\tau \left\langle \eta_N, (\theta_N + \theta_{N-1}) \right\rangle_\Phi \leq \tau \|\eta_N\|_\Phi (\|\theta_N\|_\Phi + \|\theta_{N-1}\|_\Phi).$$

Simplifying the common factor $(\|\theta_N\|_\Phi + \|\theta_{N-1}\|_\Phi)$ on both sides of the inequality, we obtain

$$\|\theta_N\|_\Phi \leq \|\theta_{N-1}\|_\Phi + \tau \|\eta_N\|_\Phi.$$

After repeated application this yields

$$\|\theta_N\|_\Phi \leq \|\theta_0\|_\Phi + \tau \sum_{j=1}^{N} (\|\eta_{j,1}\|_\Phi + \|\eta_{j,2}\|_\Phi + \|\eta_{j,3}\|_\Phi).$$

The term $\|\theta_0\|_\Phi$ can be estimated as before. For the latter sum, we have

$$\begin{aligned}
\|\eta_{j,1}\|_\Phi &= \|(P - I)\overline{\partial}_t u(t_j)\|_\Phi \\
&\leq C\tau^{-1} h_X^\sigma \int_{t_{j-1}}^{t_j} \|u_t\|_{H^{2\sigma+2}} ds.
\end{aligned}$$

Further,

$$\begin{aligned}
\|\eta_{j,2}\|_\Phi &= \|\overline{\partial}_t u(t_j) - I_X u_t(t_{j-1/2})\|_\Phi \\
&\leq \|\overline{\partial}_t u(t_j) - u_t(t_{j-1/2})\|_\Phi + \|u_t(t_{j-1/2}) - I_X u_t(t_{j-1/2})\|_\Phi \\
&= \frac{1}{2\tau}\left\|\int_{t_{j-1}}^{t_{j-1/2}} (s-t_{j-1})^2 u_{ttt}(s)ds + \int_{t_{j-1/2}}^{t_j} (s-t_j)^2 u_{ttt}(s)ds\right\|_\Phi \\
&\quad + \|u_t(t_{j-1/2}) - I_X u_t(t_{j-1/2})\|_\Phi \\
&\leq \tau \int_{t_{j-1}}^{t_j} \|u_{ttt}\|_\Phi ds + C h_X^\sigma \|u_t(t_{j-1/2})\|_{H^{2\sigma}}.
\end{aligned}$$

Let

$$\begin{aligned}
\psi &:= u(t_{j-1/2}) - \frac{u(t_j)+u(t_{j-1})}{2} \\
&= \frac{1}{2}\int_{t_{j-1}}^{t_{j-1/2}} (t_{j-1}-s)u_{tt}(s)ds + \frac{1}{2}\int_{t_{j-1/2}}^{t_j} (s-t_j)u_{tt}(s)ds.
\end{aligned}$$

Then, we have

$$\begin{aligned}
\|\eta_{j,3}\|_\Phi &= \|I_X \Delta(u(t_{j-1/2}) - \frac{1}{2}(u(t_j)+u(t_{j-1})))\|_\Phi \\
&= \|I_X \Delta\psi\|_\Phi \leq \|\Delta\psi\|_\Phi \text{ (see Lemma I.1)} \\
&\leq C_2\tau \int_{t_{j-1}}^{t_j} \|\Delta u_{tt}\|_{H^\sigma} ds, \text{ since } \|\cdot\|_\Phi \sim \|\cdot\|_{H^\sigma}.
\end{aligned}$$

Altogether, with $C_1 := C$, we have

$$\begin{aligned}
\tau\sum_{j=1}^N (\|\eta_{j,1}\|_\Phi &+ \|\eta_{j,2}\|_\Phi + \|\eta_{j,3}\|_\Phi) \\
&\leq C_1 h_X^\sigma \left(\int_0^{t_N} \|u_t\|_{H^{2\sigma+2}} ds + \tau\sum_{j=1}^N \|u_t(t_{j-1/2})\|_{H^{2\sigma}}\right) \\
&\quad + C_2\tau^2 \int_0^{t_N} (\|u_{ttt}(s)\|_{H^\sigma} + \|\Delta u_{tt}(s)\|_{H^\sigma}) ds.
\end{aligned}$$

Thus

$$\|\theta_N\|_\Phi + \|\gamma_N\|_\Phi \;\; \leq \;\; C_1 h_X^\sigma \Gamma(f, u_t) + C_2\tau^2 \left(\int_0^{t_N} \|u_{ttt}\|_{H^\sigma} + \|\Delta u_{tt}\|_{H^\sigma} ds \right),$$

where

$$\Gamma(f, u_t) \;\; := \;\; \|f\|_{H^{2\sigma}} + \|f\|_{H^{2\sigma+2}} + \int_0^{t_N} \|u_t(s)\|_{H^{2\sigma+2}} ds + \tau \sum_{j=1}^{N} \|u_t(t_{j-1/2})\|_{H^{2\sigma}}.$$

□

C. Numerical experiments on S^2

Let us consider the following function

$$G(z) = 1 - 2\ln\left(1 + \sqrt{(1-z)/2}\right).$$

We can expand $G(z)$ as a series of Legendre polynomials (cf. [21]):

$$G(z) = \sum_{\ell=1}^{\infty} \frac{1}{\ell(\ell+1)} P_\ell(z).$$

The following PDE describes the heat diffusion process from the north pole onto the surface of the unit sphere:

$$\begin{cases} \frac{\partial}{\partial t} u(x,t) &= \Delta u(x,t), \quad x \in S^2, \\ u(x,0) &= G(x \cdot p), \text{ where } p = (0,0,1)^T. \end{cases} \tag{4.22}$$

Since the initial condition $u(x, 0)$ is a zonal function which depends only on the geodesic distance from any given point on the sphere to the north pole, the solution $u(x, t)$ also depends only on the geodesic distance to the north pole. Equation (4.22) is thus reduced to

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial z}(1 - z^2)\frac{\partial u}{\partial z},$$

subject to the following initial condition:

$$u(z,0) = G(z), \qquad z \in [-1,1].$$

We know that the Legendre polynomials are eigenfunctions of the operator

$$\frac{\partial}{\partial z}(1-z^2)\frac{\partial}{\partial z}.$$

Thus, by the method of separation of variables, the exact solution of (4.22) is given as

$$u(z,t) = \sum_{\ell=1}^{\infty} \frac{e^{-\ell(\ell+1)t}}{\ell(\ell+1)} P_\ell(z).$$

We can approximate $u(z,t)$ by the truncated series of Legendre polynomials:

$$u_L(z,t) = \sum_{\ell=1}^{L} \frac{e^{-\ell(\ell+1)t}}{\ell(\ell+1)} P_\ell(z).$$

The error $u - u_L$ is estimated by using the tails of the series

$$\begin{aligned} \|u-u_L\|_\infty &= \left\| \sum_{\ell=L+1}^{\infty} \frac{e^{-\ell(\ell+1)t}}{\ell(\ell+1)} P_\ell(z) \right\|_\infty \\ &\le e^{-L(L+1)t} \int_L^\infty \frac{dx}{x(x+1)} \quad \text{since } \|P_\ell(z)\|_\infty = 1, \ \text{(see [25])} \\ &\le e^{-L(L+1)\tau} \ln\left(1+\frac{1}{L}\right). \end{aligned}$$

For time-step $\tau = 0.00125$, in order to obtain the accuracy of order 10^{-16} it is required that $L \geq 160$.

The spherical basis functions used to construct the approximate solution are derived from a class of locally supported radial basis functions proposed by Wendland [48]. These functions $\psi(x)$ are rotation invariant and thus are functions of $|x|$ only. So the corresponding convolution kernel $\psi(x-y), x, y \in S^n$, is a function of $|x-y| =$

$\sqrt{2-2x\cdot y}$. We may therefore define a function

$$\Phi(x,y)=\phi(x\cdot y):=\psi(x-y),\quad x,y\in S^n.$$

Note that $\Phi(x,y)$ inherits the property of positive definiteness from ψ, and $\widehat{\phi}(\ell)\sim(1+\lambda_\ell)^{-\sigma}$ for some $\sigma>0$ (See [30, Section 4]). For our numerical study, we use the function $\psi(r)=(1-r)_+^4(4r+1)$.

The set of points which are used in constructing the SBFs is generated according to an algorithm in [37]. These points are generated uniformly, in the sense that each point is a center of a cell on the unit sphere of area $4\pi/m$.

The iterative equation (4.16) becomes

$$(I-\tau A^{-1}B)\mathbf{c}_N=\mathbf{c}_{N-1},$$

with the initial equation

$$\mathbf{c}_0=A^{-1}f|_X.$$

Since A is positive definite and B has non-positive eigenvalues, it can be shown that all the eigenvalues of the matrix $(I-\tau A^{-1}B)$ are in the interval $(0,1]$ (see the Appendix). Hence the numerical algorithm is stable.

Table II shows the numerical errors between the iterated solution U_N obtained by backward Euler method and u_{160}. Here, $N=1.5/\tau$ and

$$E_\infty(\tau):=\max_{x\in S^2}|U_N-u_L|.$$

Table II. Backward Euler method with different sets of points and time-steps

m	h_X	q_X	$E_\infty(\tau = 0.01)$	$E_\infty(\tau = 0.005)$	$E_\infty(\tau = 0.0025)$
200	.1942	.1130	0.0224	0.0225	0.0225
400	.1288	.0731	0.0137	0.0138	0.0139
600	.1122	.0675	0.0088	0.0089	0.0090
800	.0950	.0577	0.0060	0.0061	0.0062
1000	.0849	.0516	0.0044	0.0045	0.0046
1200	.0789	.0476	0.0034	0.0036	0.0036

D. Appendix

Lemma IV.5 *(cf. [51, Chapter 1, §31]) Let A be a symmetric positive definite matrix and B be a symmetric positive semi-definite (negative semi-definite). Then all of the eigenvalues of AB are non-negative (non-positive).*

Proof. Since A is symmetric positive definite, there is an invertible matrix P such that $A = P^T P$. Let $C = PBP^T$, then C and AB have the same set of eigenvalues since $(P^T)^{-1}ABP^T = PBP^T = C$. The matrix C is symmetric since $C^T = PB^T P^T = PBP^T = C$ since B is symmetric. Now since B is positive semi-definite,

$$(P^T\mathbf{x})^T B(P^T\mathbf{x}) \geq 0 \text{ for all } \mathbf{x} \in \mathbb{R}^m.$$

Hence C is symmetric semi-positive definite. Hence all of the eigenvalues of C are non-negative, so are the eigenvalues of the matrix AB. □

Lemma IV.6 *Let A be a symmetric positive definite matrix and B be a symmetric negative semi-definite. Then for any $\epsilon > 0$, all of the eigenvalues of $(I - \epsilon A^{-1}B)^{-1}$ are in the interval $(0, 1]$.*

Proof. Let μ be an eigenvalue of $I - \epsilon A^{-1}B$, then $1/\mu$ is an eigenvalue of $(I - \epsilon A^{-1}B)^{-1}$.

It is observed that $\mu = 1 - \epsilon\delta$ where δ is an eigenvalue of $A^{-1}B$. By Lemma IV.5, $\delta \leq 0$, and therefore, $\mu \in [1, \infty)$. Thus, $1/\mu \in (0, 1]$. $\square$

CHAPTER V

CONCLUSIONS

In this dissertation we have outlined a framework for approximation of elliptic and parabolic partial differential equations on spheres. Since the theory is relatively unexplored, there are many open problems to be worked on and many improvements of the current results can be made.

For the elliptic partial differential equation $-\Delta u + \omega^2 u = f$, we have shown that the error estimate $u - u_h$, where u_h is the approximate solution obtained via finite element method, is of the following form:

$$\|u - u_h\|_{H^1} \leq Ch^{2k-n/2-1}\|u\|_{2k}, \text{ for } u \in C^{2k}(S^n).$$

Numerical experiments using MATLAB have shown that the real rate of convergence can be higher than the theoretical estimates. The error estimates in other norms such as $L^2(S^n)$ and $L^\infty(S^n)$ remain unknown.

For the heat equation on the unit sphere,

$$\begin{cases} u_t(x,t) - \Delta u(x,t) & = & F(x,t), \\ u(x,0) & = & f(x), \end{cases}$$

we have worked out only error estimates for the collocation method in the $\|\cdot\|_\Phi \sim H^\sigma(S^n)$ norm. The error estimates in other norms such as $L^2(S^n)$ and $L^\infty(S^n)$ are unknown.

REFERENCES

[1] B. Bajnok, Construction of designs on 2-sphere, European J. Combin. 12 (1991) 377-382.

[2] L. Bos, N. Levenberg, P. Milman and B. A. Taylor, Tangential Markov inequalities characterize algebraic submanifolds of R^N, Indiana Univ. Math. J. 44 (1995) 115-138.

[3] S. Brenner and L. Scott, *The Mathematical Theory of Finite Element Methods* (Springer Verlag, New York, 1994).

[4] J. Céa, Approximation variationelle des problèmes aux limites, Ann. Inst. Fourier. (Grenoble) 14 (1964) 345-444.

[5] J. Cui and W. Freeden, Equidistribution on the sphere, SIAM J. Sci. Stat. Comput. 18 (1997) 595-609.

[6] C. de Boor and B. Swartz, Collocation at Gaussian points, SIAM J. Numer. Anal. 10 (1973) 582-606.

[7] P. Delsarte, J. M. Goethals and J. J. Seidel, Spherical codes and design, Geometriae Dedicata 6 (1977) 363-388.

[8] J. Douglas Jr. and T. Dupont, *Collocation Methods for Parabolic Equations in A Single Space Variable*, Lectures Notes in Mathematics, Vol. 385 (Springer, Berlin, 1974).

[9] N. Dyn, F. J. Narcowich and J. D. Ward, Variational principles and Sobolev-type estimates for generalized interpolation on a Riemannian manifold, Constr. Approx. 15 (1999) 175-208.

[10] N. Dyn, F.J. Narcowich and J. D. Ward, A framework for interpolation and approximation on Riemannian manifolds, in: *Approximation and Optimization*, eds. M. D. Buhmann and A. Iserles (Cambridge University Press, Cambridge, 1997) pp. 133-144.

[11] G. E. Fasshauer and L. L. Schumaker, Scattered data fitting on the sphere, in: *Mathematical Methods for Curves and Surfaces II*, eds. M. Dahlen, T. Lyche and L. L. Schumaker (Vanderbilt Univ. Press, Nashville, 1998) pp. 117-166.

[12] W. Freeden, T. Gervens and M. Schreiner, *Constructive Approximation on The Sphere with Applications to Geomathematics* (Oxford University Press, Oxford, 1998).

[13] M. Golomb and H. F. Weinberger, Optimal approximation and error bounds, in: *On Numerical Approximation* ed. R. E. Langer (The Univ. of Wisconsin Press, Madison, 1959) pp. 117-190.

[14] J. Göttelmann, A Spline collocation scheme for the spherical shallow water equations, preprint.

[15] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic Press, San Diego, 1980).

[16] L. Hörmander, The boundary problems of physical geodesy, Arch. Rational Mech. Anal. 62 (1976) 1-52.

[17] K. Jetter, J. Stöckler and J. D. Ward, Error estimates for scattered data interpolation on spheres, Math. Comp. 68 (1999) 733-747.

[18] A. B. J. Kuijlaars and E. B. Saff, Asymptotics for minimal discrete energy on the sphere, Trans. Amer. Math. Soc. 350 (1998), 523-538.

[19] J. Levesley, Z. Lou and X. Sun, Norm estimates of interpolation matrices and their inverses associated with strictly positive definite functions, Proc. Amer. Math. Soc. 127 (1999), 2127-2137.

[20] J. L. Lions and E. Magenes, *Non-homogeneous Boundary Value Problems and Applications*, vol. I, (Springer-Verlag, New York, 1972).

[21] W. Magnus, F. Oberhettinger and R. P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics*, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Bd. 52, (Springer, Berlin, 1966).

[22] H. N. Mhaskar, F. J. Narcowich and J. D. Ward, Spherical Marcinkiewicz-Zygmund inequalities and positive quadrature, Math. Comp. 70 (2001), 1113-1130.

[23] T. M. Morton and M. Neamtu, Error bounds for solving pseudo-differential equations on spheres, J. Approx. Theory 114 (2002), 242-268.

[24] S. Hubbert and T. M. Morton, L_p error estimates for radial basis function interpolation on the sphere, preprint.

[25] C. Müller, *Spherical Harmonics*, Lecture Notes in Mathematics, Vol. 17, (Springer-Verlag, Berlin, 1966).

[26] C. Müller, *Analysis of Spherical Symmetries in Euclidean Spaces*, Applied Mathematical Sciences, Vol. 129, (Springer-Verlag, Berlin, 1998).

[27] R. Nagle, E. Saff and A. Snider, *Fundamentals of Differential Equations*, (Addison-Wesley, New York, 2000).

[28] F. J. Narcowich, R. Schaback and J.D. Ward, Approximation in Sobolev spaces by kernel expansions, J. Approx. Theory 114 (2002), 70-83.

[29] F. J. Narcowich, Generalized Hermite interpolation and positive definite kernels on a Riemannian manifold, J. Math. Anal. Appl. 190 (1995), 165-193.

[30] F. J. Narcowich and J. D. Ward, Scattered-data interpolation on spheres: Error estimates and locally supported basis functions, SIAM J. Math. Anal. 33 (2002), no.6, 1393-1410.

[31] F. J. Narcowich, Recent Developments in Approximation via Positive Definite Functions, in: *Approximation IX, vol. 2: Computational Aspects*, eds. C. Chui and L. Schumaker, (Vanderbilt University Press, Nashville, 1998), pp. 221-242.

[32] F. J. Narcowich, J. D. Ward and H. Wendland, Refined Error Estimates for Radial Basis Function Interpolation, preprint.

[33] F. J. Narcowich, R. Schaback and J. D. Ward, Optimal stability results for interpolation by kernel functions, preprint.

[34] S. Pawelke, Ein Satz von Jacksonschen Typ fur algebraischen Polynome, Acta Sci. Math. (Szeged) 33 (1972), 323-336.

[35] S. Pawelke, Uber die Approximationsordnung bei Kugelfunktionen und algebraischen Polynomen, Tohoku Math. Journ. 24 (1972), 473-486.

[36] J. Rauch, *Partial Differential Equations*, (Springer-Verlag, New York, 1991).

[37] E. B. Saff and A. B. J. Kuijlaars, Distributing many points on a sphere, The Mathematical Intelligencer 19 (1997), 5-11.

[38] Wu, Z. and R. Schaback, Local error estimates for radial basis function interpolation of scattered data, IMA Journ. of Numer. Analysis 13 (1993), no.1, 13-27.

[39] I. J. Schoenberg, Positive definite functions on spheres, Duke Math Jour. 9 (1942), 96-108.

[40] N. Sloane, *Encrypting by Random Rotations*, Cryptograph, Lecture Notes in Computer Science 149, (Springer-Verlag, Berlin, 1983).

[41] E. M. Stein, Interpolation in polynomial classes and Markoff's inequality, Duke Math. J. 24 (1957), 467-476.

[42] S. L. Svensson, Pseudodifferential operators - a new approach to the boundary value problems of physical geodesy, Manuscr. Geod. 8 (1983), 1-40.

[43] S. L. Svensson, Finite Elements on the sphere, J. Approx. Theory, 40 (1984), 246-260.

[44] V. Thomée, From finite differences to finite elements. A short history of numerical analysis of partial differential equations, Journal of Computational and Applied Mathematics 128 (2001), 1-54.

[45] V. Thomée, *Galerkin Finite Element Methods for Parabolic Problems*, Lecture Notes in Mathematics Vol.1054, (Springer-Verlag, Berlin, 1984).

[46] G. N. Watson, *Theory of Bessel Functions*, (Cambridge University Press, London, 1966).

[47] H. Wendland, Meshless Galerkin methods using radial basis functions, Math. Comp. 68 (1999), 1521-1531.

[48] H. Wendland, Piecewise polynomial, positive definite and compactly supported radial functions of minimal degree, Adv. in Comp. Math. 4 (1995), 389-396.

[49] H. Wendland, Error estimates for interpolation by compactly supported radial basis functions of minimal degree, J. Approx. Theory 93 (1998), 258-272.

[50] D. L. Williamson et al, A standard test for numerical approximations to the shallow water equations in spherical geometry, J. of Compt. Physics 102 (1992), 211-224.

[51] J. H. Wilkinson, *The Algebraic Eigenvalue Problem*, (Clarendon Press, Oxford, 1965).

[52] Y. Xu and E. W. Cheney, Strictly positive definite functions on spheres, Proc. Amer. Math. Soc. 116 (1992), 977-981.

APPENDIX A

MATLAB CODE FOR APPROXIMATION OF PDES ON SPHERES

```
%------------------------------------------------------------%
% Compute locally supported spherical basis function according  %
% to reference [49]                                             %
%------------------------------------------------------------%
function phi = cmp_sbf0(x,y)
% The locally supported SBF of smoothness order 0
%   phi(r) = (1-r)^2_+  where r = sqrt(2-2*x'*y)
% x,y are (3:n) matrices representing n points on the unit sphere
r = real(sqrt(2.-2.*(x'*y)));
mask = (abs(r) < 1);
phi = mask .* (1-r).^2;

function phi = cmp_sbf2(x,y)
% The locally supported SBF of smoothness of order 2
%   phi(r) = (1-r)^4_+ (4r+1)
r = real(sqrt(2.-2.*x'*y));
mask = (abs(r) < 1);
phi = mask .*(1-r).^4.*(4.*r+1);

function phi = cmp_sbf4(x,y)
% The locally supported SBF of smoothness order 4
%   phi(r) = (1-r)^6_+ (35*r^2+18*r+3)
```

```
r = real(sqrt(2-2*x'*y));

mask = (abs(r) < 1);

phi = mask .* (1-r).^6.*(35*r.^2+18*r+3);


function phi = cmp_sbf6(x,y)

% The locally supported SBF of smoothness order 6

%   phi(r) = (1-r)^8_+ (32*r^3+25*r^2+8*r+1)

r = real(sqrt(2-2*x'*y));

mask = (abs(r) < 1);

phi = mask .* (1-r).^8.*(32*r.^3+25*r.^2+8*r+1);


function phi = poisson(x,y,z)

% The Poisson kernel

% phi = (1-z^2)/(1+z^2-2z cos(x,y))^(3/2)

phi = real((1-z^2)./(1+z^2-2*z*x'*y).^(3/2));


%-------------------------------------------------------------%

% Generate various types of set of points on the sphere.     %

%-------------------------------------------------------------%


function [L,hL,qL] = saffpts(N, n)

% Generate points on the sphere according to an algorithm in

% reference [38]

% [L, hL, qL] = saffpts(N,Z)

%    N = number of points

%    n = number of zones
```

```
% Return: L is a (3:n) matrix containing Cartesian coordinates
% of the points; hL is the mesh norm; qL is the separate norm
%
if (nargin < 2)
    n = floor(sqrt(pi*N)/2)+1;
end
beta = 4/sqrt(N);
step_theta = (pi-beta)/(n-2);
theta1 = [0:step_theta:(n-3)*step_theta] + beta/2;
theta2 = [step_theta:step_theta:(n-2)*step_theta] + beta/2;
mbar = ones(1,n);
mbar(2:n-1) = N*(cos(theta1)-cos(theta2))/2;
alpha = 0;
for i=1:n
    if (mbar(i)-floor(mbar(i)+alpha)<0.5)
        m(i) = floor(mbar(i)+alpha);
    else
        m(i) = floor(mbar(i)+alpha) + 1;
    end
    alpha = alpha + mbar(i) - m(i);
end
L(:,1) = [0,0,1]';
cur_index = 2;
q = ones(1,n);
for i=2:n-1
    % generate points on one partition
```

```
    t = (i-3/2)*step_theta + beta/2;
    r = sin(t); h = cos(t);
    alpha = mod((i-2),2)/2;
    s = (2*pi*[0:m(i)-1]+alpha)/m(i);
    L(:,cur_index:cur_index+m(i)-1) =
             [r*cos(s); r*sin(s); h*ones(1,m(i))];
    % compute the separate norm by taking the geodesic distance
    % with the previous level
    tmp = L(:,cur_index)'* L(:,cur_index-m(i-1):cur_index-1);
    q(i) = min(acos([tmp L(:,cur_index)'*L(:,cur_index+1)]));
    % advance one more partition
    cur_index = cur_index+m(i);
end
L(:,cur_index)=[0 0 -1]';
tmp = L(:,cur_index)'*L(:,cur_index-m(n-1):cur_index-1);
q(n) = min(acos(tmp));
qL = min(q)/2.0;
%
% now we compute the mesh norm
%
h = zeros(1,n);
for i=2:n-1
    if (i==2)
        t = step_theta/4 + beta/2;
    else
        t = (i-3/2-1/2)*step_theta + beta/2;
```

```
    end
    r = sin(t); h = cos(t);
    alpha = mod((i-2),2)/2;
    midpt = [r*cos((alpha+pi)/m(i)); r*sin((alpha+pi)/m(i)); h];
    htmp = midpt'*L(:,cur_index-m(i-1):cur_index-1);
    h(i) = min(acos([htmp midpt'*L(:,cur_index)]));
end
hL = max(h);
%-------------------------------------------------------------%
%    An example of interpolation on spheres using SBF       %
%-------------------------------------------------------------%
function [hX,qX,err2,errsup] = interp_saff(num,sbftype)
% [hX,qX,err2,errsup] = interp_saff(num,[SBFtype])
% Choices for SBFtype
%  1 = SBF of C^0 class       2 = SBF of C^2 class
%  3 = SBF of C^4 class       4 = SBF of C^6 class
%  5 = SBF of Poisson kernel class
% The scattered points are generated based on an algorithm
% in reference [38]
[x,hX,qX] = saffpts(num);
% generate E. Saff's equi-area points
n = length(x);
tic;
if (nargin<2)
    sbftype = 2;
end
```

```
% set up the interpolation matrix
A = zeros(n,n);
switch sbftype
    case 1
        A = sparse(cmp_sbf0(x,x));
    case 2
        A = sparse(cmp_sbf2(x,x));
    case 3
        A = sparse(cmp_sbf4(x,x));
    case 4
        A = sparse(cmp_sbf6(x,x));
    case 5
        A = poisson(x,x,0.5);
end
  figure(1);
imagesc(A);

% interpolate f = (1-r)^2_+ for r=sqrt(2-2*cos(x,p)),
% where p is the north pole
b = cmp_sbf2(x,[0; 0 ;1]);

% solve the linear system Ax = b by the conjugate
% gradient method
c = cgs(A,b,1e-10,2500);

clear A;
```

```
num = 100;
figure(2);
[Sx, Sy, Sz] = sphere(num-1);
surf(Sx,Sy,Sz,zeros(num,num)); shading flat;
hold on;
plot3(x(1,:),x(2,:),x(3,:),'*');
hold off;
mS = sparse(zeros(num*num,n));
switch sbftype
case 1
    for i=1:num
        mS((i-1)*num+1:(i-1)*num+num,:) =
           cmp_sbf0([Sx(i,:);Sy(i,:);Sz(i,:)],x);
    end
case 2
    for i=1:num
        mS((i-1)*num+1:(i-1)*num+num,:) =
           cmp_sbf2([Sx(i,:);Sy(i,:);Sz(i,:)],x);
    end
case 3
    for i=1:num
        mS((i-1)*num+1:(i-1)*num+num,:) =
           cmp_sbf4([Sx(i,:);Sy(i,:);Sz(i,:)],x);
    end
case 4
    for i=1:num
```

```
        mS((i-1)*num+1:(i-1)*num+num,:) =
           cmp_sbf6([Sx(i,:);Sy(i,:);Sz(i,:)],x);
    end
case 5
    for i=1:num
        mS((i-1)*num+1:(i-1)*num+num,:) =
           poisson([Sx(i,:);Sy(i,:);Sz(i,:)],x,0.5);
    end
end
% calculate the results
sV = mS*c;...
V = zeros(num,num);...
for k=1:num
    V(k,:) = (sV((k-1)*num+1 :(k-1)*num+num))';...
end
figure(3);
surf(Sx,Sy,Sz,V);...
shading interp;
% calculate the errors
err2 = norm(V-exp(Sx));
errsup = max(max(abs(V-exp(Sx))));
err = zeros(num,num);
for k=1:num
  err(k,:) =
    abs(V(k,:)-cmp_sbf2([0;0;1],[Sx(k,:); Sy(k,:); Sz(k,:)]));
end
```

```
figure(4);
surf(Sx, Sy, Sz, err);shading flat;
err2 = norm(err);
errsup = max(max(err));
toc
%----------------------------------------------------------------%
% Generate the spherical basis functions and its derivatives     %
%----------------------------------------------------------------%
function phi = s2cmpsbf2(x,y,scale)
% s2cmpsbf2(x,y) returns the locally supported SBF of smoothness
% order 2 with scaling factor s='scale'
%     phi = (1-s*r)^4_+ (4*s*r+1), where r = sqrt(2-2*x'*y)
% x,y are two (3:n) matrices representing the Cartesian
% co-ordinates of n points on S^2
r = real(sqrt(2-2*(x'*y)));
mask = scale*r<1;
phi =  mask .* (1-scale*r).^4 .* (4*scale*r + 1);

function ans = sLcmp_sbf2(h,s,x,y)
% Lcmp_sbf2(h,x,y) returns [1- h*d/dt ((1-t^2) d/dt psi(t))]
%   where  t = cos(x,y), x, y on the unit sphere
%   and psi(t) = (1-s*r)^4_+ (4*s*r+1) for r = sqrt(2-2*t) .
%   x,y are (3:n) matrices representing n points on the unit sphere
%   ans is a (n:n) matrix
t = x'*y;
r = real(sqrt(2-2*t));
```

```
mask = (s*r<1.0)&(t>1-0.5/s);
dphi = mask.*10*s^2.*(-1+s*r).^2.*(7*s.*r.*t-4*t+3*s*r);
phi  = (s*r<1.0).*(1-s*r).^4.*(4*s*r+1);
ans = phi - h*dphi;

%----------------------------------------------------------------%
% Solve the following PDE on the sphere by the collocation method %
%  - h\Delta u + u = f                                            %
%----------------------------------------------------------------%
clear all;
num = 100;   % for plotting only
N = 400;     % dimension of SBF approximation space
h = 0.01;    % in 1 - h\Delta
X = saffsph(N);
A = sLcmp_sbf2(h,1,X,X);
t = X(3,:)';
b = exp(t).*(ones(N,1)-h*(ones(N,1)-t.^2-2*t));
c = cgs(A,b,1e-10,100);
%
[Sx, Sy, Sz] = sphere(num-1);
surf(Sx,Sy,Sz,zeros(num-1,num-1)); shading flat; hold on;
plot3(X(1,:),X(2,:),X(3,:),'*');
mS = zeros(num*num,N);
for i=1:num
    mS((i-1)*num+1:(i-1)*num+num,:) =
         s2cmpsbf2([Sx(i,:); Sy(i,:); Sz(i,:)],X,1);
```

```
end
% calculate the results
sV = mS*c;...
V = zeros(num,num);...
for k=1:num
    V(k,:) = (sV((k-1)*num+1 :(k-1)*num+num))';...
end
figure;
surf(Sx,Sy,Sz,V);...
shading interp; colorbar;
figure;
surf(Sx,Sy,Sz,abs(exp(Sz)-V));
shading interp; colorbar;
max_err = max(max(abs(exp(Sz)-V)))
L2_err =  (norm(exp(Sz)-V))/num


%---------------------------------------------------------------%
% Solve the PDE on the sphere                                   %
%  - Delta u + omega^2 u = f                                    %
%---------------------------------------------------------------%
clear all;
num = 100;   % for plotting only
omega = 0.1;
n = 800;
X = saffsph(n);
N = length(X);
```

```
tic
% [Z,W] = gauss_legendre(-1,1,6000); % the quadrature rule
load weights6000;
A = saffA2(n,omega,Z,W);          % set up the matrix
[b,al,fl] = vectorb(n,omega,Z,W); % set up vector b
c = cgs(A,b,1e-10,5000);
%
[Sx, Sy, Sz] = sphere(num-1);
surf(Sx,Sy,Sz,zeros(num-1,num-1)); shading flat; hold on;
plot3(X(1,:),X(2,:),X(3,:),'*');
mS = zeros(num*num,N);
for i=1:num
    mS((i-1)*num+1:(i-1)*num+num,:) =
      s2cmpsbf2([Sx(i,:); Sy(i,:); Sz(i,:)],X,1);
end
% calculate the results
sV = mS*c;...
V = zeros(num,num);...
for k=1:num
    V(k,:) = (sV((k-1)*num+1 :(k-1)*num+num))';...
end
figure;
surf(Sx,Sy,Sz,V);...
shading interp;colorbar;
figure;
surf(Sx,Sy,Sz,abs(exp(Sz)-V));
```

```
shading interp; colorbar;
max_err = max(max(abs(exp(Sz)-V)))
L2_err = (norm(exp(Sz)-V))/num
toc

%--------------------------------------------------------%
%  Gauss-Legendre quadrature rule                         %
%--------------------------------------------------------%
function [X,W]=gauss_legendre(x1,x2,n)
% return the abscissas and the weights of the Gaussian
% quadrature on (x1,x2)
m = (n+1)/2;
xm = 0.5*(x1+x2);
xl = 0.5*(x2-x1);
for i=1:m
   z = cos(pi*(i-0.25)/(n+0.25));
   z1 = 0;
   % refine the initial guess by Newton's method
   while (abs(z-z1)>10e-11),
       p1 = 1.0; p2 = 0.0;
       for j=1:n
          p3 = p2; p2 = p1;
          p1 = ((2.0*j-1.0)*z*p2-(j-1.0)*p3)/j;
       end
   pp = n*(z*p1-p2)/(z*z-1.0);
   z1 = z;
```

```
    z = z1 - p1/pp;
    end
    X(i) = xm-xl*z;
    X(n+1-i) = xm+xl*z;
    W(i) = 2.0*xl/((1.0-z*z)*pp*pp);
    W(n+1-i) = W(i);
end

%-----------------------------------------------------------%
% Compute the inner-product using symmetries of the points  %
%-----------------------------------------------------------%
function [A,nv] = saffA2(n,omega,Z,W)
% A is the matrix with
%    A_ij = inner_prod{(-Delta+omega^2)phi_i}{phi_j},
%    nv is the number of distinct values
[X,m] = saffsphM(n); % m is number of points on each latitude
N = length(X);       % actual number of Saff's points
nL = length(m);      % number of distinct latitudes
maxL= floor(N/2);
% number of Legendre coefficients used in the approximation
A = zeros(N,N);
lZ = length(Z);
phiz = phi2(Z);
al = leg_coeffs(phiz,Z,W,maxL);
% start to compute elements of the matrix A using as many
% symmetries as possible
```

```
[Q,V,nv] = saffV(n);
nv   % print out number of nonzeros elements in Q
nv = 1;
myA = phi_ij2(al,omega,V);
A(1,1) = myA(1);
% diagonal elements are the same
for j=2:N
    A(j,j) = A(1,1);
end
% the north pole
cur = 2; % current starting point
for i=2:floor(nL/2)+1
    % for any latitude on the northern hemisphere
    q = X(:,1)'*X(:,cur);
    if q <= -0.5
        A(1,cur) = 0.0;
    else
        nv = nv + 1
        A(1,cur) = myA(nv);
    end
    A(cur,1) = A(1,cur);
    for j=cur+1:cur+m(i)-1
        A(1,j) = A(1,cur);  A(j,1) = A(1,j);
    end
    cur = cur + m(i);
end
```

```
% the south pole
cur = N-1;
for i=nL-1:-1:floor(nL/2)-1
    % for any latitude on the southern hemisphere
    q = X(:,N)'*X(:,cur);
    if q <= -0.5
        A(N,cur) = 0.0;
    else
        nv = nv + 1
        A(N,cur) = myA(nv);
    end
    A(cur,N) = A(N,cur);
    for j=cur-1:-1:cur-m(i)+1
        A(N,j) = A(N,cur); A(j,N) = A(N,j);
    end
    cur = cur - m(i);
end
% on each latitute which contains m_i points
cur = 2;
for i=2:nL-1
    for step=1:floor(m(i)/2)+1
        q = X(:,cur)'*X(:,cur+step);
        if q <= -0.5
            A(cur,cur+step) = 0.0;
        else
            nv = nv+1
```

```
            A(cur,cur+step) = myA(nv);
        end
        A(cur+step,cur) = A(cur,cur+step);
        for k=cur+1:cur+m(i)-1
            ks = k+step;
            if ks > cur+m(i)-1
                ks = k+step-m(i);
            end
            A(k,ks) = A(cur,cur+step);
            A(ks,k) = A(k,ks);
        end
    end
    cur = cur + m(i);
end
% now for pairs of points on different latitude
for i=2:N-1
    for j=i+1:N-1
        if A(i,j) == 0    % has not yet been computed
            q = X(:,i)'*X(:,j);
            if q > -0.5
                nv = nv + 1
                A(i,j) = myA(nv);
                %A(i,j) = phi_ij1(al,omega,q);
                A(j,i) = A(i,j);
            end
        end
```

```
    end
end


%----------------------------------------------------------%
%  Compute each entry of the matrix                        %
%----------------------------------------------------------%
function pz = phi_ij2(al,omega,t)
% Compute
% int_{S^2} (-Delta+omega^2) phi(x_i . x) phi(x_j . x) dS,
%   ~= 4pi sum_{l=0}^L (l(l+1)+omega^2) a^2_l/(2l+1) P_l(t)
% where
%      a_l = (2/2l+1)int_{-1}^{1} phi(z) P_l(z)dz
%      t  = x_i . x_j
L = length(al)-1;
lt = length(t);
v = zeros(L+1,lt);
pz = zeros(1,lt);
p1 = ones(1,lt); p2=zeros(1,lt);
for i=1:lt
   v(1,i) = omega^2*(al(1))^2;
end
for l=1:L
   % v(l+1) = (l*(l+1)+omega^2)*(al(l+1))^2*mlegendre(l,t)/(2*l+1);
   p3 = p2; p2 = p1;
   p1 = ((2.0*l-1.0).*t.*p2 - (l-1.0).*p3)/l;
   v(l+1,1:lt) = ((l*(l+1) + omega^2)*(al(l+1))^2/(2*l+1)).*p1;
```

```
end
for i=1:lt
   pz(i) = 4*pi*sum(v(:,i));
end

%---------------------------------------------------------------%
% Compute the geodesic distance using symmetries                %
% Q is the matrix, nv is the number of distinct nonzero values %
%---------------------------------------------------------------%
[X,m] = saffsphM(n); % m is number of points on each latitude
N = length(X);       % actual number of Saff's points
nL = length(m);      % number of distinct latitudes
Q = zeros(N,N);
V = zeros(1,floor(N*N/2));
Q(1,1) = X(:,1)'*X(:,1);
nv = 1;
V(1) = Q(1,1);
% diagonal elements are the same
for j=2:N
    Q(j,j) = Q(1,1);
end
% the north pole
cur = 2; % current starting point
for i=2:floor(nL/2)+1
    % for any latitude on the northern hemisphere
    Q(1,cur) = X(:,1)'*X(:,cur);
```

```
    if Q(1,cur) <= -0.5
        Q(1,cur) = 0.0;
    else
        nv = nv + 1;
        V(nv) = Q(1,cur);
    end
    Q(cur,1) = Q(1,cur);
    for j=cur+1:cur+m(i)-1
        Q(1,j) = Q(1,cur);  Q(j,1) = Q(1,j);
    end
    cur = cur + m(i);
end
% the south pole
cur = N-1;
for i=nL-1:-1:floor(nL/2)-1
    % for any latitude on the southern hemisphere
    Q(N,cur) = X(:,N)'*X(:,cur);
    if Q(N,cur) <= -0.5
        Q(N,cur) = 0.0;
    else
        nv = nv + 1;
        V(nv) = Q(N,cur);
    end
    Q(cur,N) = Q(N,cur);
    for j=cur-1:-1:cur-m(i)+1
        Q(N,j) = Q(N,cur); Q(j,N) = Q(N,j);
```

```
    end
    cur = cur - m(i);
end
% on each latitute which contains m_i points
cur = 2;
for i=2:nL-1
    for step=1:floor(m(i)/2)+1
        Q(cur,cur+step) = X(:,cur)'*X(:,cur+step);
        if Q(cur,cur+step) <= -0.5
            Q(cur,cur+step) = 0.0;
        else
            nv = nv+1;
            V(nv) = Q(cur,cur+step);
        end
        Q(cur+step,cur) = Q(cur,cur+step);
        for k=cur+1:cur+m(i)-1
            ks = k+step;
            if ks > cur+m(i)-1
                ks = k+step-m(i);
            end
            Q(k,ks) = Q(cur,cur+step);
            Q(ks,k) = Q(k,ks);
        end
    end
    cur = cur + m(i);
end
```

```
% now for pairs of points on different latitude
for i=2:N-1
    for j=i+1:N-1
        if Q(i,j) == 0    % has not yet been computed
            q = X(:,i)'*X(:,j);
            if q > -0.5
                Q(i,j) = q;
                Q(j,i) = q;
                nv = nv+1;
                V(nv) = Q(i,j);
            end
        end
    end
end

%-------------------------------------------------------------------%
%  heat_cmp(d,tau,nframes)                                          %
%  solve the heat equation on the unit sphere                       %
%    u_t -  Delta_x u = 0                                           %
%    u(x,0) = f(x)                                                  %
%  time discretization scheme is backward Euler method              %
%  space discretization using SBF based on locally supported kernel, %
%  the points use to construct the SBFs are Saff's points           %
%  d number of points, tau is the timestep                          %
%  nframes is the number of iterations                              %
%-------------------------------------------------------------------%
```

```
function [errsup,err2,exact10] = heat_cmp(d,tau,nframes)
x = saffsph(d);
n = length(x);
% A(i,j) = (I - h Delta) cmp_sbf2(x_i, x_j)
A = zeros(n,n);
A = Lcmp_sbf2(tau,x,x);
B = zeros(n,n);
B = cmp_sbf2(x,x);

% f(x) is the point source, concentrated at the north pole
% b = zeros(n,1);
% b(1) = 1000;
% now f(x) = 1 - 2*ln(1+sqrt(0.5-0.5z));
b = 1 - 2*log(1+sqrt(0.5-0.5*x(3,:)));
b = b';

num = 100;
[Sx, Sy, Sz] = sphere(num-1);
surf(Sx,Sy,Sz);
plot3(x(1,:),x(2,:),x(3,:),'*');
mS = zeros(num*num,n);
for i=1:num
    mS((i-1)*num+1:(i-1)*num+num,:) = ...
      cmp_sbf2([Sx(i,:); Sy(i,:); Sz(i,:)],x);
end
```

```
% for the animation
% nframes = 300;
% Mv = moviein(nframes);

% compute the almost exact solution at equal separated points
% on the longtitute ...
%
exact10 = magnus(160, (Sz(:,1))',[tau:tau:nframes*tau]);
fprintf 'Finish intializing ... Press any key'
%pause;
newplot;

c = cgs(A,b,1e-10,500);
% next iteration
% b = v(x_j, t)
errsup = zeros(1,nframes);
err2 = zeros(1,nframes);
for t=1:nframes,...
  b = B*c;
  c = cgs(A,b,1e-10,500);...
  sV = mS*c;...
  V = zeros(num,num);...
  for k=1:num
     V(k,:) = (sV((k-1)*num+1 :(k-1)*num+num))';...
  end
  surf(Sx,Sy,Sz,V); shading flat; colorbar;
```

```
  Mv(:,t) = getframe;...
  % compute the supremum error
  errsup(t) = max(abs(exact10(t,:) - (V(:,1))'));
  err2(t) = norm(exact10(t,:) - V(:,1)')/sqrt(sqrt(num));
end
fprintf('Press any key to see the movie ...');
pause;
% Show movie 10 times 1 frame/sec
movie(Mv)
plot(errsup);
newplot;
plot(err2);
plot([1:num],V(:,1),[1:num],exact10(nframes,:));

%-------------------------------------------------------------------%
% Lcmp_sbf2(h,x,y) returns [1- h*d/dt ((1-t^2) d/dt psi(t))]        %
% where h is the timestep and t = cos(x,y), x, y on the unit sphere %
% and psi(t) = (1-r)^4_+ (4*r+1) for r = sqrt(2-2*t) .              %
% x,y can be (3:n) matrices representing n points on the unit sphere%
% if x is (3:n) and y(3:m) then the result is a (n:m) matrix        %
%-------------------------------------------------------------------%
function ans = Lcmp_sbf2(h,x,y)
t = x'*y;
r = real(sqrt(2-2*t));
mask = ((t>=0.5) & (r<1));
ans = mask .* ((1-r).^4 .*(4*r+1) - h*10*(r-1).^2.*(7*r.*t-4*t+3*r));
```

```
%-------------------------------------------------------------------%
% cmpsbf2(x,y) returns the Wendland function of smoothness order 2  %
% phi = (1-r)^4_+ (4*r+1), where r = sqrt(2-2*cos(x,y))             %
% x,y can be two matrices (3:n) representing the Cartesian          %
% coordinates of n points on S^2                                    %
%-------------------------------------------------------------------%
function phi = cmpsbf2(x,y)
r = real(sqrt(2-2*(x'*y)));
mask = (r<1);
phi = mask .* (1-r).^4 .* (4*r + 1);

%-------------------------------------------------------------------%
% magnus(N,X,T)                                                     %
%  according to a formula in [p.168,5] and [21]                     %
% f^(ell) = 1/ [4pi(2 ell+1)ell(ell + 1)]                           %
% f(x) = 1-2*ln(1+sqrt(0.5-0.5*x));                                 %
% y = Gaussian(t) convolutes with f(x)                              %
% X is a row vector, T is the time row vector,                      %
% N is the number of terms in the truncated series                  %
% return: sum_{ell=0}^{infinity}                                    %
%         exp(-ell(ell+1)t) [1/ell(ell+1)] P_ell(x)                 %
%-------------------------------------------------------------------%
function y = magnus(N,X,T)
lX = length(X);
tmp = zeros(N+1,lX);
```

```
for ell=1:N+1
   sp = legendre(ell-1,X,'sch');
   % take only Legendre of order 0
   tmp(ell,:) = sp(1,:);
end
L  = [1:N];
lenT = length(T);
yt = zeros(lenT,N+1);
for kt=1:lenT
   yt(kt,:) = [0 exp(-L.*(L+1)*T(kt))./(L.*(L+1))];
end
y = yt*tmp;
```

VITA

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