

# Multiscale approximation for functions in arbitrary Sobolev spaces by scaled radial basis functions on the unit sphere

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## Abstract

In this paper, we prove convergence results for multiscale approximation using compactly supported radial basis functions restricted to the unit sphere, for target functions outside the reproducing kernel Hilbert space of the employed kernel.

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## 1. Introduction

In the geosciences data are often collected at scattered sites on the unit sphere. Moreover, the geophysical data typically occur at many length scales: for example, the topography of central Australia varies slowly, while that of the Himalayan mountains varies rapidly. To handle multiscale data at scattered locations, we consider an approximation scheme based on radial basis

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functions of different scales, which are generated from a single underlying radial basis function (RBF)  $\Phi$  using a sequence of scales  $\delta_1, \delta_2, \dots$  with limit zero, with the scale becoming smaller as the point set becomes denser. The scaled RBF is defined by  $\Phi_\delta = c_\delta \Phi(\frac{\cdot}{\delta})$ , which is then restricted to the unit sphere. There is significant difficulty in dealing with more than one scale at the same time, namely that the associated reproducing kernel Hilbert space (RKHS) or “native space” has a different inner product for each scale. For this reason a multiresolution analysis within a single Hilbert space, of the kind familiar from wavelet analysis, does not seem possible for scaled RBFs on either a sphere or a Euclidean region.

In a recent paper [1], we resolved that issue, and constructed a multiresolution analysis for sufficiently smooth functions on the unit sphere  $\mathbb{S}^n$  based on RBFs of different scales. We have proved a convergence result for a multiscale approximation algorithm for functions from the RKHS of the given compactly supported kernel, if the RKHS is a sufficiently smooth Sobolev space.

In this paper, we shall extend our convergence results to rougher target functions. Specifically, if the employed kernel is the reproducing kernel of, say,  $H^\sigma(\mathbb{S}^n)$  then the new results will hold for all target functions from  $H^\beta(\mathbb{S}^n)$  with  $\sigma > \beta > n/2$ .

The paper is organized as follows. In Section 2, after reviewing the necessary background on spherical harmonics and spherical kernels defined from radial basis functions, we review Sobolev splines and prove a new result on the asymptotic behavior of the Fourier-Legendre coefficients of spherical kernels defined by restricting the Sobolev splines to the unit sphere. In Section 3 we review the method of multiscale approximation using scaled compactly supported radial basis functions which was analyzed in [1] for functions lying in the reproducing kernel Hilbert space defined by the spherical kernel. In Section 4 we give a convergence analysis for “rougher” target functions. The main result of the paper is stated in Theorem 4.5.

## 2. Preliminaries

### 2.1. Positive definite bizonal kernels on the unit sphere

The unit sphere  $\mathbb{S}^n$  is defined by  $\mathbb{S}^n = \{\mathbf{x} \in \mathbb{R}^{n+1} : |\mathbf{x}| = 1\}$ , where  $|\mathbf{x}|$  denotes the Euclidean norm of  $\mathbf{x}$ . Bizonal kernels defined on  $\mathbb{S}^n \times \mathbb{S}^n$  are kernels that can be represented as  $g(\mathbf{x}, \mathbf{y}) = \tilde{g}(\mathbf{x} \cdot \mathbf{y})$  for all  $\mathbf{x}, \mathbf{y} \in \mathbb{S}^n$ , where

$\tilde{g}(t)$  is a continuous function on  $[-1, 1]$ . We shall be concerned exclusively with bizonal kernels of the type

$$g(\mathbf{x}, \mathbf{y}) = \sum_{\ell=0}^{\infty} a_{\ell} P_{\ell}(n+1; \mathbf{x} \cdot \mathbf{y}), \quad a_{\ell} > 0, \quad \sum_{\ell=0}^{\infty} a_{\ell} < \infty, \quad (1)$$

where  $\{P_{\ell}(n+1; t)\}_{\ell=0}^{\infty}$  is the sequence of  $(n+1)$ -dimensional Legendre polynomials normalized to  $P_{\ell}(n+1; 1) = 1$ .

Thanks to the seminal work of Schoenberg [2] and the later work of [3], we know that such a  $g$  is (strictly) positive definite on  $\mathbb{S}^n$ , that is, the matrix  $\mathbf{A} := [g(\mathbf{x}_i, \mathbf{x}_j)]_{i,j=1}^M$  is positive definite for every set of distinct points  $\{\mathbf{x}_1, \dots, \mathbf{x}_M\}$  on  $\mathbb{S}^n$  and every positive integer  $M$ .

For mathematical analysis, sometimes it is convenient to expand the kernel  $g$  into a series of spherical harmonics. A detailed discussion on spherical harmonics can be found in [4]. In brief, spherical harmonics are the restriction to  $\mathbb{S}^n$  of homogeneous polynomials  $Y$  in  $\mathbb{R}^{n+1}$  which satisfy  $\Delta Y = 0$ , where  $\Delta$  is the Laplacian operator in  $\mathbb{R}^{n+1}$ . The space of all spherical harmonics of degree  $\ell$  on  $\mathbb{S}^n$ , denoted by  $\mathcal{H}_{\ell}$ , has an orthonormal basis

$$\{Y_{\ell k} : k = 1, \dots, N(n, \ell)\},$$

where

$$N(n, 0) = 1 \text{ and } N(n, \ell) = \frac{(2\ell + n - 1)\Gamma(\ell + n - 1)}{\Gamma(\ell + 1)\Gamma(n)} \text{ for } \ell \geq 1.$$

The space of spherical harmonics of degree  $\leq L$  will be denoted by  $\mathcal{P}_L := \bigoplus_{\ell=0}^L \mathcal{H}_{\ell}$ ; it has dimension  $N(n+1, L)$ . Every function  $f \in L_2(\mathbb{S}^n)$  can be expanded in terms of spherical harmonics,

$$f = \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(n, \ell)} \hat{f}_{\ell k} Y_{\ell k}, \quad \hat{f}_{\ell k} = \int_{\mathbb{S}^n} f \overline{Y_{\ell k}} dS,$$

where  $dS$  is the surface measure of the unit sphere. Using the addition theorem for spherical harmonics (see, for example, [4, page 18]), we can write

$$g(\mathbf{x}, \mathbf{y}) = \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(n, \ell)} \hat{g}(\ell) Y_{\ell k}(\mathbf{x}) \overline{Y_{\ell k}(\mathbf{y})}, \quad \text{with } \hat{g}(\ell) = \frac{\omega_n}{N(n, \ell)} a_{\ell}, \quad (2)$$

where  $\omega_n = \int_{\mathbb{S}^n} dS$  is the surface area of  $\mathbb{S}^n$ . We shall refer to  $\hat{g}(\ell)$  as the Fourier symbol of  $g$ .

## 2.2. Kernels defined from radial basis functions

Let  $\Phi = \rho(|\cdot|) : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$  be a compactly supported radial basis function (RBF) with associated RKHS  $H^{\sigma+1/2}(\mathbb{R}^{n+1})$  with  $\sigma > n/2$ . Examples of such RBFs are Wendland's functions (see [5]).

By restricting the function  $\Phi$  to the unit sphere  $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ , we have a positive definite, bizonal kernel on the unit sphere

$$\phi(\mathbf{x}, \mathbf{y}) = \Phi(\mathbf{x} - \mathbf{y}) = \rho(|\mathbf{x} - \mathbf{y}|), \quad \mathbf{x}, \mathbf{y} \in \mathbb{S}^n.$$

It is known (see [6]) that the Fourier symbol  $\widehat{\phi}(\ell)$  satisfies  $\widehat{\phi}(\ell) \sim (1 + \ell)^{-2\sigma}$ , i.e. there exist constants  $c_1, c_2 > 0$  such that

$$c_1(1 + \ell)^{-2\sigma} \leq \widehat{\phi}(\ell) \leq c_2(1 + \ell)^{-2\sigma}, \quad \ell \geq 0. \quad (3)$$

In the remainder of the paper, we use  $c_1, c_2, \dots$  to denote specific constants while  $c, C$  are generic constants that may take different values at each occurrence.

The reproducing kernel Hilbert space (RKHS) of the kernel  $\phi$  is given by

$$\mathcal{N}_\phi := \left\{ f \in \mathcal{D}'(\mathbb{S}^n) : \|f\|_\phi^2 := \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(n,\ell)} \frac{|\widehat{f}_{\ell k}|^2}{\widehat{\phi}(\ell)} < \infty \right\},$$

where  $\mathcal{D}'(\mathbb{S}^n)$  is the set of all tempered distributions defined on  $\mathbb{S}^n$ . Alternatively,  $\mathcal{N}_\phi$  is the completion of  $\text{span}\{\phi(\cdot, \mathbf{x}) : \mathbf{x} \in \mathbb{S}^n\}$  with respect to the norm  $\|\cdot\|_\phi$ . The norm obviously comes from the following inner product

$$(f, g)_\phi = \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(n,\ell)} \frac{\widehat{f}_{\ell k} \widehat{g}_{\ell k}}{\widehat{\phi}(\ell)}, \quad f, g \in \mathcal{N}_\phi.$$

The kernel  $\phi$  has the reproducing property with respect to this inner product, that is

$$f(\mathbf{x}) = (f, \phi(\cdot, \mathbf{x}))_\phi, \quad \mathbf{x} \in \mathbb{S}^n, f \in \mathcal{N}_\phi.$$

The Sobolev space  $H^\sigma(\mathbb{S}^n)$  with real parameter  $\sigma$  is defined by

$$H^\sigma(\mathbb{S}^n) := \left\{ f \in \mathcal{D}'(\mathbb{S}^n) : \|f\|_{H^\sigma(\mathbb{S}^n)}^2 := \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(n,\ell)} (1 + \ell)^{2\sigma} |\widehat{f}_{\ell k}|^2 < \infty \right\}.$$

For  $\sigma = 0$  the space is just  $L_2(\mathbb{S}^n)$ . Under the condition (3), the norms  $\|\cdot\|_\phi$  and  $\|\cdot\|_{H^\sigma(\mathbb{S}^n)}$  are equivalent, since

$$c_1^{1/2}\|f\|_\phi \leq \|f\|_{H^\sigma(\mathbb{S}^n)} \leq c_2^{1/2}\|f\|_\phi. \quad (4)$$

For a given  $\delta > 0$ , we define the scaled version  $\phi_\delta$  of the kernel  $\phi$  by

$$\phi_\delta(\mathbf{x}, \mathbf{y}) = \delta^{-n}\Phi((\mathbf{x} - \mathbf{y})/\delta) = \delta^{-n}\rho(|\mathbf{x} - \mathbf{y}|/\delta). \quad (5)$$

We can expand  $\phi_\delta$  into a series of spherical harmonics

$$\phi_\delta(\mathbf{x}, \mathbf{y}) = \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(n,\ell)} \widehat{\phi}_\delta(\ell) Y_{\ell k}(\mathbf{x}) \overline{Y_{\ell k}(\mathbf{y})}, \quad (6)$$

in which the Fourier coefficients satisfy the following condition (see [1, Theorem 6.2])

$$c_1(1 + \delta\ell)^{-2\sigma} \leq \widehat{\phi}_\delta(\ell) \leq c_2(1 + \delta\ell)^{-2\sigma}, \quad \ell \geq 0, \quad (7)$$

with the coefficients  $c_1$  and  $c_2$  from (3) possibly relaxed so that (7) holds for all  $0 < \delta \leq 1$ .

For a function  $f \in H^\sigma(\mathbb{S}^n)$ , we define the norm corresponding to the scaled kernel  $\phi_\delta$  by

$$\|f\|_{\phi_\delta} = \left( \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(n,\ell)} \frac{|\widehat{f}_{\ell k}|^2}{\widehat{\phi}_\delta(\ell)} \right)^{1/2}, \quad (8)$$

and the corresponding inner product is

$$(f, g)_{\phi_\delta} = \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(n,\ell)} \frac{\widehat{f}_{\ell k} \widehat{g}_{\ell k}}{\widehat{\phi}_\delta(\ell)}, \quad f, g \in \mathcal{N}_\phi. \quad (9)$$

Clearly the norms  $\|\cdot\|_{\phi_\delta}$  for different  $\delta$  are all equivalent.

From (7) and (8) it follows (see [1, Lemma 3.1]) that

$$c_1^{1/2}\|g\|_{\phi_\delta} \leq \|g\|_{H^\sigma(\mathbb{S}^n)} \leq c_2^{1/2}\delta^{-n}\|g\|_{\phi_\delta}. \quad (10)$$

### 2.3. Sobolev splines and associated spherical kernels

Our target function  $f$  will be assumed to belong to the Sobolev space  $H^\beta(\mathbb{S}^n)$  for some  $\beta$  satisfying  $\sigma > \beta > n/2$ .

We need an explicit formula for the reproducing kernel of  $H^\beta(\mathbb{S}^n)$ . This will be defined via a Sobolev spline in  $H^{\beta+1/2}(\mathbb{R}^{n+1})$ .

For  $\nu > 0$ , the Sobolev spline (or Matérn function) is defined by

$$S_\nu(\mathbf{x}) = (2\pi)^{-(n+1)/2} \frac{|\mathbf{x}|^\nu K_\nu(|\mathbf{x}|)}{2^{\nu+(n-1)/2} \Gamma(\nu + (n+1)/2)}, \quad \mathbf{x} \in \mathbb{R}^{n+1},$$

where  $K_\nu$  is the  $K$ -Bessel function of order  $\nu$ . With the Fourier transform of a function  $f \in L^1(\mathbb{R}^{n+1})$  defined by

$$\widehat{f}(\boldsymbol{\omega}) = (2\pi)^{-(n+1)/2} \int_{\mathbb{R}^{n+1}} f(\mathbf{x}) e^{-i\mathbf{x}\cdot\boldsymbol{\omega}} d\mathbf{x},$$

the Fourier transform of  $S_\nu$  is given by [5, Theorem 6.13]

$$\widehat{S}_\nu(\boldsymbol{\omega}) = (2\pi)^{-(n+1)/2} (1 + |\boldsymbol{\omega}|^2)^{-\nu-(n+1)/2}.$$

By setting  $\nu = \beta - n/2$ , the kernel  $\Psi(\mathbf{x} - \mathbf{y}) := S_\nu(\mathbf{x} - \mathbf{y})$  is hence the reproducing kernel of the Sobolev space  $H^{\beta+1/2}(\mathbb{R}^{n+1})$  with respect to the inner product

$$(f, g)_{H^{\beta+1/2}(\mathbb{R}^{n+1})} = \int_{\mathbb{R}^{n+1}} \widehat{f}(\boldsymbol{\omega}) \overline{\widehat{g}(\boldsymbol{\omega})} (1 + |\boldsymbol{\omega}|^2)^{\beta+1/2} d\boldsymbol{\omega}.$$

For  $0 < \delta \leq 1$ , we define the scaled version of  $\Psi$  by

$$\Psi_\delta(\mathbf{x}) = \delta^{-n} \Psi(\mathbf{x}/\delta), \quad \mathbf{x} \in \mathbb{R}^{n+1}. \quad (11)$$

Similarly as before, we define the kernel  $\psi$  and its scaled version by restricting  $\Psi$  on the sphere,

$$\psi(\mathbf{x}, \mathbf{y}) = \Psi(\mathbf{x} - \mathbf{y}), \quad \psi_\delta(\mathbf{x}, \mathbf{y}) = \Psi_\delta(\mathbf{x} - \mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in \mathbb{S}^n. \quad (12)$$

They can be expanded into a series of spherical harmonics as

$$\psi_\delta(\mathbf{x}, \mathbf{y}) = \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(n,\ell)} \widehat{\psi}_\delta(\ell) Y_{\ell k}(\mathbf{x}) \overline{Y_{\ell k}(\mathbf{y})}. \quad (13)$$

We define the following norm

$$\|f\|_{\psi_\delta} := \left( \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(n,\ell)} \frac{|\widehat{f}_{\ell k}|^2}{\widehat{\psi}_\delta(\ell)} \right)^{1/2}. \quad (14)$$

Then the RKHS of the kernel  $\psi_\delta$  is defined by

$$\mathcal{N}_{\psi_\delta} := \{f \in \mathcal{D}'(\mathbb{S}^n) : \|f\|_{\psi_\delta} < \infty\}.$$

The following two lemmas will give more information about this norm.

**Lemma 2.1.** *Let  $\Psi$  be the reproducing kernel of  $H^{\beta+1/2}(\mathbb{R}^{n+1})$ , and let  $\psi_\delta$  be defined by restricting  $\Psi_\delta$  to the unit sphere as in (12). Then the Fourier symbol  $\widehat{\psi}_\delta(\ell)$  satisfies*

$$c_3(1 + \delta\ell)^{-2\beta} \leq \widehat{\psi}_\delta(\ell) \leq c_4(1 + \delta\ell)^{-2\beta}, \quad \ell \geq 0, \quad (15)$$

where  $c_3, c_4$  are two positive constants independent of  $\delta$  and  $\ell$ .

**Proof.** The proof essentially follows the proof of [1, Theorem 6.2] except for one point, namely the proof that  $\widehat{\psi}_\delta(\ell) \leq c$ , for  $\ell \geq 0$ , where  $c > 0$  is independent of  $\ell$  and  $\delta$ . The corresponding proof in [1] was given under the assumption that  $\Psi$  is compactly supported with support in the unit ball, which is not the case here. A proof for the present definition of  $\Psi_\delta$  is as follows.

We have

$$\begin{aligned} \widehat{\psi}_\delta(\ell) &= \omega_{n-1} \int_{-1}^1 \Psi_\delta(\sqrt{2-2t}) P_\ell(n+1; t) (1-t^2)^{(n-2)/2} dt \\ &= C_{n,\nu} \delta^{-n} \int_{-1}^1 \left( \frac{\sqrt{2-2t}}{\delta} \right)^\nu K_\nu \left( \frac{\sqrt{2-2t}}{\delta} \right) P_\ell(n+1; t) (1-t^2)^{(n-2)/2} dt, \end{aligned}$$

where  $C_{n,\nu} := \omega_{n-1} 2^{-\nu-(n-1)/2} (2\pi)^{-(n+1)/2}$ . Using the substitution

$$r = \frac{\sqrt{2-2t}}{\delta}$$

and the bound

$$1-t^2 = (1+t)(1-t) \leq 2(1-t) = \delta^2 r^2,$$

together with  $|P_\ell(n+1; t)| \leq 1$  and  $K_\nu(r) > 0$  for  $r > 0$  and  $\nu \in \mathbb{R}$ , we have an upper bound

$$\begin{aligned}\widehat{\psi}_\delta(\ell) &\leq C_{n,\nu} \int_0^{2/\delta} r^\nu K_\nu(r) r^{n-1} dr \\ &\leq C_{n,\nu} \int_0^\infty r^\nu K_\nu(r) r^{n-1} dr \\ &= C_{n,\nu} 2^{\nu+n-2} \Gamma(n/2) \Gamma(\nu + n/2),\end{aligned}$$

where in the last step we have used [7, Formula 11.4.22, page 486]. For the remainder of the proof we refer to [1, Theorem 6.2].  $\square$

### 3. Multiscale approximation on $\mathbb{S}^n$

Let  $X = \{\mathbf{x}_1, \dots, \mathbf{x}_N\} \subset \mathbb{S}^n$  be a finite set of distinct points in  $\mathbb{S}^n$ . We define the mesh norm  $h_X$  and the separation radius  $q_X$  of this point set by

$$h_X = \sup_{\mathbf{x} \in \mathbb{S}^n} \min_{\mathbf{x}_j \in X} \theta(\mathbf{x}, \mathbf{x}_j), \quad q_X = \frac{1}{2} \min_{i \neq j} \theta(\mathbf{x}_i, \mathbf{x}_j),$$

where  $\theta(\mathbf{x}, \mathbf{y}) = \cos^{-1}(\mathbf{x} \cdot \mathbf{y})$  is the geodesic distance on  $\mathbb{S}^n$ .

We define the interpolation operator  $I_{X,\delta}$  associated with the set  $X$  and the kernel  $\phi_\delta$  by

$$I_{X,\delta} f(\mathbf{x}) = \sum_{m=1}^N b_m \phi_\delta(\mathbf{x}, \mathbf{x}_m), \quad I_{X,\delta} f(\mathbf{x}_m) = f(\mathbf{x}_m) \text{ for all } \mathbf{x}_m \in X. \quad (16)$$

We have previously obtained in [1] the following result, which extends results obtained for  $\delta = 1$  in [8] to general  $\delta \in (0, 1]$ .

**Theorem 3.1.** [1, Theorem 3.2] *Let  $h_X$  be the mesh norm of the finite set  $X \subset \mathbb{S}^n$ , and let  $g \in H^\sigma(\mathbb{S}^n)$  vanish on  $X$ . Then there exists  $c_5 > 0$  such that, for sufficiently small  $h_X$  and all  $0 < \delta \leq 1$ ,*

$$\|g\|_{L_2(\mathbb{S}^n)} \leq c_5 \left( \frac{h_X}{\delta} \right)^\sigma \|g\|_{\phi_\delta}, \quad (17)$$

and, for  $f \in H^\sigma(\mathbb{S}^n)$ ,

$$\|f - I_{X,\delta} f\|_{L_2(\mathbb{S}^n)} \leq c_5 \left( \frac{h_X}{\delta} \right)^\sigma \|f\|_{\phi_\delta}. \quad (18)$$



Suppose  $X_1, X_2, \dots \subset \mathbb{S}^n$  is a sequence of point sets with mesh norms  $h_1, h_2, \dots$  respectively. The mesh norms are assumed to satisfy  $h_{j+1} \approx \mu h_j$  for some fixed  $\mu \in (0, 1)$ .

Let  $\delta_1, \delta_2, \dots$  be a decreasing sequence of positive real numbers defined by  $\delta_j = \nu h_j$  for some  $\nu > 0$ . Taking the scale proportional to the mesh norm is desirable for both numerical stability and efficiency, since the sparsity of the interpolation matrix is maintained. For every  $j = 1, 2, \dots$  we define the scaled SBF  $\phi_j := \phi_{\delta_j}$ .

We start with a widely spread set of points and use a basis function with scale  $\delta_1$  to recover the global behavior of the function  $f$  by computing  $f_1 = s_1 := I_{X_1, \delta_1} f$ . The error, or residual, at the first step is  $e_1 = f - f_1$ . To reduce the error, at the next step we use a finer set of points  $X_2$  and a finer scale  $\delta_2$ , and compute a correction  $s_2 = I_{X_2, \delta_2} e_1$  and a new approximation  $f_2 = f_1 + s_2$ , so that the new residual is  $f - f_2 = e_1 - I_{X_2, \delta_2} e_1$ ; and so on.

Stated as an algorithm, we set  $f_0 = 0$  and  $e_0 = f$  and compute for  $j = 1, 2, \dots$ ,

$$\begin{aligned} s_j &= I_{X_j, \delta_j} e_{j-1}, \\ f_j &= f_{j-1} + s_j, \\ e_j &= e_{j-1} - s_j. \end{aligned}$$

Note that the algorithm makes  $f_j + e_j$  independent of  $j$ , from which it follows that  $f_j + e_j = f_0 + e_0 = f$ , allowing us to identify  $e_j = f - f_j$  as the error (or the residual) at stage  $j$ , and  $f_j$  as the approximation to  $f$  at the stage  $j$ . Since

$$f_j = \sum_{i=1}^j s_i = \sum_{i=1}^j I_{X_i, \delta_i} e_{i-1},$$

we see that the approximation  $f_j$  is a linear combination of spherical basis functions at all scales up to level  $j$ . We can think of  $s_j$  as adding additional “detail” to the approximation  $f_{j-1}$  to produce  $f_j$ . Also, since  $e_j|_{X_j} = 0$ , it follows that  $f_j = f - e_j$  interpolates  $f$  on  $X_j$ , or

$$f_j|_{X_j} = f|_{X_j}.$$

The next theorem is the convergence result of [1] for functions from the associated reproducing kernel Hilbert space.

**Theorem 3.2.** [1, Theorem 4.1] Let  $X_1, X_2, \dots$  be a sequence of point sets on  $\mathbb{S}^n$  with mesh norms  $h_1, h_2, \dots$  satisfying  $c\mu h_j \leq h_{j+1} \leq \mu h_j$  for  $j = 1, 2, \dots$  with fixed  $\mu, c \in (0, 1)$  and  $h_1$  sufficiently small. Let  $\phi_j = \phi_{\delta_j}$  be a kernel satisfying (7) with scale factor  $\delta_j = \nu h_j$  and  $\nu$  satisfying  $1/h_1 \geq \nu \geq \beta/\mu$  with a fixed  $\beta > 0$ . Assume that the target function  $f$  belongs to  $H^\sigma(\mathbb{S}^n)$ . Then there is a constant  $C = C(\beta)$  independent of  $j$  and  $f$  such that

$$\|e_j\|_{\phi_{j+1}} \leq \alpha \|e_{j-1}\|_{\phi_j} \quad \text{for } j = 1, 2, \dots, \quad (19)$$

with  $\alpha = C\mu^\sigma$ , and hence there exists  $c > 0$  such that

$$\|f - f_k\|_{L_2(\mathbb{S}^n)} \leq c\alpha^k \|f\|_{H^\sigma(\mathbb{S}^n)} \quad \text{for } k = 1, 2, \dots$$

Thus the multiscale approximation  $f_k$  converges linearly to  $f$  in the  $L_2$  norm if  $\mu < C^{-1/\sigma}$ .

#### 4. Approximation of rough target functions

In this section, we analyze the multistage approximation method of the last section for target functions outside the RKHS  $\mathcal{N}_\phi$ . Hence, we prove convergence of the multiscale approximation scheme under the following conditions. The interpolants are computed using the kernel  $\phi$ , which generates  $H^\sigma(\mathbb{S}^n)$  with  $\sigma > n/2$ . The target function  $f$ , however, belongs only to  $H^\beta(\mathbb{S}^n)$  for some  $\sigma > \beta > n/2$ .

##### 4.1. A convergence theorem

We need to collect a few auxiliary results before stating the convergence theorem for the multiscale approximation of rougher target functions.

**Lemma 4.1.** For  $0 < \delta \leq 1$  and all  $g \in H^\beta(\mathbb{S}^n)$  we have

$$c_3^{1/2} \|g\|_{\psi_\delta} \leq \|g\|_{H^\beta(\mathbb{S}^n)} \leq c_4^{1/2} \delta^{-\beta} \|g\|_{\psi_\delta}.$$

**Proof.** This follows from Lemma 2.1. The proof is essentially that of [1, Lemma 3.1] with  $\sigma$  replaced by  $\beta$ .  $\square$

**Lemma 4.2.** [8, Theorem 3.3] Let  $\tau > n/2$  and let  $X$  be a finite set in  $\mathbb{S}^n$  with sufficiently small mesh norm  $h_X$ . Then, there is a constant  $C > 0$ , independent of  $X$ , such that for all  $f \in H^\tau(\mathbb{S}^n)$  satisfying  $f|_X = 0$ , there holds

$$\|f\|_{H^\gamma(\mathbb{S}^n)} \leq Ch_X^{\tau-\gamma} \|f\|_{H^\tau(\mathbb{S}^n)}, \quad 0 \leq \gamma \leq \tau.$$

We also need a version of [9, Theorem 5.1], which asserts that for a sufficiently smooth function  $f$  there exists a spherical polynomial of degree at most  $L$  that interpolates  $f$  on a set  $X \subset \mathbb{S}^n$  of distinct, scattered points and, simultaneously, is a “good” approximant to  $f$ . We need a weak version of such a result for the Sobolev spaces generated by our scaled kernel  $\psi_\delta$ .

**Lemma 4.3.** *Let  $f \in H^\beta(\mathbb{S}^n)$ ,  $\beta > n/2$  and let  $X$  be a finite subset of  $\mathbb{S}^n$  with separation radius  $q_X$ . Let  $\delta \in (0, 1]$  be given. There exists a constant  $\kappa$ , which depends only on  $n$  and  $\beta$ , such that if  $L \geq \kappa \max\{\delta/q_X, 1/\delta\}$ , then there is a spherical polynomial  $p \in \mathcal{P}_L$  such that  $p|_X = f|_X$  and*

$$\|f - p\|_{\psi_\delta} \leq 5\|f\|_{\psi_\delta}.$$

We defer the proof of this lemma until the next subsection.

In the next lemma the notation is as in Theorem 3.2 with the addition that  $q_j = q_{X_j}$ . This lemma makes essential use of Lemma 4.3 in its proof.

**Lemma 4.4.** *Let  $q_j \leq h_j \leq c_q q_j$  and  $\delta_j \geq q_j$ . Let  $\psi_j := \psi_{\delta_j}$ , and let  $f \in H^\beta(\mathbb{S}^n)$  with  $\sigma > \beta > n/2$ . Then there exists a constant  $C > 0$ , independent of  $X_j$ , such that*

$$\|e_j\|_{H^\beta(\mathbb{S}^n)} \leq C\delta_j^{-\beta}\|e_{j-1}\|_{\psi_j}.$$

**Proof.** Choose  $L = \lceil \kappa \max\{\delta_j/q_j, 1/\delta_j\} \rceil$ . Then by Lemma 4.3, there is a spherical polynomial  $p \in \mathcal{P}_L$  that interpolates and approximates  $e_{j-1}$ , i.e.  $p|_{X_j} = e_{j-1}|_{X_j}$  and  $\|p - e_{j-1}\|_{\psi_j} \leq 5\|e_{j-1}\|_{\psi_j}$ . We can use  $p$  to split the error,

$$\begin{aligned} \|e_j\|_{H^\beta(\mathbb{S}^n)} &= \|e_{j-1} - I_{X_j, \delta_j} e_{j-1}\|_{H^\beta(\mathbb{S}^n)} \\ &\leq \|e_{j-1} - p\|_{H^\beta(\mathbb{S}^n)} + \|p - I_{X_j, \delta_j} e_{j-1}\|_{H^\beta(\mathbb{S}^n)}. \end{aligned} \quad (20)$$

The first term of (20) can be bounded using Lemmas 4.1 and 4.3,

$$\begin{aligned} \|e_{j-1} - p\|_{H^\beta(\mathbb{S}^n)} &\leq c_4^{1/2} \delta_j^{-\beta} \|e_{j-1} - p\|_{\psi_j} \\ &\leq 5c_4^{1/2} \delta_j^{-\beta} \|e_{j-1}\|_{\psi_j}. \end{aligned}$$

Since  $p|_{X_j} = e_{j-1}|_{X_j}$  the interpolant  $I_{X_j, \delta_j} e_{j-1}$  is identical to  $I_{X_j, \delta_j} p$ . Therefore the second term of (20) can be estimated via Lemma 4.2 and (10),

$$\begin{aligned} \|p - I_{X_j, \delta_j} e_{j-1}\|_{H^\beta(\mathbb{S}^n)} &= \|p - I_{X_j, \delta_j} p\|_{H^\beta(\mathbb{S}^n)} \leq Ch_j^{\sigma-\beta} \|p - I_{X_j, \delta_j} p\|_{H^\sigma(\mathbb{S}^n)} \\ &\leq Ch_j^{\sigma-\beta} \delta_j^{-\sigma} \|p - I_{X_j, \delta_j} p\|_{\phi_j} \leq Ch_j^{\sigma-\beta} \delta_j^{-\sigma} \|p\|_{\phi_j}, \end{aligned}$$

where in the last step we used the fact that  $I_{X_j, \delta_j} p$  is the orthogonal projection with respect to the inner product  $(\cdot, \cdot)_{\phi_j}$  onto  $\text{span}\{\phi_{\delta_j}(\cdot, \mathbf{x}) : \mathbf{x} \in X_j\}$ . Since  $p \in \mathcal{P}_L$ ,  $\delta_j \leq 1$  and  $\beta > \sigma$ , using the definition of the norm  $\|\cdot\|_{\phi_j}$  given by (8) and the condition (7) we have

$$\begin{aligned} \|p\|_{\phi_j}^2 &\leq \frac{1}{c_1} \sum_{\ell=0}^L \sum_{k=1}^{N(n, \ell)} (1 + \delta_j \ell)^{2\sigma} |\widehat{p}_{\ell, k}|^2 \\ &\leq \frac{1}{c_1} (1 + \delta_j L)^{2(\sigma - \beta)} \sum_{\ell=0}^L \sum_{k=1}^{N(n, \ell)} (1 + \delta_j \ell)^{2\beta} |\widehat{p}_{\ell, k}|^2 \\ &\leq C (\delta_j / q_j)^{2(\sigma - \beta)} \|p\|_{\psi_j}^2, \end{aligned}$$

where in the last step we used the property that  $\lceil a \rceil \leq 2a$  for  $a \geq 1$ . Thus, combining these above estimates together with the fact that  $\|p\|_{\psi_j} \leq 6\|e_{j-1}\|_{\psi_j}$  we obtain

$$\|p - I_{X_j, \delta_j} p\|_{H^\beta(\mathbb{S}^n)} \leq C \delta_j^{-\beta} (h_j / q_j)^{\sigma - \beta} \|e_{j-1}\|_{\psi_j} \leq C \delta_j^{-\beta} \|e_{j-1}\|_{\psi_j}.$$

□

In the following, we are going to prove that the multilevel algorithm also converges for target functions from  $H^\beta(\mathbb{S}^n)$ . The proof generally follows the line of Theorem 4.1 in [1] with  $\phi_j$  replaced by  $\psi_j$  but a new condition (see (ii) below) is needed, for now we need to use Lemma 4.4, which in turn rests on Lemma 4.3.

**Theorem 4.5.** *Let  $X_1, X_2, \dots$  be a sequence of point sets in  $\mathbb{S}^n$  with mesh norms  $h_1, h_2, \dots$  and separation radii  $q_1, q_2, \dots$  satisfying*

- (i)  $c\mu h_j \leq h_{j+1} \leq \mu h_j$  for  $j = 1, 2, \dots$  with  $\mu, c \in (0, 1)$ ,
- (ii)  $q_j \leq h_j \leq c_q q_j$  for  $j = 1, 2, \dots$  with  $c_q > 1$ .

Let  $\phi$  be a kernel generating  $H^\sigma(\mathbb{S}^n)$ , i.e. the Fourier symbol of  $\phi$  satisfies (3), and let  $\phi_j := \phi_{\delta_j}$  be defined by (5) with scale factor  $\delta_j = \nu h_j$  where  $1/h_1 \geq \nu \geq \gamma/\mu \geq 1$  with a fixed  $\gamma > 0$ . Let  $\psi$  be a kernel generating  $H^\beta(\mathbb{S}^n)$  with  $\sigma > \beta > n/2$  and let  $\psi_j$  be the scaled version using the same scale factor  $\delta_j$ . Assume that the target function  $f$  belongs to  $H^\beta(\mathbb{S}^n)$ . Then, there exists a constant  $C = C(\gamma) > 0$  such that, with  $\alpha = C\mu^\beta$ ,

$$\|e_j\|_{\psi_{j+1}} \leq \alpha \|e_{j-1}\|_{\psi_j} \quad \text{for } j = 1, 2, 3, \dots$$

and hence there exists a constant  $c > 0$  so that

$$\|f - f_k\|_{L_2(\mathbb{S}^n)} \leq c\alpha^k \|f\|_{H^\beta(\mathbb{S}^n)} \quad \text{for } k = 1, 2, 3, \dots$$

**Proof.** We start with, using Lemma 2.1,

$$\|e_j\|_{\psi_{j+1}}^2 \leq \frac{1}{c_3} \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(n,\ell)} |\widehat{e}_{j,\ell k}|^2 (1 + \delta_{j+1}\ell)^{2\beta} =: \frac{1}{c_3} (S_1 + S_2),$$

with

$$S_1 := \sum_{\ell \leq 1/\delta_{j+1}} \sum_{k=1}^{N(n,\ell)} |\widehat{e}_{j,\ell k}|^2 (1 + \delta_{j+1}\ell)^{2\beta},$$

$$S_2 := \sum_{\ell > 1/\delta_{j+1}} \sum_{k=1}^{N(n,\ell)} |\widehat{e}_{j,\ell k}|^2 (1 + \delta_{j+1}\ell)^{2\beta}.$$

For the first sum  $S_1$ , since  $\delta_{j+1}\ell \leq 1$ ,

$$S_1 \leq \sum_{\ell \leq 1/\delta_{j+1}} \sum_{k=1}^{N(n,\ell)} |\widehat{e}_{j,\ell k}|^2 2^{2\beta} \leq 2^{2\beta} \|e_j\|_{L_2(\mathbb{S}^n)}^2.$$

Since  $e_j$  vanishes on  $X_j$ , by Lemma 4.2 we have  $\|e_j\|_{L_2(\mathbb{S}^n)} \leq Ch_j^\beta \|e_j\|_{H^\beta(\mathbb{S}^n)}$ . Combining this with Lemma 4.4, which gives  $\|e_j\|_{H^\beta(\mathbb{S}^n)} \leq C\delta_j^{-\beta} \|e_{j-1}\|_{\psi_j}$ , we have

$$\|e_j\|_{L_2(\mathbb{S}^n)}^2 \leq C \left( \frac{h_j}{\delta_j} \right)^{2\beta} \|e_{j-1}\|_{\psi_j}^2 \leq C\mu^{2\beta} \|e_{j-1}\|_{\psi_j}^2.$$

For the sum  $S_2$ , since  $\delta_{j+1}\ell > 1$  we have,

$$(1 + \delta_{j+1}\ell)^{2\beta} < (2\delta_{j+1}\ell)^{2\beta} \leq (2\delta_{j+1})^{2\beta} (1 + \ell)^{2\beta}.$$

So

$$S_2 \leq \sum_{\ell > 1/\delta_{j+1}} \sum_{k=1}^{N(n,\ell)} (2\delta_{j+1})^{2\beta} |\widehat{e}_{j,\ell k}|^2 (1 + \ell)^{2\beta} \leq (2\delta_{j+1})^{2\beta} \|e_j\|_{H^\beta(\mathbb{S}^n)}^2.$$

Using Lemma 4.4 again, which gives  $\|e_j\|_{H^\beta(\mathbb{S}^n)} \leq C\delta_j^{-\beta} \|e_{j-1}\|_{\psi_j}$ , we obtain

$$S_2 \leq C \left( \frac{\delta_{j+1}}{\delta_j} \right)^{2\beta} \|e_{j-1}\|_{\psi_j}^2 \leq C\mu^{2\beta} \|e_{j-1}\|_{\psi_j}^2.$$

Putting all the estimates together, we obtain

$$\|e_j\|_{\psi_{j+1}} \leq C\mu^\beta \|e_{j-1}\|_{\psi_j} \leq \alpha \|e_{j-1}\|_{\psi_j},$$

which is the first statement of the theorem. Finally, since  $e_k = f - f_k$  vanishes on  $X_k$ , we have by Lemma 4.2 and Lemma 4.1 that

$$\begin{aligned} \|f - f_k\|_{L_2} &= \|e_k\|_{L_2} \leq Ch_k^\beta \|e_k\|_{H^\beta(\mathbb{S}^n)} \\ &\leq Ch_k^\beta \delta_{k+1}^{-\beta} \|e_k\|_{\psi_{k+1}} \\ &\leq C \|e_k\|_{\psi_{k+1}} \\ &\leq C\alpha \|e_{k-1}\|_{\psi_k}, \end{aligned}$$

since  $h_k/\delta_{k+1} = h_k/(\nu h_{k+1}) \leq 1/(c\gamma)$ . Now we can apply the argument  $k$  times to obtain, with the aid of Lemma 4.1,

$$\|f - f_k\|_{L_2} \leq C\alpha^k \|e_0\|_{\psi_{k+1}} = C\alpha^k \|f\|_{\psi_{k+1}} \leq C\alpha^k \|f\|_{H^\beta(\mathbb{S}^n)}.$$

□

#### 4.2. Existence of a polynomial interpolant

We are now going to prove Lemma 4.3. To this end, we have to collect a number of auxiliary results, which are mainly adaptation of results from [9]. The idea there is to “lift” the norms involving spherical basis functions (SBFs) on  $\mathbb{S}^n$  to those for RBFs on  $\mathbb{R}^{n+1}$ .

We will once again work with the bizonal kernel  $\psi(\mathbf{x}, \mathbf{y}) = \Psi(\mathbf{x} - \mathbf{y})$  for  $\mathbf{x}, \mathbf{y} \in \mathbb{S}^n$  where  $\Psi$  is the reproducing kernel of the Sobolev space  $H^{\beta+1/2}(\mathbb{R}^{n+1})$ .

For a given given  $\tau > 0$ , we define the function  $\Psi(\tau; \cdot)$  via its Fourier transform,

$$\widehat{\Psi}(\tau; \boldsymbol{\omega}) := \begin{cases} \widehat{\Psi}(\boldsymbol{\omega}), & |\boldsymbol{\omega}| \leq \tau \\ 0, & |\boldsymbol{\omega}| > \tau. \end{cases} \quad (21)$$

Hence,  $\Psi(\tau; \cdot)$  is given by

$$\begin{aligned} \Psi(\tau; \mathbf{x}) &:= (2\pi)^{-(n+1)/2} \int_{\mathbb{R}^{n+1}} \widehat{\Psi}(\tau; \boldsymbol{\omega}) e^{i\mathbf{x} \cdot \boldsymbol{\omega}} d\boldsymbol{\omega} \\ &= (2\pi)^{-(n+1)/2} \int_{|\boldsymbol{\omega}| \leq \tau} \widehat{\Psi}(\boldsymbol{\omega}) e^{i\mathbf{x} \cdot \boldsymbol{\omega}} d\boldsymbol{\omega}. \end{aligned}$$

A scaled version of  $\Psi(\tau; \mathbf{x})$  is defined by

$$\Psi_\delta(\tau; \mathbf{x}) := \delta^{-n} \Psi(\delta\tau; \mathbf{x}/\delta). \quad (22)$$

Consequently, the Fourier transform of  $\Psi_\delta(\tau; \cdot)$  scales as

$$\widehat{\Psi}_\delta(\tau; \boldsymbol{\omega}) = \delta \widehat{\Psi}(\delta\tau; \delta\boldsymbol{\omega}) = \begin{cases} \widehat{\Psi}_\delta(\boldsymbol{\omega}) = \delta \widehat{\Psi}(\delta\boldsymbol{\omega}), & |\boldsymbol{\omega}| \leq \tau \\ 0, & |\boldsymbol{\omega}| > \tau. \end{cases} \quad (23)$$

The bizonal kernel  $\psi_{\delta;\tau}(\mathbf{x}, \mathbf{y}) := \Psi_\delta(\tau; \mathbf{x} - \mathbf{y})|_{\mathbf{x}, \mathbf{y} \in \mathbb{S}^n}$  is an SBF with Fourier-Legendre coefficients given by (see [6, Theorem 4.1 and Corollary 4.3])

$$\widehat{\psi}_{\delta;\tau}(\ell) = (2\pi)^{(n+1)/2} \int_0^\tau t \widehat{\Psi}_\delta(t) J_{\ell+(n-1)/2}^2(t) dt.$$

Besides this, we will also need the truncation of  $\psi_\delta$  to  $\mathcal{P}_L$  given by

$$\psi_\delta^L(\mathbf{x}, \mathbf{y}) := \sum_{\ell=0}^L \sum_{k=1}^{N(n,\ell)} \widehat{\psi}_\delta(\ell) Y_{\ell,k}(\mathbf{x}) Y_{\ell,k}(\mathbf{y}).$$

The following result is [9, Lemma 4.2], with  $\Psi$  and  $\psi$  being replaced by  $\Psi_\delta$  and  $\psi_\delta$ , respectively.

**Proposition 4.6.** *Let  $X = \{\mathbf{x}_1, \dots, \mathbf{x}_N\}$  be a set of scattered points on  $\mathbb{S}^n$  and  $\alpha \in \mathbb{R}^N$ . Let  $\Psi$  be the reproducing kernel for  $H^{\beta+1/2}(\mathbb{R}^{n+1})$  and  $\psi$  be its associated spherical kernel. Let the scale parameter  $\delta \in (0, 1]$  be given. Then*

$$\left\| \sum_{j=1}^N \alpha_j \psi_\delta(\cdot, \mathbf{x}_j) \right\|_{\psi_\delta} = \left\| \sum_{j=1}^N \alpha_j \Psi_\delta(\cdot - \mathbf{x}_j) \right\|_{\Psi_\delta}. \quad (24)$$

*If there is a constant  $c > 0$  such that when  $\tau \leq cL$  we have  $\widehat{\psi}_{\delta;\tau}(\ell) \leq \frac{1}{2} \widehat{\psi}_\delta(\ell)$  for all  $\ell > L$ , then we have*

$$\left\| \sum_{j=1}^N \alpha_j (\psi_\delta - \psi_\delta^L)(\cdot, \mathbf{x}_j) \right\|_{\psi_\delta}^2 \leq 2 \left\| \sum_{j=1}^N \alpha_j (\Psi_\delta(\mathbf{x} - \mathbf{x}_j) - \Psi_\delta(\tau; \cdot - \mathbf{x}_j)) \right\|_{\Psi_\delta}^2. \quad (25)$$

We are now going to show that there is indeed such a constant  $c > 0$  such that when  $\tau \leq cL$  we have  $\widehat{\psi}_{\delta;\tau}(\ell) \leq \frac{1}{2} \widehat{\psi}_\delta(\ell)$  for all  $\ell > L$ . For this, we need the following lemma.

**Lemma 4.7.** *Let  $\widehat{\Psi} \in L^1(\mathbb{R}^{n+1})$  be radial and nonnegative, and let  $\ell \geq 0$ . If  $0 < \delta \leq 1, \tau > 0$  and  $\nu = \ell + (n-1)/2$ , then*

$$\begin{aligned}\widehat{\psi}_{\delta;\tau}(\ell) &= (2\pi)^{(n+1)/2} \int_0^\tau t \widehat{\Psi}_\delta(t) J_\nu^2(t) dt \\ &\leq \frac{2(2\pi)^{(n+1)/2} e \delta^{-n} \Psi(0) (e\pi)^n}{\omega_n (\nu+1)^n} \left( \frac{e\tau}{2(\nu+1)} \right)^{2\nu-n+1}.\end{aligned}$$

**Proof.** This follows from [9, Lemma 4.3] by replacing  $\Psi$  with  $\Psi_\delta$  and using  $\Psi_\delta(0) = \delta^{-n} \Psi(0)$ .  $\square$

Since  $\psi$  is a reproducing kernel for  $H^\beta(\mathbb{S}^n)$  with  $\beta > n/2$ , condition (15) and Lemma 4.7 yield

$$\frac{\widehat{\psi}_{\delta;\tau}(\ell)}{\widehat{\psi}_\delta(\ell)} \leq C_{\beta,n} (1 + \delta\ell)^{2\beta} (\delta\ell)^{-n} \left( \frac{e\tau}{2\ell} \right)^{2\ell}.$$

If we now set  $\ell_0 = \lceil 1/\delta \rceil$  then we have for all  $\ell \geq \ell_0$  that

$$\frac{\widehat{\psi}_{\delta;\tau}(\ell)}{\widehat{\psi}_\delta(\ell)} \leq C_{\beta,n} 2^{2\beta} (\delta\ell)^{2\beta-n} \left( \frac{e\tau}{2\ell} \right)^{2\ell}.$$

From this we can conclude that there is an  $L_0 \geq \ell_0$  such that for all  $\ell \geq L \geq L_0$  and  $\tau \leq e^{-1}L$  we have

$$\frac{\widehat{\psi}_{\delta;\tau}(\ell)}{\widehat{\psi}_\delta(\ell)} \leq C_{\beta,n} 2^{2\beta} (\delta\ell)^{2\beta-n} 2^{-2\ell} \leq \frac{1}{2}. \quad (26)$$

Thus the conclusions of Proposition 4.6 hold.

We now state a proposition, whose proof can be found in [10, Theorem 2.1] or [11, Prop. 3.1]. It provides the theoretical framework for proving the existence of an interpolant that is also a good approximant.

**Proposition 4.8.** *Let  $\mathcal{Y}$  be a (possibly complex) Banach space,  $\mathcal{V}$  a subspace of  $\mathcal{Y}$ , and  $Z^*$  a finite-dimensional subspace of  $\mathcal{Y}^*$ , the dual of  $\mathcal{Y}$ . If there is a  $\gamma > 1$  such that for every  $z^* \in Z^*$ ,*

$$\|z^*\|_{\mathcal{Y}^*} \leq \gamma \|z^*|_{\mathcal{V}}\|_{\mathcal{V}^*}, \quad (27)$$

*then for  $y \in \mathcal{Y}$  there exists  $v \in \mathcal{V}$  such that  $v$  interpolates  $y$  on  $Z^*$ ; that is,  $z^*(y) = z^*(v)$  for all  $z^* \in Z^*$ . In addition,  $v$  approximates  $y$  in the sense that  $\|y - v\|_{\mathcal{Y}} \leq (1 + 2\gamma) \text{dist}(y, \mathcal{V})$ .*



We will apply this result to the case in which the underlying space is the RKHS of the scaled kernel  $\psi_\delta$ , that is  $\mathcal{Y} = \mathcal{N}_{\psi_\delta}$ . We will take  $Z^* = \text{span}\{\delta_{\mathbf{x}_j}\}_{j=1}^N$ , where the points are distinct and come from a finite set  $X \subset \mathbb{S}^n$ . Finally, we take  $\mathcal{V} = \mathcal{P}_L$ , the span of the spherical harmonics of degree  $\ell \leq L$ , with  $L$  at the moment fixed.

The quantities in Proposition 4.8 can be put in terms of the reproducing kernel  $\psi_\delta$ . First, we observe that

$$\left\| \sum_{j=1}^N \alpha_j \delta_{\mathbf{x}_j} \right\|_{\mathcal{N}_{\psi_\delta}^*} = \left\| \sum_{j=1}^N \alpha_j \psi_\delta(\cdot, \mathbf{x}_j) \right\|_{\psi_\delta}.$$

Second, we have

$$\left\| \sum_{j=1}^N \alpha_j \delta_{\mathbf{x}_j} \right\|_{\mathcal{P}_L} \Big|_{\mathcal{P}_L^*} = \left\| \sum_{j=1}^N \alpha_j \psi_\delta^L(\cdot, \mathbf{x}_j) \right\|_{\psi_\delta},$$

see formula (5.3) in [9]. Moreover, since  $\psi_\delta^L(\cdot, \mathbf{x}_j)$  and  $\psi_\delta(\cdot, \mathbf{x}_k) - \psi_\delta^L(\cdot, \mathbf{x}_k)$  are orthogonal in  $\mathcal{N}_{\psi_\delta}$  for all  $j, k$ , we can use the Pythagorean theorem to obtain

$$\begin{aligned} \left\| \sum_{j=1}^N \alpha_j \psi_\delta^L(\cdot, \mathbf{x}_j) \right\|_{\psi_\delta}^2 &= \left\| \sum_{j=1}^N \alpha_j \psi_\delta(\cdot, \mathbf{x}_j) \right\|_{\psi_\delta}^2 \\ &\quad - \left\| \sum_{j=1}^N \alpha_j [\psi_\delta(\cdot, \mathbf{x}_j) - \psi_\delta^L(\cdot, \mathbf{x}_j)] \right\|_{\psi_\delta}^2. \end{aligned}$$

From this and the quantities above, it follows that Proposition 4.8 applied to our situation yields the following: If we can find a  $\gamma > 1$  such that for every linear functional  $z^* = \sum_j \alpha_j \delta_{\mathbf{x}_j}$  we have

$$\frac{\left\| \sum_{j=1}^N \alpha_j [\psi_\delta(\cdot, \mathbf{x}_j) - \psi_\delta^L(\cdot, \mathbf{x}_j)] \right\|_{\psi_\delta}^2}{\left\| \sum_{j=1}^N \alpha_j \psi_\delta(\cdot, \mathbf{x}_j) \right\|_{\psi_\delta}^2} \leq 1 - \frac{1}{\gamma^2}, \quad (28)$$

then for  $f \in \mathcal{N}_{\psi_\delta}$  there exists a polynomial  $p$  so that  $f(\mathbf{x}_j) = p(\mathbf{x}_j)$  for  $1 \leq j \leq N$ . In addition,  $p$  approximates  $f$  in the sense that  $\|f - p\|_{\psi_\delta} \leq (1 + 2\gamma)\|f\|_{\psi_\delta}$ .

An explicit value of  $\gamma$  will be given in the proof of Lemma 4.3, as follows.

**Proof of Lemma 4.3**

We have, by using Proposition 4.6 and (26) that

$$\frac{\left\| \sum_{j=1}^N \alpha_j [\psi_\delta(\cdot, \mathbf{x}_j) - \psi_\delta^L(\cdot, \mathbf{x}_j)] \right\|_{\psi_\delta}^2}{\left\| \sum_{j=1}^N \alpha_j \psi_\delta(\cdot, \mathbf{x}_j) \right\|_{\psi_\delta}^2} \leq \frac{2 \left\| \sum_{j=1}^N \alpha_j [\Psi_\delta(\cdot - \mathbf{x}_j) - \Psi_\delta(\tau; \cdot - \mathbf{x}_j)] \right\|_{\Psi_\delta}^2}{\left\| \sum_{j=1}^N \alpha_j \Psi_\delta(\cdot - \mathbf{x}_j) \right\|_{\Psi_\delta}^2}$$

where we take  $\tau = e^{-1}L$  and  $L \geq L_0$  as required for (26). Using the reproducing property of  $\Psi_\delta$ , we have

$$\begin{aligned} & \left\| \sum_{j=1}^N \alpha_j [\Psi_\delta(\cdot - \mathbf{x}_j) - \Psi_\delta(\tau; \cdot - \mathbf{x}_j)] \right\|_{\Psi_\delta}^2 \\ &= \sum_{j,k=1}^N \alpha_j \alpha_k [\Psi_\delta(\mathbf{x}_j - \mathbf{x}_k) - \Psi_\delta(\tau; \mathbf{x}_j - \mathbf{x}_k)] \\ &= \delta^{-n} \sum_{j,k=1}^N \alpha_j \alpha_k \left[ \Psi \left( \frac{\mathbf{x}_j - \mathbf{x}_k}{\delta} \right) - \Psi \left( \tau; \frac{\mathbf{x}_j - \mathbf{x}_k}{\delta} \right) \right]. \end{aligned}$$

Similarly, using the reproducing property of  $\Psi_\delta$  again, we have

$$\left\| \sum_{j=1}^N \alpha_j \Psi_\delta(\cdot - \mathbf{x}_j) \right\|_{\Psi_\delta}^2 = \sum_{j,k=1}^N \alpha_j \alpha_k \Psi_\delta(\mathbf{x}_j - \mathbf{x}_k) = \delta^{-n} \sum_{j,k=1}^N \alpha_j \alpha_k \Psi \left( \frac{\mathbf{x}_j - \mathbf{x}_k}{\delta} \right).$$

By using an estimate from the proof of [12, Lemma 3.3] on the scaled data set  $Y = X/\delta$ , we obtain the following estimate

$$\frac{\sum_{j,k=1}^N \alpha_j \alpha_k \left[ \Psi \left( \frac{\mathbf{x}_j - \mathbf{x}_k}{\delta} \right) - \Psi \left( \tau; \frac{\mathbf{x}_j - \mathbf{x}_k}{\delta} \right) \right]}{\sum_{j,k=1}^N \alpha_j \alpha_k \Psi \left( \frac{\mathbf{x}_j - \mathbf{x}_k}{\delta} \right)} \leq C(\tau Q_X / \delta)^{n+1-2(\beta+1/2)},$$

where  $C = C(\beta, n + 1)$  and  $Q_X$  is the Euclidean separation radius for  $X$  as a subset of  $\mathbb{R}^{n+1}$ . For any discrete set  $X$ , we know that  $Q_X$  is comparable

to  $q_X$ , indeed  $q_X \geq Q_X \geq (2/\pi)q_X$ . Combining all of the results and using  $\tau = e^{-1}L$  yields

$$\frac{\left\| \sum_{j=1}^N \alpha_j [\psi_\delta(\cdot, \mathbf{x}_j) - \psi_\delta^L(\cdot, \mathbf{x}_j)] \right\|_{\psi_\delta}^2}{\left\| \sum_{j=1}^N \alpha_j \psi_\delta(\cdot, \mathbf{x}_j) \right\|_{\psi_\delta}^2} \leq C' (Lq_X/\delta)^{n-2\beta},$$

where  $C' = 2^{n+1-2\beta}C/(e\pi)^{n-2\beta}$ . We can choose  $\kappa > 0$  such that  $C'\kappa^{n-2\beta} \leq \frac{3}{4}$ . Since  $\beta > n/2$  we have  $n-2\beta < 0$ , thus, if  $Lq_X/\delta \geq \kappa$ , then  $C'(Lq_X/\delta)^{n-2\beta} \leq \frac{3}{4}$ . Moreover, by enlarging  $\kappa$  if necessary, we can also ensure that the condition  $L \geq L_0$  required for (26) is satisfied. This means that (28) holds with  $\gamma = 2$  provided that  $L \geq \kappa \max\{q_X/\delta, 1/\delta\}$ .

Hence, we have shown that we can apply Proposition 4.8 in this situation, which finishes the proof.  $\square$

#### 4.3. Multiresolution analysis

For a given RBF  $\Phi$  in  $\mathbb{R}^{n+1}$  and a sequence of point sets  $X_1, X_2, \dots \subset \mathbb{S}^n$  and a sequence of scales  $\delta_1, \delta_2, \dots$  converging to 0, the multiscale approximation scheme in Section 3 can be put into the following multiresolution framework.

For  $j \geq 1$ , let  $W_j$  and  $V_j$  be the linear subspaces of  $H^\sigma(\mathbb{S}^n)$  defined by

$$W_j := \text{span}\{\phi_{\delta_j}(\cdot, \mathbf{x}) : \mathbf{x} \in X_j\}$$

and

$$V_j := \text{span}\{\phi_{\delta_i}(\cdot, \mathbf{x}) : \mathbf{x} \in X_i, \quad i \leq j\},$$

where  $\phi_\delta$  is as in (5). Thus

$$V_1 \subset V_2 \subset \dots \subset V_j \subset V_{j+1} \subset \dots \subset H^\sigma(\mathbb{S}^n).$$

In the language of wavelets,  $W_j$  is the wavelet space and  $V_j$  the scale space. The space  $V_j$  can be written as a direct sum of spaces  $W_i$ ,

$$V_j = \bigoplus_{1 \leq i \leq j} W_i, \quad j \geq 1$$

and hence if  $V_0 = \{0\}$ ,

$$V_j = V_{j-1} \bigoplus W_j, \quad j \geq 1.$$

That the sum is direct follows under appropriate conditions from the following lemma [1, Lemma 5.1].

**Lemma 4.9.** *Let  $\Phi$  be a compactly supported RBF as in [5] and let  $\phi_{\delta_i}$  for  $i = 1, 2, \dots$  be scaled SBFs constructed as in (5) where  $\delta_1, \delta_2, \dots$  are distinct scales with  $\delta_i \leq 1$ . Let  $X_i = \{\mathbf{x}_{i,1}, \dots, \mathbf{x}_{i,N_i}\} \subset \mathbb{S}^n$  for  $i \geq 1$  be a set of  $N_i$  distinct points. Then for  $j \geq 1$  and  $a_{i,k} \in \mathbb{R}$*

$$\sum_{i=1}^j \sum_{k=1}^{N_i} a_{i,k} \phi_{\delta_i}(\cdot, \mathbf{x}_{i,k}) = 0 \text{ implies } a_{i,k} = 0 \text{ for } i = 1, \dots, j; k = 1, \dots, N_i. \quad (29)$$

The sequence of spaces  $\{V_j\}$  is ultimately dense in  $L_2(\mathbb{S}^n)$ , in the sense of the following theorem [1, Theorem 5.2].

**Theorem 4.10.** *Let  $X_1, X_2, \dots$  be a sequence of point sets on  $\mathbb{S}^n$  with mesh norms  $h_1, h_2, \dots$  satisfying  $c\mu h_j \leq h_{j+1} \leq \mu h_j$  for  $j = 1, 2, \dots$  with fixed  $\mu, c \in (0, 1)$  and  $h_1$  sufficiently small. Let  $\phi_{\delta_j}$  be a kernel satisfying (7) with scale factor  $\delta_j = \nu h_j$  and  $\nu$  satisfying  $1/h_1 \geq \nu \geq \beta/\mu$  with a fixed  $\beta > 0$ . For all  $\mu$  sufficiently small.*

*the closure of  $\bigcup_{j=1}^{\infty} V_j$  with respect to the norm  $\|\cdot\|_{L_2}$  is  $L_2(\mathbb{S}^n)$ .*

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