An overlapping additive Schwarz preconditioner for interpolation on the unit sphere by spherical radial basis functions

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Abstract

The problem of interpolation of scattered data on the unit sphere has many applications in geodesy and earth science in which the sphere is taken as a model for the earth. Spherical radial basis functions provide a convenient tool to construct the interpolant. However, the underlying linear systems tend to be ill-conditioned. In this paper, we present an additive Schwarz preconditioner to accelerate the solution process. An estimate for the condition number of the preconditioned system will be discussed. Numerical experiments using MAGSAT satellite data will be presented.

1 Introduction

Let $\mathbb{S}^n$ be the unit sphere in $\mathbb{R}^{n+1}$, $n = 1, 2, \ldots$, and suppose that $X = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\}$ is a set of scattered points lying on $\mathbb{S}^n$. Given real numbers f_i, $i = 1, \ldots, N$, we want to find a smooth function u defined on $\mathbb{S}^n$ which interpolates the data, namely,

$$u(\boldsymbol{x}_i) = f_i, \quad i = 1, \ldots, N. \tag{1.1}$$

This problem arises in many areas including, e.g., geodesy and earth science in which the sphere is taken as a model for the earth. Even though in practice measured data usually contain noise which means that approximation of data is necessary, for simplicity, in this paper we consider only the interpolation problem at points determined by scattered data collected from satellites.

A review paper by Fasshauer and Schumaker [5] discusses available methods for interpolation of scattered data on the sphere. Two main methods are the one that uses spherical splines [1], and another that uses spherical radial basis functions [15, 16]. We follow the second approach in this paper.

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It is known [9, 14] that the matrix arising from this interpolation problem becomes ill-conditioned when the number of points, N, grows and the minimum geodesic distance of the data points, q_X, decreases. We show in Section 3 that for a given kernel the maximum eigenvalue of the matrix grows as $O(N)$ and the minimum eigenvalue decreases as q_X decreases, at a rate depending on the smoothness of the kernel (see Theorem 3.2). In particular, in the case of the sphere $\mathbb{S}^2$, if the Fourier-Legendre coefficients $\widehat{\phi}(\ell)$ (of the function ϕ defining the kernel) behaves like $O(\ell^{-s})$ then the minimum eigenvalue behaves like $O(q_X^{s-2})$ is s is even, and $O(q_X^{s-1})$ if s is odd.

We shall present an additive Schwarz preconditioner to accelerate the solution process and give an estimate for the condition number of the preconditioned system.

Even though domain decomposition methods have been extensively studied for finite-element and boundary-element methods, not much has been done for meshless methods using radial basis functions. Attempts to use Schwarz methods to solve interpolation problems or partial differential equations on bounded domains in $\mathbb{R}^n$ have been carried out in [2, 10, 11, 26]. These papers do not show a bound for the condition numbers of the preconditioned systems. The work [6] solves a Neumann problem by additive Schwarz methods with scaled radial basis functions and proves a bound for the condition number. However, the algorithm in [6] does not apply to the sphere where one needs to describe carefully overlapping subdomains. Moreover, the analysis in that paper cannot be carried over to the case studied in the present paper where different Sobolev spaces are used.

In [7] we study the use of additive Schwarz preconditioners for elliptic partial differential equations on the sphere, and prove a bound for the condition number. A similar approach will be used in the present paper for the interpolation problem. However, a more complicated analysis for the present case has to be carried out since the setting of the interpolation problem is in Sobolev spaces of any real order, instead of the simple space $H^1(\mathbb{S}^n)$ studied in [7].

2 Preliminaries

In this section, we will briefly review spherical harmonics and function spaces on the Euclidean unit sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$.

2.1 Spherical harmonics

Spherical harmonics are the restriction of homogeneous harmonic polynomials in $\mathbb{R}^{n+1}$ to the unit sphere $\mathbb{S}^n$. We denote an orthonormal (with respect to the $L^2(\mathbb{S}^n)$ inner product) basis for the spherical harmonics of degree ℓ by

$$\{Y_{\ell,k} : k = 1, \dots, N(n,\ell)\}, \quad \ell = 0, 1, \dots,$$

where $N(n,\ell)$ is the dimension of the space of all spherical harmonics of degree ℓ; the values of $N(n,\ell)$ are (see [13]):

$$N(n,0)=1 \text{ and } N(n,\ell)=\frac{(2\ell+n-1)\Gamma(\ell+n-1)}{\Gamma(\ell+1)\Gamma(n)} \text{ for } \ell\geq 1.$$

The asymptotic behaviour of $N(n,\ell)$ for fixed n and increasing ℓ is $O(\ell^{n-1})$. In the case $n=2$, we have $N(2,\ell)=2\ell+1$. The spherical harmonics $\{Y_{\ell,k}:\ell=0,1,\dots;k=1,\dots,N(n,\ell)\}$ form a complete orthonormal basis for $L^2(\mathbb{S}^n)$. Correspondingly, for a given function $f\in L^2(\mathbb{S}^n)$, we define its Fourier coefficients by

$$\widehat{f}_{\ell,k}=\int_{S^n} f(\boldsymbol{x})Y_{\ell,k}(\boldsymbol{x})dS(\boldsymbol{x}),$$

where dS is the surface measure of the sphere $\mathbb{S}^n$, and represent f as a Fourier series,

$$f=\sum_{\ell=0}^{\infty}\sum_{k=1}^{N(n,\ell)}\widehat{f}_{\ell,k}Y_{\ell,k},$$

in which the equal sign is understood in the $L^2(\mathbb{S}^n)$ sense. The addition formula for spherical harmonics of the same degree ℓ (see [13]) is

$$\sum_{k=0}^{N(n,\ell)} Y_{\ell,k}(\boldsymbol{x})Y_{\ell,k}(\boldsymbol{y})=\frac{1}{\omega_n}N(n,\ell)P_\ell(n+1;\boldsymbol{x}\cdot\boldsymbol{y}), \tag{2.1}$$

where $P_\ell(n+1;t)$ is the normalized Legendre polynomial of degree ℓ in $\mathbb{R}^{n+1}$ and ω_n is the surface area of the unit sphere $\mathbb{S}^n$. Recall from [13] that $P_\ell(n+1;1)=1$ and

$$\int_{-1}^{+1} P_\ell(n+1;t)P_k(n+1;t)(1-t^2)^{(n-2)/2}dt=\frac{\omega_n}{\omega_{n-1}N(n,\ell)}\delta_{\ell,k}, \tag{2.2}$$

where ω_m is the surface area of the sphere $\mathbb{S}^m$, and $\delta_{\ell,k}$ is the Kronecker delta.

2.2 Sobolev spaces on $\mathbb{S}^n$

For a given $s\geq 0$, the Sobolev space $H^s(\mathbb{S}^n)$ on the unit sphere is defined by (see [12])

$$H^s(\mathbb{S}^n):=\left\{f\in L^2(\mathbb{S}^n):\sum_{\ell=0}^{\infty}\sum_{k=1}^{N(n,\ell)}(1+\ell)^{2s}|\widehat{f}_{\ell,k}|^2<\infty\right\}. \tag{2.3}$$

We note that $H^s(\mathbb{S}^n)$ is a Hilbert space equipped with inner product

$$\langle f,g\rangle_s=\sum_{\ell=0}^{\infty}\sum_{k=1}^{N(n,\ell)}(1+\ell)^{2s}\widehat{f}_{\ell,k}\widehat{g}_{\ell,k} \qquad \forall f,g\in H^s(\mathbb{S}^n),$$

and the corresponding norm is

$$\|f\|_s = \left(\sum_{\ell=0}^{\infty} \sum_{k=1}^{N(n,\ell)} (1+\ell)^{2s} |\widehat{f}_{\ell,k}|^2 \right)^{1/2} \qquad \forall f \in H^s(\mathbb{S}^n).$$

Sobolev spaces on $\mathbb{S}^n$ can also be defined using local charts (see [12]). Here we use a specific atlas of charts, as in [7].

Let a spherical cap of radius α centered at $\boldsymbol{p} \in \mathbb{S}^n$ be defined as

$$C(\boldsymbol{p}, \alpha) := \{\boldsymbol{x} \in \mathbb{S}^n : \theta(\boldsymbol{p}, \boldsymbol{x}) < \alpha\}, \tag{2.4}$$

where $\theta(\boldsymbol{p}, \boldsymbol{x}) = \cos^{-1}(\boldsymbol{p} \cdot \boldsymbol{x})$ is the geodesic distance between two points $\boldsymbol{x}, \boldsymbol{p} \in \mathbb{S}^n$. Let $\hat{\boldsymbol{n}}$ and $\hat{\boldsymbol{s}}$ denote the north and south poles of $\mathbb{S}^n$, respectively. Then a simple cover for the sphere is provided by

$$U_1 = C(\hat{\boldsymbol{n}}, \theta_0) \quad \text{and} \quad U_2 = C(\hat{\boldsymbol{s}}, \theta_0), \text{ where } \theta_0 \in (\pi/2, 2\pi/3). \tag{2.5}$$

The stereographic projection $\sigma_{\hat{\boldsymbol{n}}}$ of the punctured sphere $\mathbb{S}^n \backslash \{\hat{\boldsymbol{n}}\}$ onto $\mathbb{R}^n$ is defined as a mapping that maps $\boldsymbol{x} \in \mathbb{S}^n \setminus \{\hat{\boldsymbol{n}}\}$ to the intersection of the equatorial hyperplane $\{z = 0\}$ and the extended line that passes through $\boldsymbol{x}$ and $\hat{\boldsymbol{n}}$. The stereographic projection $\sigma_{\hat{\boldsymbol{s}}}$ based on $\hat{\boldsymbol{s}}$ can be defined analogously. An explicit formula for the stereographic projection can be found in [17, page 112]. We set

$$\psi_1 = \frac{1}{\tan(\theta_0/2)} \sigma_{\hat{\boldsymbol{s}}}|_{U_1} \quad \text{and} \quad \psi_2 = \frac{1}{\tan(\theta_0/2)} \sigma_{\hat{\boldsymbol{n}}}|_{U_2}, \tag{2.6}$$

so that ψ_k, $k = 1, 2$, maps U_k onto $B(0, 1)$, the unit ball in $\mathbb{R}^n$. We conclude that $\mathcal{A} = \{U_k, \psi_k\}_{k=1}^2$ is a C^∞ atlas of covering coordinate charts for the sphere. It is known (see [17, page 132]) that the stereographic coordinate charts $\{\psi_k\}_{k=1}^2$ as defined in (2.6) map spherical caps to Euclidean balls, but in general concentric spherical caps are not mapped to concentric Euclidean balls. With the atlas so defined, we define the map π_k which maps a real-valued function g with compact support in U_k to a real-valued function on $\mathbb{R}^n$ by

$$\pi_k(g)(\boldsymbol{x}) = \begin{cases} g \circ \psi_k^{-1}(\boldsymbol{x}), & \text{if } \boldsymbol{x} \in B(0,1), \\ 0, & \text{otherwise .} \end{cases}$$

Let $\{\chi_k : \mathbb{S}^n \to \mathbb{R}\}_{k=1}^2$ be a partition of unity subordinated to the atlas, i.e., a pair of non-negative infinitely differentiable functions χ_k on $\mathbb{S}^n$ with compact support in U_k, such that $\sum_k \chi_k = 1$. For any function $f : \mathbb{S}^n \to \mathbb{R}$, we can use the partition of unity to write

$$f = \sum_{k=1}^{2} (\chi_k f), \text{ where } (\chi_k f)(\boldsymbol{p}) = \chi_k(\boldsymbol{p}) f(\boldsymbol{p}), \quad \boldsymbol{p} \in \mathbb{S}^n.$$

With the help of the charts, the Sobolev space $H^s(\mathbb{S}^n)$ can also be defined by

$$H^s(\mathbb{S}^n) = \{f \in L^2(\mathbb{S}^n) : \pi_k(\chi_k f) \in H^s(\mathbb{R}^n) \quad \text{for } k = 1, 2\},$$

which is equipped with the norm

$$\|f\|_{H^s(\mathbb{S}^n)} = \left(\sum_{k=1}^{2} \|\pi_k(\chi_k f)\|^2_{H^s(\mathbb{R}^n)} \right)^{1/2}. \tag{2.7}$$

Here, as usual, the $\|\cdot\|_{H^s(\mathbb{R}^n)}$ norm is defined for any function $\varphi : \mathbb{R}^n \to \mathbb{R}$ by

$$\|\varphi\|_{H^s(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\widehat{\varphi}(\xi)|^2 d\xi \right)^{1/2}.$$

The $\|\cdot\|_{H^s(\mathbb{S}^n)}$ norm is equivalent to the $\|\cdot\|_s$ norm given in Section 2.1 (see [12]).

3 Interpolation using positive definite kernels

3.1 Positive definite kernels

In this section, we will review necessary background on positive definite kernels on the unit sphere and spherical basis functions.

A real-valued kernel Φ in $C(\mathbb{S}^n \times \mathbb{S}^n)$ is termed *positive definite* on $\mathbb{S}^n$ if $\Phi(\boldsymbol{x}, \boldsymbol{y}) = \Phi(\boldsymbol{y}, \boldsymbol{x})$ and if for every finite set of distinct points $X = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\}$ on $\mathbb{S}^n$, the symmetric $N \times N$ matrix A with entries $A_{i,j} = \Phi(\boldsymbol{x}_i, \boldsymbol{x}_j)$ is positive semi-definite. If the matrix A is positive definite then Φ is called a *strictly positive definite* kernel (see [19, 25]).

Let ϕ be a univariate function defined on $[-1, 1]$ which can be expanded in terms of Legendre polynomials as

$$\phi(t) = \frac{1}{\omega_n} \sum_{\ell=0}^{\infty} N(n, \ell) \widehat{\phi}(\ell) P_\ell(n + 1; t), \tag{3.1}$$

where

$$\widehat{\phi}(\ell) = \omega_{n-1} \int_{-1}^{1} \phi(t) P_\ell(n + 1; t)(1 - t^2)^{(n-2)/2} dt. \tag{3.2}$$

Due to the addition formula (2.1), a kernel Φ defined by

$$\Phi(\boldsymbol{x}, \boldsymbol{y}) = \phi(\boldsymbol{x} \cdot \boldsymbol{y}) \tag{3.3}$$

can be represented as

$$\Phi(\boldsymbol{x}, \boldsymbol{y}) = \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(n,\ell)} \widehat{\phi}(\ell) Y_{\ell,k}(\boldsymbol{x}) Y_{\ell,k}(\boldsymbol{y}). \tag{3.4}$$

In [4], a complete characterization of strictly positive definite kernels is established: the kernel Φ is strictly positive definite if and only if $\widehat{\phi}(\ell) \geq 0$ for all $\ell \geq 0$ and $\widehat{\phi}(\ell) > 0$ for infinitely many even values of ℓ and infinitely many odd values of ℓ; see also [19] and [25]. In this paper we assume that $\widehat{\phi}(\ell) > 0$ for all $\ell \geq 0$.

The native space $\mathcal{N}_\Phi$ associated with the kernel Φ is defined by

$$\mathcal{N}_\Phi := \left\{ f \in \mathcal{D}'(\mathbb{S}^n) : \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(n,\ell)} \frac{|\widehat{f}_{\ell,k}|^2}{\widehat{\phi}(\ell)} < \infty \right\}, \tag{3.5}$$

where $\mathcal{D}'(\mathbb{S}^n)$ is the space of distributions on $\mathbb{S}^n$. This is a Hilbert space with respect to the inner product

$$\langle f, g \rangle_\Phi = \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(n,\ell)} \frac{\widehat{f}_{\ell,k} \widehat{g}_{\ell,k}}{\widehat{\phi}(\ell)} \qquad \forall f, g \in \mathcal{N}_\Phi,$$

and the corresponding norm is

$$\|f\|_\Phi = \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(n,\ell)} \frac{|\widehat{f}_{\ell,k}|^2}{\widehat{\phi}(\ell)} \qquad \forall f \in \mathcal{N}_\Phi.$$

More background on native spaces can be found in [16, 24]. If

$$c_1(1+\ell)^{-2\tau} \leq \widehat{\phi}(\ell) \leq c_2(1+\ell)^{-2\tau}, \tag{3.6}$$

for some $\tau > n/2$ and $c_1, c_2 > 0$, then the native space $\mathcal{N}_\Phi$ can be identified with the Sobolev space $H^\tau(\mathbb{S}^n)$ defined in (2.3), which in turn can be imbedded into the space of continuous functions $C(\mathbb{S}^n)$. Henceforth, the condition (3.6) is shortened to $\widehat{\phi}(\ell) \simeq (1+\ell)^{-2\tau}$.

In the sequel, c denotes a generic constant which may take different values at different occurrences.

3.2 Interpolation problem as a variational problem

With the kernel Φ given by (3.4), we can now establish a set of spherical radial basis functions $\{\Phi_1, \ldots, \Phi_N\}$ associated with a set $X = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\}$ of scattered points on $\mathbb{S}^n$, where

$$\Phi_i(\boldsymbol{x}) := \Phi(\boldsymbol{x}_i, \boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{S}^n.$$

We assume the points $\boldsymbol{x}_i$ are spread over the whole sphere so that

$$\mathbb{S}^n = \bigcup_{i=1}^{N} \operatorname{supp} \Phi_i.$$

The finite dimensional space spanned by these spherical radial basis functions is denoted by V_X:

$$V_X := \text{ span } \{\Phi_i : i = 1, \dots, N\}. \tag{3.7}$$

We note that the value of the function Φ_i at $\boldsymbol{x}$ depends only on the geodesic distance $\theta(\boldsymbol{x}, \boldsymbol{x}_i)$ between the points $\boldsymbol{x}$ and $\boldsymbol{x}_i$. The set X is characterized by its *mesh norm* h_X and *separation radius* q_X defined as

$$h_X := \sup_{\boldsymbol{y} \in \mathbb{S}^n} \min_{\boldsymbol{x}_i \in X} \theta(\boldsymbol{x}_i, \boldsymbol{y}) \quad \text{and} \quad q_X := \frac{1}{2} \min_{i \neq j} \theta(\boldsymbol{x}_i, \boldsymbol{x}_j).$$

Suppose f is a function in the native space. We seek an interpolant $I_X f \in V_X$ satisfying

$$I_X f(\boldsymbol{x}_j) = f(\boldsymbol{x}_j), \quad j = 1, \dots, N. \tag{3.8}$$

By writing $I_X f$ as $I_X f = \sum_{i=1}^N c_i \Phi_i$, we deduce from (3.8) the following linear system

$$A\mathbf{c} = \mathbf{f}, \tag{3.9}$$

where A is the matrix with entries $A_{i,j} = \Phi(\boldsymbol{x}_i, \boldsymbol{x}_j)$, $\boldsymbol{c} = [c_1, \dots, c_N]^T$ and $\mathbf{f} = [f(\boldsymbol{x}_1), \dots, f(\boldsymbol{x}_N)]^T$. Since the kernel Φ is strictly positive definite, the matrix A is positive definite, and therefore the interpolation problem in V_X of scattered data using spherical radial basis functions is always solvable.

In the following, we show how the interpolation problem can be written as a variational problem in terms of the native space inner product $\langle \cdot, \cdot \rangle_\Phi$.

Lemma 3.1 *Suppose that the kernel* Φ *is defined by* (3.3) *with* ϕ *satisfying* (3.6). *Then* $\mathcal{N}_\Phi$ *is a reproducing kernel Hilbert space, that is*

(i) $\Phi(\boldsymbol{x}, \cdot) \in \mathcal{N}_\Phi$ *for all* $\boldsymbol{x} \in \mathbb{S}^n$, *and*

(ii) $f(\boldsymbol{x}) = \langle f, \Phi(\boldsymbol{x}, \cdot) \rangle_\Phi$ *for all* $\boldsymbol{x} \in \mathbb{S}^n$ *and all* $f \in \mathcal{N}_\Phi$.

Proof. To prove property (i), we note from (3.4) that

$$\widehat{\Phi(\boldsymbol{x}, \cdot)}_{\ell,k} = \widehat{\phi}(\ell) Y_{\ell,k}(\boldsymbol{x}) \tag{3.10}$$

and hence

$$\|\Phi(\boldsymbol{x}, \cdot)\|_\Phi^2 = \sum_{\ell=0}^\infty \sum_{k=1}^{N(n,\ell)} \widehat{\phi}(\ell) |Y_{\ell,k}(\boldsymbol{x})|^2 = \sum_{\ell=0}^\infty \widehat{\phi}(\ell) \frac{N(n,\ell)}{\omega_n},$$

by the addition theorem for spherical harmonics, see (2.1). The convergence of the series on the right hand side follows from (3.6), noting that $N(n,\ell) = O(\ell^{n-1})$. Therefore $\Phi(\boldsymbol{x}, \cdot) \in \mathcal{N}_\Phi$ for all $\boldsymbol{x} \in \mathbb{S}^n$. By using (3.10) again we have

$$\langle f, \Phi(\boldsymbol{x}, \cdot) \rangle_\Phi = \sum_{\ell=0}^\infty \sum_{k=1}^{N(n,\ell)} \frac{\widehat{f}_{\ell,k} \widehat{\phi}(\ell) Y_{\ell,k}(\boldsymbol{x})}{\widehat{\phi}(\ell)} = \sum_{\ell=0}^\infty \sum_{k=1}^{N(n,\ell)} \widehat{f}_{\ell,k} Y_{\ell,k}(\boldsymbol{x}) = f(\boldsymbol{x}),$$

and thus (ii) is proved. $\square$

It follows from Lemma 3.1 that the interpolation problem (3.8) is equivalent to

$$\langle I_X f, \Phi_j \rangle_\Phi = \langle f, \Phi_j \rangle_\Phi, \quad j = 1, \dots, N,$$

or equivalently,

$$\langle I_X f, v \rangle_\Phi = \langle f, v \rangle_\Phi \quad \forall v \in V_X. \tag{3.11}$$

We note that the entries of the matrix A and right-hand side $\mathbf{f}$ in (3.9) can now be written as

$$A_{i,j} = \langle \Phi(\boldsymbol{x}_i, \cdot), \Phi(\boldsymbol{x}_j, \cdot) \rangle_\Phi \quad \text{and} \quad f(\boldsymbol{x}_j) = \langle f, \Phi_j \rangle_\Phi, \quad i, j = 1, \dots, N.$$

3.3 Estimates of the extremal eigenvalues of A

It is observed in [9, 14] that the matrix A becomes ill-conditioned when q_X tends to zero and N goes to infinity. Indeed, noting that the diagonal entries in A are $\phi(1)$ so that the trace of A is $N\phi(1)$, the maximum eigenvalue of A can be roughly estimated as

$$\lambda_{\max}(A) \le N\phi(1). \tag{3.12}$$

On the other hand, the minimum eigenvalue of A decreases as q_X goes to 0, as described in the remainder of this section.

To state the results more precisely, we first define the spherical convolution operator as follows. Given two functions $g, h \in L^2(-1,1)$, their spherical convolution is defined by

$$(g * h)(\boldsymbol{x} \cdot \boldsymbol{y}) = \int_{\mathbb{S}^n} g(\boldsymbol{x} \cdot \boldsymbol{z}) h(\boldsymbol{z} \cdot \boldsymbol{y}) dS(\boldsymbol{z}).$$

We note that by using the addition formula (2.1) one can prove that the integral on the right hand side depends only on $\boldsymbol{x} \cdot \boldsymbol{y}$. Therefore $g * h$ is well-defined as a function on $[-1,1]$. We also note that, see [9, Lemma 2.2],

$$\widehat{g * h}(\ell) = \widehat{g}(\ell)\widehat{h}(\ell), \quad \ell = 0, 1, 2, \dots. \tag{3.13}$$

With the help of this convolution, we follow [9] to define the functions $B_\nu^{(j)} \in L^2(-1,1)$ for some $\nu \in (0,1)$ and $j = 1, 2, 3, \dots$ in the following way:

$$\begin{aligned} B_\nu^{(1)}(t) &= \chi_{(\nu,1]}(t) \\ B_\nu^{(j)}(t) &= (B_\nu^{(j-1)} * B_\nu^{(1)})(t), \end{aligned} \tag{3.14}$$

for $t \in [-1,1]$, where χ_I is the characteristic function of the interval I.

Theorem 3.2 *Assume that*

$$\widehat{\phi}(0) \ge c \quad \textit{and} \quad \widehat{\phi}(\ell) \ge c(\ell + (n-1)/2)^{-s}, \quad \ell > 0, \tag{3.15}$$

for some positive integer s and some positive constant c. Then the least eigenvalue $\lambda_{\min}(A)$ satisfies

$$\lambda_{\min}(A) \geq c \left(2^s \sin^{s(n-2)} \frac{2q_X}{s} \right)^{-1} B_\nu^{(s)}(1) \quad \textit{with } \nu = \cos(2q_X/s).$$

Proof. This is a direct consequence of [9, Theorem 3.2], noting that the Fourier coefficient defined in (3.2) differs from that defined in [9] by the constant ω_{n-1}. $\square$

By adapting a technique developed in [14] (for the Euclidean space $\mathbb{R}^n$) to the sphere $\mathbb{S}^n$, we give a further bound for the term $B_\nu^{(s)}(1)$ in the theorem above.

Lemma 3.3 *Suppose that g and h are two univariate functions in $L^2(-1,1)$ satisfying*

$$g(t) \geq g_0 \chi_{(r,1]}(t) \qquad \textit{and} \qquad h(t) \geq h_0 \chi_{(s,1]}(t)$$

for some positive constants g_0, h_0, r, s. Then the spherical convolution $g * h$ is non-negative and satisfies

$$(g * h)(\boldsymbol{x} \cdot \boldsymbol{n}) \geq g_0 h_0 |C(\boldsymbol{n}, \alpha)| \chi_{(\cos\alpha, 1]}(\boldsymbol{x} \cdot \boldsymbol{n}),$$

where $\alpha = \min(\cos^{-1} r, \cos^{-1} s)/2$. Here $|C(\boldsymbol{n}, \alpha)|$ denotes the area of the spherical cap centered at $\boldsymbol{n}$ with radius α.

Proof. The required inequality is trivial when $\theta(\boldsymbol{x}, \boldsymbol{n}) > \alpha$ because then $\chi_{(\cos\alpha,1]}(\boldsymbol{x} \cdot \boldsymbol{n}) = 0$. Consider the case $\theta(\boldsymbol{x}, \boldsymbol{n}) \leq \alpha$. Let $\boldsymbol{x}_1$ be a midpoint on the geodesic between $\boldsymbol{x}$ and $\boldsymbol{n}$ so that $\theta(\boldsymbol{x}, \boldsymbol{x}_1) = \theta(\boldsymbol{x}_1, \boldsymbol{n})$. The spherical cap $C(\boldsymbol{x}_1, \alpha)$ is contained in $C(\boldsymbol{n}, \cos^{-1} s)$ since for every $\boldsymbol{y} \in C(\boldsymbol{x}_1, \alpha)$,

$$\theta(\boldsymbol{y}, \boldsymbol{n}) \leq \theta(\boldsymbol{y}, \boldsymbol{x}_1) + \theta(\boldsymbol{x}_1, \boldsymbol{n}) \leq \alpha + \frac{\theta(\boldsymbol{x}, \boldsymbol{n})}{2} \leq \cos^{-1} s.$$

Similarly, the spherical cap $C(\boldsymbol{x}_1, \alpha)$ is also contained in $C(\boldsymbol{x}, \cos^{-1} r)$. Hence

$$\begin{aligned}(g * h)(\boldsymbol{x} \cdot \boldsymbol{n}) &= \int_{\mathbb{S}^n} g(\boldsymbol{x} \cdot \boldsymbol{y}) h(\boldsymbol{y} \cdot \boldsymbol{n}) dS(\boldsymbol{y}) \\ &\geq g_0 h_0 \int_{C(\boldsymbol{x}, \cos^{-1} r) \cap C(\boldsymbol{n}, \cos^{-1} s)} dS(\boldsymbol{y}) \geq g_0 h_0 |C(\boldsymbol{x}_1, \alpha)| \\ &= g_0 h_0 |C(\boldsymbol{n}, \alpha)| \chi_{(\cos\alpha, 1]}(\boldsymbol{x} \cdot \boldsymbol{n}).\end{aligned}$$

$\square$

Recalling the definition (3.14) of $B_\nu^{(s)}$ and applying the result of Lemma 3.3 repeatedly we obtain

$$B_\nu^{(s)} \geq \left(\prod_{j=1}^{s-1} |C(\boldsymbol{n}, \mu/2^j)| \right) \chi_{\left(\cos \frac{\mu}{2^{s-1}}, 1\right]} \quad \text{where } \mu := \cos^{-1} \nu.$$

Since $|C(\boldsymbol{n}, \alpha)| \simeq \alpha^n$, we conclude that

$$B_\nu^{(s)}(1) \geq c\mu^{n(s-1)}, \tag{3.16}$$

for some constant c depending on n, s but independent of $\mu = \cos^{-1}\nu$.

By using Theorem 3.2 and inequality (3.16) we obtain the following estimate for the minimum eigenvalue of the interpolation matrix A.

$$\lambda_{\min}(A) \geq c \left(2^s \sin^{s(n-2)} \frac{2q_X}{s}\right)^{-1} \left(\frac{2q_X}{s}\right)^{n(s-1)}, \tag{3.17}$$

where c is a positive constant depending only on n, s. The implication of (3.17) is that the derived lower bound for the least eigenvalue of the matrix A depends on the separation radius q_X of the set X, which can be very small for a large set of scattered data, and also on the smoothness of the kernel Φ, which is determined by s. The smoother the kernel the smaller the lower bound for the least eigenvalue of A. We state the result for the special case when $n = 2$:

Corollary 3.4 *Suppose*

$$\widehat{\phi}(\ell) \geq c(\ell + 1/2)^{-s}, \quad \ell = 0, 1, 2, \ldots, \tag{3.18}$$

for some positive integer s. Then in the case $n = 2$ we have

$$\lambda_{\min}(A) \geq c \left(\frac{q_X}{s}\right)^{2s-2}, \tag{3.19}$$

where c is a constant depending only on s.

The above result can be improved by using a different approach to the one used in [9] to obtain Theorem 3.2. This new approach, partially described in [14], applies only to the case $n = 2$. First we note that the power of q_X/s in (3.19) is determined by the number of convolutions that define $B_\nu^{(s)}$. Therefore, these results can be improved if we replace $B_\nu^{(s)}$ by $D_\nu^{(j)}$ satisfying

$$\widehat{D}_\nu^{(j)}(\ell) \leq c(\ell + 1)^{-s} \tag{3.20}$$

for some positive integer $j < s$, where s indicates the smoothness of the kernel as given in (3.18). The function $D_\nu^{(j)}$ is defined by $D_\nu^{(j)} = D_\nu^{(j-1)} * D_\nu^{(1)}$, where for $t \in [-1, 1]$

$$D_\nu^{(1)}(t) = \begin{cases} (t - \nu)/(1 - \nu), & \nu < t \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 3.5 *The function $D_\nu^{(j)}$ satisfies*

(i) $D_\nu^{(j)}(t) \geq 0$ for all $t \in [-1, 1]$;

(ii) $D_\nu^{(j)}(t) = 0$ *if* $-1 \le t \le \cos(j \cos^{-1} \nu)$ *or equivalently*

$$D_\nu^{(j)}(\boldsymbol{x} \cdot \boldsymbol{y}) = 0 \quad \textit{if} \quad \theta(\boldsymbol{x}, \boldsymbol{y}) \ge j \cos^{-1} \nu, \quad \boldsymbol{x}, \boldsymbol{y} \in \mathbb{S}^n;$$

(iii) In the case $n = 2$, $\widehat{D_\nu^{(j)}}(\ell) \le c(\ell + 1)^{-2j}$, *where* c *is independent of* ν.

Proof.

(i) The result is clear from the definition of $D_\nu^{(j)}$.

(ii) From the definition $D_\nu^{(1)}(t) = 0$ if $-1 \le t \le \nu$. Suppose $D_\nu^{(j-1)}(t) = 0$ for $-1 \le t \le \cos((j-1)\cos^{-1} \nu)$, or equivalently $D_\nu^{(j-1)}(\boldsymbol{x} \cdot \boldsymbol{z}) = 0$ if $\theta(\boldsymbol{x}, \boldsymbol{z}) \ge (j-1)\cos^{-1} \nu$. Recall that

$$D_\nu^{(j)}(\boldsymbol{x} \cdot \boldsymbol{y}) = \int_{\mathbb{S}^n} D_\nu^{(j-1)}(\boldsymbol{x} \cdot \boldsymbol{z}) D_\nu^{(1)}(\boldsymbol{z} \cdot \boldsymbol{y}) dS(\boldsymbol{z}).$$

If $\theta(\boldsymbol{x}, \boldsymbol{y}) \ge j \cos^{-1} \nu$ then by using the triangle inequality one can prove that either $\theta(\boldsymbol{x}, \boldsymbol{z}) \ge (j-1)\cos^{-1} \nu$ or $\theta(\boldsymbol{z}, \boldsymbol{y}) \ge \cos^{-1} \nu$. Hence either $D_\nu^{(j-1)}(\boldsymbol{x} \cdot \boldsymbol{z}) = 0$ or $D_\nu^{(1)}(\boldsymbol{z} \cdot \boldsymbol{y}) = 0$ and therefore $D_\nu^{(j)}(\boldsymbol{x} \cdot \boldsymbol{y}) = 0$.

(iii) Since $\widehat{D_\nu^{(1)}}(\ell) \le c(\ell + 1)^{-2}$, see [14, page 12], by using (3.13) repeatedly we obtain $\widehat{D_\nu^{(j)}}(\ell) \le c(\ell + 1)^{-2j}$. □

We are now able to prove a result which improves the bound in (3.19).

Theorem 3.6 *If* ϕ *satisfies* (3.18) *then in the case* $n = 2$ *we have*

$$\lambda_{\min}(A) \ge \begin{cases} c q_X^{s-2} = c q_X^{2p-2} & \textit{if } s = 2p, \\ c q_X^{s-1} = c q_X^{2p} & \textit{if } s = 2p+1, \end{cases}$$

where c *is a positive constant depending on* s *but not on the data set* X.

Proof. For an arbitrary vector of coefficients $\mathbf{d} = [d_1, \ldots, d_N]^T$, we have by using (3.18)

$$\begin{aligned} \mathbf{d}^T A \mathbf{d} &= \sum_{i,j=1}^{N} d_i d_j \sum_{\ell=0}^{\infty} \sum_{k=1}^{2\ell+1} \widehat{\phi}(\ell) Y_{\ell,k}(\boldsymbol{x}_i) Y_{\ell,k}(\boldsymbol{x}_j) \\ &= \sum_{\ell=0}^{\infty} \sum_{k=1}^{2\ell+1} \widehat{\phi}(\ell) \left(\sum_{i=1}^{N} d_i Y_{\ell,k}(\boldsymbol{x}_i) \right)^2 \\ &\ge c \sum_{\ell=0}^{\infty} \sum_{k=1}^{2\ell+1} \left(\ell + \frac{1}{2} \right)^{-s} \left(\sum_{i=1}^{N} d_i Y_{\ell,k}(\boldsymbol{x}_i) \right)^2, \end{aligned} \tag{3.21}$$

where c does not depend on the data set X.

If $s = 2p$ for some positive integer p then we perform p convolutions to define $D_\nu^{(p)}$. By using the Legendre expansion of $D_\nu^{(p)}$ and the addition formula we have

$$\begin{aligned}\sum_{i,j=1}^{N} d_i d_j D_\nu^{(p)}(\boldsymbol{x}_i \cdot \boldsymbol{x}_j) &= \sum_{\ell=0}^{\infty}\sum_{k=1}^{2\ell+1} \widehat{D}_\nu^{(p)}(\ell)\left(\sum_{i=1}^{N} d_i Y_{\ell,k}(\boldsymbol{x}_i)\right)^2 \\ &\leq c\sum_{\ell=0}^{\infty}\sum_{k=1}^{2\ell+1}(1+\ell)^{-2p}\left(\sum_{i=1}^{N} d_i Y_{\ell,k}(\boldsymbol{x}_i)\right)^2, \end{aligned} \tag{3.22}$$

where in the last step we used Lemma 3.5 (iii). It follows from (3.21) and (3.22) that

$$\mathbf{d}^T A\mathbf{d} \geq c\sum_{i,j=1}^{N} d_j d_j D_\nu^{(p)}(\boldsymbol{x}_i \cdot \boldsymbol{x}_j).$$

If we choose $\nu = \cos(2q_X/p)$ then for $i \neq j$, there holds $\theta(\boldsymbol{x}_i, \boldsymbol{x}_j) \geq 2q_X = p\cos^{-1}\nu$. It follows from Lemma 3.5 (ii) that $D_\nu^{(p)}(\boldsymbol{x}_i \cdot \boldsymbol{x}_j) = 0$ if $i \neq j$. Hence if $\nu = \cos(2q_X/p)$

$$\mathbf{d}^T A\mathbf{d} \geq cD_\nu^{(p)}(1)\sum_{i=1}^{N} d_i^2 = cD_\nu^{(p)}(1)\mathbf{d}^T\mathbf{d}.$$

Hence,

$$\lambda_{\min}(A) \geq cD_\nu^{(p)}(1). \tag{3.23}$$

It remains to show $D_\nu^{(p)}(1) \geq cq_X^{2(p-1)}$. Since $D_\nu^{(1)} \geq \frac{1}{2}\chi_{\left(\frac{\nu+1}{2},1\right]}$, by applying Lemma 3.3 repeatedly we obtain

$$D_\nu^{(p)} \geq 2^{-p}\left(\prod_{m=1}^{p-1}|C(\boldsymbol{n},\mu/2^m)|\right)\chi_{\left(\cos\frac{\mu}{2^{p-1}},1\right]}, \tag{3.24}$$

where

$$\mu := \cos^{-1}\left(\frac{\nu+1}{2}\right).$$

Since on the sphere $\mathbb{S}^2$ the surface area of a spherical cap $|C(\boldsymbol{n},\alpha)| \simeq \alpha^2$, we deduce that

$$D_\nu^{(p)}(1) \geq c\mu^{2(p-1)}, \tag{3.25}$$

for some constant c independent of μ. Noting that $\nu = \cos(2q_X/p)$ we obtain

$$\mu = \cos^{-1}\left(\frac{\cos(2q_X/p)+1}{2}\right) \geq \frac{q_X}{p},$$

where the last inequality follows from elementary trigonometry. Hence

$$D_\nu^{(p)}(1) \geq cq_X^{2(p-1)}, \tag{3.26}$$

where c is a constant independent of the q_X but depends on p. Combining inequalities (3.23) and (3.26) we obtain the result for $s = 2p$.

If $s = 2p + 1$, then again we perform p convolutions to define $D_\nu^{(p)}$. By using the Legendre expansion of $D_\nu^{(p)} * B_\nu^{(1)}$ and the addition formula we have

$$\begin{aligned}\sum_{i,j=1}^{N} d_i d_j (D_\nu^{(p)} * B_\nu^{(1)})(\boldsymbol{x}_i \cdot \boldsymbol{x}_j) &= \sum_{\ell=0}^{\infty}\sum_{k=1}^{2\ell+1} \widehat{D}_\nu^{(p)}(\ell)\widehat{B}_\nu^{(1)}(\ell)\left(\sum_{i=1}^{N} d_i Y_{\ell,k}(\boldsymbol{x}_i)\right)^2 \\ &\leq c\sum_{\ell=0}^{\infty}\sum_{k=1}^{2\ell+1}(\ell+1)^{-2p-1}\left(\sum_{i=1}^{N} d_i Y_{\ell,k}(\boldsymbol{x}_i)\right)^2, \end{aligned} \tag{3.27}$$

where in the last line we used Lemma 3.5 (iii) and the fact that $\widehat{B}_\nu^{(1)}(\ell) \leq 2(\ell+1)^{-1}$ (see [9, Lemma 2.4]). Combining (3.21) and (3.27) we obtain

$$\mathbf{d}^T A\mathbf{d} \geq c\sum_{i,j=1}^{N} d_i d_j (D_\nu^{(p)} * B_\nu^{(1)})(\boldsymbol{x}_i \cdot \boldsymbol{x}_j).$$

Reasoning in the same manner as in the case $s = 2p$ with $\nu = \cos(2q_X/(p+1))$ we obtain

$$\lambda_{\min}(A) \geq c q_X^{2p},$$

where c depends only on p. □

To corroborate the results in this subsection we computed the extremal eigenvalues of a matrix A with entries

$$A_{i,j} = \Phi(\boldsymbol{x}_i, \boldsymbol{x}_j) = \phi(\boldsymbol{x}_i \cdot \boldsymbol{x}_j),$$

where

$$\phi(t) = 1 - \sqrt{(1-t)/2},$$

and the set $X = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\}$ is generated using Saff's algorithm [18]. The Fourier-Legendre coefficients of the function $\phi(t)$ are given by, see [8],

$$\widehat{\phi}(0) = \frac{2\pi}{3} \quad \text{and} \quad \widehat{\phi}(\ell) = \frac{8\pi}{(4\ell^2-1)(2\ell+3)} \geq \frac{1}{12\ell^3}, \quad \ell \neq 0.$$

Thus the value of p in Theorem 3.6 is 1. Table 1 reports on the maximum and minimum of the eigenvalues of the matrix A. We also computed the orders e_1 and e_2 such that $\lambda_{\max}(A) = O(N^{e_1})$ and $\lambda_{\min}(A) = O(q_X^{e_2})$. The values of e_1 suggest that $\lambda_{\max}(A) = O(N)$ which agrees with theoretical result (3.12). However, the values of e_2 suggest that $\lambda_{\min}(A) = O(q_X)$ which shows that the theoretical result given by Theorem 3.6 (namely $O(q_X^2)$) is not optimal. Nevertheless, it is still better than $O(q_X^4)$ as given by Corollary 3.4.

N	$\lambda_{\max}$	e_1	q_X	$\lambda_{\min}(A)$	e_2
100	33.71		0.1481	0.0743	
400	133.35	0.99	0.0731	0.0364	1.01
800	267.02	1.00	0.0577	0.0267	1.31
1600	533.34	1.00	0.0411	0.0188	1.03
3200	1066.71	1.00	0.0289	0.0133	0.98
4000	1333.33	1.00	0.0259	0.0119	1.01
6400	2133.33	1.00	0.0206	0.0094	1.03

Table 1: Eigenvalues of A with $\phi(t) = 1 - \sqrt{(1-t)/2}$, $\lambda_{\max}(A) = O(N^{e_1})$ and $\lambda_{\min}(A) = O(q_X^{e_2})$

4 Abstract framework of additive Schwarz methods

From the previous section, it can be seen the condition number of the interpolation matrix A can be very large when q_X is very small. When an iterative method such as the conjugate gradient method is applied to solve equation (3.9) a good preconditioner is needed. In the following, we shall introduce an effective preconditioner based on the additive Schwarz method in the context of the unit sphere.

4.1 Additive Schwarz operator

Additive Schwarz methods provide fast solutions to equation (3.11) by solving, at the same time, problems of smaller size. Let the space V_X be decomposed as

$$V_X = V_0 + \cdots + V_J, \tag{4.1}$$

where $V_j,\ j = 0, \ldots, J$, are subspaces of V_X. The sum on the right hand side of (4.1) does not need to be a direct sum, namely $V_i \cap V_j$ (for $i \neq j$) may be different from $\{0\}$. Let $P_j : V_X \to V_j$, $j = 0, \ldots, J$, be projections defined by

$$\langle P_j v, w \rangle_\Phi = \langle v, w \rangle_\Phi \quad \forall v \in V_X,\ \forall w \in V_j. \tag{4.2}$$

If we define

$$P := P_0 + \cdots + P_J, \tag{4.3}$$

then the additive Schwarz method for equation (3.11) consists in solving, by an iterative method, the equation

$$Pu_X = g, \tag{4.4}$$

where the right-hand side is given by $g = \sum_{j=0}^{J} g_j$, with $g_j \in V_j$ being solutions of

$$\langle g_j, w \rangle_\Phi = \langle f, w \rangle_\Phi, \text{ for any } w \in V_j. \tag{4.5}$$

The well-known equivalence of (3.11) and (4.4) was discussed explicitly in [21]. In fact, if u_X is a solution of (3.11) then from the definition of P_j and g_j we deduce

$$\langle P_j u_X, w\rangle_\Phi = \langle u_X, w\rangle_\Phi = \langle f, w\rangle_\Phi = \langle g_j, w\rangle_\Phi \text{ for any } w \in V_j,$$

i.e. $P_j u_X = g_j$. Hence $Pu_X = g$. On the other hand, if $P : V_X \to V_X$ is invertible and u_X is the solution of (4.4), then by using successively the symmetry of P and (4.2) and (4.5), we obtain

$$\begin{aligned}\langle u_X, v\rangle_\Phi &= \left\langle P^{-1} g, v\right\rangle_\Phi = \left\langle g, P^{-1} v\right\rangle_\Phi \\ &= \sum_{j=0}^{J} \left\langle g_j, P^{-1} v\right\rangle_\Phi = \sum_{j=0}^{J} \left\langle g_j, P_j P^{-1} v\right\rangle_\Phi \\ &= \sum_{j=0}^{J} \left\langle f, P_j P^{-1} v\right\rangle_\Phi = \langle f, v\rangle_\Phi \text{ for any } v \in V_X.\end{aligned}$$

A practical method to solve (4.4) is the conjugate gradient method; the additive Schwarz method (see Section 6) can be viewed as a preconditioned conjugate gradient method.

4.2 Bound on the condition number

Bounds for $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$, the minimum and maximum eigenvalues of the additive Schwarz operator P, can be obtained by using the following lemma; see [20].

Lemma 4.1 *Assume that*

(i) there exists a constant $c_1 > 0$ such that for any $u \in V_X$ satisfying $u = \sum_{j=0}^{J} u_j$ with $u_j \in V_j$ for $j = 0, \ldots, J$ the following inequality

$$\langle u, u\rangle_\Phi \le c_1 \sum_{j=0}^{J} \langle u_j, u_j\rangle_\Phi$$

holds;

(ii) there exists a constant $c_2 > 0$ such that any $u \in V_X$ has a decomposition $u = \sum_{j=0}^{J} u_j$ with $u_j \in V_j$ for $j = 0, \ldots, J$ satisfying

$$\sum_{j=0}^{J} \langle u_j, u_j\rangle_\Phi \le c_2 \langle u, u\rangle_\Phi .$$

Then

$$\lambda_{\min}(P) \ge c_2^{-1} \quad \textit{and} \quad \lambda_{\max}(P) \le c_1,$$

so that

$$\kappa(P) \le c_1 c_2.$$

Proof. The abstract theory on bounds of extremal eigenvalues of P is standard. For example the proof for the lower bound of $\lambda_{\min}(P)$ can be found in [20, Lemma 2.5]. However, we cannot find a reference for a proof of the upper bound for $\lambda_{\max}(P)$. We include the proof here for completeness.

For $u = \sum_{j=0}^{J} u_j$ with $u_j \in V_j$, noting that P is symmetric positive definite, using (4.2) and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\left\langle P^{-1}u, u\right\rangle_\Phi &= \sum_{j=0}^{J} \left\langle P^{-1}u, u_j\right\rangle_\Phi = \sum_{j=0}^{J} \left\langle P_j P^{-1}u, u_j\right\rangle_\Phi \\
&\le \left(\sum_{j=0}^{J} \left\langle P_j P^{-1}u, P_j P^{-1}u\right\rangle_\Phi\right)^{1/2} \left(\sum_{j=0}^{J} \langle u_j, u_j\rangle_\Phi\right)^{1/2} \\
&= \left(\sum_{j=0}^{J} \left\langle P^{-1}u, P_j P^{-1}u\right\rangle_\Phi\right)^{1/2} \left(\sum_{j=0}^{J} \langle u_j, u_j\rangle_\Phi\right)^{1/2} \\
&= (\left\langle P^{-1}u, u\right\rangle_\Phi)^{1/2} \left(\sum_{j=0}^{J} \langle u_j, u_j\rangle_\Phi\right)^{1/2}.
\end{aligned}$$

Thus,

$$\left\langle P^{-1}u, u\right\rangle_\Phi \le \min_{\sum u_j = u} \sum_{j=0}^{J} \langle u_j, u_j\rangle_\Phi\,,$$

and equality occurs when $u_j = P_j P^{-1}u$. Therefore

$$\lambda_{\max}(P) = \max_{v\in V_X} \frac{\langle Pv, v\rangle_\Phi}{\langle v, v\rangle_\Phi} = \max_{u\in V_X} \frac{\langle u, u\rangle_\Phi}{\langle P^{-1}u, u\rangle_\Phi} = \max_{u\in V_X} \frac{\langle u, u\rangle_\Phi}{\min_{\sum u_j = u} \sum_{j=0}^{J} \langle u_j, u_j\rangle_\Phi} \le c_1.$$

□

5 Additive Schwarz methods on the sphere

5.1 A subspace decomposition

In this section, we present a decomposition of V_X into a sum of subspaces as in (4.1), and prove the main theoretical result of the paper, namely an estimate for the condition number of the additive Schwarz operator P.

Let α be a fixed number satisfying $0 < \alpha < \pi/3$ and let $X_0 := \{\boldsymbol{p}_j : j = 1, \dots, J\}$ be a subset of X such that

$$X = \bigcup_{j=1}^{J} (\overline{C(\boldsymbol{p}_j, \alpha)} \cap X). \tag{5.1}$$

For $j = 1, \ldots, J$, the subset X_j is defined as

$$X_j := \{\boldsymbol{x}_k \in X : \theta(\boldsymbol{x}_k, \boldsymbol{p}_j) \le \alpha\} = \overline{C(\boldsymbol{p}_j, \alpha)} \cap X. \tag{5.2}$$

The sets X_j may have different numbers of elements and may overlap each other. Because of (5.1) it is clear that X is decomposed into J overlapping subsets $\{X_j : j = 1, \ldots, J\}$ of discrete points such that

$$X = \bigcup_{j=1}^{J} X_j.$$

We define $V_j = V_{X_j}$, $j = 0, \ldots, J$, i.e. $V_j =$ span $\{\Phi_k : \boldsymbol{x}_k \in X_j\}$, so that $V_X = V_0 + \cdots + V_J$. The Schwarz operator P is then defined by (4.2) and (4.3).

Assume that the support of $\Phi(\boldsymbol{p}, \cdot)$ is contained in a spherical cap centered at $\boldsymbol{p}$ and having radius γ. Then functions in V_j have supports in Γ_j, where

$$\Gamma_j := C(\boldsymbol{p}_j, \alpha + \gamma), \quad j = 1, \ldots, J.$$

We assume that:

Assumption 5.1 *We can partition the index set $\{1, \ldots, J\}$ into M (for $1 \le M \le J$) sets J_m, for $1 \le m \le M$ such that if $i, j \in J_m$ and $i \ne j$ then $\Gamma_i \cap \Gamma_j = \varnothing$.*

The partitioning problem mentioned in Assumption 5.1 is related to the graph colouring problem [3]. We can define an undirected graph $G = (V, E)$ in which the set of vertices $V = \{\nu_1, \ldots, \nu_J\}$ is identified with the set of caps Γ_j, and E is the set of edges, where if $\Gamma_i \cap \Gamma_j \ne \varnothing$ then there is an edge between ν_i and ν_j. A partition satisfying Assumption 5.1 is equivalent to a colouring of the vertices of G so that adjacent vertices have different colours. The minimal number of colours needed is called the chromatic number of G, and is denoted by $\delta(G)$. In general, it is difficult to determine the chromatic number of a graph. However, it is easy to see that

$$\delta(G) \ge \omega(G),$$

where $\omega(G)$ is the maximal order of a complete subgraph of G, that is, it is the maximal number of vertices all of which are mutually connected. In terms of the caps, every point on the sphere $\mathbb{S}^n$ lies in at most $M_1 = \omega(G)$ spherical caps Γ_j.

An upper bound of $\delta(G)$ is given in [3, Theorem 3, Chapter 5]: when G is neither a complete graph nor an odd cycle, then $\delta(G) \le \Delta(G)$, with $\Delta(G)$ being the maximal degree of G. In terms of our spherical caps, each cap Γ_j intersects at most $M_2 = \Delta(G)$ other caps.

Therefore, for a given set X_0 and parameters α, γ, we can compute the lower bound M_1 and the upper bound M_2 so that

$$1 \le M_1 = \omega(G) \le M \le M_2 = \Delta(G) \le J. \tag{5.3}$$

5.2 Bounds on the extremal eigenvalues of P

First two lemmas are presented below that establish the assumptions in Lemma 4.1. Then these lemmas and Lemma 4.1 are used to find an upper bound for the condition number of P.

Lemma 5.2 *There exists a positive constant c independent of the set X such that for any $u \in V_X$ satisfying $u = \sum_{j=0}^J u_j$ with $u_j \in V_j$ for $j = 0, \dots, J$,*

$$\langle u, u \rangle_\Phi \le cM \sum_{j=0}^{J} \langle u_j, u_j \rangle_\Phi .$$

Proof. Using the fact that $\mathcal{N}_\Phi$ is isomorphic to $H^\tau(\mathbb{S}^n)$ under assumption (3.6) it suffices to prove

$$\|u\|^2_{H^\tau(\mathbb{S}^n)} \le cM \sum_{j=0}^{J} \|u_j\|^2_{H^\tau(\mathbb{S}^n)}.$$

Using the inequality $|a+b|^2 \le 2(|a|^2 + |b|^2)$, we have

$$\|u\|^2_{H^\tau(\mathbb{S}^n)} \le 2 \left(\|u_0\|^2_{H^\tau(\mathbb{S}^n)} + \left\| \sum_{j=1}^{J} u_j \right\|^2_{H^\tau(\mathbb{S}^n)} \right).$$

Let $k = \lfloor \tau \rfloor$. Then it follows from the definition of the Sobolev norm (2.7) that

$$\left\| \sum_{j=1}^{J} u_j \right\|^2_{H^k(\mathbb{S}^n)} = \left\| \sum_{j=1}^{J} \pi_1(\chi_1 u_j) \right\|^2_{H^k(\mathbb{R}^n)} + \left\| \sum_{j=1}^{J} \pi_2(\chi_2 u_j) \right\|^2_{H^k(\mathbb{R}^n)}. \tag{5.4}$$

Now, from the fact that $u_j \in V_j$ together with Assumption 5.1 we can partition the index set $\{1, \dots, J\}$ into M sets of indices J_m so that if $i, j \in J_m$ then supp $u_i \cap$ supp $u_j = \varnothing$. Then, in this proof only, let $g_j = \pi_1(\chi_1 u_j)$ with Ω_j being the interior of supp g_j. By using the Cauchy-Schwarz inequality, we have

$$\left\| \sum_{j=1}^{J} g_j \right\|^2_{H^k(\mathbb{R}^n)} = \left\| \sum_{m=1}^{M} \sum_{j \in J_m} g_j \right\|^2_{H^k(\mathbb{R}^n)} \le M \sum_{m=1}^{M} \left\| \sum_{j \in J_m} g_j \right\|^2_{H^k(\mathbb{R}^n)}. \tag{5.5}$$

Since the supports of g_i and g_j are disjoint for $i, j \in J_m$, $i \ne j$,

$$\left\| \sum_{j \in J_m} g_j \right\|^2_{H^k(\mathbb{R}^n)} = \sum_{j \in J_m} \|g_j\|^2_{H^k(\mathbb{R}^n)}.$$

Thus,

$$\left\| \sum_{j=1}^{J} g_j \right\|_{H^k(\mathbb{R}^n)}^2 \le M \sum_{m=1}^{M} \sum_{j\in J_m} \|g_j\|_{H^k(\mathbb{R}^n)}^2 = M \sum_{j=1}^{J} \|g_j\|_{H^k(\mathbb{R}^n)}^2. \tag{5.6}$$

Similarly, we obtain

$$\left\| \sum_{j=1}^{J} g_j \right\|_{H^{k+1}(\mathbb{R}^n)}^2 \le M \sum_{m=1}^{M} \sum_{j\in J_m} \|g_j\|_{H^{k+1}(\mathbb{R}^n)}^2 = M \sum_{j=1}^{J} \|g_j\|_{H^{k+1}(\mathbb{R}^n)}^2. \tag{5.7}$$

Let the space $\widetilde{H}^k(\Omega_j)$ be defined by

$$\widetilde{H}^k(\Omega_j) := \{u \in H^k(\mathbb{R}^n) : \text{ supp } u \subset \overline{\Omega_j}\},$$

which is equipped by a norm $\|u\|_{\widetilde{H}^k(\Omega_j)} = \|u\|_{H^k(\mathbb{R}^n)}$. We consider the product space $\Pi^k = \widetilde{H}^k(\Omega_1) \times \widetilde{H}^k(\Omega_2) \times \ldots \times \widetilde{H}^k(\Omega_J)$. Each element in Π^k has the form $\mathbf{g} = (g_1, g_2, \ldots, g_J)$ with its norm given by

$$\|\mathbf{g}\|_{\Pi^k} = \left(\sum_{j=1}^{J} \|g_j\|_{\widetilde{H}^k(\Omega_j)}^2 \right)^{1/2}.$$

Defining a linear operator T from Π^k to $H^k(\mathbb{R}^n)$ by

$$T(g_1, g_2, \ldots, g_J) = \sum_{j=1}^{J} g_j,$$

inequalities (5.6)–(5.7) are equivalent to

$$\|T\mathbf{g}\|_{H^k(\mathbb{R}^n)} \le M^{1/2} \|\mathbf{g}\|_{\Pi^k} \quad \text{and} \quad \|T\mathbf{g}\|_{H^{k+1}(\mathbb{R}^n)} \le M^{1/2} \|\mathbf{g}\|_{\Pi^{k+1}}.$$

Since $k \le \tau < k+1$, by using interpolation between Sobolev spaces (see for example [22, Theorem 1.9.3]) with the operator T, we conclude that

$$\left\| \sum_{j=1}^{J} g_j \right\|_{H^\tau(\mathbb{R}^n)}^2 \le M \sum_{j=1}^{J} \|g_j\|_{H^\tau(\mathbb{R}^n)}^2.$$

By using similar arguments for $\pi_2(\chi_2 u_j)$, we deduce

$$\begin{aligned} \left\| \sum_{j=1}^{J} u_j \right\|_{H^\tau(\mathbb{S}^n)}^2 &\le M \left(\sum_{j=1}^{J} \|\pi_1(\chi_1 u_j)\|_{H^\tau(\mathbb{R}^n)}^2 + \sum_{j=1}^{J} \|\pi_2(\chi_2 u_j)\|_{H^\tau(\mathbb{R}^n)}^2 \right) \\ &= M \sum_{j=1}^{J} \|u_j\|_{H^\tau(\mathbb{S}^n)}^2. \end{aligned}$$

Therefore,

$$\|u\|_{H^\tau(\mathbb{S}^n)}^2 = \left\| \sum_{j=0}^{J} u_j \right\|_{H^\tau(\mathbb{S}^n)}^2 \leq 2M \sum_{j=0}^{J} \|u_j\|_{H^\tau(\mathbb{S}^n)}^2 .$$

□

The following lemma is proved in [7, Lemma 5.3].

Lemma 5.3 *For any $u \in V_X$ there exist $u_j \in V_j$, $j = 0, \ldots, J$, satisfying $u = \sum_{j=0}^{J} u_j$ and*

$$\sum_{j=0}^{J} \langle u_j, u_j \rangle_\Phi \leq \left(1 + \frac{J}{(1 - \|\widetilde{Q}\|_\Phi)^2} \right) \langle u, u \rangle_\Phi ,$$

where $\widetilde{Q} = Q_J \cdots Q_1$, in which Q_j is the orthogonal projection from V_X to $V_j^\perp$ with respect to $\langle \cdot, \cdot \rangle_\Phi$, and

$$\|\widetilde{Q}\|_\Phi = \sup\{\|\widetilde{Q}v\|_\Phi : v \in V_X \text{ and } \|v\|_\Phi \leq 1\}.$$

Theorem 5.4 *The condition number $\kappa(P) := \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}$ of the additive Schwarz operator P is bounded by*

$$\kappa(P) \leq cM \left(1 + \frac{J}{(1 - \|\widetilde{Q}\|_\Phi)^2} \right),$$

where c is a constant independent of M,J and the set X. The operator $\widetilde{Q}$ is defined in Lemma 5.3.

Proof. From Lemmas 5.2 and 4.1, we obtain the following estimate for the maximum eigenvalue of P:

$$\lambda_{\max}(P) \leq cM, \tag{5.8}$$

where c is a constant independent of the set X. Lemmas 5.3 and 4.1 yield the following estimate for the minimum eigenvalue of P:

$$\lambda_{\min}(P) \geq \left(1 + \frac{J}{(1 - \|\widetilde{Q}\|_\Phi)^2} \right)^{-1}. \tag{5.9}$$

Therefore the bound for $\kappa(P)$ is derived. □

Comparing this upper bound for the condition numbers with bounds proved in Subsection 3.3 shows the advantage of our preconditioners, which is illustrated in Section 7 on numerical experiments. While we are unable to relate the parameters J and $\|\widetilde{Q}\|_\Phi$ (in the lower bound for $\lambda_{\min}(P)$) to the separation radius q_X, we show numerically that the lower bound given in (5.9) is better than the lower bound for $\lambda_{\min}(A)$ given in Theorem 3.6 (see Table 7 in Section 7).

6 An algorithm to decompose the data set

Since the Earth is rotating around its own axis, and the satellite traverses from near the North pole to near the South pole and back in an elliptical path, data along the track of the satellite form a sequence of discrete points; see Figure 1. Suppose we number the scattered data following the satellite track as $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N$. These points define the data set X introduced in Section 3.2. The algorithm to decompose X in the form $X = X_0 \cup X_1 \cup \cdots \cup X_J$ can be described as follows.

(1) Select $\alpha \in (q_X, \pi/3)$ and $\beta \in [\alpha, \pi)$.

(2) The first center is $\boldsymbol{p}_1 = \boldsymbol{x}_1 \in X$.

(3) Define
$$X_1 := \{\boldsymbol{x}_k \in X : \theta(\boldsymbol{x}_k, \boldsymbol{p}_1) \leq \alpha\}.$$

(4) Suppose X_{j-1}, for $j > 1$, has been selected around center $\boldsymbol{p}_{j-1}$. The next center $\boldsymbol{p}_j$ is chosen from $X \setminus \{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_{j-1}\}$ such that $\theta(\boldsymbol{p}_{j-1}, \boldsymbol{p}_j) \geq \beta$.

(5) The subset X_j is defined as
$$X_j := \{\boldsymbol{x}_k \in X : \theta(\boldsymbol{x}_k, \boldsymbol{p}_j) \leq \alpha\}.$$

(6) Repeat (4) and (5) until every point in X is in at least one X_k or no new center can be found.

(7) If $X = \bigcup_{k=1}^{j} X_k$ then go to step (8). If not, there exists a point $\boldsymbol{x}^* \in X \setminus \bigcup_{k=1}^{j} X_k$ so that $\theta(\boldsymbol{x}^*, \boldsymbol{p}_{j-1}) =: \theta^* < \beta$. Redefine $\beta = \theta^*$ and go back to step (4).

(8) Define $X_0 = \{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_J\}$ where $J = j$.

If we set $\beta = \alpha$ the algorithm will always terminate since the set X is finite. The initial value β is chosen close to π in order to reduce the number of subproblems J.

In the following, we describe the preconditioned conjugate gradient method based on the decomposition of the scattered data $X = X_0 \cup \cdots \cup X_J$. For $k = 0, \ldots, J$, let A_k be the restriction of the matrix A onto each subspace V_k, i.e. A_k is a submatrix of size $\mathrm{card}(V_k) \times \mathrm{card}(V_k)$ given by $A_k = [\langle \Phi_i, \Phi_j \rangle_\Phi]$, where $\Phi_i, \Phi_j \in V_k$.

For $k = 0, \ldots, J$, let I_k be an ordered subset of $\{1, \ldots, N\}$ such that $\boldsymbol{x}_m \in X_k$ if and only if $m \in I_k$. The cardinality of the set I_k is denoted by s_k and the rth-element of the set I_k is denoted by $I_k(r)$. For a given vector $\boldsymbol{v} = [v_1, \ldots, v_N]^T$, the restriction map $R_k : \mathbb{R}^N \to \mathbb{R}^{s_k}$ is defined as follows:
$$R_k \boldsymbol{v} = [v_{I_k(r)}]_{r=1}^{s_k}.$$

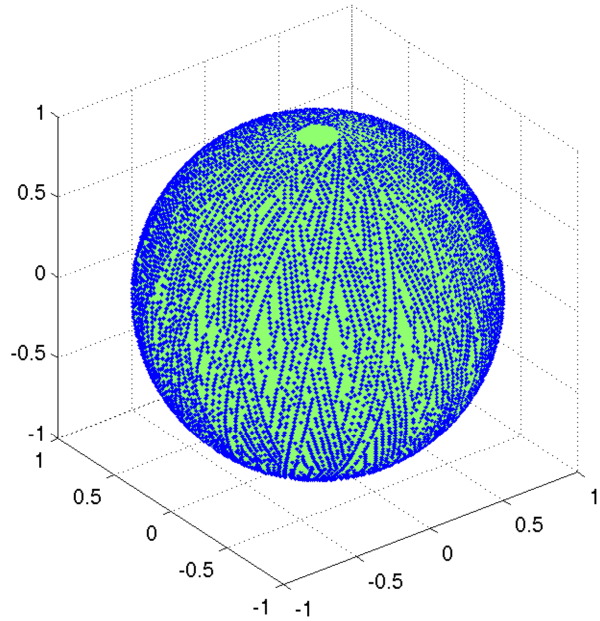

Figure 1: Global scattered MAGSAT satellite data

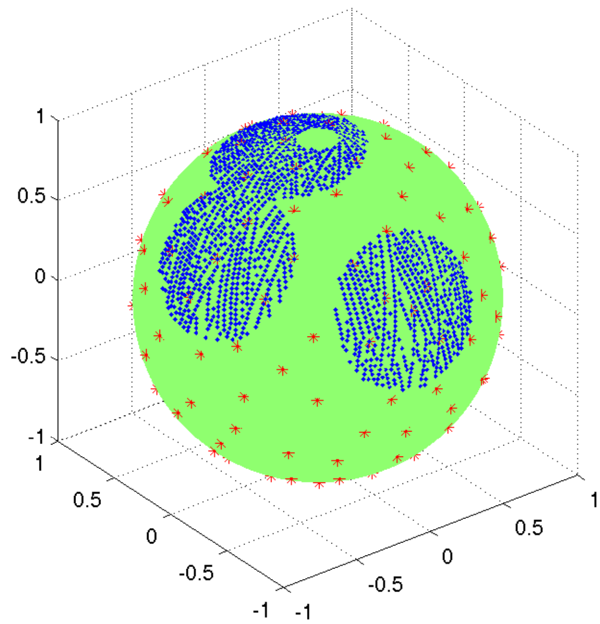

Figure 2: Three subsets X_1, X_2, X_3 produced by the partition algorithm, asterisked points are other centers $\boldsymbol{p}_j$

Conversely, for a vector $\boldsymbol{u} = [u_1, \dots, u_{s_k}]^T$, the extension map $R_k^T : \mathbb{R}^{s_k} \to \mathbb{R}^N$ is defined by $R_k^T \boldsymbol{u} = [v_j]_{j=1}^N$, where

$$v_j = \begin{cases} u_r & \text{if } j = I_k(r) \text{ for some } r \in \{1, \dots, s_k\}, \\ 0 & \text{if } j \notin I_k. \end{cases}$$

A pseudocode

INPUT

Input the scattered set X on the sphere, the right-hand side $\mathbf{f}$, and the desired accuracy ϵ.

SETUP

(1) Partition the scattered set X into $X_0 \cup \cdots \cup X_J$.

(2) The residual vector $\boldsymbol{r} = [f(\boldsymbol{x}_j)]_{j=1}^N$.

(3) The pseudo-residual vector $\mathbf{p} = 0$.

(4) The initial solution vector $\mathbf{s} = 0$.

(5) Set the iteration counter $\texttt{iter} = 0$.

ITERATIVE SOLUTION

(1) **while** $\|\boldsymbol{r}\| > \epsilon$

(2)   **for** j=1 **to** J

(3) $\mathbf{p} = \mathbf{p} + R_j^T A_j^{-1} R_j \boldsymbol{r}$.

(4)   **end for**

(5) $\mathbf{p} = \mathbf{p} + R_0^T A_0^{-1} R_0 \boldsymbol{r}$

(6)   If $\texttt{iter} > 0$ then set $\zeta_0 = \zeta_1$.

(7)   Set $\zeta_1 = \mathbf{p} \cdot \boldsymbol{r}$.

(8) $\texttt{iter} = \texttt{iter} + 1$.

(9)   If $\texttt{iter} = \texttt{1}$ then define $\mathbf{p}_1 = \mathbf{p}$ else $\mathbf{p}_1 = \mathbf{p} + (\zeta_1/\zeta_0)\mathbf{p}_1$.

(10)   Update the residual vector

$$\boldsymbol{r} = \boldsymbol{r} - \frac{\boldsymbol{r} \cdot \mathbf{p}}{\mathbf{p}_1 \cdot A\mathbf{p}_1} A\mathbf{p}_1.$$

(11)   Update the solution vector

$$\mathbf{s} = \mathbf{s} + \frac{\boldsymbol{r} \cdot \mathbf{p}}{\mathbf{p}_1 \cdot A\mathbf{p}_1} \mathbf{p}_1.$$

(12) **end while**

7 Numerical results

In this section, we present numerical experiments on $\mathbb{S}^2$ based on globally scattered data extracted from a very large data set collected by the NASA satellite MAGSAT. Given a positive real number q, different sets of scattered points X are extracted from the original data set so that the separation radius q_X is not less than q. These sets X are constructed by using a two-stage thinning process:

(1) Points are taken along the satellite track so that the geodesic distance between two successive points is not less than q.

(2) Points from stage (1) are re-selected so that the separation radius q_X is not less than q.

The number of points and the separation radius q_X of each data set are listed in Tables 3 and 4.

The locally supported spherical basis functions induced by compactly supported radial basis functions introduced by Wendland [23] for $\mathbb{R}^3$ are used. The kernel Φ, see (3.3), is defined by

$$\Phi(\boldsymbol{x}, \boldsymbol{y}) = \phi(\boldsymbol{x} \cdot \boldsymbol{y}),$$

where $\phi : [-1, 1] \to \mathbb{R}$ is defined by $\phi(t) = \rho_m(\sqrt{2 - 2t})$. Here $\rho_m(r)$ is chosen from Table 2. The native space $\mathcal{N}_\Phi$ can be identified with $H^\tau(\mathbb{S}^n)$, where $\tau = m + 3/2$, see [16, Proposition 4.6] for a more general statement of the result. For these spherical basis functions, the support radius γ described in Subsection 5.1 is $\pi/3$.

m	$\rho_m(r)$	τ
1	$(1-r)_+^4(4r+1)$	2.5
2	$(1-r)_+^6(35r^2+18r+3)$	3.5
3	$(1-r)_+^8(32r^3+25r^2+8r+1)$	4.5

Table 2: Wendland's RBFs

We solved the linear system (3.9) by the conjugate gradient method with stopping criteria

$$\frac{\|A\mathbf{s} - \mathbf{f}\|}{\|\mathbf{f}\|} \leq 10^{-7}.$$

The right hand side $\mathbf{f}$ is defined, respectively, from the values at $\boldsymbol{x}_j$ of the functions

$$f_1(\boldsymbol{x}) = \exp(x_1 + x_2 + x_3)$$

and

$$f_2(\boldsymbol{x}) = \exp(x_1 + x_2 + x_3) + [0.01 - (x_1^2 + x_2^2 + (x_3 - 1)^2)]_+^2,$$

in which $(y)_+ = y$ for $y \geq 0$ and equals 0 otherwise.

As shown in Table 3 (with $m = 1$) and Table 4, when N increases and q_X decreases, the minimal eigenvalues of the interpolation matrices A decrease, which is consistent with the estimates presented in Section 3.3. (The numbers when $m = 2$ and $m = 3$ in Table 3 do not reflect this behaviour. We believe that this is a defect in the Lanczos algorithm, used to compute the extreme eigenvalues, when f_1 is excessively smooth). As a consequence, the CPU times (in seconds) spent to solve the linear systems increase significantly. This motivates the use of the additive Schwarz preconditioner introduced in Sections 4 and 5. The results are reported in Tables 5 and 6. The condition numbers of the preconditioned systems $\kappa(P)$ are much smaller than the condition numbers of the original interpolation matrix $\kappa(A)$. As a consequence, the number of iterations and CPU times in solving the linear systems are reduced. The numbers in both tables suggest that when $\cos\alpha$ decreases both $\kappa(P)$ and the CPU time decrease then increase. With the current theory, we cannot explain this behaviour. We note that a larger value of α results in a larger size of the overlap and a smaller value of J (the number of subproblems to be solved), which in turn implies larger sizes of the subproblems. This results in a smaller condition number $\kappa(P)$ because the preconditioner is closer to the inverse of the interpolation matrix. However, for an optimal value of α in terms of CPU time, one has to balance between the number of subproblems and their sizes.

As commented at the end of Subsection 5.2, we also computed the lower bounds of $\lambda_{\min}(P)$ and $\lambda_{\min}(A)$ given in (5.9) and Theorem 3.6, respectively. The numbers, reported in Table 7, clearly show that the bound for $\lambda_{\min}(P)$ is better than that of $\lambda_{\min}(A)$. Details of this calculation have been reported in [7].

m	N	q_X	$\lambda_{\min}(A)$	$\lambda_{\max}(A)$	$\kappa(A)$	CPU	ITER
1	13897	$\pi/280$	0.1432E-03	0.5143E+03	0.3591E+07	12151	2493
	25631	$\pi/400$	0.9396E-04	0.9739E+03	0.1036E+08	66463	3719
2	13897	$\pi/280$	0.1079E-03	0.1216E+04	0.1128E+08	16777	3359
	25631	$\pi/400$	0.1714E-03	0.2321E+04	0.1354E+08	62353	3673
3	13897	$\pi/280$	0.5519E-04	0.3314E+03	0.6005E+07	13568	2667
	25631	$\pi/400$	0.1375E-03	0.6359E+03	0.4624E+07	40043	2334

Table 3: Unpreconditioned systems with $f_1(\boldsymbol{x}_j)$ as the right-hand side

Acknowledgements The support of the Australian Research Council under its Centre of Excellence Program is gratefully acknowledged. The authors are also indebted to Professor Ian Sloan for helpful suggestions, and two anonymous referees for their constructive criticisms which significantly improve the paper.

m	N	q_X	$\lambda_{\min}(A)$	$\lambda_{\max}(A)$	$\kappa(A)$	CPU	ITER
1	7663	$\pi/200$	0.2428E-03	0.2772E+03	0.1142E+07	2005.6	1638
	13897	$\pi/280$	0.1334E-03	0.5143E+03	0.3855E+07	12638.7	2596
	25631	$\pi/400$	0.7936E-04	0.9739E+03	0.1227E+08	69227.4	4121
2	7663	$\pi/200$	0.2612E-04	0.6497E+03	0.2487E+08	6294.3	5214
	13897	$\pi/280$	0.3662E-05	0.1216E+04	0.3321E+09	90163	18158
	25631	$\pi/400$				>52 hours	
3	7663	$\pi/200$	0.3336E-05	0.1756E+03	0.5265E+08	8382.1	6735
	13897	$\pi/280$				>52 hours	
	25631	$\pi/400$				>52 hours	

Table 4: Unpreconditioned systems with $f_2(\boldsymbol{x}_j)$ as the right-hand side

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m	N	$\cos\alpha$	$\cos\beta$	J	$\lambda_{\min}$	$\lambda_{\max}$	$\kappa(P)$	CPU	ITER
1	13897	0.98	0.74	217	6.6267E-03	19.5	2.9451E+03	638.7	122
	13897	0.95	0.03	90	2.1804E-02	12.9	5.9365E+02	483.6	72
	13897	0.90	-0.76	46	3.4937E-01	9.2	2.6445E+01	298.0	26
	13897	0.85	-0.75	32	9.0383E-02	8.2	9.0998E+01	552.0	28
	13897	0.80	-0.67	26	9.9896E-01	8.2	8.2570E+00	580.0	18
	13897	0.70	-0.85	17	7.2841E-01	8.0	1.0916E+01	1305.8	20
	25631	0.98	0.86	209	1.6588E-02	19.3	1.1647E+03	1769.4	93
	25631	0.95	-0.38	91	3.4405E-02	14.1	4.0911E+02	1721.0	64
	25631	0.90	-0.61	49	8.2913E-02	10.5	1.2666E+02	2674.5	45
	25631	0.85	-0.77	35	1.0195E+00	9.0	8.8340E+00	2312.3	20
	25631	0.80	-0.76	31	6.0713E-01	10.2	1.6843E+01	4901.5	23
	25631	0.70	-0.85	17	1.7102E-01	8.0	4.7048E+01	10477.1	27
2	13897	0.98	0.74	217	5.1891E-04	19.8	3.8090E+04	1501.8	280
	13897	0.95	0.03	90	1.8479E-02	13.2	7.1192E+02	548.8	80
	13897	0.90	-0.76	46	1.0180E-01	9.3	9.1298E+01	429.9	37
	13897	0.85	-0.75	32	2.3134E-02	8.3	3.5668E+02	756.2	38
	13897	0.80	-0.67	26	1.0086E+00	8.4	8.2980E+00	615.5	19
	13897	0.70	-0.85	17	1.8758E-01	8.1	4.3081E+01	1636.3	25
	25631	0.98	0.86	209	7.2039E-03	19.8	2.7516E+03	2733.8	145
	25631	0.95	-0.38	91	1.8163E-02	14.4	7.9245E+02	2364.9	86
	25631	0.90	-0.61	49	1.7743E-02	10.6	5.9768E+02	4460.6	75
	25631	0.85	-0.77	35	1.0214E+00	9.4	9.1630E+00	2313.6	20
	25631	0.80	-0.76	31	3.7871E-01	10.6	2.7980E+01	5341.0	25
	25631	0.70	-0.85	17	3.0557E-02	8.2	2.6862E+02	13974.6	36
3	13897	0.98	0.74	217	5.1520E-04	19.7	3.8311E+04	1959.6	359
	13897	0.95	0.03	90	3.4495E-03	13.2	3.8277E+03	1008.3	145
	13897	0.90	-0.76	46	2.6554E-02	9.3	3.4943E+02	667.6	57
	13897	0.85	-0.75	32	5.5097E-02	8.2	1.4928E+02	759.6	38
	13897	0.80	-0.67	26	7.0363E-01	8.4	1.1950E+01	682.6	21
	13897	0.70	-0.85	17	4.0337E-02	8.1	2.0118E+02	2625.3	40
	25631	0.98	0.86	209	1.0636E-03	20.0	1.8797E+04	5847.0	308
	25631	0.95	-0.38	91	4.9184E-03	14.5	2.9553E+03	3836.1	139
	25631	0.90	-0.61	49	2.5279E-03	10.6	4.2068E+03	8016.1	135
	25631	0.85	-0.77	35	6.4421E-01	9.5	1.4820E+01	2656.9	23
	25631	0.80	-0.76	31	2.4138E-01	10.8	4.4628E+01	6619.3	31
	25631	0.70	-0.85	17	3.7136E-02	8.3	2.2268E+02	16686.7	43

Table 5: Preconditioned systems with $f_1(\boldsymbol{x}_j)$ as the right-hand side

m	N	$\cos\alpha$	$\cos\beta$	J	$\lambda_{\min}$	$\lambda_{\max}$	$\kappa(P)$	CPU	ITER
1	13897	0.98	0.74	217	6.6267E-03	19.5	2.9451E+03	639.8	122
	13897	0.95	0.03	90	2.1804E-02	12.9	5.9365E+02	483.6	72
	13897	0.90	-0.76	46	3.4937E-01	9.2	2.6445E+01	297.4	26
	13897	0.85	-0.75	32	9.0383E-02	8.2	9.0998E+01	551.5	28
	13897	0.80	-0.67	26	9.9896E-01	8.2	8.2570E+00	579.2	18
	13897	0.70	-0.85	17	7.2841E-01	8.0	1.0916E+01	1303.6	20
	25631	0.98	0.86	209	1.6588E-02	19.3	1.1648E+03	1722.0	93
	25631	0.95	-0.38	91	3.4405E-02	14.1	4.0910E+02	1738.7	64
	25631	0.90	-0.61	49	8.2913E-02	10.5	1.2666E+02	2658.2	45
	25631	0.85	-0.77	35	1.0195E+00	9.0	8.8340E+00	2309.0	20
	25631	0.80	-0.76	31	6.0713E-01	10.2	1.6843E+01	4907.8	23
	25631	0.70	-0.85	17	1.7102E-01	8.0	4.7048E+01	10511.4	27
2	13897	0.98	0.74	217	5.1891E-04	19.8	3.8090E+04	1500.8	280
	13897	0.95	0.03	90	1.8479E-02	13.2	7.1192E+02	547.6	80
	13897	0.90	-0.76	46	1.0180E-01	9.3	9.1298E+01	428.6	37
	13897	0.85	-0.75	32	2.3134E-02	8.3	3.5668E+02	754.3	38
	13897	0.80	-0.67	26	1.0086E+00	8.4	8.2980E+00	615.6	19
	13897	0.70	-0.85	17	1.8758E-01	8.1	4.3081E+01	1634.5	25
	25631	0.98	0.86	209	7.2078E-03	19.8	2.7501E+03	2730.2	144
	25631	0.95	-0.38	91	1.8176E-02	14.4	7.9189E+02	2331.5	85
	25631	0.90	-0.61	49	1.7743E-02	10.6	5.9768E+02	4448.3	75
	25631	0.85	-0.77	35	1.0214E+00	9.4	9.1630E+00	2308.1	20
	25631	0.80	-0.76	31	3.7871E-01	10.6	2.7980E+01	5333.8	25
	25631	0.70	-0.85	17	3.0557E-02	8.2	2.6862E+02	13968.5	36
3	13897	0.98	0.74	217	5.2007E-04	19.7	3.7953E+04	1919.7	351
	13897	0.95	0.03	90	3.4495E-03	13.2	3.8277E+03	1010.0	145
	13897	0.90	-0.76	46	2.6554E-02	9.3	3.4943E+02	667.8	57
	13897	0.85	-0.75	32	5.5097E-02	8.2	1.4928E+02	759.6	38
	13897	0.80	-0.67	26	7.0364E-01	8.4	1.1950E+01	682.9	21
	13897	0.70	-0.85	17	4.0337E-02	8.1	2.0118E+02	2622.5	40
	25631	0.98	0.86	209	2.8298E-03	20.0	7.0652E+03	5033.7	264
	25631	0.95	-0.38	91	5.0009E-03	14.5	2.9065E+03	3736.6	135
	25631	0.90	-0.61	49	2.5271E-03	10.6	4.2082E+03	8058.6	135
	25631	0.85	-0.77	35	6.4420E-01	9.5	1.4820E+01	2659.3	23
	25631	0.80	-0.76	31	2.4138E-01	10.8	4.4628E+01	6628.8	31
	25631	0.70	-0.85	17	3.7136E-02	8.3	2.2268E+02	16699.9	43

Table 6: Preconditioned systems with $f_2(\boldsymbol{x}_j)$ as the right-hand side

N	q_X	q_X^{s-1}	$\cos(\alpha)$	$\cos(\beta)$	J	$\left(1+\frac{J}{(1-\|\widetilde{Q}\|_\Phi)^2}\right)^{-1}$
1344	0.0393	5.6904E-12	0.9	0.06	42	2.7020E-07
			0.8	-0.86	19	3.0785E-06
			0.7	-0.80	14	1.5421E-04
			0.6	-0.82	12	3.0157E-04
			0.5	-0.81	12	3.3190E-02
2133	0.0314	9.4501E-13	0.9	0.01	42	7.7693E-08
			0.8	-0.66	22	6.6506E-06
			0.7	-0.78	17	1.3391E-05
			0.6	-0.69	11	5.5897E-02
			0.5	-0.81	10	5.8140E-03
3458	0.0224	6.3385E-14	0.9	-0.58	41	7.0018E-07
			0.8	-0.56	26	1.0644E-02
			0.7	-0.81	14	1.5997E-03
			0.6	-0.85	11	8.9346E-04
			0.5	-0.53	11	1.9279E-02

Table 7: Lower bounds for $\lambda_{\min}(A)$ and $\lambda_{\min}(P)$ with Φ defined from $\rho_3(r)$ having $\widehat{\phi}(\ell) \simeq (1+\ell)^{-s}$ with $s=9$