Overlapping additive Schwarz preconditioners for interpolation on the unit sphere by spherical radial basis functions

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Abstract

We present an overlapping domain decomposition technique for the interpolation problem on the unit sphere using spherical basis functions.

1 Introduction

When solving the interpolation problem on the unit sphere based on scattered data, we observe a very ill-conditioned linear system. In this paper, we introduce overlapping additive Schwarz preconditioners for the interpolation problem using spherical basis functions in which the native space can be identified with a Sobolev space. Let $S^n$ be the unit sphere in $\mathbb{R}^{n+1}$, $n = 1, 2, \ldots$, and suppose that $X = \{x_1, \ldots, x_N\}$ is a set of scattered points lying on $S^n$. Given real numbers $f_i$, $i = 1, \ldots, N$, we want to find a smooth function $u$ defined on $S^n$ which interpolates the data, namely,

$$u(x_i) = f_i, \quad i = 1, \ldots, N. \quad (1.1)$$

The problem (1.1) where the underlying domain is the sphere arises in many areas including, e.g., geodesy and earth science in which the sphere is taken as a model for the earth. Even though in practice measured data usually contain noise which means that approximation of data is necessary, for simplicity, in this paper we consider only the interpolation problem, with scattered data collected from satellites.

A review paper by Fasshauer and Schumaker [5] discusses available methods for interpolation of scattered data on the sphere. Two main methods are the one that uses spherical splines [1], and another that uses spherical radial basis functions [13, 14]. We follow the second approach in this paper.

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It is perceived both theoretically [10] and numerically (Section 7) that the matrices arising from this interpolation problem are very ill-conditioned. We shall present an algorithm which uses spherical radial basis functions to construct the interpolant and an additive Schwarz preconditioner to accelerate the solution process. We shall give an estimate for the condition number of the preconditioned system. Even though domain decomposition methods have been extensively studied for finite-element and boundary-element methods, not much has been done for meshless methods using radial basis functions. For the interpolation problem in \( \mathbb{R}^n \) using radial basis functions, the idea of dividing the scattered data set into smaller subsets for the purpose of defining the Schwarz alternating algorithm is proposed in [2]. Work on applying the multiplicative Schwarz alternating algorithm using spherical splines is also carried out in [7], but in that work the data points are not scattered, and again the Schwarz method is not used as a preconditioner. None of these papers uses domain decomposition methods as preconditioners to be solved with the conjugate-gradient method, and none studies the condition numbers of the preconditioned systems. The purpose of this paper is to fill this gap. We shall study the condition number of the system preconditioned by an overlapping additive Schwarz method. In [9] we studied the use of additive Schwarz preconditioners for elliptic partial differential equations on the sphere, and proved a bound for the condition number. Similar approach will be used in the present paper for the interpolation problem.

2 Preliminaries

In this section, we will review spherical harmonics, function spaces on the unit Euclidean sphere \( S^n \subset \mathbb{R}^{n+1} \), and spherical basis functions.

2.1 Spherical harmonics

Spherical harmonics are the restriction of homogeneous harmonic polynomials in \( \mathbb{R}^{n+1} \) to the unit sphere \( S^n \). We denote an orthonormal (with respect to the \( L^2(S^n) \) inner product) basis for the spherical harmonics of degree \( \ell \) by

\[
\{ Y_{\ell,k} : k = 1, \ldots, N(n, \ell) \}, \quad \ell = 0, 1, \ldots,
\]

where \( N(n, \ell) \) is the dimension of the space of all spherical harmonics of degree \( \ell \); the values of \( N(n, \ell) \) are (see [12]):

\[
N(n, 0) = 1 \quad \text{and} \quad N(n, \ell) = \frac{(2\ell + n - 1)\Gamma(\ell + n - 1)}{\Gamma(\ell + 1)\Gamma(n)} \quad \text{for} \ \ell \geq 1.
\]

The asymptotic behaviour of \( N(n, \ell) \) for fixed \( n \) and increasing \( \ell \) is \( O(\ell^{n-1}) \). The spherical harmonics \( \{ Y_{\ell,k} : \ell = 0, 1, \ldots; k = 1, \ldots, N(n, \ell) \} \) form a complete orthonormal basis for \( L^2(S^n) \). Correspondingly, for a given function \( f \in L^2(S^n) \), we
define its Fourier coefficients by
\[
\hat{f}_{\ell,k} = \int_{S^n} f(x) Y_{\ell,k}(x) dS(x),
\]
where \(dS\) is the surface measure of the sphere \(S^n\), and represent \(f\) as a Fourier series,
\[
f = \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(n,\ell)} \hat{f}_{\ell,k} Y_{\ell,k},
\]
in which the equal sign is understood in the \(L^2(S^n)\) sense. The addition formula for spherical harmonics of the same degree \(\ell\) (see [12]) is
\[
\sum_{k=0}^{N(n,\ell)} Y_{\ell,k}(x) Y_{\ell,k}(y) = \frac{1}{\omega_n} N(n,\ell) P_\ell(n+1; x \cdot y), \quad (2.1)
\]
where \(P_\ell(n+1; t)\) is the normalized Legendre polynomial of degree \(\ell\) in \(\mathbb{R}^{n+1}\) and \(\omega_n\) is the surface area of the unit sphere \(S^n\). Recall from [12] that \(P_\ell(n+1; 1) = 1\) and
\[
\int_{-1}^{+1} P_\ell(n+1; t) P_k(n+1; t)(1 - t^2)^{(n-2)/2} dt = \frac{\omega_n}{\omega_{n-1} N(n,\ell)} \delta_{\ell,k}, \quad (2.2)
\]
where \(\omega_{n-1}\) is the surface area of the sphere \(S^{n-1}\), and \(\delta_{\ell,k}\) is the Kronecker delta.

Spherical harmonics of degree \(\ell\) are eigenfunctions of the Laplace–Beltrami operator \(\Delta^*\) on \(S^n\), with eigenvalues \(-\lambda_\ell\), with
\[
\lambda_\ell = \ell(\ell + n - 1), \quad (2.3)
\]
i.e.,
\[
\Delta^* Y_{\ell,k} = -\lambda_\ell Y_{\ell,k}.
\]

### 2.2 Sobolev spaces on \(S^n\)

For a given \(s \geq 0\), the Sobolev space \(H^s(S^n)\) on the unit sphere is defined in terms of the eigenvalues of the Laplace-Beltrami operator (see [11]),
\[
H^s(S^n) := \left\{ f \in L^2(S^n) : \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(n,\ell)} (1 + \lambda_\ell)^s |\hat{f}_{\ell,k}|^2 < \infty \right\}. \quad (2.4)
\]

The norm of a function \(f\) in this space is defined to be
\[
\|f\|_{H^s(S^n)} = \left( \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(n,\ell)} (1 + \lambda_\ell)^s |\hat{f}_{\ell,k}|^2 \right)^{1/2}.
\]
Sobolev spaces on $\mathbb{S}^n$ can also be defined using local charts (see [11]). Here we use a specific atlas of charts, as in [8].

Let a spherical cap of radius $\alpha$ centered at $p \in \mathbb{S}^n$ be defined as

$$C(p, \alpha) := \{x \in \mathbb{S}^n : \theta(p, x) < \alpha\},$$

where $\theta(p, x) = \cos^{-1}(p \cdot x)$ is the geodesic distance between two points $x, p \in \mathbb{S}^n$. Let $\hat{n}$ and $\hat{s}$ denote the north and south poles of $\mathbb{S}^n$, respectively. Then a simple cover for the sphere is provided by

$$U_1 = C(\hat{n}, \theta_0) \quad \text{and} \quad U_2 = C(\hat{s}, \theta_0), \quad \text{where} \quad \theta_0 \in (\pi/2, 2\pi/3).$$

The stereographic projection $\sigma_n$ of the punctured sphere $\mathbb{S}^n \setminus \{\hat{n}\}$ onto $\mathbb{R}^n$ is defined as a mapping that maps $x \in \mathbb{S}^n \setminus \{\hat{n}\}$ to the intersection of the equatorial hyperplane $\{z = 0\}$ and the extended line that passes through $x$ and $\hat{n}$. The stereographic projection $\sigma_s$ based on $\hat{s}$ can be defined analogously. We set

$$\psi_1 = \frac{1}{\tan(\theta_0/2)}\sigma_s|_{U_1} \quad \text{and} \quad \psi_2 = \frac{1}{\tan(\theta_0/2)}\sigma_n|_{U_2},$$

so that $\psi_k$, $k = 1, 2$, maps $U_k$ onto $B(0, 1)$, the unit ball in $\mathbb{R}^n$. We conclude that $\mathcal{A} = \{U_k, \psi_k\}_{k=1}^2$ is a $C^\infty$ atlas of covering coordinate charts for the sphere. It is known (see [15]) that the stereographic coordinate charts $\{\psi_k\}_{k=1}^2$ as defined in (2.7) map spherical caps to Euclidean balls, but in general concentric spherical caps are not mapped to concentric Euclidean balls. The projection $\psi_k$, for $k = 1, 2$, does not distort too much the geodesic distance between two points $x, y \in \mathbb{S}^n$, as shown in [6]. With the atlas so defined, we define the map $\pi_k$ which takes a real-valued function $g$ with compact support in $U_k$ into a real-valued function on $\mathbb{R}^n$ by

$$\pi_k(g)(x) = \begin{cases} g \circ \psi_k^{-1}(x), & \text{if} \ x \in B(0, 1), \\ 0, & \text{otherwise} \end{cases}.$$ 

Let $\{\chi_k : \mathbb{S}^n \to \mathbb{R}\}_{k=1}^2$ be a partition of unity subordinated to the atlas, i.e., a pair of non-negative infinitely differentiable functions $\chi_k$ on $\mathbb{S}^n$ with compact support in $U_k$, such that $\sum_k \chi_k = 1$. For any function $f : \mathbb{S}^n \to \mathbb{R}$, we can use the partition of unity to write

$$f = \sum_{k=1}^2 (\chi_k f), \quad \text{where} \quad (\chi_k f)(p) = \chi_k(p)f(p), \quad p \in \mathbb{S}^n.$$ 

The Sobolev space $H^s(\mathbb{S}^n)$ is defined to be the set

$$\{f \in L^2(\mathbb{S}^n) : \pi_k(\chi_k f) \in H^s(\mathbb{R}^n) \quad \text{for} \ k = 1, 2\},$$

which is equipped with the norm

$$\|f\|_{H^s(\mathbb{S}^n)} = \left( \sum_{k=1}^2 \|\pi_k(\chi_k f)\|_{H^s(\mathbb{R}^n)}^2 \right)^{1/2}. \quad (2.8)$$

This $H^s(\mathbb{S}^n)$ norm is equivalent to the $H^s(\mathbb{S}^n)$ norm given in Section 2.1 (see [11]).
2.3 Interpolation between Sobolev spaces

As outlined in [17], two Banach spaces $A_0$ and $A_1$ are said to be an interpolation pair $(A_0, A_1)$ if there is a continuous inclusion $A_1 \subset A_0$. For any $f \in A_0$ and $t > 0$, we define the $K$-functional as

$$K(t, f) = \inf_{g \in A_1} (\|f - g\|^2_{A_0} + t^2\|g\|^2_{A_1})^{1/2}. \quad (2.9)$$

For $\tau \in (0, 1)$ the interpolation space $A_\tau = (A_0, A_1)_\tau$ is defined to be the Banach subspace of $A_0$ for which the following norm is finite:

$$\|f\|_{A_\tau} = \|K(t, f)t^{-\tau}\|_{L^2((0,\infty), dt/t)} = \left( \int_0^\infty \left( \frac{K(t, f)}{t^\tau} \right)^2 \frac{dt}{t} \right)^{1/2}. \quad (2.10)$$

We now apply these results to the Sobolev spaces $H^k(\Omega)$, where $k$ is a non-negative integer and $\Omega$ is a bounded open set in $\mathbb{R}^n$.

$$\|f\|_{H^k(\Omega)} = \left( \sum_{0 \leq |\alpha| \leq k} \|D^\alpha f\|^2_{L^2(\Omega)} \right)^{1/2}. \quad (2.11)$$

The fractional Sobolev space $H^{k+\delta}(\Omega)$ (for $0 < \delta < 1$) is defined as follows:

$$H^{k+\delta}(\Omega) = (H^k(\Omega), H^{k+1}(\Omega))_\delta = \{ f \in H^k(\Omega) : \|f\|_{H^{k+\delta}(\Omega)} < \infty \},$$

where by (2.9)

$$\|f\|_{H^{k+\delta}(\Omega)} = \left( \int_0^\infty \left( \frac{K(t, f)}{t^\delta} \right)^2 \frac{dt}{t} \right)^{1/2} \quad (2.12)$$

and

$$K(t, f) = \inf_{g \in H^{k+1}(\Omega)} (\|f - g\|^2_{H^k(\Omega)} + t^2\|g\|^2_{H^{k+1}(\Omega)})^{1/2}. \quad (2.13)$$

3 Interpolation using spherical radial basis functions

3.1 Positive definite kernels

In this section, we will review necessary background on positive definite kernels on the unit sphere and spherical basis functions.

A real-valued kernel $\Phi$ in $C(\mathbb{S}^n \times \mathbb{S}^n)$ is termed positive definite on $\mathbb{S}^n$ if $\Phi(x, y) = \Phi(y, x)$ and if for every finite set of distinct points $X = \{x_1, \ldots, x_N\}$ on $\mathbb{S}^n$, the symmetric $N \times N$ matrix $A$ with entries $A_{i,j} = \Phi(x_i, x_j)$ is positive semi-definite. If
the matrix $A$ is positive definite then $\Phi$ is called a \textit{strictly positive definite} kernel (see [16, 22]).

Let $\phi$ be a univariate function defined on $[-1, 1]$ which can be expanded in terms of Legendre polynomials as

$$
\phi(t) = \frac{1}{\omega_n} \sum_{\ell=0}^{\infty} N(n, \ell) \hat{\phi}(\ell) P_\ell(n + 1; t),
$$

where

$$
\hat{\phi}(\ell) = \omega_{n-1} \int_{-1}^{+1} \phi(t) P_\ell(n + 1; t)(1 - t^2)^{(n-2)/2} dt.
$$

Due to the addition formula (2.1), a kernel $\Phi$ defined by

$$
\Phi(x, y) = \phi(x \cdot y)
$$

can be represented as

$$
\Phi(x, y) = \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(n, \ell)} \hat{\phi}(\ell) Y_{\ell,k}(x) Y_{\ell,k}(y).
$$

In [4], a complete characterization of strictly positive definite kernels is established: the kernel $\Phi$ is strictly positive definite if and only if $\hat{\phi}(\ell) \geq 0$ for all $\ell \geq 0$ and $\hat{\phi}(\ell) > 0$ for infinitely many even values of $\ell$ and infinitely many odd values of $\ell$; see also [16] and [22].

The native space $\mathcal{N}_\phi$ associated with the kernel $\Phi$ is defined as

$$
\mathcal{N}_\phi := \left\{ f \in L^2(S^n) : \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(n, \ell)} \left| \frac{f_{\ell,k}}{\hat{\phi}(\ell)} \right|^2 < \infty \right\}.
$$

If

$$
c_1(1 + \lambda_\ell)^{-\tau} \leq \hat{\phi}(\ell) \leq c_2(1 + \lambda_\ell)^{-\tau},
$$

where $c_1, c_2 > 0$ are some constants and $\tau > n/2$, the native space $\mathcal{N}_\phi$ can be identified with the Sobolev space $H^\tau(S^n)$ defined in (2.4). Henceforth, the condition (3.6) is shortened to $\hat{\phi}(\ell) \sim (1 + \lambda_\ell)^{-\tau}$.

### 3.2 Spherical radial basis functions

With this kernel $\Phi$, we can now establish a set of spherical basis functions (SBFs) $\{\Phi_1, \ldots, \Phi_N\}$ associated with a set $X = \{x_1, \ldots, x_N\}$ of scattered and distinct points on $S^n$, where

$$
\Phi_i(x) := \Phi(x_i, x).
$$
The finite dimensional space spanned by these SBFs is denoted by $V_X$:

$$V_X := \text{span} \{ \Phi_i : i = 1, \ldots, N \}. \quad (3.7)$$

We note that the SBFs $\Phi_i$, $i = 1, \ldots, N$, depend only on the geodesic distance between the points $x$ and $x_i$. The set $X$ is characterized by its mesh norm $h_X$ and separation radius $q_X$ defined as

$$h_X := \sup_{y \in \mathbb{S}^n} \min_{x_i \in X} \theta(x_i, y), \quad \text{and} \quad q_X := \frac{1}{2} \min_{i \neq j} \theta(x_i, x_j).$$

### 3.3 Spherical basis functions derived from compactly supported radial basis functions

In the following we define a positive definite kernel $\Phi$ from a univariate function $\phi$ satisfying (3.6) by using Wendland's compactly supported radial basis functions [18]. For any non-negative integer $j$, let

$$\rho_j(r) = \begin{cases} (1 - r)^j, & 0 < r \leq 1, \\ 0, & r > 1, \end{cases}$$

and let

$$I \rho_j(r) = \int_r^\infty s \rho_j(s) ds, \quad r \geq 0.$$ 

We define, for any non-negative integer $m$,

$$\rho_{n+1,m}(r) = I^m \rho_{n+1,m+1}(r).$$

The radial basis function $\Psi_{n+1,m}$ is defined in $\mathbb{R}^{n+1}$ as

$$\Psi_{n+1,m}(x) = \rho_{n+1,m}(\|x\|),$$

where $\| \cdot \|$ denotes the Euclidean norm in $\mathbb{R}^{n+1}$. It is shown in [19, Corollary 2.3] that $\Psi_{n+1,m} \in C^{2m}(\mathbb{R}^{n+1})$. For any given $N$ and any set of $N$ pairwise distinct points $\{x_1, \ldots, x_N\}$ in $\mathbb{R}^{n+1}$, the matrix

$$[\Psi_{n+1,m}(x_i - x_j)]_{i,j=1}^N \quad (3.8)$$

is positive definite; see [20, Theorem 9.13]. Since $\|x - y\| = \sqrt{2 - 2x \cdot y}$ for any $x, y \in \mathbb{S}^n$, the kernel $\Phi$ defined by (3.3) with

$$\phi(t) = \rho_{n+1,m}(\sqrt{2 - 2t}) \quad (3.9)$$

is related to the above radial basis function $\Psi_{n+1,m}$ by

$$\Phi(x, y) = \Psi_{n+1,m}(x - y), \quad x, y \in \mathbb{S}^n.$$
Since the matrix (3.8) is positive definite, Φ is a strictly positive definite kernel on the sphere $S^n$. Moreover, the asymptotic behaviour of the Fourier coefficients $\hat{\phi}(\ell)$ is, see [14, Proposition 4.6],

$$\hat{\phi}(\ell) = O(\ell^{-2m-n-1}).$$

Using (2.3), we deduce that $\phi$ satisfies (3.6) with

$$\tau = m + (n + 1)/2. \quad (3.10)$$

### 3.4 Interpolation problem as a variational problem

The kernel $\Phi$ being strictly positive definite, the interpolation problem in $V_X$ of scattered points using spherical basis functions is always solvable. Given a function $f$ whose values $f(x_j)$ for $j = 1, \ldots, N$ are known, the interpolant $I_X f$ of $f$ is defined as a linear combination of the SBFs which satisfies $(I_X f)(x_j) = f(x_j)$ for all $j = 1, \ldots, N$. It is observed in [10] that the matrix $A$ with entries $A_{i,j} = \Phi(x_j, \cdot)$ arising from this interpolation problem is ill-conditioned. More fully, it is shown there that the least eigenvalue of the matrix $A$ depends on the separation radius $q_X$ of the set $X$, which can be very small for a large set of scattered data, and also on the smoothness of the kernel $\Phi$. (The smoother the kernel the smaller the least eigenvalue of $A$.) In this section, we will show how the interpolation problem can be written in the variational problem in terms of inner product in the native space. First we show that $\Phi(x, y)$ is the reproducing kernel in the native space, i.e., for any function $f$ in the native space, we have

$$f(x) = \langle f, \Phi(\cdot, x) \rangle_\phi.$$ 

To see that, notice that

$$\Phi(\cdot, x)_{\ell,k} = \hat{\phi}(\ell) Y_{\ell,k}(x)$$

and hence

$$\langle f, \Phi(\cdot, x) \rangle_\phi = \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(n,\ell)} \hat{f}_{\ell,k} \hat{\phi}(\ell) Y_{\ell,k}(x) = \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(n,\ell)} \hat{f}_{\ell,k} Y_{\ell,k}(x) = f(x).$$

Suppose $f$ is a function is the native space, and we seek the spherical basis interpolant $I_X f$, which is a linear combination of $\Phi(x_j, \cdot)$, $j = 1, \ldots, N$, so that

$$I_X f(x_j) = f(x_j),$$

then by the property of the reproducing kernel $\Phi$, let $\Phi_j(\cdot) = \Phi(x_j, \cdot)$, it is equivalent to

$$\langle I_X f, \Phi_j \rangle_\phi = \langle f, \Phi_j \rangle_\phi.$$ 

Suppose $V_X$ is the finite dimensional space spanned by $\{\Phi_j : j = 1, \ldots, N\}$, the interpolation problem using spherical basis functions can be written as the following problem

Find $I_X f \in V_X$ such that $\langle I_X f, v \rangle_\phi = \langle f, v \rangle_\phi \quad \forall v \in V_X. \quad (3.11)$
Let $A$ be the matrix with entries $A_{i,j} = \Phi(x_i, x_j) = \langle \Phi(x_i, \cdot), \Phi(x_j, \cdot) \rangle_\phi$. The problem will reduce to the problem of solving the following linear system

$$Ac = f,$$

where the entries of the vector $f$ is given as $f = [f_j]_{j=1}^N$ in which $f_j = \langle f, \Phi_j \rangle_\phi = f(x_j)$ for $j = 1, \ldots, N$.

We have the following Pythagoras’ theorem for the interpolation in the native space norm:

$$\|IXf\|_\phi^2 + \|IXf - f\|_\phi^2 = \|f\|_\phi^2. \quad (3.12)$$

From (3.12) we can deduce that the interpolation operator is bounded in the $\| \cdot \|_\phi$ norm.

## 4 Additive Schwarz preconditioners

### 4.1 Additive Schwarz operator

Additive Schwarz methods provide fast solutions to equation (3.11) by solving, at the same time, problems of smaller size. Let the space $V$ be decomposed as

$$V_X = V_0 + \ldots + V_J, \quad (4.1)$$

where $V_j$, $j = 0, \ldots, J$ are subspaces of $V_X$, and let $P_j : V_X \rightarrow V_j$, $j = 0, \ldots, J$, be projections defined by

$$\langle P_j v, w \rangle_\phi = \langle v, w \rangle_\phi \quad \forall v \in V_X, \forall w \in V_j. \quad (4.2)$$

If we define

$$P := P_0 + \ldots + P_J, \quad (4.3)$$

then the additive Schwarz method for equation (3.11) consists in solving, by an iterative method, the equation

$$Pu_X = g, \quad (4.4)$$

where the righthand side is given by $g = \sum_{j=0}^J g_j$, with $g_j \in V_j$ being solutions of

$$\langle g_j, w \rangle_\phi = \langle f, w \rangle_\phi \quad \text{for any } w \in V_j, \quad (4.5)$$

A practical method to solve (4.4) is the conjugate gradient method; the additive Schwarz method (see Section 6) can be viewed as a preconditioned conjugate gradient method.
4.2 Bound on the condition number

Bounds for $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$, the minimum and maximum eigenvalues of the additive Schwarz operator $P$ can be obtained by using the following lemma, see [21].

**Lemma 4.1** (i) Assume that there exists a constant $c_1 > 0$ such that, for any $u \in V_X$ satisfying $u = \sum_{j=0}^{J} u_j$ with $u_j \in V_j$ for $j = 0, \ldots, J$ the following inequality

$$\langle u, u \rangle_\phi \leq c_1 \sum_{j=0}^{J} \langle u_j, u_j \rangle_\phi$$

holds. Then

$$\lambda_{\max}(P) \leq c_1.$$ (ii) Assume that there exists a constant $c_2 > 0$ such that any $u \in V_X$ has a decomposition $u = \sum_{j=0}^{J} u_j$ satisfying

$$\sum_{j=0}^{J} \langle u_j, u_j \rangle_\phi \leq c_2 \langle u, u \rangle_\phi.$$ Then

$$\lambda_{\min}(P) \geq c_2^{-1}.$$ 

5 Additive Schwarz method for interpolation on the unit sphere

In this section, we will present our main theoretical result of the paper, namely the estimate for condition number of the additive Schwarz method for the interpolation problem using spherical basis functions.

Let $\alpha$ and $\beta$ be fixed numbers satisfying $0 < \alpha < \beta < \pi/2$, and let $X_0 := \{p_j : j = 1, \ldots, J\}$ be a subset of $X$ such that the cap $C(p_j, \beta)$ is wholly in $U_1$ or $U_2$ (or both) of the atlas $A$ and

$$\mathbb{S}^n = \bigcup_{j=1}^{J} C(p_j, \alpha).$$ (5.1)

Assume that the support of $\Phi(p, \cdot)$, which is a spherical cap centered at $p$, has radius $\gamma$. (In the case of SBF constructed from Wendland’s radial basis functions, $\gamma = \pi/3$).

For $j = 1, \ldots, J$, the subset $X_j$ is defined as

$$X_j := \{x_k \in X : \theta(x_k, p_j) \leq \alpha\}.$$ (5.2)
The sets $X_j$ may have different numbers of elements and may overlap each other. Because of (5.1) it is clear that $X$ is decomposed into $J$ overlapping subsets $\{X_j : j = 1, \ldots, J\}$ of discrete points such that

$$X = \bigcup_{j=1}^{J} X_j.$$ 

We define $V_j = V_{X_j}$, $j = 0, \ldots, J$, i.e. $V_j = \text{span} \{\Phi_k : x_k \in X_j\}$, so that $V_X = V_0 + \ldots + V_J$. The Schwarz operator $P$ is then defined by (4.2) and (4.3). Functions in $V_j$ have supports in $\Gamma_j$, where

$$\Gamma_j := C(p_j, \alpha + \gamma), \quad j = 1, \ldots, J.$$ 

We assume that:

**Assumption 5.1** We can partition the index set $\{1, \ldots, J\}$ into $M$ (for $1 \leq M \leq J$) sets $J_m$, for $1 \leq m \leq M$ such that if $i, j \in J_m$ and $i \neq j$ then $\Gamma_i \cap \Gamma_j = \emptyset$.

The partitioning problem mentioned in Assumption 5.1 is related to the graph colouring problem [3]. We can define an undirected graph $G = (V, E)$ in which the set of vertices $V = \{\nu_1, \ldots, \nu_J\}$ is identified with the set of caps $\Gamma_j$, and $E$ is the set of edges, where if $\Gamma_i \cap \Gamma_j \neq \emptyset$ then there is an edge between $\nu_i$ and $\nu_j$. A partition satisfying Assumption 5.1 is equivalent to a colouring of the vertices of $G$ so that adjacent vertices have different colours. The minimal number of colours needed is called the chromatic number of $G$, and is denoted by $\delta(G)$. In general, it is difficult to determine the chromatic number of a graph. However, it is easy to see that

$$\delta(G) \geq \omega(G),$$

where $\omega(G)$ is the maximal order of a complete subgraph of $G$, that is, it is the maximal number of vertices all of which are mutually connected. In terms of the caps, every point on the sphere $S^n$ lies in at most $M_1 = \omega(G)$ spherical caps $\Gamma_j$.

An upper bound of $\delta(G)$ is given in [3, Theorem 3, Chapter 5]: when $G$ is neither a complete graph nor an odd cycle, then $\delta(G) \leq \Delta(G)$, with $\Delta(G)$ being the maximal degree of $G$. In terms of our spherical caps, each cap $\Gamma_j$ intersects at most $M_2 = \Delta(G)$ other caps.

Therefore, for a given set $X_0$ and parameters $\alpha, \gamma$, we can compute the lower bound $M_1$ and the upper bound $M_2$ so that

$$1 \leq M_1 = \omega(G) \leq M \leq M_2 = \Delta(G) \leq J.$$  (5.3)
5.1 A bound for $\lambda_{\text{max}}$

**Lemma 5.2** There exists a positive constant $c$ independent of the set $X$ such that for any $u \in V_X$ satisfying $u = \sum_{j=0}^{J} u_j$ with $u_j \in V_j$ for $j = 0, \ldots, J$,

$$\langle u, u \rangle_\phi \leq c M \sum_{j=0}^{J} \langle u_j, u_j \rangle_\phi.$$  

**Proof.** Using the inequality $|a + b|^2 \leq 2(|a|^2 + |b|^2)$, we have

$$\|u\|^2_{H^r(\mathbb{R}^n)} \leq 2 \left( \|u_0\|^2_{H^r(\mathbb{R}^n)} + \left| \sum_{j=1}^{J} u_j \right|^2 \right).$$

Let $k = \lceil \tau \rceil$, from the definition of the Sobolev norm (2.8),

$$\left\| \sum_{j=1}^{J} u_j \right\|_{H^k(\mathbb{R}^n)} = \left\| \sum_{j=1}^{J} \pi_1(\chi_j u_j) \right\|_{H^k(\mathbb{R}^n)} + \left\| \sum_{j=1}^{J} \pi_2(\chi_j u_j) \right\|_{H^k(\mathbb{R}^n)}$$

(5.4)

Now, from the fact that $u_j \in V_j$ together with Assumption 5.1 we can partition the index set $\{1, \ldots, J\}$ to $M$ sets of indices $J_m$ so that if $i, j \in J_m$ then $\text{supp } u_i \cap \text{supp } u_j = \emptyset$. Then, in this proof only, let $g_j = \pi_1(\chi_j u_j)$, by using the Cauchy-Schwarz inequality, we have

$$\left\| \sum_{j=1}^{J} g_j \right\|_{H^k(\mathbb{R}^n)} = \left\| \sum_{m=1}^{M} \sum_{j \in J_m} g_j \right\|_{H^k(\mathbb{R}^n)} \leq c M \sum_{m=1}^{M} \left\| \sum_{j \in J_m} g_j \right\|_{H^k(\mathbb{R}^n)}.$$  

(5.5)

Since the supports of $g_i$ and $g_j$ are disjoint for $i, j \in J_m, i \neq j$,

$$\left\| \sum_{j \in J_m} g_j \right\|_{H^k(\mathbb{R}^n)}^2 = \sum_{j \in J_m} \|g_j\|^2_{H^k(\mathbb{R}^n)}.$$  

Thus,

$$\left\| \sum_{j=1}^{J} g_j \right\|_{H^k(\mathbb{R}^n)}^2 \leq M \sum_{m=1}^{M} \sum_{j \in J_m} \|g_j\|^2_{H^k(\mathbb{R}^n)} = M \sum_{j=1}^{J} \|g_j\|^2_{H^k(\mathbb{R}^n)}.$$  

(5.6)

Similarly, we obtain

$$\left\| \sum_{j=1}^{J} g_j \right\|_{H^{k+1}(\mathbb{R}^n)}^2 \leq M \sum_{m=1}^{M} \sum_{j \in J_m} \|g_j\|^2_{H^{k+1}(\mathbb{R}^n)} = M \sum_{j=1}^{J} \|g_j\|^2_{H^{k+1}(\mathbb{R}^n)}.$$  

(5.7)
Using the definition of the $K$-functional (2.9) and (5.6)–(5.7), we conclude that

$$\left\| \sum_{j=1}^{J} g_j \right\|_{H^r(\mathbb{R}^n)}^2 \leq M \sum_{j=1}^{J} \| g_j \|_{H^r(\mathbb{R}^n)}^2.$$

Hence, by using similar arguments for $\pi_2(\chi_2 u_j)$, we conclude

$$\left\| \sum_{j=0}^{J} u_j \right\|_{H^r(\mathbb{S}^n)}^2 \leq cM \sum_{j=1}^{J} \| u_j \|_{H^r(\mathbb{S}^n)}^2.$$

Therefore,

$$\| u \|_{H^r(\mathbb{S}^n)}^2 = \left\| \sum_{j=0}^{J} u_j \right\|_{H^r(\mathbb{S}^n)}^2 \leq cM \sum_{j=1}^{J} \| u_j \|_{H^r(\mathbb{S}^n)}^2.$$

Using the fact that $\langle u, u \rangle_{\phi} \sim \| u \|_{H^r(\mathbb{S}^n)}^2$ we obtain the result. \qed

### 5.2 A bound for $\lambda_{\text{min}}$

We proceed as in [9, Lemma 5.3], in which the bilinear form $a(\cdot, \cdot)$ is re-defined by $a(\cdot, \cdot) := \langle \cdot, \cdot \rangle_{\phi}$ and the norm of a linear operator $T$ is defined by

$$\| T \|_{\phi} = \sup \{ \| Tv \|_{\phi} : v \in V_X \text{ and } \| v \|_{\phi} \leq 1 \}.$$

**Lemma 5.3** For any $u \in V_X$ there exist $u_j \in V_j$, $j = 0, \ldots, J$, satisfying $u = \sum_{j=0}^{J} u_j$ and

$$\sum_{j=0}^{J} \langle u_j, u_j \rangle_{\phi} \leq \left( 1 + \frac{J}{1 - \| \widetilde{Q} \|_{\phi}^2} \right) \langle u, u \rangle_{\phi},$$

where $\widetilde{Q} = Q_J \cdots Q_1$, in which $Q_i$ is the orthogonal projection from $V_X$ to $V^\perp$ with respect to $\langle \cdot, \cdot \rangle_{\phi}$.

The above lemma and Lemma 4.1 yeild the following estimate for the minimum eigenvalue of $P$:

$$\lambda_{\text{min}}(P) \geq \left( 1 + \frac{J}{1 - \| \widetilde{Q} \|_{\phi}^2} \right)^{-1}.$$ (5.8)

This estimate is not an optimal bound for $\lambda_{\text{min}}(P)$, as can be seen from Table 1. In that table

$$C_1 = \| P_1 (I - \widetilde{Q})^{-1} \|_{\phi} \text{ and } C_j = \| P_j Q_{j-1} \cdots Q_1 (I - \widetilde{Q})^{-1} \|_{\phi}, j = 2, \ldots, J.$$\n
The norms of the operators were computed by using their matrix representations.
Table 1: Upper bound for $\lambda_{\min}^{-1}(P)$

<table>
<thead>
<tr>
<th>$N$</th>
<th>$q_X$</th>
<th>$\cos(\alpha)$</th>
<th>$\cos(\beta)$</th>
<th>$J$</th>
<th>$\lambda_{\min}^{-1}(P)$</th>
<th>$1 + \sum_{j=1}^{J} C_j^2$</th>
<th>$1 + J/(1 - |Q|_\phi)^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1344</td>
<td>$\pi/80$</td>
<td>0.9</td>
<td>0.06</td>
<td>42</td>
<td>270.71</td>
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<td>3700951.76</td>
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<td></td>
<td></td>
<td>0.8</td>
<td>-0.86</td>
<td>19</td>
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<td>117.</td>
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<td></td>
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<td>22.01</td>
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<td></td>
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<td>0.6</td>
<td>-0.82</td>
<td>12</td>
<td>20.41</td>
<td>28.44</td>
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<td></td>
<td></td>
<td>0.5</td>
<td>-0.81</td>
<td>12</td>
<td>0.95</td>
<td>9.37</td>
<td>30.13</td>
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<tr>
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<td>0.01</td>
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<td>785.38</td>
<td>12871223.39</td>
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<tr>
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<td></td>
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<td>124.19</td>
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<td>0.6</td>
<td>-0.69</td>
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<td>1.0</td>
<td>12.01</td>
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<td>0.5</td>
<td>-0.81</td>
<td>10</td>
<td>2.5</td>
<td>11.88</td>
<td>172.0</td>
</tr>
<tr>
<td>3458</td>
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<td>-0.58</td>
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<td>463.18</td>
<td>1428201.40</td>
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<tr>
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<td></td>
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<td>-0.56</td>
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<td>2.09</td>
<td>25.18</td>
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<tr>
<td></td>
<td></td>
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<td>-0.81</td>
<td>14</td>
<td>5.25</td>
<td>19.93</td>
<td>625.10</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.6</td>
<td>-0.85</td>
<td>11</td>
<td>10.1</td>
<td>20.98</td>
<td>1119.25</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.5</td>
<td>-0.53</td>
<td>11</td>
<td>1.99</td>
<td>10.51</td>
<td>51.87</td>
</tr>
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</table>

5.3 Main result

**Theorem 5.4** The condition number of the additive Schwarz operator $P$ is bounded by

$$\kappa(P) \leq cM \left(1 + \frac{J}{(1 - \|Q\|_\phi)^2}\right),$$

where $c$ is a constant independent of $M, J$ and the set $X$, and $\tilde{Q}$ is defined as in 5.3.

6 An overlapping additive Schwarz algorithm

Since the Earth is rotating around its own axis, and the satellite traverses from near the North pole to near the South pole then back to the North pole in an elliptical path, data along the track of satellite forms of a sequence of discrete points, see Figure 1. The projection of the satellite position onto the surface of the earth is the ground track, or subsatellite track, and this path over a period of time is a measure of the coverage of the globe from which data can be acquired. Suppose we number the scattered data following the satellite track as $\{x_1, x_2, \ldots, x_N\} =: X$. The algorithm to partition $X$ in the form $X = X_0 \cup X_1 \cup \cdots \cup X_J$ can be described as the following.

1. Select $\alpha \in (0, \pi/3)$ and $\beta \in [\alpha, \pi]$.
2. The first center is $p_1 = x_1 \in X$. 

(3) Define

\[ X_1 := \{ x_k \in X : \theta(x_k, p_1) \leq \alpha \}. \]

(4) Suppose \( X_{j-1} \), for \( j > 1 \), has been selected around center \( p_{j-1} \). The next center \( p_j \) is chosen from \( X \setminus \{ p_1, \ldots, p_{j-1} \} \) such that \( \theta(p_{j-1}, p_j) \geq \beta \) and \( \theta(p_k, p_j) \geq \alpha \) for \( k = 1, \ldots, j - 2 \).

(5) The subset \( X_j \) is defined as

\[ X_j := \{ x_k \in X : \theta(x_k, p_j) \leq \alpha \}. \]

(6) Repeat (4) and (5) until every point in \( X \) is in at least one \( X_k \).

(7) Define \( X_0 = \{ p_1, \ldots, p_J \} \).

In the following, we describe the preconditioned conjugate gradient method based on the decomposition of the scattered data \( X = X_0 \cup \ldots \cup X_J \). For \( k = 0, \ldots, J \), let \( A_k \) be the restriction of the matrix \( A \) onto each subspace \( V_k \), and let \( R_k \) be matrix that transform the basis of \( V \) to the basis for \( V_k \).

**A pseudocode.**

INPUT
Input the scattered set \( X \) on the sphere, the right-hand side \( f \), and the desired accuracy \( \epsilon \).

SETUP

(1) Partition the scattered set \( X \) into \( X_0 \cup \ldots \cup X_J \).

(2) The residual vector \( r = [f(x_j)]_{j=1}^N \).

(3) The pseudo-residual vector \( p = 0 \).

(4) The initial solution vector \( s = 0 \).

(5) Set the iteration counter \( \text{iter} = 0 \).

ITERATIVE SOLUTION

(1) while \( \| r \| > \epsilon \)

(2) for \( j=1 \) to \( J \)

(3) \( p = p + R_j^T A_j^{-1} R_j r \).
Figure 1: Global scattered Magsat satellite data

Figure 2: Domain decomposition based on satellite data
(4) \textbf{end for}

(5) \[ p = p + R_0^T A_0^{-1} R_0 r \]

(6) If \( \text{iter} > 0 \) then set \( \zeta_0 = \zeta_1 \).

(7) Set \( \zeta_1 = p \cdot r \).

(8) \( \text{iter} = \text{iter} + 1 \).

(9) If \( \text{iter} = 1 \) then define \( \mathbf{p}_1 = \mathbf{p} \) else \( \mathbf{p}_1 = \mathbf{p} + (\zeta_1 / \zeta_0) \mathbf{p}_1 \).

(10) Update the residual vector

\[ r = r - \frac{r \cdot p}{\mathbf{p}_1 \cdot A \mathbf{p}_1} A \mathbf{p}_1. \]

(11) Update the solution vector

\[ s = s + \frac{r \cdot p}{\mathbf{p}_1 \cdot A \mathbf{p}_1} \mathbf{p}_1. \]

(12) \textbf{end while}

7 \textbf{Numerical results}

In this section, we present numerical experiments on \( \mathbb{S}^2 \) based on globally scattered data extracted from a very large data set collected by NASA’s satellite MAGSAT. Given a positive real number \( q \), the set of scattered point \( X \) is extracted along the satellite track from the original data set so that the separation radius \( q_X \geq q \). The set \( X \) is constructed after a two-stage thinning process:

(1) Points are taken along the satellite track so that the geodesic distance between two successive points is \( \geq q \).

(2) Points from stage (1) are re-selected so that the separation radius \( q_X \geq q \).

The number of points and their \( q_X \) of each data set are listed in Table 2.

The local supported spherical basis functions induced by compactly supported radial basis functions introduced by Wendland [18] for \( \mathbb{R}^3 \) are used. The spherical basis interpolation \( I_X f = \sum_j \alpha_j \Phi(x_j, \cdot) \), in which the kernel \( \Phi \) is defined by

\[ \Phi(x, y) = \rho_{3,m}(\sqrt{2 - 2x \cdot y}), \]

where \( \rho_{3,m}(r) \) are given in Table 3.
Table 2: Separation radius $q_X$ of different data sets

<table>
<thead>
<tr>
<th>$N$</th>
<th>$q_X$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10443</td>
<td>$\pi/240$</td>
</tr>
<tr>
<td>13897</td>
<td>$\pi/280$</td>
</tr>
<tr>
<td>17262</td>
<td>$\pi/320$</td>
</tr>
<tr>
<td>19259</td>
<td>$\pi/340$</td>
</tr>
<tr>
<td>25631</td>
<td>$\pi/400$</td>
</tr>
<tr>
<td>49377</td>
<td>$\pi/600$</td>
</tr>
</tbody>
</table>

Table 3: Wendland’s RBFs

In the numerical experiment, the basis functions used is $\Phi(x, y) = \rho_{3,3}(\sqrt{2} - 2x \cdot y)$. The right hand side are values $f(x_j)$ with $f(x) = \exp(x_1 + x_2 + x_3)$. The stopping criteria for the conjugate gradient method is

$$\frac{||A\alpha - f||}{||f||} \leq 10^{-6}.$$

References


\[
\begin{array}{cccccccccc}
N & \cos \alpha & \cos \beta & M_1 & M_2 & J & \lambda_{\text{min}} & \lambda_{\text{max}} & \kappa(P) & \text{CPU} & \text{iter} \\
13897 & 0.98 & 0.75 & 82 & 199 & 217 & 0.006 & 19.7 & 3390.3 & 775 & 140 \\
13897 & 0.95 & 0.02 & 41 & 87 & 90 & 0.018 & 13.2 & 753.2 & 528 & 75 \\
13897 & 0.90 & -0.76 & 25 & 45 & 46 & 0.028 & 9.3 & 335.0 & 544 & 46 \\
13897 & 0.85 & -0.75 & 20 & 31 & 32 & 0.071 & 8.2 & 115.6 & 582 & 29 \\
13897 & 0.80 & -0.67 & 18 & 25 & 26 & 0.730 & 8.4 & 11.5 & 554 & 17 \\
17262 & 0.98 & 0.94 & 78 & 188 & 206 & 0.004 & 19.4 & 4365.3 & 1403 & 165 \\
17262 & 0.95 & -0.17 & 27 & 46 & 47 & 0.006 & 9.9 & 1769.9 & 1768 & 85 \\
17262 & 0.85 & -0.79 & 21 & 32 & 33 & 1.048 & 8.6 & 8.2 & 587 & 16 \\
17262 & 0.80 & -0.76 & 17 & 23 & 24 & 0.517 & 9.4 & 18.1 & 1117 & 20 \\
19259 & 0.98 & 0.77 & 80 & 192 & 211 & 0.003 & 20.1 & 7225.3 & 1873 & 173 \\
19259 & 0.95 & -0.30 & 40 & 86 & 88 & 0.006 & 12.5 & 1971.2 & 1514 & 105 \\
19259 & 0.90 & -0.82 & 26 & 46 & 47 & 0.039 & 9.6 & 246.8 & 1396 & 51 \\
19259 & 0.85 & -0.69 & 20 & 29 & 30 & 0.075 & 8.5 & 112.3 & 1668 & 37 \\
19259 & 0.80 & -0.86 & 16 & 22 & 23 & 0.054 & 8.1 & 150.6 & 2753 & 37 \\
25631 & 0.98 & 0.87 & 78 & 191 & 209 & 0.004 & 20.0 & 5361.934 & 3099 & 159 \\
25631 & 0.95 & -0.38 & 42 & 88 & 91 & 0.01 & 14.5 & 1497.203 & 2321 & 82 \\
25631 & 0.90 & -0.61 & 27 & 48 & 49 & 0.006 & 10.6 & 1836.101 & 4590 & 76 \\
25631 & 0.85 & -0.77 & 22 & 34 & 35 & 0.661 & 9.5 & 14.443 & 2218 & 19 \\
25631 & 0.80 & -0.76 & 21 & 30 & 31 & 0.300 & 10.8 & 35.905 & 5016 & 23 \\
49377 & 0.98 & 0.68 & 84 & 201 & 220 & 0.012 & 21.6 & 1817.2 & 8848 & 102 \\
49377 & 0.95 & -0.49 & 44 & 91 & 94 & 0.046 & 14.1 & 308.7 & 7153 & 47 \\
49377 & 0.90 & -0.57 & 27 & 48 & 49 & 0.082 & 10.7 & 129.7 & 12293 & 32 \\
49377 & 0.85 & -0.80 & 21 & 32 & 33 & 0.043 & 9.3 & 216.0 & 27310 & 37 \\
\end{array}
\]

Table 4: Preconditioned systems


Table 5: Unpreconditioned systems

<table>
<thead>
<tr>
<th>$N$</th>
<th>$\lambda_{\text{min}}$</th>
<th>$\lambda_{\text{max}}$</th>
<th>$\kappa(A)$</th>
<th>CPU</th>
<th>iter</th>
</tr>
</thead>
<tbody>
<tr>
<td>13897</td>
<td>0.1485E-02</td>
<td>0.3314E+03</td>
<td>0.2232E+06</td>
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<td>0.4239E+03</td>
<td>0.2328E+06</td>
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