Boundary Integral Equations on the Sphere with Radial Basis Functions: Error Analysis

T. Tran * Q. T. Le Gia * I. H. Sloan * E. P. Stephan †

Abstract

Radial basis functions are used to define approximate solutions to boundary integral equations on the unit sphere. These equations arise from the integral reformulation of the Laplace equation in the exterior of the sphere, with given Dirichlet or Neumann data, and a vanishing condition at infinity. Error estimates are proved. Numerical results supporting the theoretical results are presented.

1 Introduction

Throughout this paper we denote by $\mathbb{S}$ the unit sphere in $\mathbb{R}^3$, i.e., $\mathbb{S} := \{\boldsymbol{x} \in \mathbb{R}^3 : \|\boldsymbol{x}\| = 1\}$, and by $\mathbb{B}_e$ the exterior of the sphere, i.e., $\mathbb{B}_e := \{\boldsymbol{x} \in \mathbb{R}^3 : \|\boldsymbol{x}\| > 1\}$, where $\|\boldsymbol{x}\|$ denotes the Euclidean norm in $\mathbb{R}^3$. We consider the Laplace equation

$$\Delta U = 0 \quad \text{in } \mathbb{B}_e, \tag{1.1}$$

with either a Dirichlet boundary condition

$$U = U_D \quad \text{on } \mathbb{S}, \tag{1.2}$$

or else a Neumann boundary condition

$$\partial_\nu U = Z_N \quad \text{on } \mathbb{S}, \tag{1.3}$$

where $\partial_\nu = \partial/\partial\nu$ denotes differentiation in the direction of the outward unit normal ν, and the vanishing condition at infinity for both the Dirichlet and Neumann cases is

$$U(\boldsymbol{x}) = O(1/\|\boldsymbol{x}\|) \quad \text{as } \|\boldsymbol{x}\| \to \infty. \tag{1.4}$$

*School of Mathematics and Statistics, The University of New South Wales, Sydney 2052, Australia. Email: Thanh.Tran@unsw.edu.au, qlegia@unsw.edu.au, I.Sloan@unsw.edu.au

†Institut für Angewandte Mathematik, Leibniz Universität Hannover, Welfengarten 1, 30167 Hannover, Germany. Email: stephan@ifam.uni-hannover.de

The solutions of these problems can be represented in terms of spherical harmonics. A spherical harmonic of order l on $\mathbb{S}$ is the restriction to $\mathbb{S}$ of a homogeneous harmonic polynomial of degree l in $\mathbb{R}^3$. The space of all spherical harmonics of order l is the eigenspace of the Laplace-Beltrami operator $\Delta_{\mathbb{S}}$ corresponding to the eigenvalue $\lambda_l = -l(l+1)$. The dimension of this space being $2l+1$ (see e.g. [7, page 4]), one may choose for it an orthonormal basis $\{Y_{l,m}\}_{m=-l}^{l}$. The collection of all the spherical harmonics $Y_{l,m}$, $m = -l, \ldots, l$ and $l = 0, 1, \ldots$, forms an orthonormal basis for $L^2(\mathbb{S})$. Let (r, θ, φ) be the spherical coordinates of a point $\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$, where $r = \|\boldsymbol{x}\|$, and θ and φ are the polar and azimuthal angles, so that

$$x_1 = r \sin\theta \cos\varphi, \qquad x_2 = r \sin\theta \sin\varphi, \qquad x_3 = r\cos\theta.$$

For any function $v \in L^2(\mathbb{S})$, its associated Fourier series,

$$v = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \widehat{v}_{l,m} Y_{l,m}(\theta, \varphi), \quad \text{where} \quad \widehat{v}_{l,m} = \int_{\mathbb{S}} v(\theta, \varphi) \overline{Y_{l,m}}(\theta, \varphi) d\sigma, \tag{1.5}$$

converges in $L^2(\mathbb{S})$. Here $d\sigma$ is the element of surface area. For more details on spherical harmonics, the reader is referred to Müller's book [7].

It is well-known that if the Dirichlet data U_D has an expansion as a sum of spherical harmonics

$$U_D(\theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \widehat{(U_D)}_{l,m} Y_{l,m}(\theta, \varphi),$$

then (see [9, Theorem 2.5.1]) the Dirichlet problem (1.1), (1.2) and (1.4) has the unique solution

$$U(r, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{r^{l+1}} \widehat{(U_D)}_{l,m} Y_{l,m}(\theta, \varphi). \tag{1.6}$$

Similarly, if

$$Z_N(\theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \widehat{(Z_N)}_{l,m} Y_{l,m}(\theta, \varphi),$$

then (see [9, Theorem 2.5.2]) the Neumann problem (1.1), (1.3) and (1.4) has the unique solution

$$U(r, \theta, \varphi) = -\sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{(l+1) r^{l+1}} \widehat{(Z_N)}_{l,m} Y_{l,m}(\theta, \varphi). \tag{1.7}$$

The spherical harmonic basis functions in (1.6), (1.7) are global. In contrast, in this paper we intend to use spherical basis functions obtained from compactly supported radial basis functions, which are better able to capture local properties of the solutions. Our intent is to consider the above boundary value problems as prototypes for more complicated problems for which explicit solutions may not be available (e.g., equations for Stokes flows). We shall propose a solution process in which the boundary value problems are reformulated in terms of boundary integral equations on $\mathbb{S}$, the solutions of which are then approximated by spherical basis functions.

In the next section we introduce the integral reformulation of the boundary value problems (1.1)–(1.4) into boundary integral equations. Section 3 discusses the use of radial basis functions to define the finite dimensional subspace used in the approximation. The key result of the paper (Theorem 3.7) presents the approximation property in different Sobolev norms (including non-integral and negative order norms) of this subspace. This result is employed in Section 4 to prove the convergence of the approximation schemes. Numerical experiments are presented in Section 5 which support the theoretical results.

2 Boundary integral equations and weak formulations

For $s \in \mathbb{R}$, the Sobolev space $H^s(\mathbb{S})$ is defined as usual (see e.g. [9]) with norm and Hermitian product given by

$$\|v\|_s := \left(\sum_{l=0}^{\infty} \sum_{m=-l}^{l} (l+1)^{2s} |\widehat{v}_{l,m}|^2 \right)^{1/2} \tag{2.1}$$

and

$$\langle v, w \rangle_s := \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (l+1)^{2s} \widehat{v}_{l,m} \overline{\widehat{w}_{l,m}}.$$

We note that

$$|\langle v, w \rangle_s| \le \|v\|_s \|w\|_s \quad \forall v, w \in H^s(\mathbb{S}), \forall s \in \mathbb{R}, \tag{2.2}$$

and

$$\|v\|_{s_1} = \sup_{\substack{w \in H^{s_2}(\mathbb{S}) \\ w \neq 0}} \frac{\langle v, w \rangle_{\frac{s_1+s_2}{2}}}{\|w\|_{s_2}} \quad \forall v \in H^{s_1}(\mathbb{S}), \forall s_1, s_2 \in \mathbb{R}. \tag{2.3}$$

The single-layer potential $\mathscr{S}$ and the double-layer potential $\mathscr{D}$ are defined by

$$\mathscr{S} v(\boldsymbol{x}) = \frac{1}{4\pi} \int_{\mathbb{S}} v(\boldsymbol{y}) \frac{1}{\|\boldsymbol{x} - \boldsymbol{y}\|} \, d\sigma_{\boldsymbol{y}} \quad \text{and} \quad \mathscr{D} v(\boldsymbol{x}) = \frac{1}{4\pi} \int_{\mathbb{S}} v(\boldsymbol{y}) \frac{\partial}{\partial \nu_{\boldsymbol{y}}} \frac{1}{\|\boldsymbol{x} - \boldsymbol{y}\|} \, d\sigma_{\boldsymbol{y}},$$

for $\boldsymbol{x} \in \mathbb{B}_e$. Associated with these potentials, we define the following boundary integral operators

$$\begin{aligned}
Sv(\boldsymbol{x}) &= \frac{1}{4\pi}\int_{\mathbb{S}} v(\boldsymbol{y})\frac{1}{\|\boldsymbol{x}-\boldsymbol{y}\|}\,d\sigma_{\boldsymbol{y}}\\
Dv(\boldsymbol{x}) &= \frac{1}{4\pi}\int_{\mathbb{S}} v(\boldsymbol{y})\frac{\partial}{\partial\nu_{\boldsymbol{y}}}\frac{1}{\|\boldsymbol{x}-\boldsymbol{y}\|}\,d\sigma_{\boldsymbol{y}}\\
D^*v(\boldsymbol{x}) &= \frac{1}{4\pi}\frac{\partial}{\partial\nu_{\boldsymbol{x}}}\int_{\mathbb{S}} v(\boldsymbol{y})\frac{1}{\|\boldsymbol{x}-\boldsymbol{y}\|}\,d\sigma_{\boldsymbol{y}}\\
Nv(\boldsymbol{x}) &= \frac{1}{4\pi}\frac{\partial}{\partial\nu_{\boldsymbol{x}}}\int_{\mathbb{S}} v(\boldsymbol{y})\frac{\partial}{\partial\nu_{\boldsymbol{y}}}\frac{1}{\|\boldsymbol{x}-\boldsymbol{y}\|}\,d\sigma_{\boldsymbol{y}},
\end{aligned}$$

for $\boldsymbol{x}\in\mathbb{S}$. It is well-known that for all $s\in\mathbb{R}$, the mappings

$$\begin{aligned}
S &: H^{s-1/2}(\mathbb{S})\to H^{s+1/2}(\mathbb{S}), \qquad & N &: H^{s+1/2}(\mathbb{S})\to H^{s-1/2}(\mathbb{S}),\\
D &: H^{s+1/2}(\mathbb{S})\to H^{s+1/2}(\mathbb{S}), \qquad & D^* &: H^{s-1/2}(\mathbb{S})\to H^{s-1/2}(\mathbb{S})
\end{aligned}$$

are bounded operators. (If we replace $\mathbb{S}$ by Γ, the boundary of a Lipschitz domain in $\mathbb{R}^3$, then the above operators are bounded when $s\in(-1/2,1/2)$.) The traces and normal derivatives on $\mathbb{S}$ of $\mathscr{S}$ and $\mathscr{D}$ are given by (see e.g. [4, 9], noting that the limits are taken from the exterior of $\mathbb{S}$)

$$(\mathscr{S}v)|_{\mathbb{S}} = Sv \quad\text{and}\quad \partial_\nu(\mathscr{S}v) = -\frac{1}{2}v + D^*v, \quad \text{if } v\in H^{-1/2}(\mathbb{S}),$$

and

$$(\mathscr{D}v)|_{\mathbb{S}} = \frac{1}{2}v + Dv \quad\text{and}\quad \partial_\nu(\mathscr{D}v) = Nv \quad \text{if } v\in H^{1/2}(\mathbb{S}).$$

If $U\in H^1_{\mathrm{loc}}(\mathbb{B}_e)$ satisfies (1.1) and (1.4), then using the single-layer and double-layer potentials, and Green's theorem we can represent U as (see [6, Theorems 7.12 & 8.9])

$$U = \mathscr{D}(U|_{\mathbb{S}}) - \mathscr{S}(\partial_\nu U) \quad \text{in } \mathbb{B}_e, \tag{2.4}$$

allowing us to compute U from a knowledge of both $U|_{\mathbb{S}}$ and $\partial_\nu U$. In fact, by taking the trace on both sides of (2.4) we obtain, after rearranging the equation,

$$S(\partial_\nu U) = -\frac{1}{2}U|_{\mathbb{S}} + D(U|_{\mathbb{S}}) \quad \text{on } \mathbb{S}.$$

Similarly, by taking the normal derivative of both sides of (2.4) we find

$$N(U|_{\mathbb{S}}) = \frac{1}{2}\partial_\nu U + D^*(\partial_\nu U) \quad \text{on } \mathbb{S}.$$

Therefore, the Dirichlet problem (1.1), (1.2) and (1.4) is equivalent to

$$Sz = f \quad \text{on } \mathbb{S}, \quad \text{where } f = -\frac{1}{2}U_D + DU_D, \tag{2.5}$$

and the Neumann problem (1.1), (1.3) and (1.4) is equivalent to

$$Nu = g \quad \text{on } \mathbb{S}, \quad \text{where } g = \frac{1}{2} Z_N + D^* Z_N. \tag{2.6}$$

Due to (2.4), the solution U of the Dirichlet problem can be computed from the solution z of (2.5) by

$$U = \mathscr{D} U_D - \mathscr{S} z,$$

and the solution of the Neumann problem can be computed from the solution u of (2.6) by

$$U = \mathscr{D} u - \mathscr{S} Z_N.$$

Equation (2.5) is a weakly singular integral equation and equation (2.6) is a hypersingular integral equation. In the following we will design efficient algorithms to solve these equations. We note that S and N are pseudo-differential operators of order -1 and 1, respectively. They have the following representations in terms of spherical harmonics (see [9, page 122]):

$$Sv = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \widehat{v}_{l,m} Y_{l,m}, \tag{2.7}$$

$$Nv = -\sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{l(l+1)}{2l+1} \widehat{v}_{l,m} Y_{l,m}. \tag{2.8}$$

In the remaining of this section we shall define weak solutions to (2.5) and (2.6). It is well-known [6, 9] that $S : H^{-1/2}(\mathbb{S}) \to H^{1/2}(\mathbb{S})$ and $N : H^{1/2}(\mathbb{S})/\mathbb{R} \to H^{-1/2}(\mathbb{S})$ are bijective, implying that (2.5) has a unique solution for all $f \in H^{1/2}(\mathbb{S})$, and (2.6) has a unique solution up to a constant for all $g \in H^{-1/2}(\mathbb{S})$. Defining the bilinear forms

$$a_S(v, w) := \langle Sv, w \rangle_0 \quad \forall v, w \in H^{-1/2}(\mathbb{S})$$

and

$$a_N(v, w) := -\langle Nv, w \rangle_0 \quad \forall v, w \in H^{1/2}(\mathbb{S}),$$

we seek weak solutions to equations (2.5) and (2.6) respectively as follows:

$$z \in H^{-1/2}(\mathbb{S}) : a_S(z, v) = \langle f, v \rangle_0 \quad \forall v \in H^{-1/2}(\mathbb{S}), \tag{2.9}$$

and

$$u \in H^{1/2}(\mathbb{S}) : \int_{\mathbb{S}} u(\boldsymbol{x}) \, d\sigma_{\boldsymbol{x}} = 0 \quad \text{and} \quad a_N(u, v) = -\langle g, v \rangle_0 \quad \forall v \in H^{1/2}(\mathbb{S}). \tag{2.10}$$

We note from (2.7) and (2.8) that

$$a_S(v, v) \simeq \|v\|_{-1/2}^2 \quad \forall v \in H^{-1/2}(\mathbb{S}) \quad \text{and} \quad a_N(v, v) \simeq \|v\|_{1/2}^2 \quad \forall v \in H^{1/2}(\mathbb{S})/\mathbb{R}. \tag{2.11}$$

In the following we shall approximate the solutions of the above equations by using spherical basis functions. These functions are defined via positive definite kernels.

3 Approximation by spherical basis functions

The finite dimensional subspaces that we shall use in our approximation are defined by positive definite kernels on $\mathbb{S}$ and spherical basis functions.

3.1 Positive definite kernels and native spaces

A continuous function $\Phi : \mathbb{S} \times \mathbb{S} \to \mathbb{C}$ is called a *positive definite kernel* on $\mathbb{S}$ if it satisfies

(i) $\Phi(\boldsymbol{x}, \boldsymbol{y}) = \overline{\Phi(\boldsymbol{y}, \boldsymbol{x})}$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{S}$;

(ii) for every set of distinct points $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_M\}$ on $\mathbb{S}$, the $M \times M$ matrix A with entries $A_{i,j} = \Phi(\boldsymbol{x}_i, \boldsymbol{x}_j)$ is positive semi-definite.

If the matrix A is positive definite then Φ is called a *strictly positive definite* kernel; see [12, 16].

We shall define the kernel Φ in terms of a univariate function $\phi : [-1, 1] \to \mathbb{R}$,

$$\Phi(\boldsymbol{x}, \boldsymbol{y}) = \phi(\boldsymbol{x} \cdot \boldsymbol{y}) \quad \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{S}.$$

If ϕ has a series expansion in terms of Legendre polynomials P_l,

$$\phi(t) = \frac{1}{4\pi} \sum_{l=0}^{\infty} (2l+1) \widehat{\phi}(l) P_l(t), \tag{3.1}$$

where

$$\widehat{\phi}(l) = 2\pi \int_{-1}^{1} \phi(t) P_l(t) \, \mathrm{d}t, \tag{3.2}$$

then due to the addition formula [7, 9]

$$\sum_{m=-l}^{l} Y_{l,m}(\boldsymbol{x}) \overline{Y_{l,m}(\boldsymbol{y})} = \frac{2l+1}{4\pi} P_l(\boldsymbol{x} \cdot \boldsymbol{y}) \quad \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{S}, \tag{3.3}$$

the kernel Φ can be represented as

$$\Phi(\boldsymbol{x}, \boldsymbol{y}) = \sum_{l=0}^{\infty} \widehat{\phi}(l) \sum_{m=-l}^{l} Y_{l,m}(\boldsymbol{x}) \overline{Y_{l,m}(\boldsymbol{y})}. \tag{3.4}$$

This kernel is called a *zonal* kernel. In [2], a complete characterisation of strictly positive definite kernels is established: the kernel Φ is strictly positive definite if and only if $\widehat{\phi}(l) \geq 0$ for all $l \geq 0$, and $\widehat{\phi}(l) > 0$ for infinitely many even values of l and infinitely many odd values of l; see also [12] and [16]. In the following we shall assume that $\widehat{\phi}(l) > 0$ for all $l \geq 0$.

The native space associated with ϕ is defined by

$$\mathcal{N}_\phi := \{v \in \mathcal{D}'(\mathbb{S}) : \|v\|_\phi^2 = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{|\widehat{v}_{l,m}|^2}{\widehat{\phi}(l)} < \infty\},$$

where $\mathcal{D}'(\mathbb{S})$ is the space of distributions defined on $\mathbb{S}$. This space is equipped with an inner product and a norm defined by

$$\langle v, w \rangle_\phi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{\widehat{v}_{l,m}\overline{\widehat{w}_{l,m}}}{\widehat{\phi}(l)} \quad \text{and} \quad \|v\|_\phi = \left(\sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{|\widehat{v}_{l,m}|^2}{\widehat{\phi}(l)} \right)^{1/2}.$$

If the coefficients $\widehat{\phi}(l)$ for $l = 0, 1, \dots$ satisfy

$$c_1(l+1)^{-2\tau} \le \widehat{\phi}(l) \le c_2(l+1)^{-2\tau} \tag{3.5}$$

for some positive constants c_1 and c_2, and some $\tau \in \mathbb{R}$, then the native space $\mathcal{N}_\phi$ can be identified with the Sobolev space $H^\tau(\mathbb{S})$, and the corresponding norms are equivalent. In particular, if $\tau > 1$ then the series (3.1) converges pointwise and $\mathcal{N}_\phi \subset C(\mathbb{S})$, which is essentially the Sobolev embedding theorem.

3.2 Spherical basis functions and approximation properties

Let $X = \{\boldsymbol{x}_1, \dots, \boldsymbol{x}_M\}$ be a set of data points on the sphere. Two important parameters characterising the set X are the *mesh norm* h_X and *separation radius* q_X, defined by

$$h_X := \sup_{\boldsymbol{y} \in \mathbb{S}} \min_{1 \le i \le M} \theta(\boldsymbol{x}_i, \boldsymbol{y}) \quad \text{and} \quad q_X := \frac{1}{2} \min_{i \ne j} \theta(\boldsymbol{x}_i, \boldsymbol{x}_j),$$

where $\theta(\boldsymbol{x}, \boldsymbol{y}) := \cos^{-1}(\boldsymbol{x} \cdot \boldsymbol{y})$. The *spherical basis functions* Φ_i, $i = 1, \dots, M$, associated with X and the kernel Φ are defined by

$$\Phi_i(\boldsymbol{x}) := \Phi(\boldsymbol{x}, \boldsymbol{x}_i) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \widehat{\phi}(l) \overline{Y_{l,m}(\boldsymbol{x}_i)} Y_{l,m}(\boldsymbol{x}). \tag{3.6}$$

We note that if (3.5) holds then $\Phi_i \in H^s(\mathbb{S})$ for all s satisfying $s < 2\tau - 1$.

Let

$$V_X^\phi := \operatorname{span}\{\Phi_1, \dots, \Phi_M\}. \tag{3.7}$$

In the following we will assume that (3.5) holds for some $\tau > 1$ so that $V_X^\phi \subset \mathcal{N}_\phi = H^\tau(\mathbb{S}) \subset C(\mathbb{S})$. We now study the approximation property of V_X^ϕ as a subspace of Sobolev spaces. The following lemma shows the boundedness of the interpolation operator in the native space.

Lemma 3.1 *The interpolation operator $I_X : C(\mathbb{S}) \to V_X^\phi$ defined by*

$$I_X v(\boldsymbol{x}_j) = v(\boldsymbol{x}_j), \quad j = 1, \dots, M, \quad v \in C(\mathbb{S}), \tag{3.8}$$

is well-defined, and is a bounded operator in $\mathcal{N}_\phi$. In fact, this operator is the $\mathcal{N}_\phi$-orthogonal projection from $\mathcal{N}_\phi$ onto V_X^ϕ.

Proof. Finding $I_X v$ satisfying (3.8) entails, on writing $I_X v = \sum_{i=0}^M c_i \Phi_i$, solving the system

$$\sum_{i=1}^M c_i \Phi_i(\boldsymbol{x}_j) = v(\boldsymbol{x}_j), \quad j = 1, \dots, M.$$

This has a unique solution due to the positive definiteness of the matrix $(\Phi_i(\boldsymbol{x}_j))_{i,j=1}^M$. Moreover, for any continuous function v there holds

$$\langle v, \Phi_j \rangle_\phi = \sum_{l=0}^\infty \sum_{m=-l}^l \frac{\widehat{v}_{l,m} \widehat{\phi}(l) Y_{l,m}(\boldsymbol{x}_j)}{\widehat{\phi}(l)} = v(\boldsymbol{x}_j),$$

so that (3.8) is equivalent to

$$\langle I_X v, \Phi_j \rangle_\phi = \langle v, \Phi_j \rangle_\phi, \quad j = 1, \dots, M,$$

which implies that I_X is a projection. Hence

$$\|I_X v - v\|_\phi^2 = \langle I_X v - v, I_X v - v \rangle_\phi = \langle I_X v - v, -v \rangle_\phi \le \|I_X v - v\|_\phi \|v\|_\phi,$$

implying

$$\|I_X v - v\|_\phi \le \|v\|_\phi \quad \forall v \in \mathcal{N}_\phi. \tag{3.9}$$

This proves that I_X is a bounded operator in $\mathcal{N}_\phi$. □

The following proposition is a consequence of a non-trivial result proved in [5].

Proposition 3.2 *For any $\sigma > 1$, if $w \in H^\sigma(\mathbb{S})$ satisfies $w|_X = 0$ for some set $X := \{\boldsymbol{x}_1, \dots, \boldsymbol{x}_M\} \subset \mathbb{S}$ then the following estimate holds*

$$\|w\|_t \le c h_X^{\sigma - t} \|w\|_\sigma, \quad 0 \le t \le \sigma,$$

where c may depend on t.

Proof. It is proved in [5, Theorem 3.3] that $w|_X = 0$ implies

$$\|w\|_0 \le c h_X^\sigma \|w\|_\sigma.$$

This inequality together with $\|w\|_\sigma \le \|w\|_\sigma$ and a standard interpolation argument yields the required estimate. (For this interpolation argument, the reader is referred to e.g. [1, 4, 6]. A simple proof for the case of Sobolev spaces on the sphere can be found in [3].) □

A direct consequence of the above proposition and (3.9) is the following error estimate when $v \in H^\tau(\mathbb{S})$ is approximated by its interpolant $I_X v$.

Corollary 3.3 *Assume that* (3.5) *holds for some* $\tau > 1$. *If* $v \in H^\tau(\mathbb{S})$ *then for any* $t \in [0, \tau]$ *there holds*

$$\|I_X v - v\|_t \le c h_X^{\tau - t} \|v\|_\tau.$$

Proof. This is a result of Proposition 3.2 (taking $w = I_X v - v$), of (3.9) and the equivalence of the native space norm and H^τ-norm. □

When v is smoother, the error bound can be improved using the technique developed in [11] and [15], which is modified in [13] for hybrid approximation using radial basis functions and polynomials. The crucial tool is the isomorphism defined in the following lemma.

Lemma 3.4 *Assume that* (3.5) *holds for some* $\tau > 0$. *For* $s \in \mathbb{R}$, *let* $T : H^{s-\tau}(\mathbb{S}) \to H^{s+\tau}(\mathbb{S})$ *be defined by*

$$T\psi := \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \widehat{\phi}(l) \widehat{\psi}_{l,m} Y_{l,m}.$$

Then

(i) T *is an isomorphism satisfying*

$$\|T\psi\|_{s+\tau} \simeq \|\psi\|_{s-\tau} \quad \forall \psi \in H^{s-\tau}(\mathbb{S}); \tag{3.10}$$

(ii) For any $\psi \in H^{-\tau}(\mathbb{S})$ *and* $\xi \in \mathcal{N}_\phi$ *there holds*

$$\langle T\psi, \xi \rangle_\phi = \langle \psi, \xi \rangle_0 , \tag{3.11}$$

i.e., T *is the adjoint of the embedding operator of the native space* $\mathcal{N}_\phi$ *into* $H^0(\mathbb{S})$*;*

Proof. It is clear that, under the assumption (3.5), T is well-defined for all $\psi \in H^{s-\tau}(\mathbb{S})$, and has an inverse given by

$$T^{-1}\eta = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{\widehat{\eta}_{l,m}}{\widehat{\phi}(l)} Y_{l,m}.$$

Properties (3.10)–(3.11) follow from (3.5), the definitions of various inner products and Sobolev norms, noting that $\widehat{(T\psi)}_{l,m} = \widehat{\phi}(l) \widehat{\psi}_{l,m}$. □

With the help of the above lemma, the error bound for the interpolation can be improved.

Proposition 3.5 *Assume that* (3.5) *holds for some* $\tau > 1$. *For any* $s, t \in \mathbb{R}$ *satisfying* $0 \le t \le \tau \le s \le 2\tau$, *if* $v \in H^s(\mathbb{S})$ *then the following estimate holds*

$$\|I_X v - v\|_t \le C h_X^{\min\{s-t,\, 2(\tau-t)\}} \|v\|_s.$$

Proof. Consider first the case when $v \in H^{2\tau}(\mathbb{S})$. By Lemma 3.4, we can write v as $v = T\psi$ for some $\psi \in H^0(\mathbb{S})$. It follows from Proposition 3.2 and the equivalence of the H^τ-norm and $\|\cdot\|_\phi$-norm that

$$\|I_X v - v\|_0^2 \le c h_X^{2\tau} \|I_X v - v\|_\phi^2 = c h_X^{2\tau} \langle I_X v - v, I_X v - v \rangle_\phi .$$

Using the orthogonality property of the projection I_X we have

$$\langle I_X v - v, I_X v - v \rangle_\phi = \langle v - I_X v, v \rangle_\phi ,$$

and therefore, with the help of (3.10) and (3.11) (taking $s = \tau$),

$$\begin{aligned} \|I_X v - v\|_0^2 &\le c h_X^{2\tau} \langle v - I_X v, T\psi \rangle_\phi = c h_X^{2\tau} \langle v - I_X v, \psi \rangle_0 \\ &\le c h_X^{2\tau} \|I_X v - v\|_0 \|\psi\|_0 \le c h_X^{2\tau} \|I_X v - v\|_0 \|v\|_{2\tau}, \end{aligned}$$

implying

$$\|I_X v - v\|_0 \le c h_X^{2\tau} \|v\|_{2\tau}. \tag{3.12}$$

Now assume that $v \in H^s(\mathbb{S})$, $\tau \le s \le 2\tau$. We can write s as

$$s = (1-\theta)\tau + 2\theta\tau \tag{3.13}$$

for some $\theta \in [0,1]$. More precisely, $\theta = (s-\tau)/\tau$. If $0 \le t \le 2\tau - s$ we define $t' = t/(1-\theta)$ to have

$$0 \le t' \le \tau \quad \text{and} \quad t = (1-\theta)t' + 0 \times \theta. \tag{3.14}$$

Corollary 3.3 gives

$$\|I_X v - v\|_{t'} \le c h_X^{\tau - t'} \|v\|_\tau.$$

By using interpolation, we deduce from the above estimate and (3.12), noting (3.13) and (3.14),

$$\begin{aligned} \|I_X v - v\|_t &\le c \left(h_X^{2\tau}\right)^\theta \left(h_X^{\tau - t'}\right)^{1-\theta} \|v\|_s = c h_X^{\tau + \theta\tau - t} \|v\|_s \\ &= c h_X^{s-t} \|v\|_s. \end{aligned} \tag{3.15}$$

We note that $s - t \le 2(\tau - t)$ for $t \in [0, 2\tau - s]$.

If $2\tau - s < t \le \tau$, we write

$$t = (1-\theta')\tau + \theta'(2\tau - s),$$

where $\theta' = (\tau - t)/(s - \tau)$. By interpolation there holds

$$\|I_X v - v\|_t \le \|I_X v - v\|_{2\tau - s}^{\theta'} \|I_X v - v\|_\tau^{1-\theta'}.$$

It follows from Corollary 3.3, noting $\|v\|_\tau \le \|v\|_s$, that

$$\|I_X v - v\|_\tau \le c\|v\|_s.$$

Therefore, with the help of (3.15) (with $t = 2\tau - s$) we infer

$$\|I_X v - v\|_t \le c h_X^{2\theta'(s-\tau)} \|v\|_s = c h_X^{2(\tau-t)} \|v\|_s.$$

We finish the proof by noting that $2(\tau - t) < s - t$ for $2\tau - s < t \le \tau$. □

Remark 3.6 *We note that the estimate proved in the above proposition is only optimal when* $0 \le t \le 2\tau - s$ *and* $\tau \le s \le 2\tau$.

The convergence analysis for the approximate solutions to (2.9) and (2.10) requires the approximation property of V_X^ϕ in a larger range of Sobolev norms than those considered in the above proposition, namely error bounds are sought when the approximated function v belongs to $H^s(\mathbb{S})$ for $s < \tau$ (e.g. $s = 1/2$ in the case of (2.10) and $s = -1/2$ in the case of (2.9)). The following theorem states our most general approximation result for V_X^ϕ.

Theorem 3.7 *Assume that* (3.5) *holds for some* $\tau > 1$. *For any* $s, t \in \mathbb{R}$ *satisfying* $t \le \tau$ *and* $t \le s \le 2\tau$, *if* $v \in H^s(\mathbb{S})$ *then there exists* $\eta \in V_X^\phi$ *such that*

$$\|v - \eta\|_t \le C h_X^\mu \|v\|_s, \tag{3.16}$$

where $\mu = \min\{s - t, 2(\tau - t), 2\tau + |s|\}$, *and where the constant* C *is independent of* v *and* h_X.

Proof. We prove the result by considering different cases of values of s and t.
Case 1: $0 \le t \le \tau \le s \le 2\tau$

We note that in this case $\mu = \min\{s - t, 2(\tau - t)\}$, and the result is in fact Proposition 3.5.

In the following cases, it is easy to see that $s - t \le 2(\tau - t)$ and thus $\mu = \min\{s - t, 2\tau + |s|\}$.
Case 2: $0 \le t \le s < \tau$

Let $L = \lfloor \frac{1}{h_X} \rfloor$. We define for each $v \in H^s(\mathbb{S})$ a polynomial of degree L by

$$P_L v = \sum_{l=0}^{L} \sum_{m=-l}^{l} \widehat{v}_{l,m} Y_{l,m}.$$

With $\eta = I_X P_L v$ we have

$$\begin{aligned}\|v-\eta\|_t^2 &\le 2\|v-P_Lv\|_t^2 + 2\|P_Lv - I_XP_Lv\|_t^2 \\ &\le 2\sum_{l=L+1}^{\infty}\sum_{m=-l}^{l}(l+1)^{2t}|\widehat{v}_{l,m}|^2 + ch_X^{2(\tau-t)}\|P_Lv\|_\tau^2 \\ &= 2\sum_{l=L+1}^{\infty}\sum_{m=-l}^{l}(l+1)^{2(t-s)}(l+1)^{2s}|\widehat{v}_{l,m}|^2 \\ &+ ch_X^{2(\tau-t)}\sum_{l=1}^{L}\sum_{m=-l}^{l}(l+1)^{2(\tau-s)}(l+1)^{2s}|\widehat{v}_{l,m}|^2 \\ &\le c(L+1)^{2(t-s)}\|v\|_s^2 + cL^{2(\tau-s)}h_X^{2(\tau-t)}\|v\|_s^2,\end{aligned}$$

where in the second step we have used the result given in Case 1. Here c is a generic constant which may take different values at different occurrences. Since $L \le h_X^{-1}$ and $(L+1)^{-1} \le h_X$, we deduce (3.16) with $\mu = s - t$.

Case 3: $t < 0 \le s \le 2\tau$

Let $P_0 : H^0(\mathbb{S}) \to V_X^\phi$ be the projection defined by

$$P_0 v \in V_X^\phi : \quad \langle P_0 v, w\rangle_0 = \langle v, w\rangle_0 \quad \forall w \in V_X^\phi.$$

Then $P_0 v$ is the best approximation of v from V_X^ϕ in the H^0-norm, thus the results proved in Cases 1 and 2 give

$$\|P_0 v - v\|_0 \le ch_X^\sigma \|v\|_\sigma \quad \forall v \in H^\sigma(\mathbb{S}),\ 0 \le \sigma \le 2\tau.$$

If $-2\tau \le t < 0$ we have

$$\begin{aligned}\|P_0v - v\|_t &= \sup_{w\in H^{-t}(\mathbb{S})}\frac{\langle v - P_0v, w\rangle_0}{\|w\|_{-t}} = \sup_{w\in H^{-t}(\mathbb{S})}\frac{\langle v-P_0v, w-P_0w\rangle_0}{\|w\|_{-t}} \\ &\le \|v-P_0v\|_0 \sup_{w\in H^{-t}(\mathbb{S})}\frac{\|w-P_0w\|_0}{\|w\|_{-t}} \\ &\le ch_X^s\|v\|_s h_X^{-t}.\end{aligned}$$

If $t < -2\tau$ then

$$\|P_0v - v\|_t \le \|P_0v - v\|_{-2\tau} \le ch_X^{2\tau+s}\|v\|_s.$$

Case 4: $t \le s < 0$

Let $P_s : H^s(\mathbb{S}) \to V_X^\phi$ be defined by

$$\langle P_s v, w\rangle_s = \langle v, w\rangle_s \quad \forall w \in V_X^\phi.$$

It is easily seen that

$$\|P_s v - v\|_s \le \|v\|_s. \tag{3.17}$$

If $2(s-\tau) \le t \le 2s$ so that $0 \le 2s-t \le 2\tau$ then for any $w \in H^{2s-t}(\mathbb{S})$ the result proved in Case 3 ensures the existence of $\eta_w \in V_X^\phi$ satisfying

$$\|w - \eta_w\|_s \le c h_X^{s-t} \|w\|_{2s-t}.$$

Using (2.2) and (2.3) we then deduce

$$\begin{aligned} \|P_s v - v\|_t &= \sup_{w \in H^{2s-t}(\mathbb{S})} \frac{\langle P_s v - v, w\rangle_s}{\|w\|_{2s-t}} = \sup_{w \in H^{2s-t}(\mathbb{S})} \frac{\langle P_s v - v, w - \eta_w\rangle_s}{\|w\|_{2s-t}} \\ &\le c h_X^{s-t} \|P_s v - v\|_s \le c h_X^{s-t} \|v\|_s. \end{aligned}$$

In particular, we have

$$\|P_s v - v\|_{2s} \le c h_X^{-s} \|v\|_s.$$

The result for $2s < t \le s$ is obtained by interpolation, noting (3.17).

If $t < 2(s-\tau)$ then

$$\|P_s v - v\|_t \le \|P_s v - v\|_{2(s-\tau)} \le c h_X^{2\tau - s} \|v\|_s.$$

The theorem is proved. □

The next section details how approximate solutions to (2.9) and (2.10) are sought in finite dimensional spaces of the type defined in this section.

4 Galerkin approximations

The finite dimensional space V_X^ϕ defined in (3.7) depends on the univariate function ϕ. In this section we shall choose appropriate functions ϕ for each of the two equations (2.9) and (2.10).

4.1 The hypersingular integral equation

For the approximation of (2.10) we use radial basis functions suggested by Wendland [15, page 128]. First we define a smoothing operator I on the space $C_K[0,\infty)$ of continuous functions in $[0,\infty)$ with compact supports by

$$I : C_K[0,\infty) \to C_K[0,\infty), \quad Iv(r) = \int_r^\infty s v(s)\,\mathrm{d}s, \quad r \ge 0.$$

For any non-negative integer m, let

$$\tilde{\rho}_m(r) = \begin{cases} (1-r)^{m+2}, & 0 < r \le 1, \\ 0, & r > 1, \end{cases}$$

and

$$\rho_m(r) = I^m \tilde{\rho}_m(r), \quad r \ge 0.$$

We define

$$\phi^{(N)}(t) = \rho_m(\sqrt{2-2t}), \quad t \in [-1,1], \tag{4.1}$$

and denote by $\Phi_i^{(N)}$, $i = 1, \ldots, M$, the corresponding spherical basis functions; see (3.6). Here the superscript N indicates that the functions are specifically chosen for equation (2.10) arising from the Neumann problem. We suppress the dependence on m in the notation of $\phi^{(N)}$ and $\Phi_i^{(N)}$ because m will be chosen once and for all during the whole solution process. The functions $\Phi_i^{(N)}$, $i = 1, \ldots, M$, are locally supported radial basis functions. Strict positive definiteness is proved in [15, Theorem 9.13]. It is proved in [8, Proposition 4.6] that $\widehat{\phi}^{(N)}(l)$ satisfies (3.5) with

$$\tau^{(N)} = m + 3/2. \tag{4.2}$$

In Figure 1 we plotted $l^{2m+3}\widehat{\phi}^{(N)}(l)$ to observe the asymptotic behaviour of $\widehat{\phi}^{(N)}(l)$ for $m = 0, 1, 2, 3$, with $\widehat{\phi}^{(N)}(l)$ computed by the MATLAB function `quadl` which uses an adaptive Lobatto quadrature.

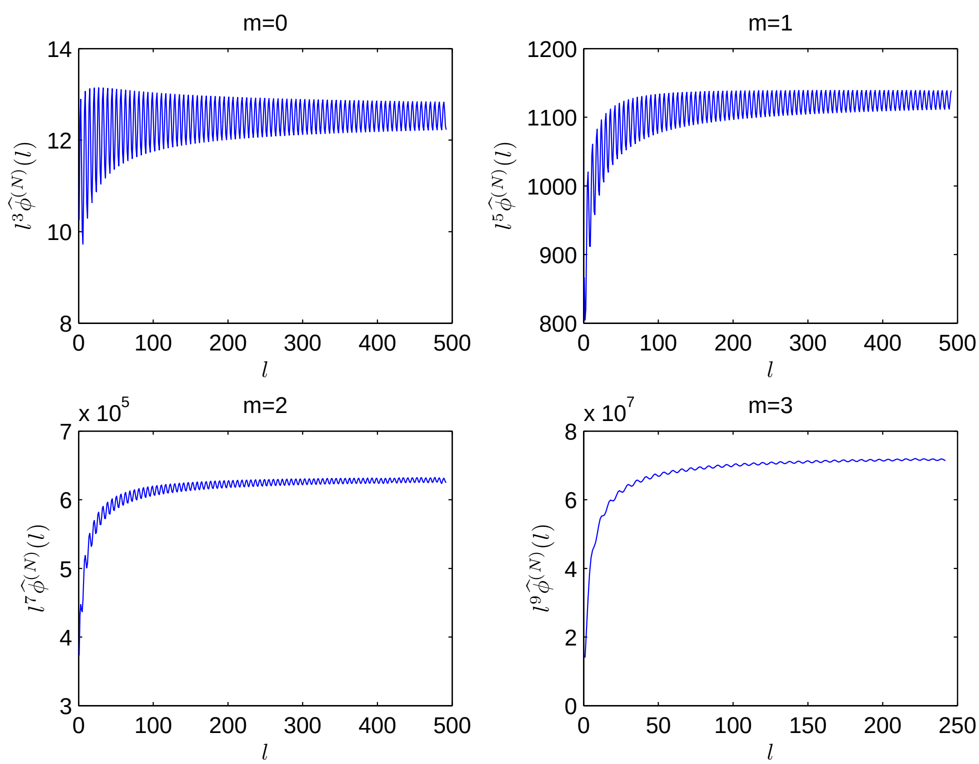

Figure 1: Asymptotic behaviour of $\widehat{\phi}^{(N)}(l)$

For given $X = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_M\} \subset \mathbb{S}$, let $V_N := V_X^{\phi^{(N)}}$. We will solve (2.10) approximately by solving instead

$$u_X \in V_N : \quad \int_{\mathbb{S}} u_X(\boldsymbol{x})\, d\sigma_{\boldsymbol{x}} = 0 \quad \text{and} \quad a_N(u_X, v_X) = -\langle g, v_X\rangle_0 \quad \forall v_X \in V_N. \tag{4.3}$$

The conformity condition $V_N \subset H^{1/2}(\mathbb{S})$ requires $\tau^{(N)} > 3/4$, and the approximation property (Theorem 3.7) requires $\tau^{(N)} > 1$. Both conditions hold for $m = 0, 1, 2, \ldots$.

By using (2.8) and (3.3), we obtain the following formula to compute the entries of the stiffness matrix resulting from equation (4.3):

$$\begin{aligned} a_N(\Phi_i^{(N)}, \Phi_j^{(N)}) &= \sum_{l=0}^{\infty} \frac{l(l+1)}{2l+1} |\widehat{\phi^{(N)}}(l)|^2 \sum_{m=-l}^{l} \overline{Y_{l,m}(\boldsymbol{x}_i)} Y_{l,m}(\boldsymbol{x}_j) \\ &= \frac{1}{4\pi} \sum_{l=0}^{\infty} l(l+1) |\widehat{\phi^{(N)}}(l)|^2 P_l(\boldsymbol{x}_i \cdot \boldsymbol{x}_j). \end{aligned} \tag{4.4}$$

The right-hand side in (4.3) is computed by using (2.6), noting $D^* = -S/2$ (see [9, page 122]), as follows:

$$\begin{aligned} \left\langle g, \Phi_i^{(N)} \right\rangle_0 &= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(\frac{1}{2} - \frac{1}{2(2l+1)} \right) \widehat{(Z_N)}_{l,m} \widehat{\phi}^{(N)}(l) \overline{Y_{l,m}(\boldsymbol{x}_i)} \\ &= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{l}{2l+1} \widehat{(Z_N)}_{l,m} \widehat{\phi}^{(N)}(l) \overline{Y_{l,m}(\boldsymbol{x}_i)}. \end{aligned} \tag{4.5}$$

Theorem 3.7 yields the following *a priori* error estimate.

Theorem 4.1 *Let $\phi^{(N)}$ be defined by* (4.1) *for some non-negative integer m, and let $\tau^{(N)} = m + 3/2$. If u is the solution to* (2.10) *satisfying $u \in H^s(\mathbb{S})$, $1/2 \le s \le 2\tau^{(N)}$, and u_X the solution to* (4.3) *then*

$$\|u - u_X\|_{1/2} \le C h_X^{\min\{s-1/2,\, 2\tau^{(N)}-1\}} \|u\|_s.$$

Proof. The condition

$$\int_{\mathbb{S}} u(\boldsymbol{x})\, d\sigma_{\boldsymbol{x}} = \int_{\mathbb{S}} u_X(\boldsymbol{x})\, d\sigma_{\boldsymbol{x}} = 0$$

yields $\widehat{u}_{0,0} = \widehat{(u_X)}_{0,0} = 0$, implying (together with (2.11))

$$\begin{aligned} \|u - u_X\|_{1/2}^2 &\simeq a_N(u - u_X, u - u_X) = a_N(u - u_X, u - w_X) \\ &\le c\|u - u_X\|_{1/2} \|u - w_X\|_{1/2} \end{aligned}$$

for all $w_X \in V_N$. This implies

$$\|u - u_X\|_{1/2} \le c \inf_{w_X \in V_N} \|u - w_X\|_{1/2},$$

and the required estimate is now a consequence of Theorem 3.7. □

4.2 The weakly singular integral equation

The same family of univariate functions defined in (4.1) can be used to define the finite dimensional subspace in the approximation of (2.9). However, for the purpose of preconditioning to be studied in a future paper, we choose to relate $\phi^{(S)}$ (the univariate function used to define the spherical basis functions for this case) to $\phi^{(N)}$ by (cf. (3.1))

$$\phi^{(S)}(t) = \frac{1}{4\pi}\sum_{l=0}^{\infty}(2l+1)(l+1)\widehat{\phi}^{(N)}(l)P_l(t), \tag{4.6}$$

and denote by $\Phi_i^{(S)}$, $i = 1, \ldots, M$, the corresponding spherical basis functions. It is clear that $\widehat{\phi}^{(S)}(l)$ satisfies (3.5) with

$$\tau^{(S)} = \tau^{(N)} - 1/2 = m + 1;$$

see (4.2).

Letting

$$V_S := V_X^{\phi^{(S)}} = \text{span}\{\Phi_1^{(S)}, \ldots, \Phi_M^{(S)}\},$$

we approximate the solution z of (2.9) by

$$z_X \in V_S : \quad a_S(z_X, v_X) = \langle f, v_X\rangle \quad \forall v_X \in V_S. \tag{4.7}$$

The resulting stiffness matrix has entries given as (cf. (4.4))

$$\begin{aligned} a_S(\Phi_i^{(S)}, \Phi_j^{(S)}) &= \sum_{l=0}^{\infty}\frac{(l+1)^2}{2l+1}|\widehat{\phi^{(N)}}(l)|^2\sum_{m=-l}^{l}\overline{Y_{l,m}(\boldsymbol{x}_i)}Y_{l,m}(\boldsymbol{x}_j) \\ &= \frac{1}{4\pi}\sum_{l=0}^{\infty}(l+1)^2|\widehat{\phi^{(N)}}(l)|^2P_l(\boldsymbol{x}_i\cdot\boldsymbol{x}_j). \end{aligned}$$

The right-hand side of (4.7) is computed by using (2.5), noting $D = -S/2$ (see [9, page 122]),

$$\begin{aligned} \left\langle f, \Phi_i^{(S)}\right\rangle &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(-\frac{1}{2} - \frac{1}{2(2l+1)}\right)(l+1)\widehat{(U_D)}_{l,m}\widehat{\phi}^{(N)}(l)\overline{Y_{l,m}(\boldsymbol{x}_i)} \\ &= -\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{(l+1)^2}{2l+1}\widehat{(U_D)}_{l,m}\widehat{\phi}^{(N)}(l)\overline{Y_{l,m}(\boldsymbol{x}_i)}. \end{aligned} \tag{4.8}$$

A priori error estimates similar to those in Theorem 4.1 can be proved.

Theorem 4.2 *Let $\phi^{(N)}$ be defined by (4.1) for some positive integer m, $\phi^{(S)}$ be defined by (4.6), and $\tau^{(S)} = m + 1$. If z is the solution to (2.9) satisfying $z \in H^s(\mathbb{S})$, $-1/2 \le s \le 2\tau^{(S)}$, and z_X the solution to (4.7), then*

$$\|z - z_X\|_{-1/2} \le Ch_X^{s+1/2}\|z\|_s.$$

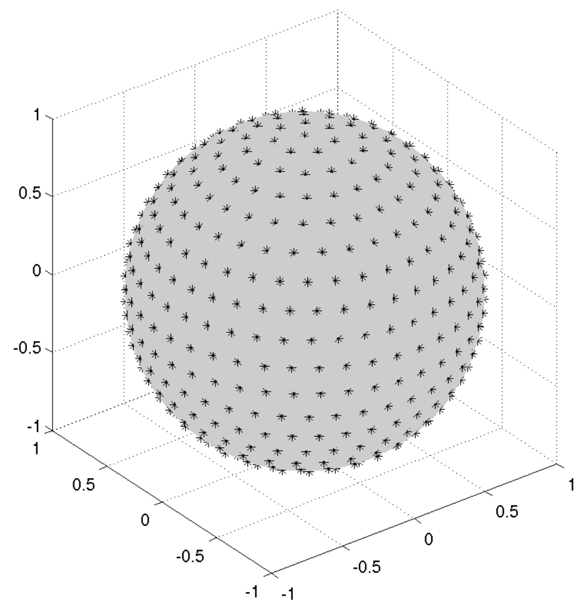

Figure 2: Scattered points in equally partitioned areas

Proof. We note that approximation property requires $m > 0$, and that

$$\min\{s + \frac{1}{2}, 2(\tau^{(S)} + \frac{1}{2}), 2\tau^{(S)} + |s|\} = s + \frac{1}{2}.$$

Since

$$\|z - z_X\|_{-1/2}^2 \simeq a_S(z - z_X, z - z_X),$$

the remainder of the proof is similar to that of Theorem 4.1, and is therefore omitted.
□

5 Numerical experiments

In this section, we present numerical results obtained from experiments with the set of scattered points X generated by a simple algorithm [10] which partitions the sphere into equal areas; see Figure 2. We chose the sets of points carefully so that the mesh norms h_X of different sets are reasonably different in order to easily observe the order of convergence. The sets of points we used have number of points $M = 20, 30, 40, 50, 100, 500,$ and 1000. Note that these experiments are purely for observing the order of convergence. In a following paper, we will experiment with sets of real data points obtained by satellites.

The spherical basis functions $\Phi_i^{(N)}$, $i = 1, \ldots, M$, are defined by (3.6) using the univariate function $\phi^{(N)}$ given by (4.1) with $m = 0, 1, 2$. The coefficients $\widehat{\phi}_N(l)$

with $l = 1, \dots, 500$ are computed by the MATLAB function `quadl` which uses an adaptive Lobatto quadrature. The spherical basis functions $\Phi_i^{(S)}$, $i = 1, \dots, M$, are defined by (3.6) with $\phi^{(S)}$ given by (4.6).

5.1 The Neumann problem

We solved the exterior Neumann problem (1.1), (1.3) and (1.4) with a boundary data given by

$$Z_N(\boldsymbol{x}) = \frac{0.5x_3 - 1}{(1.25 - x_3)^{3/2}},$$

so that the exact solution is

$$U(\boldsymbol{x}) = \frac{1}{\|\boldsymbol{x} - \boldsymbol{p}\|} \qquad \text{with } \boldsymbol{p} = (0, 0, 0.5).$$

Here $\boldsymbol{x} = (x_1, x_2, x_3)$. Due to (2.4) and (2.6), the exact solution to (2.10) is given by $u = U|_{\mathbb{S}}$. Let $\boldsymbol{n} = (0, 0, 1)$. By using the identity (see [9, page 20])

$$(1 - 2t\cos\theta + t^2)^{-1/2} = \sum_{l=0}^{\infty} t^l P_l(\cos\theta), \quad t < 1,$$

and the addition formula (3.3), we can write, for $\boldsymbol{x} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \in \mathbb{S}$,

$$\begin{aligned} u(\boldsymbol{x}) &= \frac{1}{\|\boldsymbol{x} - \boldsymbol{p}\|} = \frac{1}{\sqrt{1 - \cos\theta + 1/4}} = \sum_{l=0}^{\infty} \frac{1}{2^l} P_l(\boldsymbol{x} \cdot \boldsymbol{n}) \\ &= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2^l(2l+1)} Y_{l,m}(\boldsymbol{n}) Y_{l,m}(\boldsymbol{x}), \end{aligned}$$

so that

$$\widehat{u}_{l,m} = \frac{4\pi}{2^l(2l+1)} Y_{l,m}(\boldsymbol{n}). \tag{5.1}$$

We solved (4.3) and compared the approximate solution u_X with the exact solution u. Note that

$$\widehat{(u_X)}_{l,m} = \widehat{\phi^{(N)}}(l) \sum_{i=1}^{M} c_i \overline{Y_{l,m}(\boldsymbol{x}_i)}.$$

The error $u_X - u$ is computed by

$$\|u_X - u\|_{1/2} \approx \left(\sum_{l=1}^{500} \sum_{m=-l}^{l} (l+1) |\widehat{(u_X)}_{l,m} - \widehat{u}_{l,m}|^2 \right)^{1/2}.$$

Due to the addition formula (3.3), there holds

$$\sum_{m=-l}^{l} |\widehat{(u_X)}_{l,m} - \widehat{u}_{l,m}|^2 = \frac{2l+1}{4\pi}(\widehat{\phi^{(N)}}(l))^2 \sum_{i,j=1}^{M} c_i c_j P_l(\boldsymbol{x}_i \cdot \boldsymbol{x}_j) - \frac{\widehat{\phi^{(N)}}(l)}{2^{l-1}} \sum_{i=1}^{M} c_i P_l(\boldsymbol{x}_i \cdot \boldsymbol{n}) + \frac{4\pi}{2^{2l}(2l+1)}.$$

It is expected from our theoretical result (Theorem 4.1) that the order of convergence for the $H^{1/2}$-norm of the error is $2(m+1)$. The estimated orders of convergence (EOC) shown in Tables 1–3 appear to agree with our theoretical results.

5.2 The Dirichlet problem

We also solved the exterior Dirichlet problem (1.1), (1.2) and (1.4) with boundary data

$$U_D(\boldsymbol{x}) = \frac{1}{(1.25 - x_3)^{1/2}}.$$

The exact solution is given by

$$U(\boldsymbol{x}) = \frac{1}{\|\boldsymbol{x} - \boldsymbol{p}\|} \quad \text{with } \boldsymbol{p} = (0, 0, 0.5),$$

and hence, due to (2.4) and (2.5), the exact solution to (2.9) is

$$z(\boldsymbol{x}) = \partial_\nu U(\boldsymbol{x}) = \frac{-1 + \boldsymbol{x} \cdot \boldsymbol{p}}{\|\boldsymbol{x} - \boldsymbol{p}\|^3} = \frac{(0.5x_3 - 1)}{(1.25 - x_3)^{3/2}}.$$

It follows from (1.6), (1.7), and (5.1) that

$$\widehat{z}_{l,m} = -\frac{4\pi(l+1)}{2^l(2l+1)} Y_{l,m}(\boldsymbol{n}).$$

We solved (4.7) and compared the approximate solution z_X with the exact solution z. Note that

$$\widehat{(z_X)}_{l,m} = (l+1)\widehat{\phi^{(N)}}(l) \sum_{i=1}^{M} c_i \overline{Y_{l,m}(\boldsymbol{x}_i)}.$$

The error $z_X - z$ is approximated by

$$\|z_X - z\|_{-1/2} \approx \left(\sum_{l=1}^{500} \sum_{m=-l}^{l} \frac{|\widehat{(z_X)}_{l,m} - \widehat{z}_{l,m}|^2}{l+1} \right)^{1/2}.$$

The addition formula (3.3) yields

$$\sum_{m=-l}^{l} |\widehat{(z_X)}_{l,m} - \widehat{z}_{l,m}|^2 = \frac{(l+1)^2(2l+1)}{4\pi} (\widehat{\phi^{(N)}}(l))^2 \sum_{i,j=1}^{M} c_i c_j P_l(\boldsymbol{x}_i \cdot \boldsymbol{x}_j) + \widehat{\phi^{(N)}}(l) \frac{(l+1)^2}{2^{l-1}} \sum_{i=1}^{M} c_i P_l(\boldsymbol{x}_i \cdot \boldsymbol{n}) + \frac{4\pi(l+1)^2}{2^{2l}(2l+1)}.$$

Our theoretical result (Theorem 4.2) requires $m > 0$ and an order of convergence of $2m + 5/2$ is shown in the $H^{-1/2}$-norm. We carried out the experiment for $m = 0, 1, 2$, and observed some agreement between the estimated orders of convergence and our theoretical result (Tables 4–6). The case when $m = 0$ is the borderline case; the theory holds for all positive real values of m.

Remark 5.1 After this paper was accepted, one of us has improved [14] the error bound in Proposition 3.5, and thus that in Theorem 3.7. The exponent of h_X in these results can be of optimal value $s - t$. This results in an estimate of order $h_X^{s-1/2}$ in Theorem 4.1.

Acknowledgement

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M	h_X	$\|u_X - u\|_{1/2}$	EOC
20	0.6514	0.60872542	
40	0.4418	0.18859512	3.0179
50	0.3750	0.13264247	2.1469
100	0.2672	0.05752634	2.4649
500	0.1237	0.00738320	2.6658
1000	0.0849	0.00303414	2.3627

Table 1: Errors in the $H^{1/2}$-norm with $m = 0$

M	h_X	$\|u_X - u\|_{1/2}$	EOC
20	0.6514	0.93582688	
40	0.4418	0.17405797	4.3322
50	0.3750	0.07695943	4.9784
100	0.2672	0.02026597	3.9369
500	0.1237	0.00044098	4.9701
1000	0.0849	0.00008591	4.3459

Table 2: Errors in the $H^{1/2}$-norm with $m = 1$

M	h_X	$\|u_X - u\|_{1/2}$	EOC
20	0.6514	1.15883892	
40	0.4418	0.33819704	3.1719
50	0.3750	0.12849862	5.9031
100	0.2672	0.02401065	4.9492
500	0.1237	0.00007321	7.5219
1000	0.0849	0.00000648	6.4428

Table 3: Errors in the $H^{1/2}$-norm with $m = 2$

M	h_X	$\|u_X - u\|_{-1/2}$	EOC
20	0.6514	0.63633932	
40	0.4418	0.18846925	3.1339
50	0.3750	0.13291921	2.1301
100	0.2672	0.05752374	2.4712
500	0.1237	0.00738510	2.6654
1000	0.0849	0.00303428	2.3632

Table 4: Errors in the $H^{-1/2}$-norm with $m = 0$

M	h_X	$\|u_X - u\|_{-1/2}$	EOC
20	0.6514	0.93557363	
40	0.4418	0.18160356	4.2222
50	0.3750	0.07737696	5.2042
100	0.2672	0.02040513	3.9327
500	0.1237	0.00044099	4.9790
1000	0.0849	0.00008591	4.3460

Table 5: Errors in the $H^{-1/2}$-norm with $m = 1$

M	h_X	$\|u_X - u\|_{-1/2}$	EOC
20	0.6514	1.15254309	
40	0.4418	0.34892603	3.0774
50	0.3750	0.13191711	5.9335
100	0.2672	0.02401536	5.0261
500	0.1237	0.00007324	7.5217
1000	0.0849	0.00000648	6.4429

Table 6: Errors in the $H^{-1/2}$-norm with $m = 2$