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Preconditioners for pseudodifferential equations on the sphere with radial basis functions

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Abstract In a previous paper a preconditioning strategy based on overlapping domain decomposition was applied to the Galerkin approximation of elliptic partial differential equations on the sphere. In this paper the methods are extended to more general pseudodifferential equations on the sphere, using as before spherical radial basis functions for the approximation space, and again preconditioning the ill-conditioned linear systems of the Galerkin approximation by the additive Schwarz method. Numerical results are presented for the case of hypersingular and weakly singular integral operators on the sphere $\mathbb{S}^2$.

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1 Introduction

In this paper we present preconditioning techniques for pseudodifferential equations on the unit sphere $\mathbb{S}^n := \{\boldsymbol{x} \in \mathbb{R}^{n+1} : \|\boldsymbol{x}\| = 1\}$, where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^{n+1}$. These equations arise, for example, from geodesy; see [3].

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In [20] we studied the approximation of pseudodifferential equations by spherical radial basis functions. The reason for choosing radial basis functions is that they allow unstructured grids, and that radial basis functions are a promising tool in scattered data approximation [24]. However, it is also well-known [10] that these functions result in ill-conditioned systems. This paper presents preconditioning techniques based on additive Schwarz methods for these systems.

In [7] we introduced a preconditioning technique based on the domain decomposition of the sphere into overlapping spherical caps; and applied it to a particular equation namely the Laplace–Beltrami equation. Since the Laplace–Beltrami operator is a local operator, the stiffness matrix resulting from the Galerkin scheme applied to this equation is sparse.

Here we apply the same preconditioning technique to a general class of pseudodifferential equations on the sphere. Pseudodifferential operators in general include both local and nonlocal operators. In particular the hypersingular and weakly singular integral operators considered in the numerical experiments in this paper are nonlocal. As a consequence, the stiffness matrices are dense, and therefore a direct extension of the preconditioner from sparse matrices is not obvious. Moreover, for these nonlocal operators one has to deal with fractional Sobolev spaces.

In the boundary element context, we have successfully extended [21,22] preconditioning techniques developed for sparse matrices arising from differential operators to dense matrices arising from boundary integral operators. Here the extension is for meshless methods with radial basis functions.

The paper is organised as follows. In Sect. 2, we report on spherical harmonics and Sobolev spaces on the sphere. In Sect. 3 we introduce pseudodifferential operators on the sphere via their spherical symbols, and the Galerkin scheme with radial basis functions. In Sect. 4, we consider our overlapping Schwarz method and show that the condition number of the additive Schwarz operator is only mildly growing with the number of subspaces. In Sect. 5, we present numerical results which underline our theory.

2 Preliminaries

2.1 Spherical harmonics

Spherical harmonics are the restriction of homogeneous harmonic polynomials in $\mathbb{R}^{n+1}$ to the unit sphere $\mathbb{S}^n$. We denote an orthonormal (with respect to the $L^2(\mathbb{S}^n)$ inner product) basis for the spherical harmonics of degree ℓ by

$$\{Y_{\ell,m} : m = 1, \ldots, N(n,\ell)\}, \quad \ell = 0, 1, \ldots,$$

where $N(n,\ell)$ is the dimension of the space of all spherical harmonics of degree ℓ; the values of $N(n,\ell)$ are (see [9]):

$$N(n,0) = 1 \quad \text{and} \quad N(n,\ell) = \frac{(2\ell+n-1)\Gamma(\ell+n-1)}{\Gamma(\ell+1)\Gamma(n)} \quad \text{for } \ell \geq 1.$$

The asymptotic behaviour of $N(n, \ell)$ for fixed n and increasing ℓ is $O(\ell^{n-1})$. For a given function $f \in L^2(\mathbb{S}^n)$, we define its Fourier coefficients by

$$\widehat{f}_{\ell,m} = \int\limits_{\mathbb{S}^n} f(\boldsymbol{x}) Y_{\ell,m}(\boldsymbol{x}) dS(\boldsymbol{x}),$$

where dS is the surface measure of the sphere $\mathbb{S}^n$. We can represent f as a Fourier series,

$$f = \sum_{\ell=0}^{\infty} \sum_{m=1}^{N(n,\ell)} \widehat{f}_{\ell,m} Y_{\ell,m},$$

in which the equal sign is understood in the $L^2(\mathbb{S}^n)$ sense. The addition formula for spherical harmonics of the same degree ℓ, see [9], is

$$\sum_{m=1}^{N(n,\ell)} Y_{\ell,m}(\boldsymbol{x}) Y_{\ell,m}(\boldsymbol{y}) = \frac{1}{\omega_n} N(n, \ell) P_\ell(n + 1; \boldsymbol{x} \cdot \boldsymbol{y}), \tag{2.1}$$

where $P_\ell(n + 1; t)$ is the Legendre polynomial of degree ℓ in $\mathbb{R}^{n+1}$ normalized by $P_\ell(n + 1; 1) = 1$ and ω_n is the surface area of the unit sphere $\mathbb{S}^n$.

2.2 Sobolev spaces

For $s \in \mathbb{R}$, the Sobolev space H^s on $\mathbb{S}^n$ is defined as usual, see e.g. [12], by

$$H^s := \left\{ v \in \mathcal{D}'(\mathbb{S}^n) : \sum_{\ell=0}^{\infty} \sum_{m=1}^{N(n,\ell)} (\ell + 1)^{2s} |\widehat{v}_{\ell,m}|^2 < \infty \right\},$$

where $\mathcal{D}'(\mathbb{S}^n)$ is the space of distributions on $\mathbb{S}^n$. The space H^s is equipped with the following norm and inner product:

$$\begin{aligned} \|v\|_s &:= \left(\sum_{\ell=0}^{\infty} \sum_{m=1}^{N(n,\ell)} (\ell + 1)^{2s} |\widehat{v}_{\ell,m}|^2 \right)^{1/2} \\ \langle v, w \rangle_s &:= \sum_{\ell=0}^{\infty} \sum_{m=1}^{N(n,\ell)} (\ell + 1)^{2s} \widehat{v}_{\ell,m} \widehat{w}_{\ell,m}. \end{aligned} \tag{2.2}$$

When $s = 0$ we write $\langle \cdot, \cdot \rangle$ instead of $\langle \cdot, \cdot \rangle_0$, since this is the L_2-inner product on $\mathbb{S}^n$. For the analysis we can alternatively define the space H^s using local charts (see [8, Chap. 1, Sect. 7.3]) with a specific atlas of charts chosen as in [5]. Let a spherical cap of radius α centered at $\boldsymbol{p} \in \mathbb{S}^n$ be defined by

$$C(\boldsymbol{p}, \alpha) := \{\boldsymbol{x} \in \mathbb{S}^n : \theta(\boldsymbol{p}, \boldsymbol{x}) < \alpha\},$$

where $\theta(\boldsymbol{p}, \boldsymbol{x}) = \cos^{-1}(\boldsymbol{p} \cdot \boldsymbol{x})$ is the geodesic distance between two points $\boldsymbol{x}, \boldsymbol{p} \in \mathbb{S}^n$. Let $\widehat{\boldsymbol{n}}$ and $\widehat{\boldsymbol{s}}$ denote the north and south poles of $\mathbb{S}^n$, respectively. Then a simple cover for the sphere is provided by

$$U_1 = C(\widehat{\boldsymbol{n}}, \theta_0) \quad \text{and} \quad U_2 = C(\widehat{\boldsymbol{s}}, \theta_0), \;\; \text{where } \theta_0 \in (\pi/2, 2\pi/3).$$

The stereographic projection $\sigma_{\widehat{\boldsymbol{n}}}$ of the punctured sphere $\mathbb{S}^n \setminus \{\widehat{\boldsymbol{n}}\}$ onto $\mathbb{R}^n$ is defined as a mapping that maps $\boldsymbol{x} \in \mathbb{S}^n \setminus \{\widehat{\boldsymbol{n}}\}$ to the intersection of the equatorial hyperplane $\{z = 0\}$ and the extended line that passes through $\boldsymbol{x}$ and $\widehat{\boldsymbol{n}}$. The stereographic projection $\sigma_{\widehat{\boldsymbol{s}}}$ based on $\widehat{\boldsymbol{s}}$ can be defined analogously. We set

$$\psi_1 = \frac{1}{\tan(\theta_0/2)} \sigma_{\widehat{\boldsymbol{s}}}|_{U_1} \quad \text{and} \quad \psi_2 = \frac{1}{\tan(\theta_0/2)} \sigma_{\widehat{\boldsymbol{n}}}|_{U_2}, \tag{2.3}$$

so that ψ_k, $k = 1, 2$, maps U_k onto $B(0, 1)$, the unit ball in $\mathbb{R}^n$. We conclude that $\mathcal{A} = \{U_k, \psi_k\}_{k=1}^2$ is a C^∞ atlas of covering coordinate charts for the sphere. It is known [14] that the stereographic coordinate charts $\{\psi_k\}_{k=1}^2$ as defined in (2.3) map spherical caps to Euclidean balls, but in general concentric spherical caps are not mapped to concentric Euclidean balls. The projection ψ_k, for $k = 1, 2$, does not distort too much the geodesic distance between two points $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{S}^n$, as shown in [6]. With the atlas so defined, we define the map π_k which takes a real-valued function g with compact support in U_k into a real-valued function on $\mathbb{R}^n$ by

$$\pi_k(g)(\boldsymbol{x}) = \begin{cases} g \circ \psi_k^{-1}(\boldsymbol{x}), & \text{if } \boldsymbol{x} \in B(0, 1), \\ 0, & \text{otherwise} . \end{cases}$$

Let $\{\chi_k : \mathbb{S}^n \to \mathbb{R}\}_{k=1}^2$ be a partition of unity subordinated to the atlas, i.e., a pair of non-negative infinitely differentiable functions χ_k on $\mathbb{S}^n$ with compact support in U_k, such that $\sum_k \chi_k = 1$. For any function $f : \mathbb{S}^n \to \mathbb{R}$, we can use the partition of unity to write

$$f = \sum_{k=1}^{2} (\chi_k f), \;\; \text{where } (\chi_k f)(\boldsymbol{p}) = \chi_k(\boldsymbol{p}) f(\boldsymbol{p}), \quad \boldsymbol{p} \in \mathbb{S}^n.$$

The Sobolev space H^s is defined to be the set

$$\{f \in L^2(\mathbb{S}^n) : \pi_k(\chi_k f) \in H^s(\mathbb{R}^n) \quad \text{for } k = 1, 2\},$$

which is equipped with the norm

$$\|f\|_{H^s(\mathbb{S}^n)} = \left(\sum_{k=1}^{2} \|\pi_k(\chi_k f)\|^2_{H^s(\mathbb{R}^n)}\right)^{1/2}. \tag{2.4}$$

This $H^s(\mathbb{S}^n)$-norm is equivalent to the H^s norm given in (2.2); see [8].

3 Galerkin approximations for pseudodifferential equations

3.1 Pseudodifferential equations

Let $\{\widehat{L}(\ell)\}_{\ell\geq 0}$ be a sequence of real numbers. The pseudodifferential operator considered in this paper is a linear operator L that assigns to any $v \in \mathcal{D}'(\mathbb{S}^n)$ a distribution

$$Lv := \sum_{\ell=0}^{\infty} \sum_{m=1}^{N(n,\ell)} \widehat{L}(\ell)\widehat{v}_{\ell,m} Y_{\ell,m}. \tag{3.1}$$

The sequence $\{\widehat{L}(\ell)\}_{\ell\geq 0}$ is referred to as the *spherical symbol* of L. For simplicity of presentation, we assume that L is injective, so that $\widehat{L}(\ell) \neq 0$ for all $\ell = 0, 1, \ldots$. We also assume that

$$C_1(\ell+1)^{2\sigma} \leq \widehat{L}(\ell) \leq C_2(\ell+1)^{2\sigma} \quad \forall \ell = 0, 1, 2, \ldots. \tag{3.2}$$

for some $\sigma \in \mathbb{R}$ and some positive constants C_1 and C_2. Then $L : H^\sigma \to H^{-\sigma}$ is bounded, and L is a pseudodifferential operator of order 2σ.

Remark 3.1 More general pseudodifferential operators can be defined via Fourier transforms by using local charts; see e.g., [4,13].

When $n = 2$, some commonly seen operators of the form (3.1) satisfying (3.2) (at least for $\ell \geq 1$) are

1. The Laplace–Beltrami operator is an operator of order 2 and has as symbol $\widehat{L}(\ell) = \ell(\ell+1)$. This operator is the restriction of the Laplacian on the sphere.
2. The hypersingular integral operator (without the minus sign) is an operator of order 1 and has as symbol $\widehat{L}(\ell) = \ell(\ell+1)/(2\ell+1)$. This operator arises from the boundary-integral reformulation of the Neumann problem with the Laplacian in the interior or exterior of the sphere.
3. The weakly singular integral operator is an operator of order -1 and has as symbol $\widehat{L}(\ell) = 1/(2\ell+1)$. This operator arises from the boundary-integral reformulation of the Dirichlet problem with the Laplacian in the interior or exterior of the sphere.

Given $f \in H^{-\sigma}$, we want to solve the equation $Lu = f$ using its weak formulation

$$a(u, v) = \langle f, v\rangle \quad \forall v \in H^\sigma \tag{3.3}$$

by the Galerkin method, where

$$a(u, v) := \langle Lu, v\rangle.$$

3.2 Finite dimensional subspaces

The finite dimensional subspaces that we shall use in our approximation are defined by positive definite kernels on $\mathbb{S}^n$ and spherical radial basis functions.

A continuous function $\Phi : \mathbb{S}^n \times \mathbb{S}^n \to \mathbb{C}$ is called a *positive definite kernel* on $\mathbb{S}^n$ if it satisfies

(i) $\Phi(\boldsymbol{x}, \boldsymbol{y}) = \Phi(\boldsymbol{y}, \boldsymbol{x})$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{S}^n$;
(ii) for every positive integer K and every set of distinct points $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_K\}$ on $\mathbb{S}^n$, the $K \times K$ matrix A with entries $A_{i,j} = \Phi(\boldsymbol{x}_i, \boldsymbol{x}_j)$ is positive semi-definite.

If the matrix A is positive definite then Φ is called a *strictly positive definite* kernel; see [15,25].

We define the kernel Φ from a univariate function $\phi : [-1, 1] \to \mathbb{R}$ by

$$\Phi(\boldsymbol{x}, \boldsymbol{y}) = \phi(\boldsymbol{x} \cdot \boldsymbol{y}) \quad \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{S}^n, \tag{3.4}$$

where ϕ has a series expansion in terms of Legendre polynomials $P_\ell(n+1; \cdot)$ as

$$\phi(t) = \frac{1}{\omega_n} \sum_{\ell=0}^{\infty} N(n, \ell)\widehat{\phi}(\ell) P_\ell(n+1; t). \tag{3.5}$$

Here

$$\widehat{\phi}(\ell) = \omega_{n-1} \int_{-1}^{+1} \phi(t) P_\ell(n+1; t)(1-t^2)^{(n-2)/2} dt. \tag{3.6}$$

Due to the addition formula (2.1), a kernel Φ defined by

$$\Phi(\boldsymbol{x}, \boldsymbol{y}) = \phi(\boldsymbol{x} \cdot \boldsymbol{y}) \tag{3.7}$$

can be represented as

$$\Phi(\boldsymbol{x}, \boldsymbol{y}) = \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(n,\ell)} \widehat{\phi}(\ell) Y_{\ell,k}(\boldsymbol{x}) Y_{\ell,k}(\boldsymbol{y}). \tag{3.8}$$

In [2], a complete characterisation of strictly positive definite kernels is established: the kernel Φ is strictly positive definite if and only if $\widehat{\phi}(\ell) \geq 0$ for all $\ell \geq 0$ and $\widehat{\phi}(\ell) > 0$ for infinitely many even values of ℓ and infinitely many odd values of ℓ; see also [15] and [25]. In the following we shall assume that $\widehat{\phi}(\ell) > 0$ for all $\ell \geq 0$.

Let $X = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\}$ be a set of data points on the sphere. The *spherical radial basis functions* Φ_i, $i = 1, \ldots, N$, associated with X and the kernel Φ are defined by

$$\Phi_i(\boldsymbol{x}) := \Phi(\boldsymbol{x}, \boldsymbol{x}_i) = \sum_{\ell=0}^{\infty} \sum_{m=1}^{N(n,\ell)} \widehat{\phi}(\ell) Y_{\ell,m}(\boldsymbol{x}_i) Y_{\ell,m}(\boldsymbol{x}). \tag{3.9}$$

We assume that the coefficients $\widehat{\phi}(\ell)$ for $\ell = 0, 1, \ldots$ satisfy

$$c_1(\ell+1)^{-2\tau} \le \widehat{\phi}(\ell) \le c_2(\ell+1)^{-2\tau} \tag{3.10}$$

for some positive constants c_1 and c_2, and some $\tau \in \mathbb{R}$. As a consequence, $\Phi_i \in H^s(\mathbb{S}^n)$ for all s satisfying $s < 2\tau - 1$.

Let

$$V_X^{\phi} := \operatorname{span}\{\Phi_1, \ldots, \Phi_N\}. \tag{3.11}$$

The finite dimensional space V_X^{ϕ} defined in (3.11) depends on the univariate function ϕ and the set X. There being no confusion, we write V_X^{ϕ} as V for simplicity.

3.3 Approximation by Galerkin methods

The solution u of (3.3) is approximated by $\widetilde{u} \in V$ satisfying

$$a(\widetilde{u}, v) = \langle f, v \rangle \quad \forall v \in V. \tag{3.12}$$

This results in a matrix equation

$$\boldsymbol{A}\boldsymbol{c} = \boldsymbol{f}, \tag{3.13}$$

where $\boldsymbol{c} = (c_j)_{j=1}^N$ is the vector of coefficients of $\widetilde{u}$ in terms of Φ_j, $\boldsymbol{A} = (a(\Phi_i, \Phi_j))_{i,j=1}^N$ is the stiffness matrix, and $\boldsymbol{f} = (\langle f, \Phi_j \rangle)_{j=1}^N$ is the load vector. By using (2.1), we obtain the following formula to compute the entries of the stiffness matrix $\boldsymbol{A}$:

$$\begin{aligned} a(\Phi_i, \Phi_j) &= \sum_{\ell=0}^{\infty} \widehat{L}(\ell) |\widehat{\phi}(\ell)|^2 \sum_{m=1}^{N(n,\ell)} Y_{\ell,m}(\boldsymbol{x}_i) Y_{\ell,m}(\boldsymbol{x}_j) \\ &= \frac{1}{\omega_n} \sum_{\ell=0}^{\infty} N(n,\ell) \widehat{L}(\ell) |\widehat{\phi}(\ell)|^2 P_\ell(\boldsymbol{x}_i \cdot \boldsymbol{x}_j). \end{aligned} \tag{3.14}$$

4 An overlapping additive Schwarz method

In the following we shall design additive Schwarz preconditioners for the matrix systems arising from (3.12). At the heart of the design of preconditioners by additive

Schwarz methods is a subspace decomposition of the trial space V:

$$V = V_0 + V_1 + \cdots + V_J \tag{4.1}$$

for some positive integer J. We follow the notations of [7].

4.1 Subspace decomposition

As before, we assume that we are given a finite set $X = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\}$ of points on $\mathbb{S}^n$. To define the subspace decomposition (4.1), we first decompose the data set X in the form $X = X_0 \cup \cdots \cup X_J$ as follows.

(1) Select $\alpha \in (0, \pi/3)$ and $\beta \in [\alpha, \pi]$.
(2) Choose first centre $\boldsymbol{p}_1 = \boldsymbol{x}_1 \in X$.
(3) Define $X_1 := \{\boldsymbol{x} \in X : \cos^{-1}(\boldsymbol{x} \cdot \boldsymbol{p}_1) \le \alpha\}$.
(4) Suppose X_{j-1}, $j > 1$, has been defined. Then $\boldsymbol{p}_j$ is chosen from $X \setminus \{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_{j-1}\}$ such that $\cos^{-1}(\boldsymbol{p}_{j-1} \cdot \boldsymbol{p}_j) \ge \beta$ and $\cos^{-1}(\boldsymbol{p}_l \cdot \boldsymbol{p}_j) \ge \alpha$ for $l = 1, \ldots, j-2$.
(5) Define $X_j := \{\boldsymbol{x} \in X : \cos^{-1}(\boldsymbol{x} \cdot \boldsymbol{p}_j) \le \alpha\}$.
(6) Repeat (4) and (5) until every point in X is in at least one X_j.
(7) Define $X_0 := \{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_J\}$.

The second condition in Step (4) ensures that the geodesic distance between centres is no less than α, guaranteeing that the algorithm terminates. We note that the subsets X_j overlap. The subspaces V_j can now be defined by

$$V_j = \operatorname{span}\{\Phi_k : \boldsymbol{x}_k \in X_j\}, \quad j = 0, \ldots, J.$$

Assume that the support of $\Phi(\boldsymbol{p}, \cdot)$, which is a spherical cap centered at $\boldsymbol{p}$, has radius γ. (In case the spherical basis functions are unscaled Wendland's radial basis functions [24], we have $\gamma = \pi/3$). Then, functions in V_j have supports in Γ_j, where

$$\Gamma_j := C(\boldsymbol{p}_j, \alpha + \gamma), \quad j = 1, \ldots, J.$$

We assume that:

Assumption 4.1 *We can partition the index set* $\{1, \ldots, J\}$ *into* M *(with* $1 \le M \le J$*) sets* J_m*, for* $1 \le m \le M$*, such that if* $i, j \in J_m$ *and* $i \ne j$ *then* $\Gamma_i \cap \Gamma_j = \emptyset$.

Technically, M is the least number of colours that we have to use to colour the subdomains Γ_j, $j = 1, \ldots, J$, so that two intersecting subdomains are coloured differently. Note that if M_1 is the maximal number of subdomains Γ_j that a point on the sphere resides in, and M_2 is the maximal number of subdomains Γ_j that any subdomain intersects, then

$$1 \le M_1 \le M \le M_2 \le J. \tag{4.2}$$

4.2 Additive Schwarz operator and algorithm

Let $P_j : V \to V_j$, $j = 0, \ldots, J$, be projections defined by

$$a(P_j v, w) = a(v, w) \quad \forall v \in V, \ \forall w \in V_j. \tag{4.3}$$

Then the additive Schwarz operator is defined by

$$P := P_0 + \cdots + P_J. \tag{4.4}$$

The additive Schwarz method for equation (3.12) consists of solving, by an iterative method, the equation

$$P\widetilde{u} = g, \tag{4.5}$$

where the right-hand side is given by $g = \sum_{j=0}^{J} g_j$, with $g_j \in V_j$ being solutions of

$$a(g_j, w) = \langle f, w \rangle, \ \text{for any } w \in V_j. \tag{4.6}$$

Solving (4.5) is equivalent to solving (3.12), and in turn equivalent to solving

$$P_j \widetilde{u} = g_j, \quad j = 0, \ldots, J.$$

The operator P can be considered as the preconditioned version of L, i.e., $P = BL$ for some preconditioner B. The preconditioning technique is in practice performed by computing the action of B^{-1} on a residual $r \in V$. This consists of the solution of independent problems on each of the subspaces involved in the decomposition.

(i) Correction on the global coarse set X_0:

$$\text{Find } u_0 \in V_0 \text{ satisfying} \quad a(u_0, v) = \langle r, v \rangle \quad \forall v \in V_0.$$

(ii) Corrections on the local sets X_j, $j = 1, \ldots, J$:

$$\text{Find } u_j \in V_j \text{ satisfying} \quad a(u_j, v) = \langle r, v \rangle \quad \forall v \in V_j.$$

(iii) The residual r in the conjugate gradient is preconditioned by:

$$B^{-1} r := \sum_{j=0}^{J} u_j.$$

For the implementation details, see the pseudocode in [7, p. 17].

4.3 Convergence result

The main task in the analysis of the additive Schwarz preconditioner is to show a bound for the condition number $\kappa(P) := \lambda_{\max}(P)/\lambda_{\min}(P)$ of the operator P defined in (4.4), where $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ are the maximum and minimum eigenvalues of P. This bound should be given in terms of quantities associated with the decomposition. The following result is well-known; see e.g. [16,18].

Lemma 4.2 *(i) Assume that there exists a constant $c_1 > 0$ such that, for any $v \in V$ satisfying $v = \sum_{j=0}^{J} v_j$ with $v_j \in V_j$ for $j = 0, \ldots, J$, the following inequality*

$$a(v, v) \le c_1 \sum_{j=0}^{J} a(v_j, v_j)$$

holds. Then

$$\lambda_{\max}(P) \le c_1.$$

(ii) Assume that there exists a constant $c_2 > 0$ such that every $v \in V$ has a decomposition $v = \sum_{j=0}^{J} v_j$, $v_j \in V_j$, satisfying

$$\sum_{j=0}^{J} a(v_j, v_j) \le c_2 a(v, v).$$

Then

$$\lambda_{\min}(P) \ge c_2^{-1}.$$

4.3.1 A bound for the maximum eigenvalue of P

In the subsequent proof of Lemma 4.4 (on a bound for the maximum eigenvalue of P), we need the following fundamental result on Sobolev norms for domains with smooth boundary, which is a direct consequence of a more general result for Lipschitz domains proved in [23]; see also [1, Theorem 4.1]. The result concerns the global and local norms of the following Sobolev space of functions supported in the closure of a bounded open subset B of $\mathbb{R}^n$:

$$\widetilde{H}^s(B) := \{v \in H^s(\mathbb{R}^n) : \operatorname{supp} v \subset \overline{B}\}, \quad s \in \mathbb{R},$$

which is equipped with a norm defined by

$$\|v\|_{\widetilde{H}^s(B)} := \|v\|_{H^s(\mathbb{R}^n)}.$$

Extending the definition, a function defined on B whose zero extension to $\mathbb{R}^n$ belongs to $H^s(\mathbb{R}^n)$ is a function in $\widetilde{H}^s(B)$.

Lemma 4.3 *Let Ω be a bounded open domain in $\mathbb{R}^n$ with a smooth boundary and let $\Omega_1, \ldots, \Omega_N$ be a partitioning of Ω into nonoverlapping subdomains with smooth boundaries. For any $s \in \mathbb{R}$, if $v|_{\Omega_j} \in \widetilde{H}^s(\Omega_j)$ for $j = 1, \ldots, J$ then $v \in \widetilde{H}^s(\Omega)$ and there holds*

$$\|v\|^2_{\widetilde{H}^s(\Omega)} \leq \sum_{j=1}^{N} \|v|_{\Omega_j}\|^2_{\widetilde{H}^s(\Omega_j)}.$$

Lemma 4.4 *Let L be a pseudodifferential operator of order 2σ. Under Assumption 4.1, there exists a positive constant c independent of the set X such that for every $v \in V$ satisfying $v = \sum_{j=0}^{J} v_j$ with $v_j \in V_j$ for $j = 0, \ldots, J$ there holds*

$$a(v, v) \leq cM \sum_{j=0}^{J} a(v_j, v_j).$$

Proof Using the inequality $|a + b|^2 \leq 2(|a|^2 + |b|^2)$, we have

$$\|v\|^2_{H^\sigma(\mathbb{S}^n)} \leq 2\left(\|v_0\|^2_{H^\sigma(\mathbb{S}^n)} + \left\|\sum_{j=1}^{J} v_j\right\|^2_{H^\sigma(\mathbb{S}^n)}\right).$$

Recalling the definition of the Sobolev norm (2.4), we have

$$\left\|\sum_{j=1}^{J} v_j\right\|^2_{H^\sigma(\mathbb{S}^n)} = \left\|\sum_{j=1}^{J} \pi_1(\chi_1 v_j)\right\|^2_{H^\sigma(\mathbb{R}^n)} + \left\|\sum_{j=1}^{J} \pi_2(\chi_2 v_j)\right\|^2_{H^\sigma(\mathbb{R}^n)}.$$

Now, from the fact that $v_j \in V_j$ together with Assumption 4.1 we can partition the index set $\{1, \ldots, J\}$ to M sets of indices J_m so that if $i, j \in J_m$ then $\operatorname{supp} v_i \cap \operatorname{supp} v_j = \emptyset$. In this proof only, let $g_j = \pi_1(\chi_1 v_j)$ with $\Omega_j := \operatorname{supp} g_j$. By using successively the triangle inequality and the Cauchy–Schwarz inequality, we have

$$\left\|\sum_{j=1}^{J} g_j\right\|^2_{H^\sigma(\mathbb{R}^n)} = \left\|\sum_{m=1}^{M} \sum_{j \in J_m} g_j\right\|^2_{H^\sigma(\mathbb{R}^n)} \leq \left(\sum_{m=1}^{M} \left\|\sum_{j \in J_m} g_j\right\|_{H^\sigma(\mathbb{R}^n)}\right)^2$$
$$\leq M \sum_{m=1}^{M} \left\|\sum_{j \in J_m} g_j\right\|^2_{H^\sigma(\mathbb{R}^n)}.$$

With the partition $\{\Omega_j : j \in J_m\}$ of Ω where $\Omega = \operatorname{supp}\left(\sum_{j \in J_m} g_j\right)$ we have, due to Lemma 4.3,

$$\|\sum_{j \in J_m} g_j\|^2_{H^\sigma(\mathbb{R}^n)} = \|\sum_{j \in J_m} g_j\|^2_{\widetilde{H}^\sigma(\Omega)} \leq \sum_{j \in J_m} \|g_j\|^2_{\widetilde{H}^\sigma(\Omega_j)} = \sum_{j \in J_m} \|g_j\|^2_{H^\sigma(\mathbb{R}^n)}.$$

Thus,

$$\left\| \sum_{j=1}^{J} g_j \right\|_{H^\sigma(\mathbb{R}^n)}^2 \le M \sum_{m=1}^{M} \sum_{j \in J_m} \|g_j\|_{H^\sigma(\mathbb{R}^n)}^2 = M \sum_{j=1}^{J} \|g_j\|_{H^\sigma(\mathbb{R}^n)}^2. \tag{4.7}$$

Hence, by using similar arguments for $\pi_2(\chi_2 v_j)$, we conclude

$$\begin{aligned} \left\| \sum_{j=1}^{J} v_j \right\|_{H^\sigma(\mathbb{S}^n)}^2 &\le cM \left(\sum_{j=1}^{J} \|\pi_1(\chi_1 v_j)\|_{H^\sigma(\mathbb{R}^n)}^2 + \sum_{j=1}^{J} \|\pi_2(\chi_2 v_j)\|_{H^\sigma(\mathbb{R}^n)}^2 \right) \\ &= cM \sum_{j=1}^{J} \|v_j\|_{H^\sigma(\mathbb{S}^n)}^2. \end{aligned}$$

Therefore,

$$\|v\|_{H^\sigma(\mathbb{S}^n)}^2 = \left\| \sum_{j=0}^{J} v_j \right\|_{H^\sigma(\mathbb{S}^n)}^2 \le cM \sum_{j=0}^{J} \|v_j\|_{H^\sigma(\mathbb{S}^n)}^2.$$

Using the fact that $a(v, v) \simeq \|v\|_{H^\sigma(\mathbb{S}^n)}^2$ we obtain the result. □

The above lemma and Lemma 4.2 yield the following estimate for the maximum eigenvalue of P

$$\lambda_{\max}(P) \le cM. \tag{4.8}$$

4.3.2 A bound for the minimum eigenvalue of P

The approach to be used in this paper is similar to that in [7], and is included here for completeness. We note that the present bilinear form $a(\cdot, \cdot)$ has a more general property that $a(v, v) \simeq \|v\|_{H^\sigma}^2$ for $\sigma \in \mathbb{R}$, whereas $a(v, v) \simeq \|v\|_{H^1}^2$ in [7].

Firstly, we recall the following definition of angle between closed subspaces of a general Hilbert space.

Definition 4.5 Let $\mathcal{V}$ be a Hilbert space with inner product and norm denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Assume that $\mathcal{U}_1$ and $\mathcal{U}_2$ are two closed subspaces of $\mathcal{V}$. The angle α between $\mathcal{U}_1$ and $\mathcal{U}_2$ is the angle in $[0, \pi/2]$ whose cosine is given by

$$\cos \alpha = \sup\{\langle v, w \rangle : v \in \mathcal{U}_1 \cap \mathcal{U}^\perp, \quad w \in \mathcal{U}_2 \cap \mathcal{U}^\perp, \quad \|v\| \le 1, \ \|w\| \le 1\},$$

where $\mathcal{U} = \mathcal{U}_1 \cap \mathcal{U}_2$, and $\mathcal{U}^\perp$ is its orthogonal complement, namely,

$$\mathcal{U}^\perp := \{f \in \mathcal{V} : \langle f, v \rangle = 0 \quad \forall v \in \mathcal{U}\}.$$

It follows from the definition of orthogonal complement that

$$
\begin{aligned}
(\mathcal{U}_1 + \mathcal{U}_2)^{\perp} &= \{f \in \mathcal{V} : \langle f, v\rangle = 0 \quad \forall v \in \mathcal{U}_1 + \mathcal{U}_2\} \\
&= \{f \in \mathcal{V} : \langle f, v_1\rangle = 0 = \langle f, v_2\rangle \quad \forall v_1 \in \mathcal{U}_1 \quad \text{and} \quad v_2 \in \mathcal{U}_2\} \\
&= \mathcal{U}_1^{\perp} \cap \mathcal{U}_2^{\perp}.
\end{aligned} \tag{4.9}
$$

The following theorem [17, Theorem 2.2] is crucial in our estimate of the minimum eigenvalue of P.

Theorem 4.6 *Let $\mathcal{V}_1$, ..., $\mathcal{V}_J$ be closed subspaces of a Hilbert space $\mathcal{V}$, and $\mathcal{W}_i := \cap_{j=i}^{J} \mathcal{V}_j$, $i = 1, \dots, J$. If $Q_i : \mathcal{V} \to \mathcal{V}_i$ is the orthogonal projection onto $\mathcal{V}_i$, $i = 1, \dots, J$, and $Q : \mathcal{V} \to \mathcal{W}_1$ is the orthogonal projection onto $\mathcal{W}_1$, then*

$$
\|\widetilde{Q}^l f - Qf\| \le c^l \|f - Qf\|, \quad \forall f \in \mathcal{V}, \quad l = 1, 2, \dots,
$$

where $\widetilde{Q} := Q_J \cdots Q_1$ and

$$
c^2 = 1 - \prod_{i=1}^{J-1} \sin^2 \alpha_i,
$$

with α_i being the angle between $\mathcal{V}_i$ and $\mathcal{W}_{i+1}$.

We shall apply Theorem 4.6 with $\mathcal{V}$ being V, which is equipped with the inner product $a(\cdot, \cdot)$ and induced norm $\|\cdot\|_a$, and $\mathcal{V}_j$ being $V_j^{\perp}$, $j = 1, \dots, J$. If T is a linear operator on V we denote by $\|T\|_a$ the norm of T defined by $\|\cdot\|_a$, i.e.,

$$
\|T\|_a = \sup_{\substack{v \in V \\ \|v\|_a \le 1}} \|Tv\|_a.
$$

Proposition 4.7 *Let $\widetilde{Q} := Q_J \cdots Q_1$ where Q_i is the orthogonal projection from V onto $V_i^{\perp}$, and let $W_i := \cap_{j=i}^{J} V_j^{\perp}$, $i = 1, \dots, J$. Then*

$$
\|\widetilde{Q}\|_a \le \left(1 - \prod_{i=1}^{J-1} \sin^2 \alpha_i\right)^{1/2} < 1,
$$

where α_i is the angle between $V_i^{\perp}$ and W_{i+1}.

Proof First we note that since $X_0 \subset X_1 \cup \cdots \cup X_J$, there holds $V = V_1 + \cdots + V_J$. It follows from (4.9) that

$$
W_1^{\perp} = (V_1 + \cdots + V_J)^{\perp\perp} = V^{\perp\perp} = V.
$$

Hence $W_1 = \{0\}$ which implies that the orthogonal projection Q from V onto W_1 is identically zero. Theorem 4.6 then yields

$$\|\widetilde{Q}\|_a \leq \left(1 - \prod_{i=1}^{J-1} \sin^2 \alpha_i\right)^{1/2}.$$

It remains to show that $\alpha_i \neq 0$ for all $i = 1, \ldots, J-1$. Suppose that $\alpha_i = 0$ for some $i \in \{1, \ldots, J-1\}$. Then noting that

$$(V_i^\perp \cap W_{i+1})^\perp = W_i^\perp,$$

we obtain from Definition 4.5

$$\sup\{a(v, w) : v \in V_i^\perp \cap W_i^\perp,\ w \in W_{i+1} \cap W_i^\perp,\ \|v\|_a \leq 1,\ \|w\|_a \leq 1\} = 1.$$

The spaces being finite dimensional, by compactness there exist $v \in V_i^\perp \cap W_i^\perp$ and $w \in W_{i+1} \cap W_i^\perp$ satisfying

$$\|v\|_a = \|w\|_a = 1 \quad \text{and} \quad a(v, w) = 1.$$

The condition for equality to occur in the Cauchy–Schwarz inequality implies $v = w$. Thus $v \in V_i^\perp \cap W_{i+1} = W_i$. On the other hand $v \in W_i^\perp$, which implies $v = 0$. This contradicts the fact that $\|v\|_a = 1$, proving the proposition. □

Lemma 4.8 *For any $v \in V$ there exist $v_j \in V_j$, $j = 0, \ldots, J$, satisfying $v = \sum_{j=0}^{J} v_j$ and*

$$\sum_{j=0}^{J} a(v_j, v_j) \leq \left(1 + \frac{J}{(1 - \|\widetilde{Q}\|_a)^2}\right) a(v, v),$$

where $\widetilde{Q}$ is defined in Proposition 4.7.

Proof It follows from Proposition 4.7 that $I - \widetilde{Q}$ is invertible and satisfies

$$\|(I - \widetilde{Q})^{-1}\|_a \leq \frac{1}{1 - \|\widetilde{Q}\|_a},$$

where I is the identity operator on V. We define, for any $v \in V$,

$$\begin{aligned}
v_0 &= P_0 v, \quad w = v - v_0, \\
v_1 &= P_1 (I - \widetilde{Q})^{-1} w, \\
v_j &= P_j Q_{j-1} \cdots Q_1 (I - \widetilde{Q})^{-1} w,\ \ j = 2, \ldots, J,
\end{aligned}$$

where $P_j = I - Q_j$ is the orthogonal projection from V onto V_j, $j = 0, \ldots, J$, defined in (4.3). It is easy to check that $\sum_{j=1}^{J} v_j$ (being a telescoping sum) equals w, and therefore $\sum_{j=0}^{J} v_j = v$. A crude estimate yields, for $j = 1, \ldots, J$,

$$a(v_j, v_j) \leq \|(I - \widetilde{Q})^{-1}\|_a^2 a(v, v) \leq \frac{1}{(1 - \|\widetilde{Q}\|_a)^2} a(v, v),$$

resulting in

$$\begin{aligned} \sum_{j=0}^{J} a(v_j, v_j) &= a(v_0, v_0) + \sum_{j=1}^{J} a(v_j, v_j) \\ &\leq \left(1 + \frac{J}{(1 - \|\widetilde{Q}\|_a)^2}\right) a(v, v), \end{aligned}$$

proving the lemma. □

The above lemma and Lemma 4.2 yield the following estimate for the minimum eigenvalue of P

$$\lambda_{\min}(P) \geq \left(1 + \frac{J}{(1 - \|\widetilde{Q}\|_a)^2}\right)^{-1}. \tag{4.10}$$

This estimate is by no means sharp. In fact the right hand side is not an optimal lower bound for $\lambda_{\min}(P)$; see [7].

4.3.3 Main result

Theorem 4.9 *The condition number of the additive Schwarz operator P is bounded by*

$$\kappa(P) \leq cM \left(1 + \frac{J}{(1 - \|\widetilde{Q}\|_a)^2}\right).$$

where c is a constant independent of M and the set X.

Proof The result is a direct consequence of (4.8) and (4.10). □

5 Numerical experiments

In this section, we present numerical experiments based on globally scattered data extracted from a very large data set collected by NASA satellite MAGSAT. The scattered data sets $X = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_N\}$ are extracted so that the separation radius q_X defined by

$$q_X := \frac{1}{2} \min_{i \neq j} \cos^{-1}(\boldsymbol{x}_i \cdot \boldsymbol{x}_j)$$

satisfies $q_X = \pi/240, \pi/280$ and $\pi/320$. The corresponding sets have cardinality $N = 10{,}443$, $13{,}897$ and $17{,}262$.

The code was written in FORTRAN 90 and run on computers equipped with dual Opteron 2.0 GHz CPU and 4 GB RAM.

We solved the Dirichlet and Neumann problems in the unbounded domain $\mathbb{B} := \{\boldsymbol{x} \in \mathbb{R}^3 : \|\boldsymbol{x}\| > 1\}$, namely,

$$\begin{aligned} \Delta U &= 0 \quad \text{in } \mathbb{B} \\ U &= U_D \quad \text{on } \mathbb{S}^2 \end{aligned}$$

and

$$\begin{aligned} \Delta U &= 0 \quad \text{in } \mathbb{B} \\ \partial_\nu U &= U_N \quad \text{on } \mathbb{S}^2, \end{aligned}$$

with a vanishing condition at infinity for both problems:

$$U(\boldsymbol{x}) = O(1/\|\boldsymbol{x}\|) \quad \text{as} \quad \|\boldsymbol{x}\| \to \infty.$$

Here ν is the outer normal vector to $\mathbb{S}^2$, and U_D and U_N are chosen such that the exact solutions to both problems are

$$U(\boldsymbol{x}) = \frac{1}{\|\boldsymbol{x} - \boldsymbol{p}\|} \quad \text{where } \boldsymbol{p} = (0, 0, p) \text{ with } p \in (0, 1). \tag{5.1}$$

Thus, for $\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{S}^2$, we chose

$$U_D(\boldsymbol{x}) = \frac{1}{(1 + p^2 - 2px_3)^{1/2}} \quad \text{and} \quad U_N(\boldsymbol{x}) = \frac{px_3 - 1}{(1 + p^2 - 2px_3)^{3/2}}.$$

It is well known that the Dirichlet problem can be reformulated into a weakly singular integral equation on $\mathbb{S}^2$, which can then be rewritten [19] as a pseudodifferential equation $Lu = f$ with

$$\widehat{L}(\ell) = \frac{1}{2\ell + 1}$$

and

$$
\begin{aligned}
f(\boldsymbol{x}) &= -\frac{1}{2}U_D(\boldsymbol{x}) + \frac{1}{4\pi}\int\limits_{\mathbb{S}^2} \frac{\partial}{\partial\nu(\boldsymbol{y})}\frac{U_D(\boldsymbol{y})}{\|\boldsymbol{x}-\boldsymbol{y}\|}\,dS(\boldsymbol{y}) \\
&= \frac{1}{2}\sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell}\left(-1-\frac{1}{2\ell+1}\right)\widehat{(U_D)}_{\ell,m}Y_{\ell,m}(\boldsymbol{x}) \\
&= -\sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell}\frac{\ell+1}{2\ell+1}\widehat{(U_D)}_{\ell,m}Y_{\ell,m}(\boldsymbol{x}).
\end{aligned}
$$

In this case L is a pseudodifferential operator of order -1 and hence $\sigma = -1/2$.

Similarly, the Neumann problem can be reformulated into a hypersingular integral equation on $\mathbb{S}^2$, which can then be rewritten [19] as a pseudodifferential equation $Lu = f$ with

$$
\widehat{L}(\ell) = \frac{\ell(\ell+1)}{2\ell+1}
$$

and

$$
\begin{aligned}
f(\boldsymbol{x}) &= \frac{1}{2}U_N(\boldsymbol{x}) + \frac{1}{4\pi}\frac{\partial}{\partial\nu(\boldsymbol{x})}\int\limits_{\mathbb{S}^2} \frac{U_N(\boldsymbol{y})}{\|\boldsymbol{x}-\boldsymbol{y}\|}\,dS(\boldsymbol{y}) \\
&= \frac{1}{2}\sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell}\left(1-\frac{1}{2\ell+1}\right)\widehat{(U_N)}_{\ell,m}Y_{\ell,m}(\boldsymbol{x}) \\
&= \sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell}\frac{\ell}{2\ell+1}\widehat{(U_N)}_{\ell,m}Y_{\ell,m}(\boldsymbol{x}).
\end{aligned}
$$

In this case L is a pseudodifferential operator of order 1 and hence $\sigma = 1/2$.

The univariate function ϕ defining the kernel Φ, see (3.4), is given by $\phi(t) = \rho_m(\sqrt{2-2t})$, where ρ_m are Wendland's functions [24, page 128]. It is proved in [11, Proposition 4.6] that (3.10) holds with $\tau = m + 3/2$. Table 1 details the functions ρ_m used in our experiments and the corresponding values for τ. The spherical radial basis functions Φ_i, $i = 1, \ldots, N$, are computed by

$$
\Phi_i(\boldsymbol{x}) = \rho_m(\sqrt{2-2\boldsymbol{x}\cdot\boldsymbol{x}_i}), \quad \boldsymbol{x} \in \mathbb{S}^2. \tag{5.2}
$$

Table 1 Wendland's RBFs

m	$\rho_m(r)$	τ
0	$(1-r)_+^2$	1.5
1	$(1-r)_+^4(4r+1)$	2.5

Table 2 Unpreconditioned CG for the Dirichlet problem with $p = 0.5$

m	N	$\lambda_{\min}$	$\lambda_{\max}$	$\kappa(\boldsymbol{A})$	CPU	ITER
0	10,443	2.729E-10	227.937	8.353E+11	168.7	259
	13,897	1.182E-10	303.418	2.568E+12	98.2	85
	17,262	5.359E-11	377.013	7.035E+12	227.7	128
1	10,443	−3.131E-14	167.478	n/a	133.2	209
	13,897	−1.866E-13	222.963	n/a	143.3	124

Table 3 Unpreconditioned systems for the Dirichlet problem with $p = 0.95$

m	N	$\lambda_{\min}$	$\lambda_{\max}$	$\kappa(\boldsymbol{A})$	CPU	ITER
0	10,443	2.729E-10	227.937	8.353E+11	>23,000	>41,000
	13,897	1.182E-10	303.418	2.568E+12	>40,000	>33,000
	17,262	5.359E-11	377.013	7.035E+12	>73,000	>48,000
1	10,443	−3.131E-14	167.478	n/a	>23,000	>41,000
	13,897	−1.866E-13	222.963	n/a	>65,000	>55,000

Table 4 Unpreconditioned systems for the Neumann problem with $p = 0.5$

m	N	$\lambda_{\min}$	$\lambda_{\max}$	$\kappa(\boldsymbol{A})$	CPU	ITER
0	10,443	5.449E-06	149.848	2.750E+07	775.0	1,397
	13,897	3.305E-06	204.544	6.189E+07	339.7	299
	17,262	2.040E-06	259.912	1.274E+08	453.7	286
1	10,443	1.207E-10	135.598	1.124E+12	447.3	611
	13,897	3.694E-11	185.949	5.034E+12	195.6	195
	17,262	1.217E-11	237.437	1.951E+13	400.6	237

Table 5 Unpreconditioned systems for the Neumann problem with $p = 0.95$

m	N	$\lambda_{\min}$	$\lambda_{\max}$	$\kappa(\boldsymbol{A})$	CPU	ITER
0	10,443	5.449E-06	149.848	2.750E+07	4,587.4	6,398
	13,897	3.305E-06	204.544	6.189E+07	6,402.2	5,772
	17,262	2.040E-06	259.912	1.274E+08	9,736.9	5,743
1	10,443	1.207E-10	135.598	1.124E+12	>23,000	>41,000
	13,897	3.694E-11	185.949	5.034E+12	>55,000	>55,000
	17,262	1.217E-11	237.437	1.951E+13	>123,000	>69,000

We solved the matrix equation (3.13) by the conjugate gradient method with relative tolerance 10^{-6}, i.e. the stopping criterion is

$$\frac{\|\boldsymbol{A}\boldsymbol{c}^{(m)} - \boldsymbol{g}\|_{l^2}}{\|\boldsymbol{g}\|_{l^2}} \leq 10^{-6}.$$

Table 6 Preconditioned systems for the Dirichlet problem with $p = 0.5$

m	N	$\cos\alpha$	$\cos\beta$	J	$\lambda_{\min}$	$\lambda_{\max}$	$\kappa(P)$	CPU	ITER
0	10,443	0.95	−0.07	86	2.110E-02	52.945	2.509E+03	86.0	62
	10,443	0.90	−0.63	45	3.048E-01	31.130	1.021E+02	101.0	29
	10,443	0.80	−0.87	20	1.199E-01	15.992	1.334E+02	316.4	31
	10,443	0.60	−0.87	11	5.191E-01	10.365	1.997E+01	716.0	18
	13,897	0.95	0.03	90	6.043E-02	55.499	9.184E+02	168.1	59
	13,897	0.90	−0.76	46	1.961E-01	31.905	1.627E+02	271.4	35
	13,897	0.80	−0.67	26	1.081E+00	20.528	1.898E+01	555.5	19
	13,897	0.60	−0.69	13	1.088E+00	12.031	1.105E+01	1,723.9	16
	17,262	0.95	−0.17	90	8.102E-02	55.624	6.866E+02	254.3	50
	17,262	0.90	−0.62	47	1.468E-02	32.666	2.226E+03	1118.4	76
	17,262	0.80	−0.76	24	9.821E-01	19.087	1.943E+01	1006.8	20
	17,262	0.60	−0.85	12	2.635E-01	11.240	4.266E+01	4,507.6	23
1	10,443	0.99	0.99	424	1.371E-02	223.980	1.634E+04	145.1	202
	10,443	0.98	0.95	201	9.979E-03	113.033	1.133E+04	136.7	179
	10443	0.97	0.69	138	1.446E-02	81.024	5.605E+03	121.2	132
	10,443	0.95	−0.07	86	1.155E-02	53.828	4.658E+03	163.5	122
	10,443	0.60	−0.87	11	5.299E-02	10.402	1.963E+02	1,202.7	31
	13,897	0.99	0.98	395	8.076E-03	210.291	2.604E+04	327.7	259
	13,897	0.98	0.74	217	1.169E-02	122.993	1.052E+04	248.8	168
	13,897	0.97	0.68	138	3.374E-03	81.772	2.424E+04	414.9	234
	13,897	0.95	0.03	90	9.140E-03	56.687	6.202E+03	389.4	134
	13,897	0.60	−0.69	13	1.054E+00	12.085	1.147E+01	1,823.4	17

Here $\boldsymbol{c}^{(m)}$ is the mth iterate. The eigenvalues of the matrix are computed using the LAPACK subroutine DSYEV.

Tables 2–5 show the ℓ_2-condition number $\kappa(\boldsymbol{A})$, the CPU time in seconds, and the number of iterations ITER for two different values of p, see (5.1), namely $p = 0.5$ and $p = 0.95$. The numbers suggest that when the point $\boldsymbol{p}$ is deeply buried in the sphere ($p = 0.5$), the unpreconditioned method can solve the problems with $m = 0$ but not with $m = 1$, though the solutions are not reliable due to large condition numbers of the stiffness matrix $\boldsymbol{A}$. The results are even worse when the point $\boldsymbol{p}$ is closer to the surface ($p = 0.95$); the unpreconditioned method can hardly solve the problems.

We then tested the overlapping additive Schwarz method as a preconditioner for the conjugate gradient method, with different values of α and β, see Subsect. 4.1, and hence different values of J, the number of subproblems. The numbers in Tables 6–9 show the efficiency of the preconditioner. (In these tables $\kappa(P)$ is the condition number of the preconditioned systems.) The numbers suggest that the algorithm is not affected by the smoothness of the kernel. In all cases, for both the weakly singular and hypersingular operators, the condition numbers and the CPU times are both much

Table 7 Preconditioned systems for the Dirichlet problem with $p = 0.95$

m	N	$\cos\alpha$	$\cos\beta$	J	$\lambda_{\min}$	$\lambda_{\max}$	$\kappa(P)$	CPU	ITER
0	10,443	0.99	0.99	424	3.954E-03	223.130	5.644E+04	160.3	252
	10,443	0.98	0.95	201	4.652E-03	111.679	2.400E+04	109.8	162
	10,443	0.97	0.69	138	5.381E-02	79.763	1.482E+03	64.2	78
	10,443	0.95	−0.07	86	1.280E-02	52.945	4.135E+03	122.2	98
	10,443	0.90	−0.63	45	2.976E-01	31.130	1.046E+02	109.8	34
	13,897	0.99	0.98	395	1.782E-02	208.653	1.171E+04	219.7	173
	13,897	0.98	0.74	217	2.665E-02	120.889	4.536E+03	161.4	110
	13,897	0.97	0.68	138	8.361E-03	80.043	9.574E+03	233.9	134
	13,897	0.95	0.03	90	7.944E-02	55.499	6.987E+02	178.9	63
	13,897	0.90	−0.76	46	1.914E-01	31.905	1.667E+02	298.9	40
	17,262	0.99	0.98	405	5.336E-02	214.422	4.019E+03	172.2	102
	17,262	0.98	0.94	206	2.755E-02	115.125	4.179E+03	221.4	108
	17,262	0.97	0.70	140	8.905E-03	81.374	9.139E+03	394.8	151
	17,262	0.95	−0.17	90	6.687E-02	55.624	8.318E+02	275.4	58
	17,262	0.90	−0.62	47	1.449E-02	32.666	2.255E+03	1,209.6	84
1	10,443	0.99	0.99	424	2.337E-03	224.019	9.585E+04	371.6	584
	10,443	0.98	0.95	201	1.679E-03	113.144	6.740E+04	289.9	428
	10,443	0.97	0.69	138	1.534E-03	81.042	5.284E+04	231.9	282
	10,443	0.95	−0.07	86	1.408E-03	53.828	3.822E+04	306.4	246
	10,443	0.90	−0.63	45	4.726E-02	31.444	6.653E+02	203.5	63
	13,897	0.99	0.98	395	3.300E-04	210.991	6.395E+05	1,228.8	1,019
	13,897	0.98	0.74	217	2.545E-03	122.884	4.829E+04	456.4	325
	13,897	0.97	0.68	138	7.030E-04	82.394	1.172E+05	869.3	506
	13,897	0.95	0.03	90	1.050E-02	56.681	5.401E+03	405.1	143
	13,897	0.90	−0.76	46	1.381E-02	32.438	2.349E+03	833.3	111

better than in the unpreconditioned case. The benefit is particularly great for the harder case $p = 0.95$.

It should be noted that when $\cos\alpha$ decreases (meaning that α increases) then $\lambda_{\max}$ decreases (whereas the behaviour of $\lambda_{\min}$ is influenced by the LANCZOS algorithm), but the CPU time first decreases then increases. We note that the larger value of α results in the larger size of the overlap and a smaller value of J (the number of subproblems to be solved), which in turn implies larger sizes of the subproblems. As in the case of the differential operator considered in [7], this results in a smaller condition number $\kappa(P)$ because the preconditioner is closer to the inverse of the stiffness matrix. However, for an optimal value of α in terms of CPU time, one has to balance between the number of subproblems and their sizes. As our experiments show, any value of α so that $\cos\alpha \le 0.6$ is not recommended.

Table 8 Preconditioned systems for the Neumann problem with $p = 0.5$

m	N	$\cos\alpha$	$\cos\beta$	J	$\lambda_{\min}$	$\lambda_{\max}$	$\kappa(P)$	CPU	ITER
0	10,443	0.99	0.99	424	1.998E-03	56.451	2.825E+04	105.6	165
	10,443	0.98	0.95	201	3.350E-04	31.681	9.446E+04	132.1	194
	10,443	0.97	0.69	138	1.300E-04	24.397	1.870E+05	105.7	128
	10,443	0.95	−0.07	86	4.700E-05	18.374	3.917E+05	123.8	99
	10,443	0.90	−0.63	45	6.888E-01	12.711	1.845E+01	71.4	22
	13,897	0.99	0.98	395	1.707E-02	52.756	3.091E+03	125.5	100
	13,897	0.98	0.74	217	5.593E-02	34.114	6.100E+02	93.3	64
	13,897	0.97	0.68	138	4.843E-02	24.713	5.102E+02	108.7	62
	13,897	0.95	0.03	90	2.361E-02	19.316	8.182E+02	179.2	63
	13,897	0.90	−0.76	46	3.453E-01	12.978	3.758E+01	188.6	25
	17,262	0.99	0.98	405	2.565E-02	54.441	2.122E+03	168.2	95
	17,262	0.98	0.94	206	3.661E-02	32.891	8.984E+02	143.5	67
	17,262	0.97	0.70	140	3.908E-02	25.302	6.475E+02	176.0	63
	17,262	0.95	−0.17	90	1.113E-01	19.347	1.738E+02	203.0	41
	17,262	0.90	−0.62	47	7.370E-02	13.570	1.841E+02	701.4	47
1	10,443	0.99	0.99	424	1.204E-02	64.025	5.319E+03	128.3	156
	10,443	0.98	0.95	201	5.453E-03	35.481	6.506E+03	147.1	170
	10,443	0.97	0.69	138	5.919E-03	27.191	4.594E+03	154.6	152
	10,443	0.95	−0.07	86	8.853E-03	20.464	2.312E+03	146.4	101
	10,443	0.90	−0.63	45	1.181E-01	13.787	1.168E+02	121.8	35
	13,897	0.99	0.98	395	1.110E-02	59.929	5.398E+03	184.9	165
	13,897	0.98	0.74	217	1.145E-02	38.469	3.360E+03	165.2	125
	13,897	0.97	0.68	138	5.694E-03	27.781	4.879E+03	255.5	158
	13,897	0.95	0.03	90	1.569E-02	21.614	1.378E+03	253.0	93
	13,897	0.90	−0.76	46	3.217E-02	14.137	4.395E+02	376.3	51
	17,262	0.99	0.98	405	1.402E-02	62.282	4.442E+03	272.2	145
	17,262	0.98	0.94	206	8.439E-03	37.329	4.423E+03	355.7	159
	17,262	0.97	0.70	140	1.098E-02	28.467	2.593E+03	341.5	120
	17,262	0.95	−0.17	90	2.380E-02	21.584	9.068E+02	350.9	72

Table 9 Preconditioned systems for the Neumann problem with $p = 0.95$

m	N	$\cos\alpha$	$\cos\beta$	J	$\lambda_{\min}$	$\lambda_{\max}$	$\kappa(P)$	CPU	ITER
0	10,443	0.99	0.99	424	1.982E-03	56.452	2.849E+04	181.3	224
	10,443	0.98	0.95	201	3.340E-04	31.682	9.474E+04	162.9	222
	10,443	0.97	0.69	138	1.300E-04	24.398	1.873E+05	115.8	126
	10,443	0.95	−0.07	86	4.700E-05	18.374	3.924E+05	148.7	105
	10,443	0.90	−0.63	45	1.100E-05	12.772	1.176E+06	173.4	50
	13,897	0.99	0.98	395	9.992E-03	52.758	5.280E+03	144.9	121
	13,897	0.98	0.74	217	9.638E-02	34.115	3.540E+02	88.5	62
	13,897	0.97	0.68	138	4.898E-02	24.714	5.046E+02	119.5	66
	13,897	0.95	0.03	90	2.462E-02	19.317	7.845E+02	170.0	59
	13,897	0.90	−0.76	46	3.435E-01	12.981	3.779E+01	223.8	29

Table 9 continued

m	N	$\cos\alpha$	$\cos\beta$	J	$\lambda_{\min}$	$\lambda_{\max}$	$\kappa(P)$	CPU	ITER
0	17,262	0.99	0.98	405	3.767E-02	54.443	1.445E+03	160.8	85
	17,262	0.98	0.94	206	2.676E-02	32.892	1.229E+03	209.8	93
	17,262	0.97	0.70	140	8.573E-02	25.303	2.951E+02	175.4	61
	17,262	0.95	−0.17	90	8.531E-02	19.348	2.268E+02	230.5	46
	17,262	0.90	−0.62	47	8.307E-02	13.570	1.634E+02	636.0	43
1	10,443	0.99	0.99	424	6.942E-05	64.025	9.223E+05	1,467.0	2,316
	10,443	0.98	0.95	201	6.208E-06	35.481	5.715E+06	1,087.1	1,612
	10,443	0.97	0.69	138	8.176E-07	27.191	3.326E+07	589.6	721
	10,443	0.95	−0.07	86	1.332E-07	20.464	1.537E+08	545.1	440
	10,443	0.90	−0.63	45	1.300E-08	13.787	1.061E+09	331.7	103
	13,897	0.99	0.98	395	2.819E-03	59.929	2.126E+04	343.0	311
	13,897	0.98	0.74	217	7.676E-03	38.469	5.011E+03	184.3	141
	13,897	0.97	0.68	138	4.683E-03	27.781	5.932E+03	276.3	171
	13,897	0.95	0.03	90	2.381E-02	21.614	9.078E+02	222.5	80
	13,897	0.90	−0.76	46	6.049E-02	14.137	2.337E+02	384.9	50
	17,262	0.99	0.98	405	1.701E-02	62.282	3.662E+03	296.8	151
	17,262	0.98	0.94	206	7.684E-03	37.329	4.858E+03	383.9	165
	17,262	0.97	0.70	140	5.230E-03	28.467	5.443E+03	406.4	138
	17,262	0.95	−0.17	90	2.162E-02	21.584	9.981E+02	379.0	76

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