A QMC-spectral method for elliptic PDEs with random coefficients on the unit sphere

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Abstract We present a quasi-Monte Carlo spectral method for a class of elliptic partial differential equations (PDEs) with random coefficients defined on the unit sphere. The random coefficients are parametrised by the Karhunen-Loève expansion, while the exact solution is approximated by the spherical harmonics. The expectation of the solution is approximated by a quasi-Monte Carlo integration rule. A method for obtaining error estimates between the exact and the approximate solution is also proposed. Some numerical experiments are provided in the last section.

1 Introduction

Let S be the unit sphere in $\mathbb{R}^3$, i.e. $S = \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}| = 1\}$. Let (Ω, Σ, P) be a probability space and assume that $a(\cdot, \omega) : \Omega \to L^\infty(S)$ is a P-measurable map. We assume

$$a \in L^2(\Omega, dP; L^\infty(S)), \tag{1}$$

which renders the mean and variance of the random field a as elements of $L^\infty(S)$ and respectively of $L^\infty(S \times S)$, finite.

Given a random diffusion coefficient $a(\mathbf{x}, \omega)$, a prediction of the concentration $u(\mathbf{x}, \omega)$ for $\mathbf{x} \in S$ requires a solution of a stochastic differential equation such as

$$\begin{cases} -\mathrm{Div}\,(a(\mathbf{x},\omega)\mathrm{Grad}\,u(\mathbf{x},\omega)) & = f(\mathbf{x}) \quad \text{on } S, \\ \int_S u(\mathbf{x},\omega)dS(\mathbf{x}) & = 0, \quad \omega \in \Omega, \end{cases} \tag{2}$$

where Div and Grad are the surface divergence and surface gradient on the sphere respectively.

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These equations can be used to describe a diffusion on the sphere in which the diffusivity contains random noises coming from measurements. This situation can occur when turbulent diffusivity of the atmosphere is inferred from radar measurements [11].

To ensure (2) has a unique solution, we assume further that $a \in L^\infty(S \times \Omega)$ is strictly positive, with lower and upper bound $a_{\min} > 0$ and $a_{\max} < \infty$ respectively, i.e.

$$a_{\min} \le \text{ess inf } a(\mathbf{x}, \omega) \text{ and ess sup } a(\mathbf{x}, \omega) \le a_{\max} \quad \text{P-a.s,} \tag{3}$$

where the essential infimum and supremum are taken with respect to the Lebesgue measure in S.

In this work, we propose an approximation scheme for (2) using the Karhunen-Loève expansion on the unit sphere of the random coefficient. A similar approximation model for elliptic PDEs with random coefficients on bounded domains in $\mathbb{R}^n$ has been proposed recently [4].

The paper is organised as follows. In Section 2, we review background materials on spherical harmonics, Karhunen-Loève expansion on the unit sphere, quasi Monte Carlo method using lattice rules. In Section 3, we describe the parametric variational formulation of the PDE and discuss the regularity of the solution, QMC integration for the exact solution of the PDE. The spectral method on the sphere and the combined error estimates are presented in the last two sections of the paper.

2 Preliminaries

2.1 Spherical harmonics

Spherical harmonics are the restriction to S of homogeneous polynomials Y in $\mathbb{R}^3$ which satisfy $\Delta Y = 0$, where Δ is the Laplacian operator in $\mathbb{R}^3$. The space of all spherical harmonics of degree ℓ on S, denoted by $\mathscr{H}_\ell$, has an orthonormal basis

$$\{Y_{\ell,m} : m = -\ell, \dots, \ell\}.$$

The space of spherical harmonics of degree $\le L$ will be denoted by $\mathscr{P}_L := \bigoplus_{\ell=0}^{L} \mathscr{H}_\ell$; it has dimension $(\ell+1)^2$. Every function $f \in L^2(S)$ can be expanded in terms of spherical harmonics,

$$f = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \widehat{f}_{\ell,m} Y_{\ell,m}, \quad \widehat{f}_{\ell,m} = \int_S f \overline{Y_{\ell,m}} dS,$$

where dS is the surface measure of the unit sphere. The following formulas are the addition theorems for spherical harmonics [10, page 223].

$$\sum_{m=-\ell}^{\ell} Y_{\ell,m}(\mathbf{x})\overline{Y_{\ell,m}}(\mathbf{y}) = \frac{2\ell+1}{4\pi} P_\ell(\mathbf{x}\cdot\mathbf{y}), \tag{4}$$

$$\sum_{m=-\ell}^{\ell} \mathrm{Grad}\, Y_{\ell,m}(\mathbf{x})\cdot\overline{\mathrm{Grad}\, Y_{\ell,m}}(\mathbf{y}) = \frac{(2\ell+1)\ell(\ell+1)}{4\pi} P_\ell(\mathbf{x}\cdot\mathbf{y}), \tag{5}$$

where P_ℓ is the Legendre polynomial of degree ℓ normalised so that $P_\ell(1) = 1$.

Lemma 1. *Let Y_ℓ be a spherical harmonic of degree ℓ. Then*

$$|Y_\ell(\mathbf{x})| \le \sqrt{\frac{2\ell+1}{4\pi}} \left(\int_S |Y_\ell(\mathbf{x})|^2 dS\right)^{1/2} \tag{6}$$

and

$$|\mathrm{Grad}\, Y_\ell(\mathbf{x})| \le \sqrt{\frac{2\ell+1}{4\pi}} \left(\int_S |\mathrm{Grad}\, Y_\ell(\mathbf{x})|^2 dS\right)^{1/2} \tag{7}$$

Proof. Inequality (6) is a result of [6, Lemma 8, p.14]. In order to prove inequality (7), suppose $Y_\ell(\mathbf{x}) = \sum_{m=-\ell}^{\ell} d_m Y_{\ell,m}(\mathbf{x})$, where $d_m = (Y_\ell, Y_{\ell,m})_{L^2(S)}$. We use the orthogonality [10, page 227] $\int_S \mathrm{Grad}\, Y_{\ell,m}\cdot\overline{\mathrm{Grad}\, Y_{\ell',m'}} dS = \ell(\ell+1)\delta_{\ell,\ell'}\delta_{m,m'}$, to obtain

$$\int_S |\mathrm{Grad}\, Y_\ell|^2 dS = \ell(\ell+1) \sum_{m=-\ell}^{\ell} (d_m)^2. \tag{8}$$

Applying Cauchy-Schwarz's inequality and (5) we have

$$|\mathrm{Grad}\, Y_\ell(\mathbf{x})|^2 \le \sum_{m=-\ell}^{\ell} (d_m)^2 \sum_{m=-\ell}^{\ell} |\mathrm{Grad}\, Y_{\ell,m}(\mathbf{x})|^2 = \frac{(2\ell+1)\ell(\ell+1)}{4\pi} \sum_{m=-\ell}^{\ell} (d_m)^2.$$

Combining this with (8), we obtain (7). □

2.2 Karhunen-Loève expansion on the unit sphere

To define the Karhunen-Loève (KL) expansion of $a(\mathbf{x},\omega)$, we assume the mean field and two-point correlation of $a(\mathbf{x},\omega)$ are known, i.e. that

$$\bar{a}(\mathbf{x}) := \int_\Omega a(\mathbf{x},\omega)dP(\omega) \text{ and } C_a(\mathbf{x},\mathbf{y}) := \int_\Omega a(\mathbf{x},\omega)a(\mathbf{y},\omega)dP(\omega) \tag{9}$$

are known. An equivalent assumption is that the mean field $\bar{a}$ and its covariance V_a are known, since by definition,

$$V_a(\mathbf{x},\mathbf{y}) = C_a(\mathbf{x},\mathbf{y}) - \bar{a}(\mathbf{x})\bar{a}(\mathbf{y}). \tag{10}$$

The 2-point correlation of $a(\mathbf{x},\omega)$ is well-defined and belongs to $L^\infty(S\times S)$ due to (1). Associated with V_a we can define a compact, self-adjoint operator $\mathscr{V}_a : L^2(S)\to L^2(S)$ by

$$(\mathscr{V}_a u)(\mathbf{x}) = \int_S V_a(\mathbf{x},\mathbf{y})u(\mathbf{y})dS(\mathbf{y}), \tag{11}$$

where dS is the surface measure of the unit sphere S.

A covariance function $V_a(\mathbf{x},\mathbf{y})\in L^2(S\times S)$ given by (10) is said to be *admissible* if it is symmetric and positive definite in the sense that

$$\sum_{k=1}^{n}\sum_{j=1}^{n} a_k V_a(\mathbf{x}_k,\mathbf{x}_j)\overline{a}_j \geq 0, \quad \forall \mathbf{x}_j,\mathbf{x}_k\in S, \quad a_k,a_j\in\mathbb{C}.$$

Using a characterisation of positive definite functions on the unit sphere by Schoenberg [8, Theorem 1], we conclude that an admissible covariance kernel $V_a(\mathbf{x},\mathbf{y})$ admits the following expansion into spherical harmonics

$$V_a(\mathbf{x},\mathbf{y}) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\widehat{v}_\ell Y_{\ell,m}(\mathbf{x})\overline{Y_{\ell,m}}(\mathbf{y}), \quad \mathbf{x},\mathbf{y}\in S. \tag{12}$$

where

$$\sum_{\ell=0}^{\infty}(2\ell+1)\widehat{v}_\ell < \infty, \quad \widehat{v}_\ell > 0, \quad \ell = 0,1,\ldots \tag{13}$$

From the addition theorem for spherical harmonics (4),

$$V_a(\mathbf{x},\mathbf{y}) = \sum_{\ell=0}^{\infty}\frac{2\ell+1}{4\pi}\widehat{v}_\ell P_\ell(\mathbf{x}\cdot\mathbf{y}).$$

Therefore, in view of condition (13), the series (12) converge uniformly by Weierstrass M-test.

Using the orthogonality of the spherical harmonics, we have

$$\int_S V_a(\mathbf{x},\mathbf{y})Y_{\ell,m}(\mathbf{y})dS(\mathbf{y}) = \widehat{v}_\ell Y_{\ell,m}(\mathbf{x}).$$

Therefore $\{(\widehat{v}_\ell, Y_{\ell,m}) : \ell = 0,1,\ldots; m = -\ell,\ldots,\ell\}$ is the sequence of eigenpairs of the integral operator $\mathscr{V}_a$.

Using the Loève representation theorem, since S is a compact set and the spherical harmonics form an orthonormal basis of $L^2(S)$, the random field (1) takes the form

$$a(\mathbf{x},\omega) = \bar{a}(\mathbf{x}) + \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\sqrt{\widehat{v}_\ell}Y_{\ell,m}(\mathbf{x})X_{\ell,m}(\omega), \tag{14}$$

where $X_{\ell,m}(\omega)$ are centred at 0, pairwise uncorrelated random variables on probability spaces $(\Omega_{\ell,m},\Sigma_{\ell,m},P_{\ell,m})$ for $\ell = 0,1,2,\ldots; m = -\ell,\ldots,\ell$.

We now assume that

$$\bar{a} \in W^{1,\infty}(S), \quad \sum_{\ell=1}^{\infty} \sqrt{\ell(\ell+1)(2\ell+1)}\sqrt{\widehat{v}_\ell} < \infty, \tag{15}$$

where $\|v\|_{W^{1,\infty}(S)} = \max\{\|v\|_{L^\infty(S)}, \|\text{Grad}\, v\|_{L^\infty(S)}\}$.

Since $Y_{\ell,m}$ is an element of an orthonormal basis, we deduce from (6),

$$|Y_{\ell,m}(\mathbf{x})| \leq \sqrt{\frac{2\ell+1}{4\pi}} \quad \forall \mathbf{x} \in S. \tag{16}$$

From assumption (15) and estimate (16) we obtain

$$\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \sqrt{\widehat{v}_\ell} \|Y_{\ell,m}(\mathbf{x})\|_{L^\infty(S)} < \infty. \tag{17}$$

We sometimes make a stronger assumption that

$$\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \sqrt{(\widehat{v}_\ell)^p} \|Y_{\ell,m}(\mathbf{x})\|_{L^\infty(S)}^p < \infty, \quad 0 < p < 1. \tag{18}$$

Using the orthogonality of $\text{Grad}\, Y_{\ell,m}$, we deduce from (7) that

$$|\text{Grad}\, Y_{\ell,m}(\mathbf{x})| \leq \sqrt{\frac{\ell(2\ell+1)(\ell+1)}{4\pi}}. \tag{19}$$

So from (19) and (15) we deduce that

$$\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \sqrt{\widehat{v}_\ell} \|Y_{\ell,m}(\mathbf{x})\|_{W^{1,\infty}(S)} < \infty, \tag{20}$$

which will guarantee the convergence of the approximate solution.

Following [1], we assume that the diffusion coefficient $a(\mathbf{x}, \omega)$ satisfies (3), the covariance kernel V_a is admissible and $a(\mathbf{x}, \omega)$ admits a Karhunen-Loève expansion (14). Furthermore, we make the following assumptions on the random variables $X_{\ell,m}$ in the KL-expansion (14) of the random coefficient.

Assumption 2.1 *(i) The family* $\{X_{\ell,m} : \ell = 0, 1, 2, \ldots; m = -\ell, \ldots, \ell\}$ *is independent.*

(ii)The KL-expansion (14) *of the random coefficient is finite, i.e. there exists* $\overline{M} < \infty$ *such that* $X_{\ell,m} = 0$ *for all* $\ell > \overline{M}$.

(iiiEach $X_{\ell,m}(\omega)$ *in* (14) *is associated with a probability space* $(\Omega_{\ell,m}, \Sigma_{\ell,m}, P_{\ell,m})$ *with the following properties:*

(a)the range of $X_{\ell,m}$, $U_{\ell,m} := Range(X_{\ell,m}) \subset \mathbb{R}$, *is assumed to be compact,*

(b)the probability measure $P_{\ell,m}$ *admits a probability density function* $\rho_{\ell,m} : U_{\ell,m} \to [0, \infty)$ *such that* $dP_{\ell,m}(\omega) = \rho_{\ell,m}(y_{\ell,m}) dy_{\ell,m}$, $y_{\ell,m} \in U_{\ell,m}$, *and*

(c)the sigma algebras $\Sigma_{\ell,m}$ *are subsets of the Borel sets of the interval* $U_{\ell,m}$, *i.e.* $\Sigma_{\ell,m} \subset \mathscr{B}(U_{\ell,m})$

In the sequel, we let $\Lambda := \{\ell = 0,1,2,\ldots,\overline{M}; m = -\ell,\ldots,\ell\}$.

Assumption 2.1 ii) is made so that we can represent the measure P on the space of input data as an $\overline{M}-$fold product measure and to avoid technical issues related to countable product measures on the space $L^\infty(S)$ of input data. We have

$$\Sigma = \bigotimes_{(\ell,m)\in\Lambda} \Sigma_{\ell,m}, \quad dP = \bigotimes_{(\ell,m)\in\Lambda} dP_{\ell,m}, \quad U = \bigotimes_{(\ell,m)\in\Lambda} U_{\ell,m}.$$

While the double index (ℓ,m) follows the convention of spherical harmonics, it is inconvenient in subsequent analysis. We introduce a single index via the map

$$j(\ell,m) = \ell(\ell+1)+1+m, \quad \ell = 0,1,2,\ldots; m = -\ell,\ldots,\ell.$$

In subsequent sections, we assume that $X_{\ell,m}(\omega) = \omega_{\ell,m}$ uniformly distributed and $U = \left[-\frac{1}{2},\frac{1}{2}\right]^s$, for some $s \le \overline{M}$.

2.3 Quasi-Monte Carlo integration in weighted spaces

Let s be a positive integer, we consider integrals over the s-dimensional cube $\left[-\frac{1}{2},\frac{1}{2}\right]^s$ of the form

$$\mathscr{I}_s(F) := \int_{\left[-\frac{1}{2},\frac{1}{2}\right]^s} F(\mathbf{y})d\mathbf{y}.$$

An N-point QMC approximation to this integral is an equal weight quadrature of the form

$$Q_{s,N}(F) := \frac{1}{N}\sum_{i=1}^{N} F(\mathbf{y}^{(i)}),$$

with carefully chosen points $\mathbf{y}^{(1)},\ldots,\mathbf{y}^{(N)} \in [-\frac{1}{2},\frac{1}{2}]^s$. We shall assume that the integrand F belongs to weighted and anchored Sobolev space $\mathscr{W}_{s,\gamma}$, which is a Hilbert space equipped with the norm

$$\|F\|_{\mathscr{W}_{s,\gamma}} := \left(\sum_{\mathfrak{u}\subseteq\{1:s\}} \gamma_{\mathfrak{u}}^{-1} \int_{[-\frac{1}{2},\frac{1}{2}]^{|\mathfrak{u}|}} \left| \int_{[-\frac{1}{2},\frac{1}{2}]^{s-|\mathfrak{u}|}} \frac{\partial^{|\mathfrak{u}|}F}{\partial \mathbf{y}_{\mathfrak{u}}}(\mathbf{y}_{\mathfrak{u}};\mathbf{y}_{\{1:s\}\setminus\mathfrak{u}};0)d\mathbf{y}_{\{1:s\}\setminus\mathfrak{u}} \right|^2 d\mathbf{y}_{\mathfrak{u}} \right)^{1/2} \quad (21)$$

The norm of $\mathscr{I}_s$ as a linear functional on the function space $\mathscr{W}_{s,\gamma}$ is, from [9],

$$\|\mathscr{I}_s\| := \sup_{\|F\|_{\mathscr{W}_{s,\gamma}}\le 1} |\mathscr{I}_s(F)| = \left(\sum_{\mathfrak{u}\subseteq\{1:s\}} \gamma_{\mathfrak{u}} \left(\frac{1}{12}\right)^{|\mathfrak{u}|}\right)^{1/2}.$$

We shall assume that we have a sequence of positive weights $\gamma = (\gamma_{\mathfrak{u}})_{|\mathfrak{u}|<\infty}$ satisfying

$$\sum_{|\mathfrak{u}|\leq\infty} \gamma_{\mathfrak{u}} \left(\frac{1}{12}\right)^{|\mathfrak{u}|} < \infty. \tag{22}$$

For the moment, we focus on a family of QMC rules known as "shifted rank-1 lattice rules", whose quadrature points are given by the following formula

$$\mathbf{y}^{(i)} = \operatorname{frac}\left(\frac{i\mathbf{z}}{N} + \Delta\right) - \left(\frac{1}{2}, \dots, \frac{1}{2}\right), \quad i = 1, \dots, N,$$

where $\mathbf{z} \in \mathbb{Z}^s$ is known as the *generating vector*, $\Delta \in [0,1]^s$ is the *shift*, and $\operatorname{frac}(\cdot)$ means to take the fractional part of each component in the vector.

An application of $Q_{s,N}$ with a realization for a draw of the shift Δ will be denoted by $Q_{s,N}(\cdot;\Delta)$.

Theorem 1. *[4, Theorem 2.1] Let $s, N \in \mathbb{N}$ be given, and assume $F \in \mathscr{W}_{s,\gamma}$ for a particular choice of weights γ. Then a randomly shifted lattice rule can be constructed using a component-by-component algorithm such that the root-mean-square error for approximating the s-dimensional integral $\mathscr{I}_s(F)$ satisfies, for all $\lambda \in (1/2, 1]$,*

$$\sqrt{\mathbb{E}[|\mathscr{I}_s(F) - Q_{s,N}(F;\cdot)|^2]} \tag{23}$$

$$\leq \left(\sum_{|\mathfrak{u}|<\infty} \gamma_{\mathfrak{u}}^{\lambda} \left(\frac{2\zeta(2\lambda)}{(2\pi^2)^{\lambda}} + \frac{1}{12^{\lambda}} \right)^{|\mathfrak{u}|} \right)^{1/(2\lambda)} (N-1)^{-1/(2\lambda)} \|F\|_{\mathscr{W}_{s,\gamma}}, \tag{24}$$

where $\mathbb{E}[\cdot]$ denotes the expectation with respect to the random shift Δ which is uniformly distributed over $[0,1]^s$.

3 Parametric variational formulation

3.1 Function spaces

Define the following Sobolev space

$$H^1(S) := \{u \in L^2(S) : \operatorname{Grad} u \in \mathbb{L}^2(S)\}.$$

That is, $H^1(S)$ consists of functions $u \in L^2(S)$, whose weak gradient $\operatorname{Grad} u$ exists and is in the space $\mathbb{L}^2(S)$, which contains all vector fields $\mathbf{u}$ so that $\int_S \mathbf{u}\cdot\mathbf{u} < \infty$.

Let V be a subspace of $H^1(S)$ which contains all functions with zero mean over S, i.e.

$$V := \left\{ u \in H^1(S) : \int_S u = 0 \right\}.$$

It can be shown that V is a Hilbert space with the following inner product and norm

$$(u,v)_V = (\mathrm{Grad}\, u, \mathrm{Grad}\, v)_{\mathbb{L}^2(S)}, \quad \|u\|_V = \|\mathrm{Grad}\, u\|_{\mathbb{L}^2(S)}.$$

Let V^* be the dual space of V with respect to the $L^2(S)$ inner product $(\cdot,\cdot)$, i.e., the space of all continuous linear functionals defined on V. Since $V \subset H^1(S)$, we have $H^{-1}(S) \subset V^*$. We also consider the following function space

$$Z := \{v \in V : \Delta^* v \in L^2(S)\},$$

where Δ^* is the Laplace-Beltrami operator on S. The space $Z \subset V$ is a closed subspace which, when equipped with the norm

$$\|v\|_Z := \left(\|v\|^2_{L^2(S)} + \|\Delta^* v\|^2_{L^2(S)}\right)^{1/2},$$

is a Hilbert space. The space Z is a subspace of $H^2(S)$, see [5] for definitions of Sobolev spaces on manifolds based on the Laplace–Beltrami operator.

We also define the weighted spaces $\mathscr{W}_{s,\gamma}(U;V)$, which are the Bochner versions of the weighted spaces $\mathscr{W}_{s,\gamma}$, with the norm

$$\|u\|_{\mathscr{W}_{s,\gamma}(U;V)} := \left(\sum_{|\mathfrak{u}|\subseteq\{1:s\}} \gamma_{\mathfrak{u}}^{-1} \int_{[-\frac{1}{2},\frac{1}{2}]^{|\mathfrak{u}|}} \left\| \int_{[-\frac{1}{2},\frac{1}{2}]^{s-|\mathfrak{u}|}} \frac{\partial^{|\mathfrak{u}|} u}{\partial \mathbf{y}_{\mathfrak{u}}}(\cdot,(\mathbf{y}_{\mathfrak{u}};\mathbf{y}_{\{1:s\}\setminus\mathfrak{u}};0))\, d\mathbf{y}_{\{1:s\}\setminus\mathfrak{u}} \right\|_V^2 d\mathbf{y}_{\mathfrak{u}} \right)^{1/2}. \quad (25)$$

In this paper, we wish to compute

$$\int_U F(\mathbf{y})\,d\mathbf{y}, \quad \text{with} \quad F(\mathbf{y}) = G(u(\cdot,\mathbf{y})), \quad G \in V^*. \quad (26)$$

Then for $s \geq 1$ and for $u \in \mathscr{W}_{s,\gamma}(U;V)$, using (21) and

$$\frac{\partial^{|\mathfrak{u}|} F}{\partial \mathbf{y}_{\mathfrak{u}}}(\mathbf{y}) = G\left(\frac{\partial^{|\mathfrak{u}|} u}{\partial \mathbf{y}_{\mathfrak{u}}}(\cdot,\mathbf{y})\right),$$

we have

$$\|F\|_{\mathscr{W}_\gamma} \leq \|G(\cdot)\|_{V^*} \|u\|_{\mathscr{W}_{s,\gamma}(U;V)} < \infty. \quad (27)$$

3.2 The parametric variational formulation

The weak formulation of (2) is given by

$$E\left[\int_S a(\mathbf{x},\omega)\mathrm{Grad}\,u(\mathbf{x},\omega)\cdot\mathrm{Grad}\,v(\mathbf{x},\omega)dS(\mathbf{x})\right]=E\left[\int_S f(\mathbf{x})v(\mathbf{x},\omega)dS(\mathbf{x})\right],$$

where E denotes the expectation with respect to the random variable ω.

As a consequence of the independence in Assumption 2.1, the multivariate probability density on U is given by

$$\rho(\mathbf{y}) := \Pi_{(\ell,m)\in\Lambda}\rho_{\ell,m}(y_{\ell,m}).$$

We substitute $X_{\ell,m}(\omega)$ by $y_{\ell,m}$ and equip the range U of the vector $\mathbf{y}$ with the product measure $dP(\omega)=\bigotimes\rho_{\ell,m}(y_{\ell,m})dy_{\ell,m}$. Here we assume that the random variable $X_{\ell,m}$ and the random numbers $y_{\ell,m}$ have the same probability distribution. Changing measure from $dP(\omega)$ to $\Pi_{(\ell,m)\in\Lambda}\rho_{\ell,m}(y_{\ell,m})dy_{\ell,m}$, problem (2) is equivalent to the parametric, deterministic problem:

$$\text{Find } u(\cdot,\mathbf{y})\in V \text{ such that } -\mathrm{Div}\,a(\mathbf{x},\mathbf{y})\mathrm{Grad}\,u(\mathbf{x},\mathbf{y})=f(\mathbf{x}),\quad \mathbf{x}\in S,\ \mathbf{y}\in U. \tag{28}$$

For a fixed $\mathbf{y}\in U$, the parametric variational formulation of the PDE (28) is obtained by multiplying it with a test function and integrating by part: find $u(\cdot,\mathbf{y})\in V$ such that

$$\int_S a(\mathbf{x},\mathbf{y})\mathrm{Grad}\,u(\mathbf{x},\mathbf{y})\cdot\mathrm{Grad}\,v(\mathbf{x},\mathbf{y})dS(\mathbf{x})=\int_S f(\mathbf{x})v(\mathbf{x})dS\quad\forall v\in V.$$

Let us define the parametric bilinear form $b(\mathbf{y};v,w)$ by

$$b(\mathbf{y};v,w):=\int_S a(\cdot,\mathbf{y})\mathrm{Grad}\,v\cdot\mathrm{Grad}\,w dS\quad v,w\in V. \tag{29}$$

In view of (3) the bilinear form $b(\cdot,\cdot)$ is continuous and coercive on $V\times V$, i.e., for all $\mathbf{y}\in U$ and all $v,w\in V$ we have

$$b(\mathbf{y};v,v)\ge a_{\min}\|v\|_V^2\quad\text{and}\quad|b(\mathbf{y};v,w)|\le a_{\max}\|v\|_V\|w\|_V.$$

The variational form now reads:

$$\text{Find } u(\cdot,\mathbf{y})\in V \text{ such that } b(\mathbf{y};u(\cdot,\mathbf{y}),v)=\langle f,v\rangle\quad\forall v\in V. \tag{30}$$

By the Lax-Milgram Lemma we conclude that for every $f\in V^*$, there exists a unique solution to the parametric weak problem (30).

Theorem 2. *Under Assumptions* (3) *and* (17)*, for every $f\in V^*$ and every $\mathbf{y}\in U$, there exists a unique solution $u(\cdot,\mathbf{y})$ of the parametric weak problem* (30)*, which satisfies*

$$\|u(\cdot,\mathbf{y})\|_V\le\frac{1}{a_{\min}}\|f\|_{V^*}.$$

3.3 Regularity of the PDE solution

Assume that $f \in L^2(S)$, we want to obtain a bound on the Z norm of $u(\cdot,\mathbf{y})$ for each value of the parameter $\mathbf{y}$.

Theorem 3. *Under Assumptions* (3) *and* (15)*, there exists a constant $C > 0$ such that for every $f \in L^2(S)$ and every $\mathbf{y} \in U$, the solution $u(\cdot,\mathbf{y}) \in V$ of the parametric weak problem* (30) *satisfies*

$$\|u(\cdot,\mathbf{y})\|_Z \le C\|f\|_{L^2(S)}. \tag{31}$$

Proof. Using Assumption (15), which implies (20), for every $\mathbf{y} \in U$, we have,

$$\|a(\cdot,\mathbf{y})\|_{W^{1,\infty}(S)} \le \|\bar{a}\|_{W^{1,\infty}(S)} + \frac{1}{2}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\sqrt{\widehat{\nu}_\ell}\|Y_{\ell,m}\|_{W^{1,\infty}(S)} < \infty.$$

We apply the identity

$$\operatorname{Div}(\alpha(\mathbf{x})\operatorname{Grad} w(\mathbf{x})) = \alpha(\mathbf{x})\Delta^* w(\mathbf{x}) + \operatorname{Grad}\alpha(\mathbf{x})\cdot\operatorname{Grad} w(\mathbf{x}),$$

to (2) in order to obtain

$$\begin{aligned}
&-a(\cdot,\mathbf{y})\Delta^* u(\cdot,\mathbf{y}) = \operatorname{Grad} a(\cdot,\mathbf{y})\cdot\operatorname{Grad} u(\cdot,\mathbf{y}) + f(\cdot) \quad \text{on S},\\
&\int_S u(\mathbf{x},\mathbf{y})dS(\mathbf{x}) = 0.
\end{aligned}$$

This implies that for every $\mathbf{y} \in U$ there holds

$$a_{\min}\|\Delta^* u(\cdot,\mathbf{y})\|_{L^2(S)} \le \|a(\cdot,\mathbf{y})\|_{W^{1,\infty}(S)}\|u(\cdot,\mathbf{y})\|_V + \|f\|_{L^2(S)},$$

and this yields

$$\|u(\cdot,\mathbf{y})\|_Z^2 \le \|u(\cdot,\mathbf{y})\|_{L^2(S)}^2 + \frac{1}{a_{\min}^2}(\|a(\cdot,\mathbf{y})\|_{W^{1,\infty}(S)}\|u(\cdot,\mathbf{y})\|_V + \|f\|_{L^2(S)})^2.$$

Using the Poincaré inequality $\|u\|_{L^2(S)} \le C_P\|u\|_V$ for all $u \in V$, we obtain

$$\|u(\cdot,\mathbf{y})\|_Z^2 \le \left(\frac{1}{C_P^2} + \frac{2}{a_{\min}^2}\sup_{\mathbf{z}\in U}\|a(\cdot,\mathbf{z})\|_{W^{1,\infty}(S)}^2\right)\|u(\cdot,\mathbf{y})\|_V^2 + \frac{2}{a_{\min}^2}\|f\|_{L^2(S)}^2.$$

The proof is completed by using Theorem 2. □

In the following, we will discuss the regularity of $u(\mathbf{x},\mathbf{y})$ with respect to the $\mathbf{y}$ variable. Firstly, we introduce a multi-index notation. For $\mu = (\mu_j)_{j\ge1} \in \mathbb{N}_0^{\mathbb{N}}$, where $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$, we define $|\mu| := \mu_1 + \mu_2 + \cdots$, and we refer to μ as a "multi-index" and $|\mu|$ as the length of μ. For $\mu \subset \mathbb{N}$ of finite cardinality, we denote by

$$\partial_{\mathbf{y}}^{\boldsymbol{\mu}} u := \frac{\partial^{|\boldsymbol{\mu}|}}{\partial_{y_1}^{\mu_1} \partial_{y_2}^{\mu_2} \cdots} u$$

the partial derivative of order $|\boldsymbol{\mu}|$ of u with respect to $\mathbf{y}$.

Theorem 4. *Under Assumptions* (3) *and* (17)*, for every $f \in V^*$, every $\mathbf{y} \in U$ and every $\boldsymbol{\mu} \subset \mathbb{N}$ of finite cardinality, the solution $u(\cdot,\mathbf{y})$ of the parametric weak problem* (30) *satisfies*

$$\|\partial_{\mathbf{y}}^{\boldsymbol{\mu}} u(\cdot,\mathbf{y})\|_V \leq |\boldsymbol{\mu}|! \frac{\|f\|_{V^*}}{a_{\min}} \prod_{j \in \boldsymbol{\mu}} b_j, \tag{32}$$

where the sequence $\mathbf{b} = (b_j)_{j\geq 1} \in \ell^1(\mathbb{N})$ is defined by

$$b_j := \frac{1}{a_{\min}} \sqrt{\widetilde{v}_j} \|Y_j\|_{L^\infty(S)}. \tag{33}$$

Remark. As mentioned earlier, the relationship between j, ℓ, m is given by

$$j(\ell,m) = \ell(\ell+1)+1+m, \quad \ell = 0,1,2,\ldots; m = -\ell,\ldots,\ell.$$

The coefficients $\widetilde{v}_j$ are defined by

$$\widetilde{v}_j = \widehat{v}_\ell, \quad \text{for} \quad \ell(\ell+1)+1-\ell \leq j \leq \ell(\ell+1)+1+\ell.$$

Proof. The proof of this theorem can be adapted from the proof of [4, Theorem 4.2] with a slight modification, namely the ∇ is replaced by the surface gradient Grad on the sphere. The key idea is the following recurrence relation deduced from (30)

$$\begin{aligned}&\int_S a(\mathbf{x},\mathbf{y}) \mathrm{Grad}\,(\partial_{\mathbf{y}}^{\boldsymbol{\mu}} u(\mathbf{x},\mathbf{y})) \cdot \mathrm{Grad}\, v(\mathbf{x}) dS(\mathbf{x}) \\ &\quad + \sum_{j \in \mathrm{supp}(\boldsymbol{\mu})} \mu_j \int_S \sqrt{\widetilde{v}_j} Y_j(\mathbf{x}) \mathrm{Grad}\,(\partial_{\mathbf{y}}^{\boldsymbol{\mu}-\mathbf{e}_j} u(\mathbf{x},\mathbf{y})) \cdot \mathrm{Grad}\, v(\mathbf{x}) dS(\mathbf{x}) = 0,\end{aligned} \tag{34}$$

for every $v \in V$, $\mathbf{y} \in U$ and $\boldsymbol{\mu} \subset \mathbb{N}$ with $0 \neq |\boldsymbol{\mu}| < \infty$. Here $\mathbf{e}_j \in \mathbb{N}^{\mathbb{N}}$ denotes the multi-index with entry 1 in the jth position and zeros elsewhere, and where $\mathrm{supp}(\mu) = \{j \in \mathbb{N} : \nu_j \neq 0\}$ denotes the "support" of μ. We now select in (34) the function $v(\mathbf{x}) = \partial_{\mathbf{y}}^{\boldsymbol{\mu}} u(\mathbf{x},\mathbf{y}) \in V$ to estimate $\partial_{\mathbf{y}}^{\boldsymbol{\mu}} u(\mathbf{x},\mathbf{y})$ in appropriate norms. □

3.4 Dimensional truncation

Theorem 5. *Under Assumptions* (3) *and* (17)*, for every $f \in V^*$, every $\mathbf{y} \in U$ and every $s \in \mathbb{N}$, the solution $u(\cdot,(\mathbf{y}_{\{1:s\}};0))$ of the truncated parametric weak problem* (30) *satisfies*

$$\|u(\cdot,\mathbf{y})-u(\cdot,(\mathbf{y}_{\{1:s\}};0))\|_V \le \frac{\|f\|_{V^*}}{2a_{\min}^2}\sum_{j\ge s+1}\sqrt{\widetilde{v}_j}\|Y_j\|_{L^\infty(S)}.$$

Proof. The proof of this theorem can also be adapted from the proof of [4, Theorem 5.1] when the ∇ is replaced by the surface gradient Grad and the eigenfunctions $\psi_j = \sqrt{\widetilde{v}_j}Y_j$. □

4 Spectral method on the sphere

Let $\mathscr{P}_L^*$ be the space of all spherical harmonics of degree $\le L$ excluding the constant function, i.e.

$$\mathscr{P}_L^* = \operatorname{span}\{Y_{\ell,m} : \ell = 1,\dots,L;\ m = -\ell,\dots,\ell\}.$$

The space $\mathscr{P}_L^*$ is a subspace of V since the orthogonality of the spherical harmonics implies

$$\int_S p dS = 0 \quad \forall p \in \mathscr{P}_L^*.$$

For a given function $u \in H^m(S)$ with $m \ge 1$, there is a constant $C > 0$ such that [3]

$$\inf_{p\in\mathscr{P}_L^*}\|u-p\|_{H^1(S)} \le CL^{1-m}\|u\|_{H^m(S)}. \tag{35}$$

Consequently, for a given function $u \in Z \subset H^2(S)$, there is a positive constant C such that

$$\inf_{p\in\mathscr{P}_L^*}\|u-p\|_V \le CL^{-1}\|u\|_Z. \tag{36}$$

For any $\mathbf{y} \in U$, we define the parametric spectral approximation $u_L(\cdot,\mathbf{y})$ as the spectral solution of the parametric deterministic problem: for $f \in V^*$ and $\mathbf{y} \in U$, find

$$u_L(\cdot,\mathbf{y}) \in \mathscr{P}_L^* : \quad b(\mathbf{y}; u_L(\cdot,\mathbf{y}), p) = \langle f, p\rangle \quad \forall p \in \mathscr{P}_L^*. \tag{37}$$

Theorem 6. *Under Assumptions* (3) *and* (15)*, for every $f \in V^*$ and every $\mathbf{y} \in U$, the spectral approximations $u_L(\cdot,\mathbf{y})$ are stable in the sense that*

$$\|u_L(\cdot,\mathbf{y})\|_V \le \frac{\|f\|_{V^*}}{a_{\min}}. \tag{38}$$

Moreover, for every $f \in L^2(S)$, as $L \to \infty$, there exists a constant $C > 0$ independent of L such that

$$\|u(\cdot,\mathbf{y})-u_L(\cdot,\mathbf{y})\|_V \le CL^{-1}\|f\|_{L^2(S)}. \tag{39}$$

Proof. The proof follows from Cea's lemma and the approximation property (36).

Since we are interested in estimating the error in approximating functionals (26), we also assume that $G(\cdot) \in L^2(S)$.

Theorem 7. *Under Assumptions* (3) *and* (15)*, for every* $f \in L^2(S)$*, every* $G(\cdot) \in L^2(S)$*, and every* $\mathbf{y} \in U$*, the spectral approximation* $G(u_L(\cdot,\mathbf{y}))$ *satisfy the asymptotic estimate*

$$|G(u(\cdot,\mathbf{y})) - G(u_L(\cdot,\mathbf{y}))| \leq CL^{-2}\|f\|_{L^2(S)}\|G(\cdot)\|_{L^2(S)},$$

where the constant $C > 0$ *is independent of* $\mathbf{y} \in U$.

Proof. For $G(\cdot) \in L^2(S)$ and any $\mathbf{y} \in U$, we define $v_G(\cdot,\mathbf{y}) \in V$ as the unique solution to the adjoint problem

$$b(\mathbf{y}; w, v_G(\cdot,\mathbf{y})) = G(w) \quad \forall w \in V. \tag{40}$$

Since b is symmetric, $b(\mathbf{y}; w, v) = b(\mathbf{y}; v, w)$ for all $v, w \in V$, we also have

$$b(\mathbf{y}; v_G(\cdot,\mathbf{y}), w) = G(w) \quad \forall w \in V.$$

So, by the regularity estimate (31), there is a constant $C > 0$, which is independent of $\mathbf{y}$ such that

$$\|v_G(\cdot,\mathbf{y})\|_Z \leq C\|G(\cdot)\|_{L^2(S)}. \tag{41}$$

Using the orthogonality property and (40), we may write for every $\mathbf{y} \in U$ and every $v_L \in \mathscr{P}_L^*$,

$$\begin{aligned}
|G(u(\cdot,\mathbf{y})) - G(u_L(\cdot,\mathbf{y}))| &= |G(u(\cdot,\mathbf{y}) - u_L(\cdot,\mathbf{y}))| \\
&= |b(\mathbf{y}; u(\cdot,\mathbf{y}) - u_L(\cdot,\mathbf{y}), v_G(\cdot,\mathbf{y})| \\
&= |b(\mathbf{y}; u(\cdot,\mathbf{y}) - u_L(\cdot,\mathbf{y}), v_G(\cdot,\mathbf{y}) - p)| \\
&\leq C\|u(\cdot,\mathbf{y}) - u_L(\cdot,\mathbf{y})\|_V \|v_G(\cdot,\mathbf{y}) - p\|_V.
\end{aligned}$$

Finally, we apply (36), (38) and (41) to obtain

$$|G(u(\cdot,\mathbf{y}) - G(u_L(\cdot,\mathbf{y}))| \leq CL^{-2}\|f\|_{L^2(S)}\|v_G\|_Z \leq CL^{-2}\|f\|_{L^2(S)}\|G(\cdot)\|_{L^2(S)}.$$

□

5 Combined error estimates

We now present the error analysis for the combined QMC spectral approximation for the integral (26) using a randomly shifted lattice rule with N points in s dimensions. A realization for a draw of the shift Δ will be denoted by $Q_{s,N}(\cdot;\Delta)$ and for each evaluation of the integrand, the exact solution $u(\cdot,\mathbf{y})$ of the parametric weak problem (30) is replaced by its spectral approximation $u_L \in \mathscr{P}_L^*$.

Theorem 8. *Under the same assumptions and definitions as in Theorems 5, 6, and 7 and* (18)*, if we approximate the integral over* U *by the randomly shifted lattice rule* *from Theorem 1 with N points in s dimensions, and for each shifted lattice point we solve the approximate elliptic problem* (38) *by a spectral method then we have the root-mean-square error bound*

$$\sqrt{\mathbb{E}[|\mathscr{I}(G(u)) - Q_{s,N}(G(u_L);\cdot)|^2]}$$
$$\leq C(\kappa(s,N)\|f\|_{V^*}\|G(\cdot)\|_{V^*} + CL^{-2}\|f\|_{L^2(S)}\|G\|_{L^2(S)}),$$

where

$$\kappa(s,N) = \begin{cases} s^{2(1/p-1)} + N^{-(1-\delta)} & \text{when } p \in (0,2/3], \\ s^{-2(1/p-1)} + N^{-(1/p-1/2)} & \text{when } p \in (2/3,1), \\ (\sum_{j\geq s+1} b_j)^2 + N^{-1/2} & \text{when } p = 1, \end{cases}$$

and $\mathbb{E}[\cdot]$ *denotes the expectation with respect to the random shift* Δ *which is uniformly distributed over* $[0,1]^s$.

Proof. We express the overall error as the sum of a dimensional truncation error, a QMC quadrature error, and a spectral approximation error:

$$\mathscr{I}(G(u)) - Q_{s,N}(G(u_L);\Delta)$$
$$= (\mathscr{I} - \mathscr{I}_s)(G(u)) + (\mathscr{I}_s(G(u)) - Q_{s,N}(G(u);\Delta)) + Q_{s,N}(G(u-u_L);\Delta).$$

The mean-square error with respect to the random shift can then be bounded by

$$\begin{aligned} \mathbb{E}[|\mathscr{I}(G(u)) - Q_{s,N}(G(u_L);\cdot)|^2] \leq 3|(\mathscr{I} - \mathscr{I}_s)(G(u))|^2 + \\ 3\mathbb{E}\,[|I_s(G(u)) - Q_{s,N}(G(u);\cdot)|^2] + \\ 3\mathbb{E}\,[|Q_{s,N}(G(u-u_L);\cdot)|^2]. \end{aligned} \tag{42}$$

For the truncation error, i.e., the first term in (42), we use the estimate

$$\begin{aligned} |(\mathscr{I} - \mathscr{I}_s)(G(u))| &= \left| \int_U G(u(\cdot,\mathbf{y}) - u(\cdot,(\mathbf{y}_{\{1:s\}};0)))d\mathbf{y} \right| \\ &\leq \sup_{\mathbf{y}\in U} |G(u(\cdot,\mathbf{y}) - u(\cdot,(\mathbf{y}_{\{1:s\}};0)))| \\ &\leq \|G(\cdot)\|_{V^*} \sup_{\mathbf{y}\in U} \|u(\cdot,\mathbf{y}) - u(\cdot,(\mathbf{y}_{\{1:s\}};0))\|_V, \end{aligned}$$

and then apply Theorem 5. The QMC error is already analysed in [4, Section 6]. Finally, for the spectral error, i.e., the third term in (42), using the property that the QMC quadrature weights $1/N$ are positive and sum to 1, we obtain

$$\mathbb{E}[|Q_{s,N}(G(u-u_L);\cdot)|^2] \leq \sup_{\mathbf{y}\in U} |G(u(\cdot,\mathbf{y}) - u_L(\cdot,\mathbf{y}))|^2,$$

and then apply Theorem 7.

6 Numerical experiments

Let the unit sphere S by parametrised by

$$\mathbf{x} = (x_1, x_2, x_3) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \quad \theta \in [0,\pi], \phi \in [0,2\pi).$$

Let $U = \left[-\frac{1}{2}, \frac{1}{2}\right]^2$, we consider the following simplified problem

$$\begin{cases} -\mathrm{Div}\,(a(\mathbf{x},\mathbf{y})\mathrm{Grad}\,u(\mathbf{x},\mathbf{y})) & = f(\mathbf{x},\mathbf{y}), \\ \int_S u(\mathbf{x},\mathbf{y})dS(\mathbf{x}) & = 0, \quad \forall \mathbf{y} = (y_1, y_2) \in \left[-\frac{1}{2}, \frac{1}{2}\right]^2 \end{cases} \tag{43}$$

with

$$a(\mathbf{x},\mathbf{y}) = 3 + y_1 + y_2 x_3,$$

and

$$f(\mathbf{x},\mathbf{y}) = 2y_1 x_1 (8y_2 x_3^2 + 18x_3 + 6x_3 y_1 - y_2).$$

It can be shown that the exact solution is given by

$$u(\mathbf{x},\mathbf{y}) = y_1 \cos(\phi)\sin(2\theta) = 2y_1 x_1 x_3.$$

The spherical harmonics and their gradients are computed explicitly using formulas in Varshalovich's book [10], see also [2]. Integration of a function f on the sphere is approximated by a quadrature of the form

$$\int_S f dS \approx \frac{2\pi}{M} \sum_{p=1}^{M/2} w_p \sum_{q=0}^{M-1} f(\sin\theta_p \cos\phi_q, \sin\theta_p \sin\phi_q, \cos\theta_p),$$

for an even number $M \geq 2$, where $\int_{-1}^{1} g(z)dz \approx \sum_{p=1}^{M/2}$ is a Gauss-Legendre rule and $\phi_q = 2\pi q/R$.

For fixed value $\mathbf{y} = (1/2, \ -1/4)^T$, Table 1 shows the values of the quantities

$$e_{\max} = \max_{\mathbf{x}\in\mathscr{Q}} |u(\mathbf{x},\mathbf{y}) - u_L(\mathbf{x},\mathbf{y})| \text{ and } e_2 = \left(\sum_{\mathbf{x}\in\mathscr{Q}} w_{\mathbf{x}} |u(\mathbf{x},\mathbf{y}) - u_L(\mathbf{x},\mathbf{y})|^2\right)^{1/2},$$

where $\mathscr{Q}$ is the set of quadrature points.

L	1	2	3	5
$e_{\max}$	0.2581	3.3307e-16	3.2613e-16	4.1633e-16
e_2	0.4583	3.3729e-16	3.4730e-16	3.8967e-16

Table 1 Errors of the PDE solvers for fixed $\mathbf{y} = (1/2, \ -1/4)^T$

Table 1 shows spectral approximation errors are neglectable for $L \geq 2$ in this example.

We used Fibonacci lattice point sets for our QMC rule since these are optimal for any choice of weights [7, Chapter 5]. The Fibonacci point set $\{\mathbf{y}^{(1)},\dots,\mathbf{y}^{(N)}\}$ of size $N = F_n$, where F_n is the nth Fibonacci number, has the generating vector $(1,\ F_{n-1})$, so that $\mathbf{y}^{(k)} = \left(\frac{k}{F_n},\ \frac{kF_{n-1}}{F_n}\right) \mod 1$. The random shifts Δ are drawn uniformly from $[0,1]^2$.

We let $n = 10, 11, \dots, 20$ and hence $N = 55, 89, \dots, 6765$. Figure 1 shows the plot of quantities

$$\mathrm{err0}(N) = \left(\mathbb{E}\left|\frac{1}{N}\sum_{k=1}^{N} u_5(\mathbf{x}_0,\mathbf{y}_k+\Delta) - \int_{\left[\frac{-1}{2},\frac{1}{2}\right]^3} u(\mathbf{x}_0,\mathbf{y})d\mathbf{y}\right|^2\right)^{1/2},$$

where $\mathbf{x}_0 = (-0.9994,\ 0.0314,\ -0.0156) \in S$ and $\mathbb{E}$ is taken over 10 random shifts Δ's. The plot is in the log-log scale compared with $0.03/N$. Since the truncation error is also neglectable in this example, the numerical results are consistent with Theorem 8.

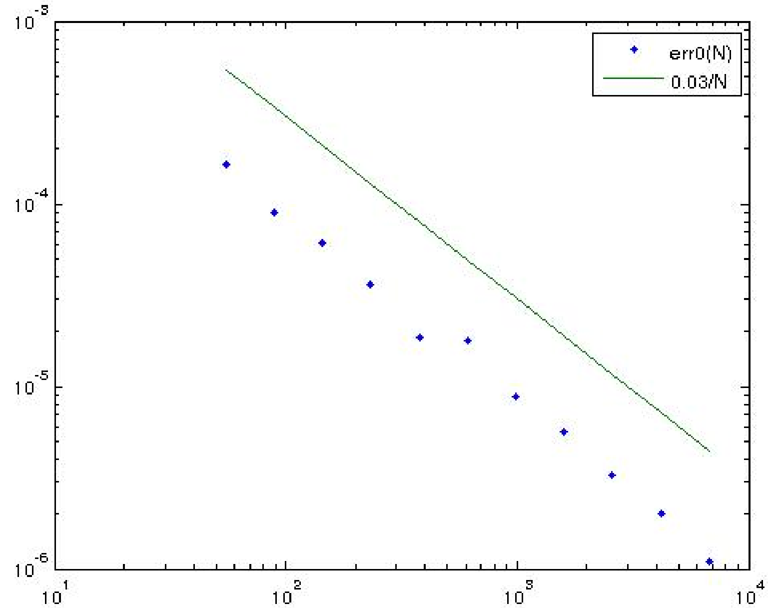

Fig. 1 Errors plot of $\mathrm{err0}(N)$

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