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CONSTRUCTIVE APPROXIMATION

The Uniform Norm of Hyperinterpolation on the Unit Sphere in an Arbitrary Number of Dimensions

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Abstract. In this paper we study the order of growth of the uniform norm of the hyperinterpolation operator on the unit sphere $S^{r-1} \subset \mathbf{R}^r$. The hyperinterpolation approximation $L_n f$, where $f \in C(S^{r-1})$, is derived from the exact L^2 orthogonal projection $\Pi_n f$ onto the space $P_n^r(S^{r-1})$ of spherical polynomials of degree n or less, with the Fourier coefficients approximated by a positive weight quadrature rule that integrates exactly all polynomials of degree $\le 2n$. We extend to arbitrary r the recent $r = 3$ result of Sloan and Womersley [9], by proving that under an additional "quadrature regularity" assumption on the quadrature rule, the order of growth of the uniform norm of the hyperinterpolation operator on the unit sphere is $O(n^{r/2-1})$, which is the same as that of the orthogonal projection Π_n, and best possible among all linear projections onto $P_n^r(S^{r-1})$.

1. Introduction

In this paper we study the order of growth of the uniform norm of the hyperinterpolation operator on the unit sphere $S^{r-1} \subset \mathbf{R}^r$ for arbitrary space dimension r, thereby extending to general r the recent $r = 3$ results proved in [9].

Hyperinterpolation, introduced by Sloan in [7], is a constructive approximation method for continuous functions over some general region Ω, where Ω is a bounded region of $\mathbf{R}^r$ which is either the closure of a connected open domain, or a smooth, closed lower-dimensional manifold in $\mathbf{R}^r$. The region Ω is also assumed to have finite measure with respect to a positive measure $d\mu$, that is,

$$\int_\Omega d\mu = V < \infty.$$

The hyperinterpolation approximation on S^{r-1} is an approximation from the space $P_n^r(S^{r-1})$ of all polynomials of degree $\le n$, i.e., the restriction to S^{r-1} of the space of polynomials of degree $\le n$ in $\mathbf{R}^r$. Roughly speaking, it is the approximation obtained from the L^2 orthogonal projection by evaluating the Fourier coefficients with a positive weight quadrature rule that integrates exactly all polynomials of degree $\le 2n$.

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Let $L_n : C(S^{r-1}) \to P_n^r(S^{r-1})$ denote the hyperinterpolation operator, and let the L^2-norm of the operator be defined as $\|L_n\|_2 := \sup\{\|L_n f\|_2 : f \in C(S^{r-1}), \|f\|_\infty \leq 1\}$. It was shown in [7] that L_n is a projection on $P_n^r(S^{r-1})$ and that $\|L_n\|_2$ is bounded and indeed that $\|L_n\|_2 = \sqrt{\omega_{r-1}}$, where ω_{r-1} is the surface area of S^{r-1}.

Both hyperinterpolation and interpolation over the unit sphere S^{r-1} were studied in [8], again in the L^2 setting. There it was shown that whereas $\|L_n\|_2 = \sqrt{\omega_{r-1}}$, for the classical interpolation operator $\mathcal{L}_n$ (which is denoted by Λ_n in [8]), there holds

$$\|\mathcal{L}_n\|_2 > \sqrt{\omega_{r-1}} \qquad \text{if} \quad r \geq 3 \quad \text{and} \quad n \geq 3.$$

In this paper we consider the uniform norm of the hyperinterpolation operator

$$\|L_n\|_\infty := \sup\{\|L_n f\|_\infty : f \in C(S^{r-1}), \|f\|_\infty \leq 1\}.$$

Recently, it was proved in [9], under a mild additional assumption (the "quadrature regularity" property, see Section 3), that for $r = 3$ the exact order of growth of $\|L_n\|_\infty$ on S^2 is $O(n^{1/2})$, which is the same as the order of growth of the uniform norm of the L^2 orthogonal projection, and best possible among all linear projections onto the same space.

The main purpose of this paper is to extend the arguments of [9] to arbitrary values of r, by proving that $\|L_n\|_\infty = O(n^{r/2-1})$ as $n \to \infty$ if the quadrature rule has the quadrature regularity property. This is the same order of growth as the uniform norm of the L^2 orthogonal projection, which has the least possible uniform norm among all linear projections on $P_n^r(S^{r-1})$, see [5]. Thus the result to be proved here shows that hyperinterpolation has the optimal order of growth for its uniform norm as $n \to \infty$.

The argument is based on the explicit construction of points on S^{r-1} that provide centers simultaneously for both a covering and a packing of the sphere (or, more precisely, of a subset of the sphere consisting of a spherical collar) by spherical caps of suitable size.

Since this work was completed we have learned of recent and as yet unpublished work by Reimer [6] that obtains the same generalization $\|L_n\|_\infty = O(n^{r/2-1})$ of the Sloan and Womersley $r = 3$ result (without a quadrature regularity assumption), by an argument based on Riemann–Stieltjes integration. The present approach is more elementary in nature, and may be of independent interest because of its construction of the simultaneous covering and packing.

2. Preliminaries

Let $H_\ell^r(S^{r-1})$ be the space of all spherical harmonics of exact order ℓ, and let $P_n^r(S^{r-1})$ be the space of all spherical polynomials of degree $\leq n$ on S^{r-1}. It is known that

$$P_n^r(S^{r-1}) = H_0^r(S^{r-1}) \oplus H_1^r(S^{r-1}) \oplus \cdots \oplus H_n^r(S^{r-1}). \tag{1}$$

The space of all spherical harmonics on S^{r-1} is dense in $L^2(S^{r-1})$, that is,

$$L^2(S^{r-1}) = \bigoplus_{\ell=0}^{\infty} H_\ell^r(S^{r-1}).$$

A popular orthonormal basis for $P_n^r(S^{r-1})$ is the set, as defined in [2], of normalized spherical harmonics $\{Y_{\ell k} : 1 \le k \le N(r,\ell), 0 \le \ell \le n\}$, satisfying

$$\int_{S^{r-1}} Y_{\ell k}(x) Y_{\ell' k'}(x)\, d\omega_{r-1} = \delta_{\ell\ell'}\delta_{kk'},$$

where $d\omega_{r-1}$ is the surface measure on S^{r-1}. Here, $N(r,\ell)$ is the dimension of the polynomial space $H_\ell^r(S^{r-1})$, given by (see [2]):

$$N(r,0) = 1, \qquad N(r,\ell) = \frac{(2\ell + r - 2)\Gamma(\ell + r - 2)}{\Gamma(\ell+1)\Gamma(r-1)}.$$

The dimension of the space $P_n^r(S^{r-1})$ is denoted by

$$d_n \equiv d_n^{(r)} = \sum_{\ell=0}^{n} N(r,\ell) = N(r+1, n).$$

Equivalently,

$$d_n = \frac{1}{\Gamma(r)}(2n + r - 1)(n + r - 2)(n + r - 3)\cdots(n+2)(n+1).$$

A useful lower bound is $d_n \ge 2n^{r-1}/\Gamma(r)$. An upper bound, if $n \ge r - 1$, is $d_n \le 3.2^{r-2} n^{r-1}/\Gamma(r)$.

For arbitrary $x, y \in S^{r-1}$ and $\ell = 0, 1, \ldots, n$, we define the kernel $G_\ell(x, y)$ as

$$G_\ell(x,y) = \sum_{k=1}^{N(r,\ell)} Y_{\ell k}(x) Y_{\ell k}(y).$$

Theorem 2.1 (Addition Theorem).

$$G_\ell(x,y) = \frac{1}{\omega_{r-1}} N(r,\ell) P_\ell^{(r)}(x \cdot y),$$

where $x \cdot y = x_1 y_1 + \cdots + x_r y_r$, *and* $P_\ell^{(r)}(t)$ *is the Legendre polynomial of degree* ℓ *in* $\mathbf{R}^r$.

It can be seen that the value of the kernel depends only on the geodesic distance $\phi_r(x,y) = \cos^{-1}(x \cdot y)$ between the two points, so we can write $G_\ell(x,y) = G_\ell(x \cdot y)$. The following result, in which $\langle \cdot, \cdot \rangle$ denotes the inner product,

$$\langle f, g\rangle := \int_{S^{r-1}} fg\, d\omega_{r-1},$$

is well known.

Lemma 2.1. $G_\ell(x, y)$ *is a reproducing kernel in* $H_\ell^r(S^{r-1})$. *In particular,*

$$\langle p, G_\ell(\cdot, x)\rangle = p(x) \qquad \forall\, p \in H_\ell^r(S^{r-1}).$$

Proof. From the definition of the kernel $G_\ell(\cdot,\cdot)$, we have

$$\langle p, G_\ell(\cdot, x)\rangle = \sum_{k=1}^{N(r,\ell)} \langle p, Y_{\ell k}\rangle\, Y_{\ell k}(x) = p(x),$$

the last step being just the Fourier expansion of $p \in H_\ell^r(S^{r-1})$. ■

By the decomposition relation between the spaces in (1), we deduce that the reproducing kernel in $P_n^r(S^{r-1})$ is

$$K_n(x\cdot y) = \sum_{\ell=0}^{n} G_\ell(x\cdot y), \qquad x, y \in S^{r-1}.$$

The following lemmas, expressing the kernel $K_n(x\cdot y)$ in terms of Gegenbauer polynomials, facilitate our subsequent estimate of the uniform norm of the hyperinterpolation operator. The idea of the proof is adapted from Chapter 7 of [5]. We use the definition of the Gegenbauer polynomials from [10].

Lemma 2.2. *For $r \ge 3$, $\ell \in \mathbf{N}$ and $\lambda = (r-2)/2$:*

$$G_\ell(t) = \frac{\ell+\lambda}{\lambda\omega_{r-1}} C_\ell^\lambda(t),$$

where $C_\ell^\lambda(t)$ is the Gegenbauer polynomial.

Proof. First, we recall the generating function definition of the Gegenbauer polynomial as

$$\sum_{\ell=0}^{\infty} C_\ell^\lambda(t) z^\ell = \frac{1}{(1-2zt+z^2)^\lambda}. \tag{2}$$

By applying the operator $(1/\lambda)(z(\partial/\partial z)+\lambda)$ to both sides of (2) we obtain

$$\sum_{\ell=0}^{\infty} \frac{\ell+\lambda}{\lambda} C_\ell^\lambda(t) z^\ell = \frac{1-z^2}{(1-2zt+z^2)^{\lambda+1}}. \tag{3}$$

The Poisson identity, see [3, p. 46], is

$$\sum_{\ell=0}^{\infty} N(r,\ell) z^\ell P_\ell^{(r)}(t) = \frac{1-z^2}{(1-2zt+z^2)^{r/2}}. \tag{4}$$

We compare the coefficients of (3) and (4) with $\lambda = (r-2)/2$, and then use the addition theorem to get the result. ■

Lemma 2.3. *For $r \ge 2$ and n a positive integer, the reproducing kernel of $P_n^r(S^{r-1})$ is*

$$K_n(x\cdot y) = \frac{1}{\omega_{r-1}} \left(C_n^{r/2}(x\cdot y) + C_{n-1}^{r/2}(x\cdot y)\right),$$

where ω_{r-1} is the surface area of S^{r-1}.

Proof. Let us expand (3) as the following:

$$\sum_{\ell=0}^{\infty} \frac{\ell+\lambda}{\lambda} C_\ell^\lambda(t) z^\ell = \frac{1-z^2}{(1-2zt+z^2)^{\lambda+1}} = \sum_{\ell=0}^{\infty} C_\ell^{\lambda+1}(t) z^\ell - \sum_{\ell=0}^{\infty} C_\ell^{\lambda+1}(t) z^{\ell+2}.$$

By equating the coefficients, we obtain

$$\frac{\ell+\lambda}{\lambda} C_\ell^\lambda(t) = C_\ell^{\lambda+1}(t) - C_{\ell-2}^{\lambda+1}(t)$$

for $\ell \in \mathbf{N}$, if $C_{-1}^{\lambda+1} := C_{-2}^{\lambda+1} := 0$. Now we insert this equation in the previous lemma to get

$$G_\ell(t) = \frac{1}{\omega_{r-1}} (C_\ell^{r/2}(t) - C_{\ell-2}^{r/2}(t)).$$

Thus

$$K_n(t) = \sum_{\ell=0}^{n} G_\ell(t) = \frac{1}{\omega_{r-1}} (C_n^{r/2}(t) + C_{n-1}^{r/2}(t)). \qquad \blacksquare$$

3. Hyperinterpolation and the Reproducing Kernel

On the unit sphere $S^{r-1} \subset \mathbf{R}^r$, we start with an m-point quadrature rule $\sum_{k=1}^m w_k f(t_k)$, where $\{t_1, \dots, t_m\} \subset S^{r-1}$ is some finite set of quadrature points on the unit sphere, and where $w_k > 0$ for $k = 1, \dots, m$. The rule is assumed to have the exactness property

$$\int_{S^{r-1}} p(x)\, d\omega_{r-1} = \sum_{k=1}^{m} w_k p(t_k)$$

for p an arbitrary spherical polynomial of degree $\leq 2n$. It is known, see [7], that the number of quadrature points must be at least as large as the dimension of the polynomial space, i.e., $m \geq d_n$. The semi-inner product of two continuous functions $f, g \in C(S^{r-1})$ is then defined as

$$\langle f, g \rangle_m := \sum_{k=1}^{m} w_k f(t_k) g(t_k). \tag{5}$$

Let $\{p_1, \dots, p_{d_n}\}$ be an orthonormal basis for the set $P_n^r(S^{r-1})$ of spherical polynomials of degree $\leq n$. Then the hyperinterpolant of a function $f \in C(S^{r-1})$ is defined as

$$L_n f := \sum_{j=1}^{d_n} \langle f, p_j \rangle_m p_j.$$

Lemma 3.1. *Let $f \in C(S^{r-1})$. Then*

$$\langle L_n f, \chi \rangle_m = \langle f, \chi \rangle_m \qquad \forall\, \chi \in P_n^r(S^{r-1}).$$

Proof. Since $\{p_1, \dots, p_{d_n}\}$ is a basis for $P_n^r(S^{r-1})$, it is enough to show the result for $\chi = p_k$, $k = 1, \dots, d_n$. We have

$$\langle L_n f, p_k \rangle_m = \left\langle \sum_{j=1}^{d_n} \langle f, p_j \rangle_m p_j, p_k \right\rangle_m = \langle f, p_k \rangle_m,$$

where we have used $\langle p_j, p_k \rangle_m = \langle p_j, p_k \rangle = \delta_{jk}$, which follows because $p_j p_k$ is a polynomial of degree at most $2n$. ∎

Theorem 3.1. *L_n is a linear projection, that is, $L_n^2 = L_n$.*

Proof. Suppose that $p \in P_n^r(S^{r-1})$, so p can be expressed as $p = \sum_{k=1}^{d_n} c_k p_k$. As above, we have $\langle p_j, p_k \rangle_m = \langle p_j, p_k \rangle = \delta_{jk}$. Hence, from the definition,

$$L_n p = \sum_{j=1}^{d_n} \left\langle \sum_{k=1}^{d_n} c_k p_k, p_j \right\rangle_m p_j = \sum_{j=1}^{d_n} c_j p_j = p.$$

Since $L_n f \in P_n^r(S^{r-1})$, the last result implies $L_n(L_n f) = L_n f$, or $L_n^2 = L_n$. ∎

On the sphere, we can choose the orthonormal basis of $P_n^r(S^{r-1})$ to be a set of orthonormal spherical harmonics $Y_{\ell k}$ of degree $\ell \le n$. The hyperinterpolation approximation of a function $f \in C(S^{r-1})$ now takes the following form:

$$L_n f(x) = \sum_{\ell=0}^{n} \sum_{k=0}^{N(r,\ell)} \langle f, Y_{\ell k} \rangle_m Y_{\ell k}(x). \tag{6}$$

As in [7], $L_n f$ can also be expressed in terms of the reproducing kernel $K_n(x, y)$: for by the reproducing kernel property, and the exactness of the quadrature rule for polynomials of degree $\le 2n$, we have, using Lemma 3.1:

$$\begin{aligned} L_n f(x) &= \langle L_n f, K_n(\cdot, x) \rangle = \langle L_n f, K_n(\cdot, x) \rangle_m = \langle f, K_n(\cdot, x) \rangle_m \\ &= \sum_{j=1}^{m} w_j f(t_j) K_n(t_j, x). \end{aligned} \tag{7}$$

Alternatively, the result can be obtained by writing out explicitly the semi-inner product in (6), and then changing the order of summation.

4. Spherical Caps and Spherical Collars

The parametrization of the unit sphere and its corollaries can be found in [2] or [4]. Here we develop the machinery needed in the next section for defining quadrature regularity and working out its properties.

With S^{r-1} denoting, as usual, the unit sphere in $\mathbf{R}^r$, let ϕ_r be the geodesic distance on S^{r-1}; that is, if $x, y \in S^{r-1}$, then

$$\phi_r(x, y) = \cos^{-1}(x \cdot y),$$

where $x \cdot y$ is the usual dot product in $\mathbf{R}^r$. Let $\{e_1, \ldots, e_r\}$ be the standard unit-vector basis for $\mathbf{R}^r$. If $x \in S^{r-1}$, then x can be expressed as

$$x = te_r + \sqrt{1-t^2}\nu_r, \qquad -1 \le t \le 1, \quad t := e_r \cdot x,$$

where ν_r is a unit vector in the span of $e_1, \ldots, e_{r-1}$. Equivalently,

$$x = \cos\theta\, e_r + \sin\theta\, \nu_r, \qquad 0 \le \theta \le \pi, \tag{8}$$

where $\theta = \cos^{-1}(e_r \cdot x)$.

For $0 \le \alpha \le \pi$ and $a \in S^{r-1}$, the spherical cap $A(\alpha) = A(a; \alpha)$ is defined as

$$A(\alpha) := \{x \in S^{r-1} : \phi_r(x, a) \le \alpha\}.$$

The point a is termed the center of the cap, and α is the spherical radius of the cap. We let $|A(\alpha)|$ denote the surface area of the spherical cap $A(\alpha) \subset S^{r-1}$.

Lemma 4.1. *We have*

$$|A(\alpha)| = \omega_{r-2} \int_0^\alpha \sin^{r-2}\theta\, d\theta.$$

Proof. Without loss of generality, we may assume that the cap is centered at $e_r = (0, 0, \ldots, 0, 1)$. The surface area element $d\omega_{r-1}$ is given recursively by (see [2, p. 1]):

$$d\omega_{r-1} = \sin^{r-2}\theta\, d\theta\, d\omega_{r-2},$$

where θ is as in (8). Since $A(\alpha)$ is centered at e_r, and θ ranges from 0 to α, the surface area of $A(\alpha)$ is given by

$$|A(\alpha)| = \int_{S^{r-2}} \int_0^\alpha \sin^{r-2}\theta\, d\theta\, d\omega_{r-2},$$

from which the result follows. ∎

A simple upper bound is

$$|A(\alpha)| \le \frac{\omega_{r-2}}{r-1}\alpha^{r-1}.$$

Now, let us assume that $0 \le \alpha \le \gamma \le \pi$. A spherical collar $B(\alpha, \gamma) = B(a; \alpha, \gamma)$ is defined as

$$B(\alpha, \gamma) := \{x \in S^{r-1} : \alpha < \phi_r(x, a) \le \gamma\},$$

where a is some fixed point on S^{r-1}. We observe that $B(\alpha, \gamma)$ is the difference between two spherical caps $A(\gamma)$ and $A(\alpha)$ which have the same center; the difference $\gamma - \alpha$ is termed the spherical height of the collar. By the previous lemma, the surface area of $B(\alpha, \gamma)$, which is denoted by $|B(\alpha, \gamma)|$, is given by

$$|B(\alpha, \gamma)| = \omega_{r-2} \int_\alpha^\gamma \sin^{r-2}\theta\, d\theta. \tag{9}$$

It turns out to be useful to construct a set of points on the spherical collar that serve simultaneously as the centers for the covering and packing of a spherical collar by spherical caps of the appropriate spherical radii. Specifically, we consider a spherical collar $B(\alpha, \alpha+h)$ of spherical height h and center e_r, with $0 \le \alpha < \alpha+h \le \pi$. Defining $\theta_r := \alpha + h/2$, let $\Lambda_r = \Lambda(\theta_r)$ be the latitude of S^{r-1} with polar angle θ_r with respect to e_r, i.e.,

$$\Lambda_r := \{x \in S^{r-1} : x \cdot e_r = x_r = \cos\theta_r\}.$$

Alternatively, Λ_r is the set of points expressible as

$$x = \cos\theta_r e_r + \sin\theta_r \nu_r,$$

for ν_r a unit vector in the span of $e_1, \dots, e_{r-1}$. We observe that Λ_r is a hypersphere of dimension $r-2$ and radius $\sin\theta_r$.

Now we take the parametrization to a further stage, by observing, for $r \ge 3$, that each point $x \in \Lambda_r$ can be written as

$$x = \cos\theta_r e_r + \sin\theta_r (\cos\theta_{r-1} e_{r-1} + \sin\theta_{r-1}\nu_{r-1}), \tag{10}$$

where ν_{r-1} is a unit vector in the span of $\{e_1, \dots, e_{r-2}\}$ and $0 \le \theta_{r-1} \le \pi$. Consider the locus of a point x on Λ_r with fixed values of both θ_r and θ_{r-1}: from (10) this is a hypersphere of dimension $r-3$ and radius $\sin\theta_r \sin\theta_{r-1}$. We refer to this locus as a latitude of Λ_r with respect to the axis e_{r-1}, and denote it either by Λ_{r-1} or by the more explicit notation $\Lambda(\theta_r, \theta_{r-1})$.

Suppose now that $\Lambda'_{r-1} := \Lambda(\theta_r, \theta'_{r-1})$ is another latitude of Λ_r, parallel to Λ_{r-1}, so that a general point of Λ'_{r-1} is expressible as

$$x' = \cos\theta_r e_r + \sin\theta_r (\cos\theta'_{r-1} e_{r-1} + \sin\theta'_{r-1}\nu'_{r-1}),$$

with ν'_{r-1} also a unit vector in span $\{e_1, \dots, e_{r-2}\}$. Then for arbitrary $x \in \Lambda_{r-1}$ and $x' \in \Lambda'_{r-1}$ the distance between x and x' in the metric ϕ_r is

$$\phi_r(x, x') = \cos^{-1}(\cos^2\theta_r + \sin^2\theta_r (\cos\theta_{r-1}\cos\theta'_{r-1} + \sin\theta_{r-1}\sin\theta'_{r-1}\nu_{r-1}\cdot\nu'_{r-1})),$$

from which it follows easily that

$$\phi_r(x, \Lambda'_{r-1}) = \phi_r(\Lambda_{r-1}, \Lambda'_{r-1}) = \cos^{-1}(\cos^2\theta_r + \sin^2\theta_r \cos(\theta_{r-1} - \theta'_{r-1})),$$

with the minimum distance being achieved when $\nu'_{r-1} = \nu_{r-1}$.

We see here an important fact for the subsequent development, that every point $x \in \Lambda_{r-1}$ is at the minimum distance $\phi_r(\Lambda_{r-1}, \Lambda'_{r-1})$ from the latitude Λ'_{r-1}. In the subsequent discussion we shall recursively consider latitudes of latitudes, without pausing to remark that the analogous property is always true.

With that background, we now partition the latitude $\Lambda_r \in S^{r-1}$ by parallel latitudes $\Lambda^{(k)}_{r-1} = \Lambda(\theta_r, \theta^{(k)}_{r-1})$ with respect to e_{r-1}, and with k running from 1 to K_1. The successive parallel latitudes are required to have polar angles $\theta^{(k)}_{r-1}$ increasing with k, and such that

$$\Lambda^{(1)}_{r-1} = \Lambda(\theta_r, 0), \qquad \phi_r(\Lambda^{(k)}_{r-1}, \Lambda^{(k+1)}_{r-1}) = \frac{h}{2r} \qquad \text{for} \quad k = 1, \dots, K_1 - 1,$$

and with $K_1 \geq 1$ determined by the condition

$$\phi_r(\Lambda_{r-1}^{(K_1)}, \Lambda(\theta_r, \pi)) \leq \frac{h}{2r}.$$

Note that the latitude $\Lambda_{r-1}^{(1)}$, for which $\theta_{r-1}^{(1)} = 0$, is degenerate, consisting of just the single point $\cos\theta_r e_r + \sin\theta_r e_{r-1}$. (The latitude $\Lambda(\theta_r, \pi)$ which includes the south pole is similarly degenerate, but is in general **not** one of the latitudes in the partition.) We also remark that if $\phi_r(\Lambda(\theta_r, 0), \Lambda(\theta_r, \pi)) < h/2r$ then the degenerate latitude $\Lambda_{r-1}^{(1)} = \Lambda(\theta_r, 0)$ is the **only** latitude $\Lambda_{r-1}^{(k)}$, i.e., $K_1 = 1$, and the subsequent recursive construction of latitudes described below is vacuous.

It will be important to us later that every point of Λ_r is at a distance (in the sense of the metric ϕ_r) of not more than $h/2r$ from one of the latitudes $\Lambda_{r-1}^{(k)}$. This holds because a point between $\Lambda_{r-1}^{(k)} = \Lambda(\theta_r, \theta_{r-1}^{(k)})$ and an adjacent latitude $\Lambda_{r-1}^{(k+1)} = \Lambda(\theta_r, \theta_{r-1}^{(k+1)})$, i.e., a point of the form (10), with $\theta_{r-1}^{(k)} < \theta_{r-1} < \theta_{r-1}^{(k+1)}$, satisfies

$$\phi_r(\Lambda_{r-1}^{(k)}, x) < \phi_r(\Lambda_{r-1}^{(k)}, \Lambda_{r-1}^{(k\pm1)}) \leq \frac{h}{2r}. \tag{11}$$

We now proceed to the next level in our construction of latitudes, partitioning each latitude $\Lambda_{r-1}^{(k_1)} = \Lambda(\theta_r, \theta_{r-1}^{(k_1)})$ into latitudes $\Lambda_{r-2}^{(k_1,k_2)} := \Lambda(\theta_r, \theta_{r-1}^{(k_1)}, \theta_{r-2}^{(k_1,k_2)})$ with respect to the axis e_{r-2}, such that

$$\begin{aligned}
\Lambda_{r-2}^{(k_1,1)} &= \Lambda(\theta_r, \theta_{r-1}^{(k_1)}, 0),\\
\phi_r(\Lambda_{r-2}^{(k_1,k_2)}, \Lambda_{r-2}^{(k_1,k_2+1)}) &= \frac{h}{2r} \qquad \text{for} \quad k_2 = 1, \ldots, K_2(k_1) - 1,\\
\phi_r(\Lambda_{r-2}^{(k_1,K_2(k_1))}, \Lambda(\theta_r, \theta_{r-1}^{(k_1)}, \pi)) &\leq \frac{h}{2r}.
\end{aligned}$$

Clearly, we can continue this way, recursively defining latitudes, until we reach

$$\Lambda_3^{(k_1,\ldots,k_{r-3})} = \Lambda(\theta_r, \theta_{r-1}^{(k_1)}, \ldots, \theta_3^{(k_1,\ldots,k_{r-3})}),$$

each of which is a (one-dimensional) circle. On each such circle $C \subset S^{r-1}$, we place n_C points $t_1, \ldots, t_{n_C}$, with $n_C \geq 1$, the first point $t_1 \in C$ being placed arbitrarily. If $\phi_r(t_1, x) < h/2r$ for all $x \in C$ then t_1 is the only point on C, and $n_C = 1$. In particular, this is the case if C is a degenerate circle, i.e., a point. Otherwise, we place points progressively in one direction around the circle, so that

$$\phi_r(t_\ell, t_{\ell+1}) = \frac{h}{2r} \qquad \text{for} \quad \ell = 1, \ldots, n_C,$$

stopping before the point t_{n_C+1} wraps around, and choosing n_C so that $\phi_r(t_{n_C}, t_1) \geq h/2r$ and $\phi_r(t_{n_C+1}, t_1) < h/2r$. (To be sure that the metric ϕ_r displays no pathological behavior on the circle C, note that for $t, t' \in C$, with angles θ_2 and θ_2' around the circle C:

$$\begin{aligned}
\phi_r(t, t') = \cos^{-1}(&\cos^2\theta_r + \sin^2\theta_r \\
&(\cos^2\theta_{r-1} + \sin^2\theta_{r-1}(\cos^2\theta_{r-2} + \cdots + \sin^2\theta_3\cos(\theta_2 - \theta_2')\ldots))),
\end{aligned}$$

which increases monotonically in $|\theta_2 - \theta_2'|$ for $0 \leq |\theta_2 - \theta_2'| \leq \pi$.)

Notice that the construction ensures that the points $t_1, \ldots, t_{n_C}$ on a single circle C are well spaced, in the sense that

$$\phi_r(t_\ell, t_m) \ge \frac{h}{2r} \qquad \text{for} \quad 1 \le \ell, m \le n_C, \tag{12}$$

and also

$$\min_{\ell=1,\ldots,n_C} \phi_r(x, t_\ell) \le \frac{h}{r} \qquad \text{for} \quad x \in C. \tag{13}$$

We denote the union of points $\{t_1, \ldots, t_{n_C}\} \in C$ by T_C, and the union of the points on all circles C by T:

$$T = \bigcup_C T_C.$$

The set of points $T \subset S^{r-1}$ has been constructed so that no two points of T are too close, in the sense of the metric ϕ_r, and on the other hand that every point of $B(\alpha, \alpha + h)$ is close to a point of T. Precisely, we establish the two results in the following lemma. The first says that the set of all spherical caps with centers in T and spherical radius $h/4r$ forms a "packing" on $B(\alpha, \alpha + h)$. The second says that the set of all spherical caps with centers in T and spherical radius h "covers" $B(\alpha, \alpha + h)$.

Lemma 4.2. *Let T be defined as above. Then*:

(a)

$$\bigcup_{t \in T} A\left(t; \frac{h}{4r}\right) \subset B(\alpha, \alpha + h),$$

and the spherical caps $A(t; h/4r)$, for $t \in T$, are pairwise "disjoint" except perhaps for single points of intersection.

(b)

$$B(\alpha, \alpha + h) \subset \bigcup_{t \in T} A(t; h).$$

Proof. (a) That the union of the sets $A(t; h/4r)$ is contained in $B(\alpha, \alpha + h)$ is trivial, since $T \subset \Lambda_r$, and $B(\alpha, \alpha + h)$ contains every point on S^{r-1} whose geodesic distance from Λ_r is less than $h/2$.

To prove the disjointness of the sets $A(t; h/4r)$, suppose the contrary, that there exist $t, t' \in T$ with $t \neq t'$ and $x \in S^{r-1}$ such that x belongs to both $A(t; h/4r)$ and $A(t'; h/4r)$, with x in the interior of at least one of the spherical caps. Then from the triangle inequality we have

$$\phi_r(t, t') \le \phi_r(t, x) + \phi_r(x, t') < \frac{h}{4r} + \frac{h}{4r} = \frac{h}{2r}. \tag{14}$$

In the construction we have pointed out that all pairs of points t and t' in T, and on the same circle C, are separated by a distance of at least $h/2r$ in the metric ϕ_r (see (12)). And if t, t' are on different circles then there must be some $j \in \{0, \ldots, r-3\}$ such that t, t' belong to the same latitude $\Lambda(\theta_r, \ldots, \theta_{r-j}^{(k_1,\ldots,k_j)})$ but belong to different latitudes at the next level, and in every case the latitudes are separated by a distance (in the metric ϕ_r) of at least $h/2r$. Thus (14) is a contradiction and (a) is proved.

(b) Let x be any point of $B(\alpha, \alpha + h)$, and let $x_1 \in \Lambda_r$ be a nearest point to x in the metric ϕ_r. Then we know $\phi_r(x, x_1) \le h/2$. Now let $x_2 \in \Lambda_{r-1}^{(k)}$ for some $k \in \{1, \ldots, K_1\}$ be such that $\phi_r(x_1, x_2) \le h/2r$, which we know is possible by (11). We may continue in this way, successively defining a sequence $\{x_i\} \subset S^{r-1}$ such that $\phi_r(x_i, x_{i+1}) \le h/2r$, until we reach $x_{r-2} \in C$ for some circle C. We have observed already (see (13)) that there exists $t \in T \cap C$ such that $\phi_r(x_{r-2}, t) \le h/r$. In total we have, therefore, by repeated use of the triangle inequality,

$$\begin{aligned}\phi_r(x, t) &\le \phi_r(x, x_1) + \phi_r(x_1, x_2) + \cdots + \phi_r(x_{r-3}, x_{r-2}) + \phi_r(x_{r-2}, t)\\ &\le \frac{h}{2} + (r-3)\frac{h}{2r} + \frac{h}{r} < h,\end{aligned}$$

from which it follows that $x \in A(t; h)$. Thus the union of all such spherical caps covers $B(\alpha, \alpha + h)$, and the result is proved. ■

Corollary 4.1. *Let T be as in Lemma* 4.2. *Then there exists $c > 0$, with c depending only on r, such that*

$$c.\mathrm{card}(T)|A(h)| \le |B(\alpha, \alpha + h)| \le \mathrm{card}(T)|A(h)|.$$

Proof. Immediately from the lemma, we have

$$\mathrm{card}(T)\left|A\left(\frac{h}{4r}\right)\right| \le |B(\alpha, \alpha + h)| \le \mathrm{card}(T)|A(h)|.$$

We also have, from Lemma 4.1:

$$\frac{|A(h)|}{|A(h/4r)|} = \frac{\omega_{r-2}\int_0^h \sin^{r-2}\theta\, d\theta}{\omega_{r-2}\int_0^{h/4r} \sin^{r-2}\theta\, d\theta} \le \frac{\omega_{r-2}\int_0^h \theta^{r-2}\, d\theta}{\omega_{r-2}\int_0^{h/4r} (2\theta/\pi)^{r-2}\, d\theta} = 2^r \pi^{r-2} r^{r-1},$$

since $2\theta/\pi \le \sin\theta \le \theta$ for $0 \le \theta \le \pi/2$. Thus, the result follows. ■

5. Quadrature Regularity

Assumption (Quadrature Regularity). The infinite family of positive weight m-point quadrature rules $\{Q_m\}$ is said to satisfy the quadrature regularity assumption if there exists a positive constant γ, with γ independent of m, such that for every spherical cap $A(h)$ with spherical radius $h = h_m := m^{-1/(r-1)}$, we have

$$\sum_{t_j \in A(h)} w_j \le \gamma |A(h)|. \tag{15}$$

That the property is a reasonable one may be seen by observing that $|A(h)| \approx [\omega_{r-2}/(r-1)]h^{r-1}$ for small h, so that if h is proportional to $m^{-1/(r-1)}$ then the expected number of points falling in $A(h)$ is of order 1. In the case $r = 3$, we obtain $h = 1/\sqrt{m}$, as in [9].

We shall also need analogous regularity results for spherical collars, and for spherical caps of larger spherical radius.

Lemma 5.1. *Let $\{Q_m\}$ be an infinite family of positive weight m-point quadrature rules on S^{r-1} which satisfies the quadrature regularity assumption. Then we have*:

(a) *For every spherical cap $A(\alpha)$, with spherical radius $\alpha \geq h_m$, there exists a positive constant c_1 which is independent of m and α such that*

$$\sum_{t_j \in A(\alpha)} w_j \leq c_1 |A(\alpha)|.$$

(b) *For every spherical collar, with spherical height $\beta \geq h_m$, there exists a positive constant c_2 which is independent of m such that*

$$\sum_{t_j \in B(\alpha,\alpha+\beta)} w_j \leq c_2 |B(\alpha, \alpha+\beta)|.$$

Proof. Without loss of generality we may assume that $e_r = (0, \ldots, 0, 1)$ is the axis for both $A(\alpha)$ and $B(\alpha, \alpha+\beta)$. We now prove the lemma, starting with part (b). Set $h = h_m = m^{-1/(r-1)}$. For a spherical collar $B(\alpha, \alpha+h)$, let T be the set constructed as described in Lemma 4.2. Because $B(\alpha, \alpha+h)$ is covered by spherical caps of radius h with centers at the points of T (see part (b) of Lemma 4.2), it follows from (15) that

$$\sum_{t_j \in B(\alpha,\alpha+h)} w_j \leq \sum_{t \in T} \sum_{t_j \in A(t;h)} w_j \leq \mathrm{card}(T) \gamma |A(h)| \leq \frac{c_2}{2} |B(\alpha, \alpha+h)|,$$

with the second inequality following from Corollary 4.1, and with c_2 a constant dependent only on r. This proves part (b) for the case $\beta = h = h_m$.

We now consider an arbitrary spherical collar $B(\alpha, \alpha+\beta)$ with $\beta \geq h$. If β is an integer multiple of h then the result extends trivially with the same constant c_2 as above. Otherwise, let $k := \lfloor \beta/h \rfloor$ and consider the subcollars

$$B_1 := B(\alpha, \alpha+kh), \qquad B_2 := B(\alpha+\beta-kh, \alpha+\beta).$$

Since $B_j \subset B(\alpha, \alpha+\beta)$ for $j = 1, 2$, and since every point of $B(\alpha, \alpha+\beta)$ is in at least one of B_1 and B_2, by the result already obtained, we have

$$\sum_{t_j \in B(\alpha,\alpha+\beta)} w_j \leq \sum_{t_j \in B_1} w_j + \sum_{t_j \in B_2} w_j \leq \frac{c_2}{2}(|B_1| + |B_2|) \leq c_2 |B(\alpha, \alpha+\beta)|.$$

Thus the quadrature regularity for spherical collars stated in part (b) is proved.

Finally, to prove part (a), let $A(h)$ be the spherical cap which has spherical radius $h = h_m$ and the same center as $A(\alpha)$, and let $B_3 := B(\alpha - kh, \alpha)$, where $k = \lceil \alpha/h \rceil - 1$, be a spherical collar that has the same axis as $A(\alpha)$. Both $A(h)$ and B_3 are subsets of $A(\alpha)$, and every point of $A(\alpha)$ is in at least one of $A(h)$ or B_3. Thus

$$\begin{aligned}\sum_{t_j \in A(\alpha)} w_j &\leq \sum_{t_j \in A(h)} w_j + \sum_{t_j \in B_3} w_j \\ &\leq \gamma |A(h)| + c_2 |B_3| \leq (\gamma + c_2) |A(\alpha)|.\end{aligned}$$

Thus the spherical regularity property for spherical caps stated in part (a) is proved. ■

6. The Uniform Norm of the Hyperinterpolation Operator

We begin, as in [9], by developing an expression for the uniform norm of the hyperinterpolation operator on S^{r-1}, namely $\|L_n\|_\infty$, via the reproducing kernel K_n in $P_n^r(S^{r-1})$. The main result of the paper is Theorem 6.2, in which we prove that the order of growth of $\|L_n\|_\infty$ as $n \to \infty$ is $O(n^{r/2-1})$ if the quadrature regularity assumption holds. The arguments in this section follow closely the model of [9]. The first result, shown in [9], is a simple consequence of (7).

Theorem 6.1 ([9]). *The uniform norm of the hyperinterpolation operator is given by*

$$\|L_n\|_\infty = \max_{x \in S^{r-1}} \sum_{j=1}^{m} w_j |K_n(t_j, x)|.$$

We are now ready to state the main theorem of this paper.

Theorem 6.2. *Assume that $\{Q_m\}$ is a sequence of quadrature-regular m-point rules, and that $m \geq d_n$. Then there exist constants γ_1 and γ_2, depending only on r, such that*

$$\gamma_1 n^{r/2-1} \leq \|L_n\|_\infty \leq \gamma_2 n^{r/2-1}.$$

Proof. Let $x_0 \in S^{r-1}$ be a point at which $\sum_{j=1}^{m} w_j |K_n(t_j, x)|$ attains its maximum value. Using the previous result and the results in Section 2, we can express the uniform norm in terms of Gegenbauer polynomials

$$\begin{aligned}(16)\quad \|L_n\|_\infty = \sum_{j=1}^{m} w_j |K_n(t_j \cdot x_0)| &= \frac{1}{\omega_{r-1}} \sum_{j=1}^{m} w_j |C_n^{r/2}(t_j \cdot x_0) + C_{n-1}^{r/2}(t_j \cdot x_0)| \\ &\leq \frac{1}{\omega_{r-1}} \sum_{j=1}^{m} w_j (|C_n^{r/2}(t_j \cdot x_0)| + |C_{n-1}^{r/2}(t_j \cdot x_0)|).\end{aligned}$$

We split the sum in (16) into two parts: G_+ containing all the terms such that $z_j \geq 0$, and G_- containing all the terms such that $z_j < 0$ where $z_j = t_j \cdot x_0 = \cos(\phi_r(t_j, x_0))$. We will carry out the estimate for the first term G_+; the same bound also holds for the second term G_-, since $|C_n^{r/2}(z)| = |C_n^{r/2}(-z)|$.

To bound G_+ we use an analogous argument to that in [9]. From Szegő [10, Theorem 7.33.6], we know that

$$(17)\qquad |C_n^{r/2}(\cos\theta)| \leq c \min(n^{r-1}, \theta^{-r/2} n^{r/2-1}), \qquad 0 \leq \theta \leq \pi/2,$$

where $c > 0$ is some constant independent of θ and n. Thus, if we denote by $u_n(\cos\theta)$ the right-hand side of (17), then u_n is a monotone increasing function of $z = \cos\theta$ on $[0, 1]$, and

$$(18)\qquad G_+ = \sum_{z_j \geq 0} w_j |K_n(z_j)| \leq c_1 \sum_{z_j \geq 0} w_j u_n(z_j).$$

With z_0 satisfying $0 < z_0 < 1$, we write $\sum_{z_j \geq 0} w_j u_n(z_j) = M + R$, where M is the contribution to the sum from points z_j for which $0 \leq z_j < z_0$, and R is the contribution from $z_0 \le z_j \le 1$; the value of z_0 will be chosen later. The main term M will be bounded by a Riemann sum and then a one-dimensional integral. Let us introduce a change of variable from z to τ, defined by

$$\tau = \tau(z) = \int_0^z (1 - z'^2)^{(r-3)/2}\, dz'.$$

We then define T by

$$T = \tau(z_0) = \int_0^{z_0} (1 - z^2)^{(r-3)/2}\, dz,$$

and partition the interval $[0, T]$ into equal intervals by defining

$$\tau_k = T - \frac{N-k}{d_n^{1/(r-1)}}, \qquad k = 1, \ldots, N,$$

where $N := \lceil d_n^{1/(r-1)} T \rceil$. So $\tau_1 > 0, \tau_N = T$, and $\tau_{k+1} - \tau_k = d_n^{-1/(r-1)}$ for $k = 1, \ldots, N-1$. We also define $\tau_0 = \tau_1 - d_n^{-1/(r-1)} < 0$ and $\tau_{N+1} = T + d_n^{-1/(r-1)}$, at the same time introducing a constraint on the choice of z_0, namely that it must be small enough to ensure

$$\int_{z_0}^1 (1 - z^2)^{(r-3)/2}\, dz \ge d_n^{-1/(r-1)}. \tag{19}$$

We then define $\xi_k,\ k = 0, \ldots, N+1$, by

$$\int_0^{\xi_k} (1 - z^2)^{(r-3)/2}\, dz = \tau_k,$$

so that $\tau_k = \tau(\xi_k)$. The uniform spacing of the points τ_k then ensures

$$\begin{aligned}\int_{\xi_k}^{\xi_{k+1}} &(1 - z^2)^{(r-3)/2} dz \\ &= \int_{\xi_{k+1}}^{\xi_{k+2}} (1 - z^2)^{(r-3)/2} dz = d_n^{-1/(r-1)} \qquad \text{for} \quad k = 0, \ldots, N-1.\end{aligned}$$

Now we can write

$$M = \sum_{0 \le z_j < z_0} w_j u_n(z_j) \le \sum_{k=0}^{N-1} \sum_{\max(\xi_k, 0) \le z_j < \xi_{k+1}} w_j u_n(z_j) \le \sum_{k=0}^{N-1} \left(\sum_{\xi_k \le z_j < \xi_{k+1}} w_j \right) u_n(\xi_{k+1}),$$

the last step following from the monotonicity of u_n.

The next step is to make use of the quadrature regularity property for the spherical collar defined by the hyperplanes $z = \xi_k$ and $z = \xi_{k+1}$. First, however, we need to make sure that the assumption in Lemma 5.1(b) is satisfied for each such collar. Let $\alpha_k = \cos^{-1} \xi_k$ for $k = 0, \ldots, N-1$, and change the variable z to $\phi = \cos^{-1} z$, so that

$$d_n^{-1/(r-1)} = \int_{\xi_k}^{\xi_{k+1}} (1 - z^2)^{(r-3)/2}\, dz = \int_{\alpha_{k+1}}^{\alpha_k} \sin^{r-2} \phi\, d\phi \qquad \text{for} \quad k = 0, \ldots, N-1,$$

from which follows

$$d_n^{-1/(r-1)} \leq (\alpha_k - \alpha_{k+1}) \max_{\alpha_{k+1} \leq \phi \leq \alpha_k} |\sin^{r-2}\phi| \leq \alpha_k - \alpha_{k+1}.$$

The assumption $m \geq d_n$ now gives

$$\alpha_k - \alpha_{k+1} \geq d_n^{-1/(r-1)} \geq m^{-1/(r-1)},$$

so that we can use the quadrature regularity results of Lemma 5.1 on the spherical collars defined by the partition ξ_k for $k = 0, \ldots, N$. Therefore, we have, by part (b) of Lemma 5.1 and (9):

$$\sum_{\xi_k \leq z_j < \xi_{k+1}} w_j \ \leq c \int_{\alpha_{k+1}}^{\alpha_k} \sin^{r-2}\phi \, d\phi = c \int_{\xi_k}^{\xi_{k+1}} (1 - z^2)^{(r-3)/2} \, dz,$$

where c from now on denotes a generic constant which may depend on r. Thus,

$$\begin{aligned} M \ &\leq \ c \sum_{k=0}^{N-1} u_n(\xi_{k+1}) \int_{\xi_k}^{\xi_{k+1}} (1 - z^2)^{(r-3)/2} \, dz \\ &\leq \ c \sum_{k=0}^{N-1} u_n(\xi_{k+1}) \int_{\xi_{k+1}}^{\xi_{k+2}} (1 - z^2)^{(r-3)/2} \, dz \\ &\leq \ c \sum_{k=0}^{N-1} \int_{\xi_{k+1}}^{\xi_{k+2}} u_n(z)(1 - z^2)^{(r-3)/2} \, dz \\ &= \ c \int_{\xi_1}^{\xi_{N+1}} u_n(z)(1 - z^2)^{(r-3)/2} \, dz \\ &\leq \ c \int_0^1 u_n(z)(1 - z^2)^{(r-3)/2} \, dz \\ &\leq \ cn^{r/2-1} \int_0^{\pi/2} \theta^{-r/2} \sin^{r-2}\theta \, d\theta. \end{aligned}$$

Hence $M \leq cn^{r/2-1}$ for an appropriate positive constant c.

For the estimation of the term R it is useful to define

$$\nu_r := \frac{r}{2(r-1)},$$

and to observe that $\frac{1}{2} < \nu_r \leq \frac{3}{4}$ for $r \geq 3$. To estimate R, which is the contribution to the sum $\sum_j w_j u_n(z_j)$ from points x_j belonging to the spherical cap $A(x_0; \cos^{-1} z_0)$, we partition the sum into the contributions from a smaller spherical cap $A_n := A(x_0; d_n^{-\nu_r/(r-1)})$ and a finite sequence of spherical collars

$$B_{n,j} := B(x_0; \ f_{n,j}, \ f_{n,j+1}), \qquad j = 1, 2, \ldots, J_r,$$

where

$$f_{n,j} := \sum_{k=1}^{j} d_n^{-\nu_r^k/(r-1)}, \qquad j = 1, 2, \ldots, J_r + 1, \tag{20}$$

and z_0 is determined by

$$\cos^{-1} z_0 := f_{n,J_r+1},$$

with J_r chosen sufficiently large to ensure that (19) holds. Corresponding to the partition $A(\cos^{-1} z_0) = A_n \cup B_{n,1} \cup \cdots \cup B_{n,J_r}$, we write, in an obvious notation,

$$R = R(A_n) + \sum_{j=1}^{J_r} R(B_{n,j}),$$

and estimate the contribution from each term separately.

First, however, we note that the quadrature regularity results in Lemma 5.1 are applicable to each of $A_n, B_{n,1}, \ldots, B_{n,J_r}$. In the case of A_n, this is clear because the spherical radius satisfies, since $\nu_r < 1$:

$$d_n^{-\nu_r/(r-1)} > d_n^{-1/(r-1)} \geq m^{-1/(r-1)} = h_m.$$

For the spherical collar $B_{n,j}$, $j = 1, \ldots, J_r$, the spherical height of the collar is

$$f_{n,j+1} - f_{n,j} = d_n^{-\nu_r^{j+1}/(r-1)} > d_n^{-1/(r-1)},$$

which is again bounded below by h_m. (By a similar argument, the terms in the sum (20) are monotone increasing.)

Using then the quadrature regularity property and the definition of u_n, we have

$$\begin{aligned} R(A_n) \; &\leq \; cu_n(1)|A_n| \leq cn^{r-1}(d_n^{-\nu_r/(r-1)})^{r-1} \\ &= \; cn^{r-1}d_n^{-\nu_r} \leq cn^{r-1}n^{-\nu_r(r-1)} = cn^{r/2-1}, \end{aligned}$$

while for $j = 1, \ldots, J_r$ we have

$$R(B_{n,j}) \; \leq \; cu_n(\cos f_{n,j})|B_{n,j}| \leq c f_{n,j}^{-r/2} n^{r/2-1} f_{n,j+1}^{r-1},$$

and because

$$f_{n,j} > d^{-\nu_r^j/(r-1)}, \qquad f_{n,j+1} < (j+1)d_n^{-\nu_r^{j+1}/(r-1)},$$

we obtain

$$R(B_{n,j}) \; \leq \; cn^{r/2-1} d_n^{(\nu_r^j/(r-1))r/2} d_n^{(-\nu_r^{j+1}/(r-1))(r-1)} = cn^{r/2-1}$$

since $\nu_r^j r/2(r-1) = \nu_r^{j+1}$.

Putting all terms together, we find $R \leq cn^{r/2-1}$. Together with the lower bound of the same order achieved by the (optimal) L^2 orthogonal projection (see Section 1), this completes the proof. ∎

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