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# Polynomial operators and local approximation of solutions of pseudo-differential equations on the sphere

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**Abstract** We study the solutions of an equation of the form $Lu = f$, where $L$ is a pseudo-differential operator defined for functions on the unit sphere embedded in a Euclidean space, $f$ is a given function, and $u$ is the desired solution. We give conditions under which the solution exists, and deduce local smoothness properties of $u$ given corresponding local smoothness properties of $f$, measured by local Besov spaces. We study the global and local approximation properties of the spectral solutions, describe a method to obtain approximate solutions using values of $f$ at points on the sphere and polynomial operators, and describe the global and local rates of approximation provided by our polynomial operators.

## 1 Introduction

Many applications in geodesics, oceanography, and potential theory give rise to pseudo-differential equations on the unit sphere, embedded in a finite dimensional Euclidean space. For example, in physical geodesy, one seeks to find the gravitational potential of the earth from certain observables, such as geoid undulation, gravity anomaly, or the radial derivative of the potential at the surface of the earth or at satellite height. The mathematical model consists of solving a pseudo-differential equation [6,23,5]. Similar problems arise in crust modelling in geomagnetism

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and oceanography [11,17, and references therein ]. Recently, we have treated the problem of constructing certain neural network approximations on the sphere as the problem of solving suitable pseudo-differential equations in high dimensions [14]. While the definition of the pseudo-differential equation requires functions defined on the whole sphere, it is a challenge to construct local solutions to the equation. For example, the compatibility conditions in the altimetry–gravimetry boundary value problem are formulated with part of the boundary conditions defined on the land and the other part defined on the sea [9]. One approach to solve such equations is to consider somewhat artificial extensions of the two boundary conditions, so that the problem splits into the solution of two globally defined boundary value problems. In most of the above cited papers, spherical wavelets are used, which in turn, are defined using spherical harmonics. In this paper, we study the local smoothness properties of the solutions of pseudo-differential equations and the local approximation capability of their approximate solutions, measured in terms of Besov norms, using global data in the form of spherical harmonic coefficients. An essential tool is given by the polynomial frames developed in [13] using spherical polynomials.

To describe the concept of pseudo–differential equations, we need some notations. Let $q \ge 1$ be an integer, $\mathbb{S}^q$ denote the unit sphere of the Euclidean space $\mathbb{R}^{q+1}$, $\mu_q^*$ be the volume element of $\mathbb{S}^q$, and

$$\omega_q := \int_{\mathbb{S}^q} d\mu_q^* = \frac{2\pi^{(q+1)/2}}{\Gamma((q+1)/2)}.$$

For a fixed integer $\ell \ge 0$, the restriction to $\mathbb{S}^q$ of a homogeneous harmonic polynomial of degree $\ell$ is called a spherical harmonic of degree $\ell$. The class of all spherical harmonics of degree $\ell$ will be denoted by $\mathbf{H}_\ell^q$, and its dimension is given by

$$d_\ell^q := \dim \mathbf{H}_\ell^q = \begin{cases} \dfrac{2\ell+q-1}{\ell+q-1}\dbinom{\ell+q-1}{\ell}, & \text{if } \ell \ge 1, \\ 1, & \text{if } \ell = 0. \end{cases} \tag{1.1}$$

We choose an orthonormal basis $\{Y_{\ell,k} : k = 1, \dots, d_\ell^q\}$ for each $\mathbf{H}_\ell^q$. Let $\mathcal{S}$ be the space of all infinitely often differentiable functions on $\mathbb{S}^q$, endowed with the locally convex topology induced by the supremum norms of all the derivatives of such functions, and $\mathcal{S}^*$ be the dual of this space. For $x^* \in \mathcal{S}^*$, we define

$$\hat{x^*}(\ell, k) := x^*(Y_{\ell,k}),\ k = 1, \dots, d_\ell^q,\ \ell = 0, 1, \dots. \tag{1.2}$$

A most common example is when $x^*$ is defined by $g \mapsto \int_{\mathbb{S}^q} g(\xi) f(\xi) d\mu_q^*(\xi)$ for some integrable function $f$ on $\mathbb{S}^q$. In this case, we identify $x^*$ with $f$ and use the notation $\hat{f}(\ell, k)$ to denote the corresponding $\hat{x^*}(\ell, k)$. Let $\lambda_\ell$ be a sequence of nonnegative numbers. The pseudo-differential operator $L$ with *eigenvalues* $\lambda_\ell$ is defined (when possible) by

$$\widehat{Lu}(\ell, k) = \lambda_\ell \hat{u}(\ell, k), k = 1, \dots, d_\ell^q,\ \ell = 0, 1, \dots,\ u \in \mathcal{S}^*. \tag{1.3}$$

A pseudo-differential equation is an equation of the form $Lu = f$. Clearly, if $\hat{f}(\ell, k)$'s are all known (and $\hat{f}(\ell, k) = 0, k = 1, \dots, d_\ell^q$ for all $\ell$ for which

$\lambda_\ell = 0$), then the solution to such an equation is given formally by $u \in \mathcal{S}^*$ that satisfies for integer $k = 1, \ldots, d_\ell^q$, $\ell = 0, 1, \ldots,$

$$\hat{u}(\ell, k) = \begin{cases} \lambda_\ell^{-1} \hat{f}(\ell, k), & \text{if } \lambda_\ell \neq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{1.4}$$

We write formally $u = L^{[-1]}(f)$. If only finitely many of the coefficients $\hat{f}(\ell, k)$'s are known, then one obtains a *spectral approximation* to the solution, defined by requiring the equations (1.4) only for those $\ell$ and $k$ for which the coefficients are known.

Many modern applications cited above require that a solution $u$ of an equation of the form $Lu = f$ be obtained given values of $f$ at certain sites on the sphere, rather than the spherical harmonic coefficients of $f$. If these values are known at a set $\mathcal{C}$ of $N$ points, the *collocation method* consists of fixing a function space $V_N$ having dimension equal to $N$, and finding a function $v_N \in V_N$ that satisfies

$$L(v_N, \xi) = f(\xi), \qquad \xi \in \mathcal{C}. \tag{1.5}$$

The collocation method has several advantages, which makes it applicable in more general situations; for example, when the equations involved have variable coefficients. Also, the evaluation of the left hand side of (1.5) is numerically less expensive than computing an inner product with spherical harmonics. However, it also has some disadvantages, especially in the context of the sphere. Typically, the approximation $v_N$ is found by solving a large matrix equation. In this method, we cannot use $V_N$ to be the space of all spherical polynomials of a given degree, since it is not known in general whether the given data admits an interpolatory polynomial, and even if it does, it is not clear what the condition number of the matrices involved would be. Morton and Neamtu [19] have recently introduced spaces spanned by spherical basis functions for this purpose, and have investigated the convergence of such methods in a suitable reproducing kernel Hilbert space. However, the spherical basis functions depend upon the operator $L$, and to the best of our knowledge, there are no nontrivial estimates for the error in uniform approximation.

In all of the examples of pseudo-differential equations in [7] as well as in most of the situations in the applications cited in the first paragraph, all the eigenvalues of the operator are known in advance. This suggests the following method. We may compute a judicious discretization of the integral defining $\hat{f}(\ell, k)$; i.e., define discrete spherical harmonic coefficients. We may then obtain an approximate solution similar to the spectral approximation, but using these discrete coefficients. In this paper, we will follow Gottlieb and Tadmor [8], and refer to this method as *pseudo-spectral method*, although the actual methods we will discuss will be necessarily different from those discussed in [8] in their context.

To distinguish between spectral and pseudo-spectral methods, we note that a straightforward discretization of the spherical harmonic coefficients, for example, using a Monte-Carlo method, does not yield satisfactory results (cf. Table 4). One needs to use quadrature formulas exact for high degree polynomials, and in theory, satisfying certain additional conditions. Many such quadrature formulas are known explicitly if the data can be chosen at specific sites; for example, the product Gaussian rule described in the book [22] of Stroud, Driscoll-Healy formulas [4], and some newer formulas by Brown, Feng, and Sheng [2], Potts, Steidle, and Tasche [20], etc. Thus, in the case when all the eigenvalues of the operator are explicitly known, and one can use one of these formulas, the pseudo-spectral method eliminates the need to solve any matrix equation. Moreover, the resulting methods are universal; i.e., the same class of approximants can be used for all the operators and all the functions.

We discuss briefly the situations when one is either dealing with scattered data, or when the eigenvalues of the operator are not known. In the case when one does not have a choice on where to sample the function, it is instinctive to solve an obvious system of equations to generate a quadrature formula. However, this is a one time task, if the same equation needs to be solved for many functions $f$, sampled at the same sites. On the other hand, the resulting quadrature formulas may not be adequate from a theoretical point of view; for example, the sum of the absolute values of the weights might be too large. We have proved in [15] that if the set of sites is sufficiently dense, there exist positive quadrature formulas based on these sites, that are exact for high degree polynomials, and satisfy the theoretical conditions to be described in this paper. We hope to report on an effective procedure for generating such formulas, perhaps without the positivity requirement, in near future. Finally, if all the eigenvalues are not known explicitly, then the pseudo-spectral method would require the solution of a matrix equation with a symmetric positive definite matrix; and the norm of the inverse of this matrix can be estimated if one has some estimates on the eigenvalues of the operator.

In this paper, we establish certain local as well as global convergence results for spectral and pseudo-spectral methods. Our results are valid in all the $L^p$ norms on the sphere, including the uniform norm. We describe our theoretical results in the next section, and illustrate some aspects of this theory in Section 3. The proofs of the results in Section 2 will be given in Section 4.

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## 2 Main results

In order to state our results, we need to introduce some further notation. First, we introduce some sequence spaces. Let $\mathbf{a} = \{a_n\}_{n=0}^{\infty}$ be a sequence of real numbers. The forward difference operators are defined by

$$\Delta a_n := \Delta^1 a_n := a_{n+1} - a_n,\ \Delta^r a_n := \Delta(\Delta^{r-1} a_n),\ r = 2, 3, \dots,$$

and $\Delta^0 a_n := a_n$. If $K \geq 1$ is an integer and $\alpha \in \mathbb{R}$, we define

$$|||\mathbf{a}|||_{\alpha,K} := \sup_{0\leq r\leq K,\ \nu\geq 0} (\nu+1)^{\alpha+r}|\Delta^r a_\nu|,$$
$$|||\mathbf{a}|||^*_{\alpha,K} := \sum_{r=1}^{K}\sum_{\nu=0}^{\infty}(\nu+1)^{\alpha+r-1}|\Delta^r a_\nu|, \tag{2.1}$$

and denote by $\mathcal{B}_{\alpha,K}$ (respectively, $\mathcal{B}^*_{\alpha,K}$) the set of all sequences $\mathbf{a}$ for which $|||\mathbf{a}|||_{\alpha,K} < \infty$ (respectively, $|||\mathbf{a}|||_{\alpha,K} + |||\mathbf{a}|||^*_{\alpha,K} < \infty$). If $\alpha = 0$, we will omit it from the notation here. We note that $|||\mathbf{a}|||_{\alpha,K}$ involves a bound on the sequence itself, while $|||\mathbf{a}|||^*_{\alpha,K}$ involves only the forward differences of the sequence.

We will need another sequence space. If $\gamma > 0$ and $0 < \rho \le \infty$, we define

$$\|\mathbf{a}\|_{\rho,\gamma} := \begin{cases} \left\{ \sum_{n=0}^{\infty} 2^{n\gamma\rho} |a_n|^\rho \right\}^{1/\rho}, & \text{if } 0 < \rho < \infty, \\ \sup_{n\ge 0} 2^{n\gamma} |a_n|, & \text{if } \rho = \infty. \end{cases} \tag{2.2}$$

The space of sequences $\mathbf{a}$ for which $\|\mathbf{a}\|_{\rho,\gamma} < \infty$ will be denoted by $\mathsf{b}_{\rho,\gamma}$.

Next, we introduce the function spaces which we will work with. If $\nu$ is any (possibly signed) measure on $\mathbb{S}^q$, its total variation measure will be denoted by $|\nu|$. If $1 \le p \le \infty$, $A \subseteq \mathbb{S}^q$ is $\nu$-measurable, and $f$: $A \to \mathbb{R}$ is $\nu$-measurable, we define the $L^p(\nu; A)$ norms of $f$ by

$$\|f\|_{\nu;A,p} := \begin{cases} \left\{ \int_A |f(\xi)|^p d|\nu|(\xi) \right\}^{1/p}, & \text{if } 1 \le p < \infty, \\ |\nu| - \operatorname{ess\,sup}_{\xi\in A} |f(\xi)|, & \text{if } p = \infty. \end{cases} \tag{2.3}$$

The space of all $\nu$-measurable functions on $A$ such that $\|f\|_{\nu;A,p} < \infty$ will be denoted by $L^p(\nu; A)$, with the usual convention that two functions are considered equal as elements of this space if they are equal $|\nu|$-almost everywhere. The space of all uniformly continuous, bounded functions on $A$ will be denoted by $C(A)$, and the symbol $X^p(\nu; A)$ will denote $L^p(\nu; A)$ if $1 \le p < \infty$ and $C(A)$ if $p = \infty$ (equipped with the norm of $L^\infty(\nu; A)$). We will omit the mention of the measure from the notations if the measure is $\mu_q^*$. Thus, for example, $\|f\|_{A,p}$ will mean $\|f\|_{\mu_q^*;A,p}$, $X^p(A)$ will mean $X^p(\mu_q^*; A)$, etc.

We will be interested in the relationship between the smoothness of a function $f$ and that of $L^{[-1]}f$. This smoothness will be measured by Besov spaces, which we now introduce. There are many ways to define Besov spaces on the sphere [10,21]. We find it convenient to adopt the following definition, motivated by the equivalence theorem in [10, Theorem 3.2]. For $x \ge 0$, we denote by $\Pi_x^q$ the class of all spherical polynomials of degree at most $x$; i.e., $\Pi_x^q$ comprises the restrictions to $\mathbb{S}^q$ of polynomials in $q+1$ real variables, with total degree not exceeding $x$. We note that $\Pi_x^q = \Pi_{\lfloor x \rfloor}^q$ for any real $x$, since the degree of a polynomial is always an integer. However, we will need to determine the degrees of different polynomials from the support of a function $h$. It is then convenient to use the notation for all real $x$, rather than using the more cumbersome notation with integer parts in the subscript. For any $x \ge 0$, $1 \le p \le \infty$ and $f \in L^p(\mathbb{S}^q)$, we write

$$E_{x,p}(f) := \min_{P\in\Pi_x^q} \|f - P\|_{\mathbb{S}^q,p}. \tag{2.4}$$

If $1 \le p \le \infty$, the Besov space $B_{p,\rho,\gamma}$ consists of all functions $f \in X^p(\mathbb{S}^q)$ for which $\{E_{2^n,p}(f)\} \in \mathsf{b}_{\rho,\gamma}$. A spherical cap centered at a point $\mathbf{x}_0 \in \mathbb{S}^q$, and radius $\alpha \in [0, \pi]$ is defined by

$$\mathbb{S}_\alpha^q(\mathbf{x}_0) := \{\mathbf{x} \in \mathbb{S}^q \ : \ \mathbf{x}\cdot\mathbf{x}_0 \ge \cos\alpha\} = \{\mathbf{x} \in \mathbb{S}^q \ : \ \|\mathbf{x}-\mathbf{x}_0\| \le 2\sin(\alpha/2)\}, \tag{2.5}$$

where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^{q+1}$. For a cap $C$, the space $C_0^\infty(C)$ consists of infinitely differentiable functions $\phi$ on $\mathbb{S}^q$ such that $\phi(\mathbf{x}) = 0$ if $\mathbf{x} \notin C$. If $\mathbf{x}_0 \in \mathbb{S}^q$, the local Besov space $B_{p,\rho,\gamma}(\mathbf{x}_0)$ consists of functions $f \in X^p(\mathbb{S}^q)$ for which there exists a cap $C$ centered at $\mathbf{x}_0$ such that for every $\phi \in C_0^\infty(C)$, $f\phi \in B_{p,\rho,\gamma}$. (Here, and in the sequel, the notation $fg$ will denote the pointwise product of the functions $f$ and $g$.) For example, the function

$$\mathbf{x} \mapsto (\mathbf{x}\cdot\mathbf{n} - 0.75)_+^{3/4} = \begin{cases} (\mathbf{x}\cdot\mathbf{n} - 0.75)^{3/4}, & \text{if } \mathbf{x}\cdot\mathbf{n} \geq 0.75, \\ 0, & \text{otherwise,} \end{cases}$$

where $\mathbf{n} = (0, \cdots, 0, 1)$, is in $B_{\infty,\infty,3/4}(\mathbf{x}_0)$ for all $\mathbf{x}_0 \in \mathbb{S}^q$; the cap involved may be chosen to be the whole sphere. On the other hand, if $\gamma > 3/4$ is arbitrary, the function is in $B_{\infty,\infty,\gamma}(\mathbf{x}_0)$ for every point $\mathbf{x}_0$ not on the "circle" $x_{q+1} = 0.75$, and the cap for any such point can be any cap that does not intersect the circle.

In the sequel, if $f \in \mathcal{S}^*$ and $\beta \in \mathbb{R}$, then $\partial_\beta f \in \mathcal{S}^*$ is defined by

$$\widehat{\partial_\beta f}(\ell, k) := (\ell+1)^\beta \hat{f}(\ell, k), \qquad k = 1, \ldots, d_\ell^q,\ \ell = 0, 1, \ldots.$$

In the remainder of this paper, we adopt the following convention regarding constants. The letters $c, c_1, \cdots$ will denote positive constants depending only on the dimension $q$, the pseudo-differential operator $L$, the corresponding parameter $\beta$, the different norms involved in the formula, and other explicitly mentioned quantities, if any. Their value will be different at different occurrences, even within the same formula.

Let $L$ be a pseudo-differential operator defined as in (1.3) with eigenvalues $\{\lambda_\ell\}_{\ell=0}^\infty$. We define

$$\mathcal{N}_L := \{f \in \mathcal{S}^* \ :\ \lambda_\ell = 0 \Rightarrow \hat{f}(\ell, k) = 0,\ k = 1, \ldots, d_\ell^q\}. \tag{2.6}$$

In the sequel, we will write $\lambda_\ell^{[-1]}$ to denote $1/\lambda_\ell$ if $\lambda_\ell \neq 0$ and 0 if $\lambda_\ell = 0$.

Our first theorem gives sufficient conditions for the existence and uniqueness of a solution $u$ of the equation $Lu = f$, as well as describes the smoothness of this solution given the smoothness of $f$. We note that each of the operators mentioned in examples 1–5 in [7, pp. 84–85] satisfy the conditions of Theorem 2.1 below for suitable values of $\beta$.

**Theorem 2.1** *Let $\beta \in \mathbb{R}$, $L$ be a pseudo-differential operator with eigenvalues $\{\lambda_\ell\}_{\ell=0}^\infty$. Let $K > (q+1)/2$ be an integer, and the sequence $\{(\ell+1)^{-\beta}\lambda_\ell^{[-1]}\}_{\ell=0}^\infty \in \mathcal{B}_K^*$. Let $1 \leq p \leq \infty$, $\partial_\beta f \in X^p(\mathbb{S}^q) \cap \mathcal{N}_L$.*
(a) *There exists a unique $u := L^{[-1]}f \in X^p(\mathbb{S}^q)$ such that $Lu = f$, and*

$$\|u\|_{\mathbb{S}^q,p} \leq c\|\partial_\beta f\|_{\mathbb{S}^q,p}. \tag{2.7}$$

(b) *Let $\gamma > \max(\beta, 0)$, $0 < \rho \leq \infty$, $\mathbf{x}_0 \in \mathbb{S}^q$, $K > \beta + q + 1 + \max(\gamma, \gamma - \beta, 1)$. If $f \in B_{p,\rho,\gamma}(\mathbf{x}_0)$ then $L^{[-1]}f \in B_{p,\rho,\gamma-\beta}(\mathbf{x}_0)$.*

The proof of part (a) of the above theorem follows standard arguments, but we are unable to find a reference for this statement. The proof of the global smoothness properties analogous to part (b) would also be similarly standard. The novelty here is that the smoothness of $f$ on a cap determines that of $L^{[-1]}f$ in a smaller concentric cap. We note that $L^{[-1]}$ is not a local operator in the sense that even if $f(\mathbf{x}) \equiv 0$ for $\mathbf{x}$ on some cap, $L^{[-1]}f$ may fail to be identically equal to zero on this cap. Nevertheless, Theorem 2.1 allows one to study the solutions of a mixed boundary value problem as in [9], by applying the theorem to different operators on different regions of the sphere. The proof of the theorem will show that any smaller cap will work here, but the constants involved will depend upon the cap in the definition of the local smoothness class, and the cap where the smoothness of the solution is desired.

Next, we discuss approximate solutions of $Lu = f$, given the spherical harmonic coefficients of $f$. Towards this end, we introduce certain summability operators. We recall first the addition formula [18]:

$$\sum_{k=1}^{d_\ell^q} Y_{\ell,k}(\mathbf{x})Y_{\ell,k}(\mathbf{y}) = \frac{d_\ell^q}{\omega_q}\mathcal{P}_\ell(q+1;\mathbf{x}\cdot\mathbf{y}), \qquad \ell = 0, 1, \ldots, \tag{2.8}$$

where $\mathcal{P}_\ell(q+1;x)$ is the degree-$\ell$ Legendre polynomial in $q+1$-dimensions. (We note that Müller's $\omega_q$ and $N(q,\ell)$ are the same as our $\omega_{q+1}$ and $d_\ell^{q+1}$.)

The Legendre polynomials are normalized so that $\mathcal{P}_\ell(q+1;1) = 1$, and satisfy the orthogonality relations [18, Lemma 10]

$$\int_{-1}^{1} \mathcal{P}_\ell(q+1;x)\mathcal{P}_k(q+1;x)(1-x^2)^{\frac{q}{2}-1}dx = \frac{\omega_q}{\omega_{q-1}d_\ell^q}\delta_{\ell,k}. \tag{2.9}$$

Let $h : [0,\infty) \to \mathbb{R}$, and for some constant $c > 0$ (depending on $h$), $h(x) = 0$ if $x > c$. We extend $h$ by setting $h(x) = 0$ if $x < 0$. We then define the kernel

$$\Phi_y(h,t) := \begin{cases} \displaystyle\sum_{\ell=0}^{\infty} h(\ell/y)\frac{d_\ell^q}{\omega_q}\mathcal{P}_\ell(q+1;t), & t \in \mathbb{R},\ y \geq 1, \\ 0, & t \in \mathbb{R},\ y < 1, \end{cases} \tag{2.10}$$

and the *summability operator* by

$$\sigma_y^*(h,f,\mathbf{x}) := \int_{\mathbb{S}^q} \Phi_y(h,\mathbf{x}\cdot\xi)f(\xi)d\mu_q^*(\xi). \tag{2.11}$$

We observe that if $n \geq 1$ is an integer,

$$\sigma_n^*(h,f,\mathbf{x}) = \sum_{0\leq\ell\leq cn} h(\ell/n)\sum_{k=1}^{d_\ell^q} \hat{f}(\ell,k)Y_{\ell,k}(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{S}^q.$$

In our theory of polynomial frames (cf. [16]), the kernels $\Phi_{2^n}$ $(n = 0, 1, 2, \cdots)$ play the role of a scaling function. They are also similar to the scaling function in the theory discussed in [7, Chapter 10]. In contrast to that scaling function, the kernels $\Phi_{2^n}$ are not positive functions. On the other hand, with a proper choice of $h$, they reproduce polynomials of degree proportional to $2^n$, rather than a fixed degree. They are necessarily polynomials (band limited), and with a proper choice of $h$, their span is equal to $\Pi_{2^n-1}^q$. Their space localization properties are studied in a great detail (cf. Proposition 4.2 below.) With proper choices of $h$, they lead to asymptotically best uncertainty products (cf. [16]).

The polynomial $L^{[-1]}\sigma_n^*(h, f)$ will be called a *spectrum-based approximation* to $L^{[-1]}f$. In Theorem 2.2, we will give sufficient conditions on $h$ to guarantee the convergence of these approximations, and estimate the rates of convergence. While the spectral approximation $u_n$ is obtained by requiring that $\widehat{Lu_n}(\ell, k) = \hat{f}(\ell, k)$ for $k = 1, \ldots, d_\ell^q$, $\ell = 0, 1, \ldots, n$, the spectrum-based approximation $v_n$ is obtained by requiring that $\widehat{Lv_n}(\ell, k) = h(\ell/n)\hat{f}(\ell, k)$ for $k = 1, \ldots, d_\ell^q$, $\ell = 0, 1, \ldots, n$. Of course, if $h(x) = 1$ if $0 \le x \le 1$ and 0 for other values of $x$, then the two solutions coincide. However, we will show that by choosing a smooth masking function $h$ instead, the spectrum-based approximation exhibits better localization properties.

In [13], we have described several connections between the smoothness of $h$ and the operators $\sigma_n^*(h)$. Let $K \ge 1$ be an integer. We will write $h \in \mathcal{A}_K$ if each of the following conditions is satisfied: (i) $h : [0, \infty) \to [0, \infty)$, (ii) $h$ is a $K-1$ times iterated integral of a function of bounded variation, (iii) $h'(x) = 0$ if $x \le c_1$, and (iv) $h(x) = 0$ if $x > c$. Here, $c, c_1$ depend upon $h$. Further, we will write $h \in \mathcal{A}_K^*$ if $h \in \mathcal{A}_K$, $h(x) = 1$ for $x \in [0, 1/2]$ and $h(x) = 0$ for $x > 1$. (Our notation in [13] is different.) In the remainder of this section, all unspecified constants will depend upon $h$ and the sequence of measures $\{\mu_n\}$ to be introduced later, in addition to the other parameters mentioned earlier.

In the following theorem, we estimate the rate of convergence of spectrum-based approximations.

**Theorem 2.2** *Let $\beta \in \mathbb{R}$, $L$ be a pseudo-differential operator with eigenvalues $\{\lambda_\ell\}_{\ell=0}^\infty$. Let $K > (q+1)/2$ be an integer, $h \in \mathcal{A}_K^*$, and the sequence $\{(\ell + 1)^{-\beta}\lambda_\ell^{[-1]}\}_{\ell=0}^\infty \in \mathcal{B}_K^*$. Let $1 \le p \le \infty$, $\partial_\beta f \in X^p(\mathbb{S}^q) \cap \mathcal{N}_L$.*
(a) *We have*

$$\|L^{[-1]}f - L^{[-1]}\sigma_n^*(h, f)\|_{\mathbb{S}^q, p} \le cE_{n/2,p}(\partial_\beta f). \tag{2.12}$$

(b) *Let $\gamma > \max(\beta, 0)$, $0 < \rho \le \infty$, $\mathbf{x}_0 \in \mathbb{S}^q$, $K > \beta + q + 1 + \max(1, \gamma, \gamma - \beta)$. If $f \in B_{p,\rho,\gamma}(\mathbf{x}_0)$ then there exists a cap $C$, centered at $\mathbf{x}_0$ such that*

$$\{\|L^{[-1]}f - L^{[-1]}\sigma_{2^n}^*(h, f)\|_{C,p}\}_{n=0}^\infty \in \mathsf{b}_{\rho,\gamma-\beta}.$$

We observe (Proposition 4.3(b) below) that if $\beta < 0$ then $E_{n,p}(\partial_\beta f) \le cn^{-|\beta|}E_{n,p}(f)$. Even though the approximations $L^{[-1]}\sigma_{2^n}^*(h, f)$ are constructed using global information on $f$ contained its spherical harmonic coefficients, Theorem 2.2(b) shows that the rate of convergence adjusts itself on each cap, according to the smoothness of $f$ on that cap.

Next, we discuss pseudo-spectral approximations. In the case of uniform approximation, such solutions may be defined using values of the function $f$. However, since we wish to develop an $L^p$ theory, and function evaluation is not defined if $p < \infty$, we wish to allow as our starting points other kinds of information, such as averages over small caps around certain points on the sphere. To codify this information, we find it convenient to use the notation of Lebesgue–Stieltjes integral. We recall that if $\mathcal{C}$ is a finite set of points on $\mathbb{S}^q$, a sum of the form $\sum_{\xi\in\mathcal{C}} w_\xi f(\xi)$ may be expressed as the integral $\int_{\mathbb{S}^q} f(\xi)d\nu(\xi)$, where $\nu$ is the measure that associates the mass $w_\xi$ with $\xi$, $\xi \in \mathcal{C}$. In this case, $\|f\|_{\nu;\mathbb{S}^q,p}$ is the corresponding discrete norm. An important property of these norms is that $\|f\|_{\nu;\mathbb{S}^q,\infty} \le \|f\|_{\mathbb{S}^q,\infty}$ for each $f \in C(\mathbb{S}^q)$. In general, if $\{\mu_n\}$ is a sequence of measures, we will write $\{\mu_n\} \preceq_p \mu_q^*$ if for each integer $n \ge 0$, every $\mu_q^*$-measurable function is also $\mu_n$-measurable, and $\|f\|_{\mu_n;\mathbb{S}^q,p} \le c(p, \{\mu_n\})\|f\|_{\mathbb{S}^q,p}$ for every $f \in X^p(\mathbb{S}^q)$.

We define the *quasi-interpolatory operator* by discretizing the operator $\sigma_y(h)$ using a quadrature formula. Formally, if $\nu$ is a signed, or positive measure with bounded variation, we define

$$\sigma_y(\nu; h, f, \mathbf{x}) := \int_{\mathbb{S}^q} \Phi_y(h, \mathbf{x}\cdot\xi) f(\xi) d\nu(\xi), \qquad \mathbf{x} \in \mathbb{S}^q,\ y \ge 0. \tag{2.13}$$

In [13], we have shown that if $\{\mu_n\}$ satisfies certain technical conditions, then the operators $\sigma_n(\mu_n; h)$ share many properties with their continuous version. To state these conditions, we now recall a definition from [13].

Let $x \ge 0$. A (possibly signed) measure $\nu$ will be called an *M–Z (Marcinkiewicz–Zygmund) quadrature measure of order* $x$ if each of the following conditions is satisfied.

$$\|P\|_{\nu;\mathbb{S}^q,p} \le M_{\nu;q,p}(x)\|P\|_{\mathbb{S}^q,p}, \qquad P \in \Pi_x^q,\ 1 \le p \le \infty, \tag{2.14}$$

for a constant $M_{\nu;q,p}(x) \ge 1$ independent of the polynomial $P$, and

$$\int_{\mathbb{S}^q} P(\xi) d\mu_q^*(\xi) = \int_{\mathbb{S}^q} P(\xi) d\nu(\xi), \qquad P \in \Pi_x^q. \tag{2.15}$$

The constant $M_{\nu;q,p}(x)$ will be called the M–Z constant for $\nu$ ($x$, and $p$). We note that $\mu_q^*$ itself is an M–Z quadrature measure of order $x$ for each $x \ge 0$, with $M_{\mu_q^*;q,p}(x) = 1$. In [15], we have established the existence of discrete M–Z quadrature measures with uniformly bounded M–Z constants, supported on sufficiently dense scattered data sets.

The polynomial $L^{[-1]}\sigma_n(\mu_n; h, f)$ will be called the *pseudo-spectral approximation* to the solution of $Lu = f$. It is the same as the spectrum-based approximation, except that the spherical harmonic coefficients of $f$ are approximated first by a "discretization" with the measures $\mu_n$. One important difference is that the spherical harmonic coefficients of $f$ are always defined using the measure $\mu_q^*$, independently of $n$, while these discretizations depend upon the desired $n$. Since there are no known aliasing formulas on the sphere, this makes the analysis of these solutions much harder than in the case of the circle $\mathbb{S}^1$. The following theorem summarizes a partial analogue of Theorem 2.2.

**Theorem 2.3** *Let $\beta \in \mathbb{R}$, $L$ be a pseudo-differential operator with eigenvalues $\{\lambda_\ell\}_{\ell=0}^\infty$. Let $K > (q+1)/2$ be an integer, $h \in \mathcal{A}_K^*$, and the sequence $\{(\ell+1)^{-\beta}\lambda_\ell^{[-1]}\}_{\ell=0}^\infty \in \mathcal{B}_K^*$. Let $1 \le p \le \infty$, $\partial_\beta f \in X^p(\mathbb{S}^q) \cap \mathcal{N}_L$. For each integer $n \ge 0$, let $\mu_n$ be an M–Z quadrature measure of order* $6(2^n)$, $M_{\mu_n;q,1}(6(2^n)) \le c$, *and $\{\mu_n\} \preceq_p \mu_q^*$.*
(a) *We have*

$$\|L^{[-1]}(f - \sigma_{2^n}(\mu_n; h, f))\|_{\mathbb{S}^q,p} \le c \begin{cases} E_{2^{n-1},p}(\partial_\beta f), & \textit{if } \beta \ge 0, \\ E_{2^{n-2},p}(f), & \textit{if } \beta < 0. \end{cases} \tag{2.16}$$

(b) *Let $\beta \ge 0$, $0 < \rho \le \infty$, $\gamma > \beta$, $K > \beta + q + 1 + \max(1, \gamma)$. Suppose that there exists a polynomial $R^*$ such that $\lambda_\ell^{[-1]} = R^*(\ell(\ell + q - 1))$. Let $\mathbf{x}_0 \in \mathbb{S}^q$, and $C$ be a cap centered at $\mathbf{x}_0$ such that for every $\phi \in C_0^\infty(C)$, $f\phi \in B_{p,\rho,\gamma}$. Let $C'$ be a cap, centered at $\mathbf{x}_0$ and having radius less than that of $C$, and $\phi^* \in C_0^\infty(C)$ be chosen so that $\phi^*(\mathbf{x}) = 1$ if $\mathbf{x} \in C'$. There exists a cap $C''$, centered at $\mathbf{x}_0$ such that $\{\|L^{[-1]}(f - \sigma_{2^n}(\mu_n; h, f\phi^*))\|_{C'',p}\}_{n=0}^\infty \in \mathsf{b}_{\rho,\gamma-\beta}$.*

## 3 Numerical experiments

The purpose of this section is to illustrate and supplement the theoretical results in Section 2. Our intention is to illustrate general purpose methods, rather than solve any specific problem arising in practice. For the most of this section, we will consider the function

$$f(x_1, x_2, x_3) = (x_1 - 0.9)_+^{3/4} + (x_3 - 0.9)_+^{3/4}, \qquad (x_1, x_2, x_3) \in \mathbb{S}^2. \quad (3.1)$$

The function is clearly in $B_{\infty,\infty,3/4}(\mathbf{x}_0)$ for every $\mathbf{x}_0 \in \mathbb{S}^2$. For any $\mathbf{x}_0 \in \mathbb{S}^2$ which is not on the circles $x_1 = 0.9$ or $x_3 = 0.9$, it is also in $B_{\infty,\infty,\gamma}(\mathbf{x}_0)$ for every $\gamma > 3/4$. We are interested in solving an equation of the form $Lu = f$ on different caps of $\mathbb{S}^2$ for different operators $L$, and using different choices for the function $h$ appearing in (2.11). For the sake of example, we will consider the following operators. The operator $L_{NEU}$ is defined by

$$L_{NEU}u = \frac{\partial u}{\partial n} + (1/2)u, \quad (3.2)$$

where $\frac{\partial u}{\partial n}$ denotes the outward normal derivative of $u$. The operator $L_{POT}$ is defined by

$$L_{POT}u(\mathbf{x}) = u(\mathbf{x}) - \frac{1}{2\pi}\int_{\mathbb{S}^2} u(\mathbf{y}) \frac{\partial}{\partial n(\mathbf{y})} \frac{1}{|\mathbf{x} - \mathbf{y}|} d\mu_2^*(\mathbf{y}). \quad (3.3)$$

Finally, the operator $L_{GR}$ is defined using the Laplace–Beltrami operator $\Delta^*$, by

$$L_{GR}u(\mathbf{x}) := (-\Delta^* + 1/4)^{-1}u(\mathbf{x}). \quad (3.4)$$

The operator $L_{NEU}$ arises in the conversion of the classical inner boundary value problem for potential theory into pseudo–differential equations ([7, p. 87]), and the operator $L_{POT}$ is related to the integral operator of the double layer potential [7, p. 85]. The inverse of the operator $L_{GR}$ is related to the operator of second radial derivative in geodesy. From our point of view, the interest lies in the fact that their symbols, given by

$$\lambda_{\ell,NEU} = \ell + 1/2, \quad \lambda_{\ell,POT} = \frac{2\ell + 2}{2\ell + 1}, \quad \lambda_{\ell,GR} = (\ell(\ell + 1) + 1/4)^{-1}, \quad (3.5)$$

correspond to different values of the parameter $\beta$. The operator $L_{GR}$ satisfies the conditions of Theorem 2.3(b).

For the function $h$, we will choose

$$h_m(x) = \sum_{k=-m}^{m} B_m(2mx - k), \qquad m = 1, 2, \ldots, \tag{3.6}$$

where the $B$ spline $B_m$ of order $m$ is defined recursively [1, p. 131] by

$$B_m(x) := \begin{cases} 1, & \text{if } m = 1, 0 < x \leq 1, \\ 0, & \text{if } m = 1, x \in \mathbb{R} \setminus (0, 1], \\ \dfrac{x}{m-1} B_{m-1}(x) + \dfrac{m-x}{m-1} B_{m-1}(x-1), & \text{if } m > 1, x \in \mathbb{R}. \end{cases} \tag{3.7}$$

It is known [1] that each $h_m$ is a $m - 2$ times continuously differentiable function on $\mathbb{R}$, $h_m(x) = 1/2$ if $x \in [0, 1/2]$, and $h_m(x) = 0$ if $x > 1$. We also considered another function

$$h_{m,P}(x) = 1 - A_m \int_{1/2}^{x} (t - 1/2)^{m-1}(1 - t)^{m-1} dt,$$

where $A_m$ is chosen so that $h_{m,P}(1) = 0$, that corresponds to $h_{m+1}$. However, we did not find any perceptible difference in the results. Therefore, we will report only on the experiments with $h_m$.

To apply the spectral and spectrum-based approximations, we define, in this section only,

$$b_\ell = 2\pi \int_{0.9}^{1} (t - 0.9)^{3/4} \mathcal{P}_\ell(3; t) dt. \tag{3.8}$$

Then (2.9) and (2.8) imply that

$$\hat{f}(\ell, k) = b_\ell (Y_{\ell,k}((1, 0, 0)) + Y_{\ell,k}((0, 0, 1))). \tag{3.9}$$

We computed the coefficients $b_\ell$ using the Gauss quadrature formula based on the zeros of the Legendre polynomials of degree 1024. In the all the subsequent experiments, we took the resulting spectral or spectrum-based approximations $L_{NEU}^{[-1]}\sigma_{1024}^*(h_m, f)$ (respectively, $L_{POT}^{[-1]}\sigma_{1024}^*(h_m, f)$, $L_{GR}^{[-1]}\sigma_{1024}^*(h_m, f)$) as the "ideal" solutions to compare with our approximations. In the remainder of this section, we will write for $C \subseteq \mathbb{S}^2$,

$$E_{NEU}(C, m, n) := \max_{\mathbf{x} \in C} |L_{NEU}^{[-1]}\sigma_{1024}^*(h_m, f, \mathbf{x}) - L_{NEU}^{[-1]}\sigma_n^*(h_m, f, \mathbf{x})|$$

and similarly for the other pseudo-differential operators. The tables below are representatives of our experiments with different values of $m$, $n$, and $C$.

In Table 1, we tabulate the global errors, computed by taking 10000 random points on the whole sphere $\mathbb{S}^2$. We note that the case $m = 1$ corresponds to the classical spectral method. It is clear that the solutions become more accurate with the increasing values of $m$, although they seem to stabilize for any given degree after some stage.

**Table 1** The maximum error with the spectrum based methods on 10000 random points on $\mathbb{S}^2$

| $m$ | $E_{NEU}(\mathbb{S}^2, m, 128)$ | $E_{NEU}(\mathbb{S}^2, m, 256)$ | $E_{POT}(\mathbb{S}^2, m, 128)$ | $E_{POT}(\mathbb{S}^2, m, 256)$ |
|---|---|---|---|---|
| 1 | $5.7740 * 10^{-8}$ | $3.3697 * 10^{-9}$ | $3.9648 * 10^{-6}$ | $1.7819 * 10^{-6}$ |
| 2 | $1.0119 * 10^{-8}$ | $9.8496 * 10^{-10}$ | $2.4124 * 10^{-6}$ | $2.2242 * 10^{-7}$ |
| 3 | $1.4520 * 10^{-8}$ | $1.9113 * 10^{-11}$ | $1.0709 * 10^{-6}$ | $1.1790 * 10^{-8}$ |
| 4 | $1.3458 * 10^{-8}$ | $6.3574 * 10^{-11}$ | $3.0627 * 10^{-7}$ | $1.0818 * 10^{-8}$ |

The localization of the methods is illustrated by Table 2, where we compute the maximum errors at 1000 randomly chosen points on the cap

$$C := \{\mathbf{x} \in \mathbb{S}^2 \ : \ \mathbf{x} \cdot (-1/\sqrt{2}, 0, -1/\sqrt{2}) \geq 0.9\}. \tag{3.10}$$

The results for $L_{GR}$ are shown in Table 3. The errors here are substantially smaller than those for the whole sphere, reflecting the fact that $f$ is a smooth function on this cap. For each degree of the approximating polynomial, the errors decrease with increasing $m$ as before. Also, the higher the value of $m$, the greater is the decrease in the error with the increasing degree. The same phenomena were observed qualitatively also for the cap $\{\mathbf{x} \in \mathbb{S}^2 \ : \ \mathbf{x} \cdot (0, 0, 1) \geq 0.95\}$.

Next, we discuss the pseudo-spectral method, supplementing our theoretical discussions with numerical experiments. Here, we used the Driscoll-Healy formulas [4] instead of (3.9) to compute the spherical harmonic coefficients of $f$. Using the notation $D_N$ to denote the Driscoll-Healy formula based on $N$ points, we write

$$E_{NEU}(N; C, m, n) = \max_{\mathbf{x} \in C} |L_{NEU}^{[-1]} \sigma_{1024}^*(h_m, f, \mathbf{x}) - L_{NEU}^{[-1]} \sigma_n(D_N; h_m, f, \mathbf{x})|$$

and similarly for the other pseudo-differential operators. Table 3 gives the results of our experiments with the three operators, with $N = 65536$, $n = 64$, and $C$ as in (3.10), for different values of $m$.

**Table 2** The maximum error with the spectrum based methods on 1000 random points on $C := \{\mathbf{x} \in \mathbb{S}^2 \ : \ \mathbf{x} \cdot (-1/\sqrt{2}, 0, -1/\sqrt{2}) \geq 0.9\}$

| $m$ | $E_{NEU}(C, m, 128)$ | $E_{NEU}(C, m, 256)$ | $E_{POT}(C, m, 128)$ | $E_{POT}(C, m, 256)$ |
|---|---|---|---|---|
| 1 | $7.3461 * 10^{-10}$ | $4.4260 * 10^{-11}$ | $6.2609 * 10^{-8}$ | $2.5284 * 10^{-8}$ |
| 2 | $8.7449 * 10^{-11}$ | $4.7681 * 10^{-12}$ | $7.8477 * 10^{-9}$ | $2.9409 * 10^{-9}$ |
| 3 | $9.3389 * 10^{-12}$ | $9.7739 * 10^{-14}$ | $6.3541 * 10^{-10}$ | $3.0158 * 10^{-11}$ |
| 4 | $1.7298 * 10^{-12}$ | $7.5876 * 10^{-15}$ | $1.1632 * 10^{-10}$ | $1.3011 * 10^{-12}$ |

**Table 3** The maximum error with pseudo-spectral methods on 1000 random points on $C := \{\mathbf{x} \in \mathbb{S}^2 \ : \ \mathbf{x} \cdot (-1/\sqrt{2}, 0, -1/\sqrt{2}) \geq 0.9\}$

| $m$ | $E_{NEU}(256^2; C, m, 64)$ | $E_{POT}(256^2; C, m, 64)$ | $E_{GR}(256^2; C, m, 64)$ | $E_{GR}(C, m, 64)$ |
|---|---|---|---|---|
| 1 | $3.6941 * 10^{-6}$ | $2.0752 * 10^{-4}$ | $8.2827 * 10^{-1}$ | $8.2623 * 10^{-1}$ |
| 2 | $5.2936 * 10^{-7}$ | $5.7567 * 10^{-6}$ | $1.5719 * 10^{-2}$ | $1.5923 * 10^{-2}$ |
| 3 | $4.5244 * 10^{-7}$ | $1.2154 * 10^{-6}$ | $3.6640 * 10^{-3}$ | $3.5548 * 10^{-3}$ |
| 4 | $4.4980 * 10^{-7}$ | $6.8164 * 10^{-7}$ | $1.9791 * 10^{-3}$ | $1.9303 * 10^{-3}$ |

We note that the qualitative behavior of the approximate solution is the same as in the case of spectrum-based solutions. In particular, the pseudo-spectral method seems to exhibit the same local behavior in the case of $L_{NEU}$ and $L_{POT}$, even though they do not satisfy the conditions of Theorem 2.3(b).

Finally, to distinguish between the pseudo-spectral method and the spectral method with a discretization used to compute the spherical harmonic coefficients further, we considered the solution of

$$L_{NEU}u = (x_1 \cos x_3 - x_3 \sin x_3 + 0.5 \cos x_3)e^{x_1},$$

given explicitly by

$$u_{NEU}(f_1, \mathbf{x}) = e^{x_1} \cos x_3, \qquad \mathbf{x} \in \mathbb{S}^2.$$

It is clear from Table 4 that a careful discretization using Driscoll-Healy formulas gives the solution with nearly a machine precision accuracy, while a Monte-Carlo method with the same number of points gives very poor results. In Table 4, the notation $E_{MC}(N; \mathbb{S}^2, m, n)$ denotes the errors in which the spherical harmonic coefficients of the approximate solution of $u_{NEU}(f_1, x)$ are generated using Monte-Carlo quadrature. The maximum errors are measured on a grid of $51^2$ points on the unit sphere. Here, the errors $E_{NEU}(256^2; \mathbb{S}^2, m, 32)$ etc. are measured from the known ideal solution rather than the spectral approximation as in the other tables.

## 4 Proofs

Let $p_\ell$ denote the Jacobi polynomial with positive leading coefficient, and orthonormalized on $[-1, 1]$ with respect to $(1-t^2)^{q/2-1}$. If $\mathbf{b} = \{b_\ell\}_{\ell=0}^\infty$ is any sequence such that $b_\ell = 0$ for all sufficiently large $\ell$ (depending upon $\mathbf{b}$), we write

$$\Phi(\mathbf{b}, t) := \omega_{q-1}^{-1} \sum_{\ell=0}^{\infty} b_\ell p_\ell(1) p_\ell(t) = \sum_{\ell=0}^{\infty} b_\ell \frac{d_\ell^q}{\omega_q} \mathcal{P}_\ell(q+1; t), \qquad t \in \mathbb{R}.$$

The starting point of our proofs is a general estimate on these kernels. We recall the following lemma (cf. [12, Lemma 4.6, 4.10]).

**Lemma 4.1** *Let $\mathbf{b} = \{b_\ell\}_{\ell=0}^\infty$ be any sequence such that $b_\ell = 0$ for all sufficiently large $\ell$ (depending upon $\mathbf{b}$). We have for integer $K > (q+1)/2$,*

$$\int_{-1}^{1} |\Phi(\mathbf{b}, t)|(1-t^2)^{q/2-1} dt \le c|||\mathbf{b}|||_K^*. \tag{4.1}$$

**Table 4** The maximum errors on $\mathbb{S}^2$ using Monte Carlo methods and Drisco-Healy formulas with indicated number of points

| $m$ | $E_{NEU}(256^2; \mathbb{S}^2, m, 32)$ | $E_{NEU}(256^2; \mathbb{S}^2, m, 64)$ | $E_{MC}(256^2; \mathbb{S}^2, m, 64)$ |
|---|---|---|---|
| 1 | $1.3211 * 10^{-14}$ | $4.3298 * 10^{-15}$ | 2.1575 |
| 2 | $1.3211 * 10^{-14}$ | $4.3298 * 10^{-15}$ | 2.0400 |
| 3 | $1.3211 * 10^{-14}$ | $4.3298 * 10^{-15}$ | 2.0675 |
| 4 | $1.3211 * 10^{-14}$ | $4.3298 * 10^{-15}$ | 2.0498 |

*Moreover, for each $\eta > 0$ and integer $K \ge 1$, there exists a constant $c(\eta) > 0$ such that*

$$|\Phi(\mathbf{b}, t)| \le c(\eta)|||\mathbf{b}|||^*_{q-K,K}, \qquad -1 \le t \le 1-\eta. \tag{4.2}$$

*Proof* The estimate (4.1) follows from Lemma 4.6 in [12], (4.2) follows from Lemma 4.10 in [12]. □

**Proposition 4.1** *Let $N \ge 0$, $\mathbf{b} = \{b_\ell\}_{\ell=0}^\infty$ be any sequence such that $b_\ell = 0$ for all $\ell \ge N$ (depending upon $\mathbf{b}$), $\nu$ be an M–Z quadrature measure of order $N$. We have for integer $K > (q+1)/2$,*

$$\sup_{\mathbf{x}\in\mathbb{S}^q} \int_{\mathbb{S}^q} |\Phi(\mathbf{b}, \mathbf{x}\cdot\mathbf{y})| d|\nu|(\mathbf{y}) \le cM_{\nu;q,1}(N)|||\mathbf{b}|||^*_K. \tag{4.3}$$

*Let $1 \le p \le \infty$, and $f \in L^p(\nu; \mathbb{S}^q)$. Then for each $\mathbf{x} \in \mathbb{S}^q$,*

$$\left\| \int_{\mathbb{S}^q} \Phi(\mathbf{b}, \mathbf{x}\cdot\mathbf{y}) f(\mathbf{y}) d\nu(\mathbf{y}) \right\|_{\mathbb{S}^q,p} \le cM_{\nu;q,1}(N)|||\mathbf{b}|||^*_K \|f\|_{\nu;\mathbb{S}^q,p}. \tag{4.4}$$

*Moreover, if $\mathbf{x}_0 \in \mathbb{S}^q$, $0 < \alpha < \alpha'$, then there exists a positive constant $c(\alpha, \alpha')$ such that*

$$\max_{\mathbf{x}\in\mathbb{S}^q_\alpha(\mathbf{x}_0)} |\Phi(\mathbf{b}, \mathbf{x}\cdot\mathbf{y})| \le c(\alpha, \alpha')|||\mathbf{b}|||^*_{q-K,K}, \qquad \mathbf{y} \in \mathbb{S}^q \setminus \mathbb{S}^q_{\alpha'}(\mathbf{x}_0). \tag{4.5}$$

*Proof* In view of (2.14) and the rotation invariance of $\mu^*_q$, the estimate (4.1) implies that

$$\sup_{\mathbf{x}\in\mathbb{S}^q} \int_{\mathbb{S}^q} |\Phi(\mathbf{b}, \mathbf{x}\cdot\mathbf{y})| d|\nu|(\mathbf{y}) \le M_{\nu;q,1}(N) \sup_{\mathbf{x}\in\mathbb{S}^q} \int_{\mathbb{S}^q} |\Phi(\mathbf{b}, \mathbf{x}\cdot\mathbf{y})| d\mu^*_q(\mathbf{y})$$
$$= cM_{\nu;q,1}(N) \int_{-1}^1 |\Phi(\mathbf{b}, t)|(1-t^2)^{q/2-1} dt \le cM_{\nu;q,1}(N)|||\mathbf{b}|||^*_K.$$

This proves (4.3). We note that this estimate is now proved also for $\mu^*_q$ in place of $\nu$. The estimate (4.4) follows using the Riesz–Thorin interpolation theorem (cf. [12, Lemma 4.1]). It is not difficult to check that for $\mathbf{x} \in \mathbb{S}^q_\alpha(\mathbf{x}_0)$ and $\mathbf{y} \in \mathbb{S}^q \setminus \mathbb{S}^q_{\alpha'}(\mathbf{x}_0)$, $-1 \le \mathbf{x}\cdot\mathbf{y} \le 1 - 2(\sin(\alpha'/2) - \sin(\alpha/2))^2$. Hence, (4.5) follows from (4.2). □

Next, we investigate the bounds in Proposition 4.1 when each $b_\ell$ is replaced by $h(\ell/n)\lambda_\ell^{[-1]}$. For the convenience of the reader, we summarize a few very elementary facts in the following lemma.

**Lemma 4.2** *Let $\mathbf{a}$, $\mathbf{h}$ be sequences of real numbers. We have*

$$\Delta^r(a_\nu h_\nu) = \sum_{\ell=0}^r \binom{r}{\ell} \Delta^\ell a_\nu \Delta^{r-\ell} h_{\nu+\ell}, \qquad r \ge 1. \tag{4.6}$$

*Suppose that $h_\nu = 0$ for all sufficiently large $\nu$. We write $S_m = \sum_{\nu=m}^\infty a_\nu$, $s_m = \sum_{\nu=0}^m a_\nu$, and assume that each $S_m$ is finite. Then*

$$\sum_{\nu=0}^\infty a_\nu h_\nu = -\sum_{\nu=0}^\infty s_\nu \Delta h_\nu = a_0 S_0 + \sum_{\nu=0}^\infty S_{\nu+1} \Delta h_\nu. \tag{4.7}$$

*Proof* The formula (4.6) is the Leibniz formula for differences, and can be proved easily using induction. The formulas in (4.7) follow by a summation by parts. □

The next lemma (applied with $a_\ell = \lambda_\ell^{[-1]}$) gives the necessary bounds on the right hand sides in (4.4) and (4.5) when each $b_\ell$ is replaced by $h(\ell/n)\lambda_\ell^{[-1]}$.

**Lemma 4.3** *Let* $\{h_\ell\}_{\ell=0}^\infty$ *be a sequence of nonnegative numbers such that* $h_\ell = 0$ *for all sufficiently large* $\ell$. *Let* $\alpha, \beta \in \mathbb{R}$, $K \geq 1$, *and* $\mathbf{a}_\beta := \{(\ell+1)^{-\beta} a_\ell\}_{\ell=0}^\infty \in \mathcal{B}_K$.
(a) *If* $\alpha + \beta > 0$ *then*

$$|||\{a_\nu h_\nu\}_{\nu=0}^\infty|||_{\alpha,K}^* \leq c|||\mathbf{a}_\beta|||_K \cdot |||\mathbf{h}|||_{\alpha+\beta,K}^*. \tag{4.8}$$

(b) *If* $\alpha + \beta < 0$ *then*

$$|||\{a_\nu h_\nu\}_{\nu=0}^\infty|||_{\alpha,K}^* \leq c|||\mathbf{a}_\beta|||_K \left\{h_0 + |||\mathbf{h}|||_{\alpha+\beta,K}^*\right\}. \tag{4.9}$$

(c) *If* $\alpha + \beta = 0$, *we have*

$$|||\{a_\nu h_\nu\}_{\nu=0}^\infty|||_{\alpha,K}^* \leq c|||\mathbf{a}_\beta|||_K \left\{\sum_{\nu=0}^\infty \frac{h_\nu}{\nu+1} + |||\mathbf{h}|||_K^*\right\}. \tag{4.10}$$

(d) *In the special case when* $\alpha = \beta = 0$ *and* $\mathbf{a} \in \mathcal{B}_K^*$, *we have*

$$|||\{a_\nu h_\nu\}_{\nu=0}^\infty|||_K^* \leq c|||\mathbf{a}|||_K^* (\max h_\nu) + |||\mathbf{a}|||_K \cdot |||\mathbf{h}|||_K^*. \tag{4.11}$$

*Proof* Let $1 \leq r \leq K$. Using (4.6), we obtain that

$$\sum_{\nu=0}^\infty (\nu+1)^{\alpha+r-1} |\Delta^r(a_\nu h_\nu)| \leq c \sum_{\nu=0}^\infty \sum_{m=0}^r (\nu+1)^{\alpha+r-1} |\Delta^{r-m} a_{\nu+m}||\Delta^m h_\nu| \tag{4.12}$$

Using (4.6) again, we obtain for any integer $m$, $0 \leq m \leq r$, and $\nu \geq 0$, that

$$\begin{aligned}|\Delta^{r-m} a_{\nu+m}| &= \left|\Delta^{r-m}\{(\nu+m+1)^\beta (\nu+m+1)^{-\beta} a_{\nu+m}\}\right| \\ &\leq \sum_{\ell=0}^{r-m} |\Delta^{r-m-\ell}(\nu+m+\ell+1)^\beta||\Delta^\ell((\nu+m+1)^{-\beta} a_{\nu+m})| \\ &\leq c(\nu+1)^{\beta-r+m} \sum_{\ell=0}^{r-m} (\nu+m+1)^\ell |\Delta^\ell((\nu+m+1)^{-\beta} a_{\nu+m})|\end{aligned} \tag{4.13}$$

Thus,

$$|\Delta^{r-m} a_{\nu+m}| \leq c(\nu+1)^{\beta-r+m} |||\mathbf{a}_\beta|||_K. \tag{4.14}$$

Therefore, we obtain

$$\begin{aligned}
&\sum_{\nu=0}^{\infty}\sum_{m=0}^{r}(\nu+1)^{\alpha+r-1}|\Delta^{r-m}a_{\nu+m}||\Delta^m h_\nu| \\
&\le c|||\mathbf{a}_\beta|||_K \sum_{\nu=0}^{\infty}\sum_{m=0}^{r}(\nu+1)^{\alpha+\beta+m-1}|\Delta^m h_\nu| \\
&= c|||\mathbf{a}_\beta|||_K \left\{\sum_{\nu=0}^{\infty}(\nu+1)^{\alpha+\beta-1}h_\nu + |||\mathbf{h}|||^*_{\alpha+\beta,K}\right\}. \qquad (4.15)
\end{aligned}$$

If $\alpha+\beta>0$, we use the first equation in (4.7) to obtain

$$\left|\sum_{\nu=0}^{\infty}(\nu+1)^{\alpha+\beta-1}h_\nu\right| \le c\sum_{\nu=0}^{\infty}(\nu+1)^{\alpha+\beta}|\Delta h_\nu|.$$

Along with (4.15), this implies (4.8).

If $\alpha+\beta<0$, we use the second equation in (4.7) to obtain

$$\left|\sum_{\nu=0}^{\infty}(\nu+1)^{\alpha+\beta-1}h_\nu\right| \le c\left\{h_0+\sum_{\nu=0}^{\infty}(\nu+1)^{\alpha+\beta}|\Delta h_\nu|\right\}.$$

Along with (4.15), this implies (4.9).

If $\alpha+\beta=0$, then (4.15) reduces to (4.10). If $\alpha=\beta=0$ then (4.11) follows immediately from (4.12). □

For the convenience of reference in the later proofs, we record a consequence of Lemma 4.3 and Proposition 4.1.

**Proposition 4.2** *Let $K>(q+1)/2$ be an integer, $\beta\in\mathbb{R}$, $D>0$, $h\in\mathcal{A}_K$, $h(x)=0$ if $x>D$, and $\mathbf{a}_\beta:=\{(\ell+1)^{-\beta}a_\ell\}_{\ell=0}^{\infty}\in\mathcal{B}_K$. Let $V$ be the total variation of $h^{(K-1)}$. We write*

$$\Psi_n(\mathbf{a},h,t):=\sum_{\ell=0}^{\infty}a_\ell h(\ell/n)\frac{d_\ell^q}{\omega_q}\mathcal{P}_\ell(q+1;t), \qquad t\in\mathbb{R},\ n=1,2,\ldots.$$

*For each integer $n\ge 0$, let $\mu_n$ be an M–Z quadrature measure of order $Dn$, and $M_{\mu_n;q,1}(Dn)\le c$. Let $1\le p\le\infty$, and $f\in X^p(\mu_n;\mathbb{S}^q)$.*
(a) *Suppose that one of the following three conditions holds: (i) $\beta>0$, (ii) $\beta<0$ and $h(0)=0$, (iii) $\beta=0$ and $\mathbf{a}\in\mathcal{B}_K^*$. Then for any $\mathbf{x}\in\mathbb{S}^q$,*

$$\left\|\int_{\mathbb{S}^q}f(\mathbf{y})\Psi_n(\mathbf{a},h,\mathbf{x}\cdot\mathbf{y})d\mu_n(\mathbf{y})\right\|_{\mathbb{S}^q,p}\le c(\mathbf{a})n^\beta V\|f\|_{\mu_n;\mathbb{S}^q,p}. \qquad (4.16)$$

(b) *If $\beta<0$,*

$$\left\|\int_{\mathbb{S}^q}f(\mathbf{y})\Psi_n(\mathbf{a},h,\mathbf{x}\cdot\mathbf{y})d\mu_n(\mathbf{y})\right\|_{\mathbb{S}^q,p}\le c(\mathbf{a})V\|f\|_{\mu_n;\mathbb{S}^q,p}. \qquad (4.17)$$

(c) *If* $Q \ge 1$, $K \ge \max(1, Q+q+\beta+1)$, $0 < \alpha < \alpha'$, $\mathbf{x}_0 \in \mathbb{S}^q$, *and* $h(0) = 0$, *then*

$$\max_{\mathbf{x}\in\mathbb{S}^q_\alpha(\mathbf{x}_0)} |\Psi_n(\mathbf{a}, h, \mathbf{x}\cdot\mathbf{y})| \le c(\mathbf{a}, \alpha, \alpha')Vn^{-Q}, \qquad \mathbf{y} \in \mathbb{S}^q \setminus \mathbb{S}^q_{\alpha'}(\mathbf{x}_0). \quad (4.18)$$

*Proof* Since $h'(x) = 0$ for $x \le c_1$, we obtain by the mean value theorem that

$$\sum_{\nu=0}^{\infty}(\nu+1)^{s+r-1}|\Delta^r h(\nu/n)| \le cVn^s, \qquad s \in \mathbb{R},\ 1 \le r \le K.$$

Therefore,

$$|||\{h(\nu/n)\}_{\nu=0}^{\infty}|||^*_{s,K} \le cVn^s, \qquad s \in \mathbb{R}.$$

In this proof only, let $h_\nu := h(\nu/n)$, $\mathbf{b}$ be defined by $b_\nu = a_\nu h_\nu$. The estimates (4.17), (4.16) now follow from (4.4), and Lemma 4.3 used with $\alpha = 0$. The estimate (4.18) follows from (4.5), where we need to use only Lemma 4.3(b). □

The following proposition notes a number of consequences of the above results in approximation theory. The results in this proposition are probably not new, but we find it easier to prove them than finding a reference. In the remainder of this section, all unspecified constants may depend upon $h$ as well as $q$, $L$, $\beta$, $K$, $\gamma$, $\rho$, $p$, and the sequence $\{\mu_n\}$.

**Proposition 4.3** *For each integer* $n \ge 0$, *let* $\mu_n$ *be an M–Z quadrature measure of order* $2n$, *and* $M_{\mu_n;q,1}(2n) \le c$. *Let* $\beta \ge 0$, $K > (q+1)/2$ *be an integer,* $h \in \mathcal{A}^*_K$, $1 \le p \le \infty$, *and* $f \in X^p(\mathbb{S}^q)$.
(a) *We have*

$$E_{n,p}(f) \le \|f - \sigma_n(\mu_n; h, f)\|_{\mathbb{S}^q,p} \le cE_{n/2,p}(f). \quad (4.19)$$

(b) *If* $\partial_\beta f \in L^p(\mathbb{S}^q)$,

$$E_{n,p}(f) \le cn^{-\beta}E_{n,p}(\partial_\beta f). \quad (4.20)$$

(c) *If* $P \in \Pi^q_n$ *then*

$$\|\partial_\beta P\|_{\mathbb{S}^q,p} \le cn^\beta\|P\|_{\mathbb{S}^q,p}. \quad (4.21)$$

(d) (Simultaneous approximation) *If* $P \in \Pi^q_n$ *and* $\|f - P\|_{\mathbb{S}^q,p} \le cE_{n,p}(f)$ *then* $\|\partial_\beta(f - P)\|_{\mathbb{S}^q,p} \le cE_{n,p}(\partial_\beta f)$.

*Proof* Part (a) is proved in the same way as [13, Proposition 4.1]. To prove part (b), we use the duality principle [3, Chapter 3, Theorem 1.3]. It is enough to prove this part when $n = 2^\ell$ for some integer $\ell \ge 1$. Let $p'$ be defined by $1/p + 1/p' = 1$, $\|g\|_{\mathbb{S}^q,p'} = 1$,

$$\int_{\mathbb{S}^q} g(\xi)P(\xi)d\mu^*_q(\xi) = 0, \qquad P \in \Pi^q_{2^\ell}. \quad (4.22)$$

Then for any integer $m$, $\sigma^*_{2^m}(h, g)$ satisfies (4.22) as well. Let $m \ge \ell$ and $h(\ell; x) = h(x) - h(2^{m-\ell}x)$. Then (4.22) implies that $\sigma^*_{2^m}(h, g) = \sigma^*_{2^m}(h(\ell; \circ), g)$. Moreover,

$h(\ell; \nu/2^m) = 0$ if $\nu \le 2^{\ell-1}$. So, denoting by $V$ the total variation of $h^{(K-1)}$, we obtain for any $r$, $1 \le r \le K$, that

$$\begin{aligned}
&\sum_{\nu=0}^{\infty}(\nu+1)^{-\beta+r-1}|\Delta^r h(\ell;\nu/2^m)| \\
&\le c2^{-\beta\ell}\left\{\sum_{\nu=0}^{\infty}(\nu+1)^{r-1}|\Delta^r h(\nu/2^m)| + \sum_{\nu=0}^{\infty}(\nu+1)^{r-1}|\Delta^r h(\nu/2^\ell)|\right\} \\
&\le c2^{-\beta\ell}V.
\end{aligned}$$

Therefore, keeping in mind that the constants may depend upon $h$, we obtain that

$$|||\{(\nu+1)^{-\beta}h(\ell;\nu/2^m)\}_{\nu=0}^{\infty}|||_K^* \le c2^{-\beta\ell}, \qquad m = \ell, \ell+1, \cdots.$$

Hence, for all $m \ge \ell$, (4.4) (used with $b_\nu = (\nu+1)^{-\beta}h(\ell;\nu/2^m)$ and $p'$ in place of $p$) shows that

$$\|\partial_{-\beta}\sigma_{2^m}^*(h, g)\|_{\mathbb{S}^q,p'} = \|\partial_{-\beta}\sigma_{2^m}^*(h(\ell;\circ), g)\|_{\mathbb{S}^q,p'} \le c2^{-\beta\ell}\|g\|_{\mathbb{S}^q,p'}.$$

Hence, recalling that $\|g\|_{\mathbb{S}^q,p'} = 1$,

$$\begin{aligned}
\left|\int_{\mathbb{S}^q} f(\xi)\sigma_{2^m}^*(h, g, \xi)d\mu_q^*(\xi)\right| &= \left|\int_{\mathbb{S}^q} \partial_\beta f(\xi)\partial_{-\beta}\sigma_{2^m}^*(h, g, \xi)d\mu_q^*(\xi)\right| \\
&\le c2^{-\beta\ell}\|\partial_\beta f\|_{\mathbb{S}^q,p}.
\end{aligned}$$

Since $\sigma_{2^m}^*(h, g) \to g$ in $X^{p'}(\mathbb{S}^q)$ norm as $m \to \infty$, we conclude that for all $g$ with $\|g\|_{\mathbb{S}^q,p'} = 1$, and satisfying (4.22),

$$\left|\int_{\mathbb{S}^q} f(\xi)g(\xi)d\mu_q^*(\xi)\right| \le c2^{-\beta\ell}\|\partial_\beta f\|_{\mathbb{S}^q,p}.$$

In view of the duality principle, this implies $E_{2^\ell,p}(f) \le c2^{-\beta\ell}\|\partial_\beta f\|_{\mathbb{S}^q,p}$. If $P \in \Pi_{2^\ell}^q$ is chosen such that $\|\partial_\beta f - P\|_{\mathbb{S}^q,p} \le 2E_{2^\ell,p}(\partial_\beta f)$, and $R \in \Pi_{2^\ell}^q$ is such that $\partial_\beta R = P$, then

$$E_{2^\ell,p}(f) = E_{2^\ell,p}(f - R) \le c2^{-\beta\ell}\|\partial_\beta f - P\|_{\mathbb{S}^q,p} \le c2^{-\beta\ell}E_{2^\ell,p}(\partial_\beta f).$$

This completes the proof of part (b).

If $P \in \Pi_n^q$, then $\sigma_{2n}^*(h, P) = P$. The estimate (4.21) follows from (4.16) with $a_\nu = (\nu+1)^\beta$ and $\mu_n = \mu_q^*$.

Finally, if $P \in \Pi_n^q$ and $\|f - P\|_{\mathbb{S}^q,p} \le cE_{n,p}(f)$, then (4.19) implies that

$$\|\sigma_{2n}^*(h, f) - P\|_{\mathbb{S}^q,p} \le cE_{n,p}(f).$$

Consequently, (4.21) and (4.20) show that

$$\begin{aligned}
\|\sigma_{2n}^*(h, \partial_\beta f) - \partial_\beta P\|_{\mathbb{S}^q,p} = \|\partial_\beta(\sigma_{2n}^*(h, f) - P)\|_{\mathbb{S}^q,p} &\le cn^\beta E_{n,p}(f) \\
&\le cE_{n,p}(\partial_\beta f).
\end{aligned}$$

Consequently, (4.19) used with $\partial_\beta f$ in place of $f$ implies that

$$\begin{aligned}\|\partial_\beta(f-P)\|_{\mathbb{S}^q,p} &\le \|\partial_\beta f-\sigma^*_{2n}(h,\partial_\beta f)\|_{\mathbb{S}^q,p}+\|\sigma^*_{2n}(h,\partial_\beta f)-\partial_\beta P\|_{\mathbb{S}^q,p}\\ &\le cE_{n,p}(\partial_\beta f).\end{aligned}$$

□

We will need the properties of another operator in our proofs. For $f \in \mathcal{S}^*$ and integer $n \ge 0$, we define the *polynomial frame operator* by

$$\tau^*_n(h, f) = \sigma^*_{2^n}(h, f) - \sigma^*_{2^{n-1}}(h, f). \tag{4.23}$$

Our notation implies that $\sigma^*_n(h, f) \in \Pi^q_{cn}$, while $\tau^*_n(h, f) \in \Pi^q_{c2^n}$. The operator $\tau^*_n$ is one of the frame operators (cf. [16]) in our theory of polynomial frames. It is also similar to the wavelet operator discussed in [7, Chapter 10]. However, the number of vanishing moments is not fixed here; it is proportional to $2^n$. Also, the kernel of this operator is band limited. They do not yield a Parseval identity as in [7, Eqn. (10.2.9)], but have bounded frame bounds (cf. [13]). Their space localization has been studied in a great detail in [13]. In particular, the following proposition summarizes some facts from [13] concerning the characterization of local Besov spaces using the operators $\tau^*_n(h)$.

**Proposition 4.4** *Let $1 \le p \le \infty$, $f \in X^p(\mathbb{S}^q)$, $K > (q+1)/2$, $h \in \mathcal{A}^*_K$. Then*

$$f = \sum_{n=0}^{\infty} \tau^*_n(h, f), \tag{4.24}$$

*with the series converging in the sense of $X^p(\mathbb{S}^q)$. Let $\gamma > 0$, $0 < \rho \le \infty$, $K > \max(1,\gamma)+q+1$, $h \in \mathcal{A}^*_K$, and $\mathbf{x}_0 \in \mathbb{S}^q$. Then the following statements are equivalent.*

(a) $f \in B_{p,\rho,\gamma}(\mathbf{x}_0)$.

(b) *There exists a cap $C$, centered at $\mathbf{x}_0$, such that for every $\phi \in C^\infty_0(C)$, $\{\|\tau^*_n(h, f\phi)\|_{\mathbb{S}^q,p}\}_{n=0}^\infty \in \mathsf{b}_{\rho,\gamma}$.*

(c) *There exists a cap $C$, centered at $\mathbf{x}_0$, such that $\{\|\tau^*_n(h, f)\|_{C,p}\}_{n=0}^\infty \in \mathsf{b}_{\rho,\gamma}$.*

*Proof* The expansion (4.24) follows from the definitions and Proposition 4.3(a) (with $\mu^*_q$ in place of $\mu_n$). The equivalence of the statements (a)–(c) is a part of [13, Theorem 3.3], applied with $\mu^*_q$ in place of $\mu_n$ there. □

The next proposition will be used to obtain the connection between the smoothness of $f$ and $L^{[-1]}f$.

**Proposition 4.5** *Let $K > (q+1)/2$ be an integer, and $h \in \mathcal{A}^*_K$. Let $\beta \in \mathbb{R}$, $\mathbf{a}$ be a sequence such that $\mathbf{a}_\beta = \{(\nu+1)^{-\beta}a_\nu\}_{\nu=0}^\infty \in \mathcal{B}^*_K$. For $f \in \mathcal{S}^*$, let $T_{\mathbf{a}}f \in \mathcal{S}^*$ be defined by $\widehat{T_{\mathbf{a}}f}(\ell,k) = a_\ell \hat{f}(\ell,k)$, $k = 1,\ldots,d^q_\ell$, $\ell = 0,1,\ldots$. Then for $1 \le p \le \infty$,*

$$\|\tau^*_n(h, T_{\mathbf{a}}f)\|_{\mathbb{S}^q,p} \le c2^{n\beta}\|\tau^*_n(h, f)\|_{\mathbb{S}^q,p}. \tag{4.25}$$

*In particular, if $\gamma > \max(\beta,0)$, $0 < \rho \le \infty$, $K > \max(1,\gamma,\gamma-\beta)+q+\beta+1$, and $f \in B_{p,\rho,\gamma}$ then $T_{\mathbf{a}}f \in B_{p,\rho,\gamma-\beta}$.*

*Proof* In this proof only, let $g_1(x) = h(x) - h(2x)$, and $g_2(x) = h(x/2) - h(4x)$. Then $g_2 \in \mathcal{A}_K$, and $g_2(0) = 0$. Moreover, $g_2(x) = 1$ if $g_1(x) \neq 0$. Using Proposition 4.2(a) with $\mu_n = \mu_q^*$, and $g_2$ in place of $h$, we get

$$\|\tau_n^*(h, T_{\mathbf{a}} f)\|_{\mathbb{S}^q,p} = \left\| \sum_{\ell,k} a_\ell g_2(\ell/2^n) g_1(\ell/2^n) \hat{f}(\ell, k) Y_{\ell,k} \right\|_{\mathbb{S}^q,p}$$
$$\leq c 2^{n\beta} \left\| \sum_{\ell,k} g_1(\ell/2^n) \hat{f}(\ell, k) Y_{\ell,k} \right\|_{\mathbb{S}^q,p} = c 2^{n\beta} \|\tau_n^*(h, f)\|_{\mathbb{S}^q,p}.$$

The last statement follows from Proposition 4.4. □

We can now prove the theorems in Section 2. In the sequel, $Q$ will be fixed such that $\max(1, \gamma, \gamma - \beta) < Q \leq K - \beta - q - 1$.

*Proof of Theorem 2.1* Let $L^{[-1]} f \in \mathcal{S}^*$ be defined by $\widehat{L^{[-1]} f}(\ell, k) = \lambda_\ell^{[-1]} \hat{f}(\ell, k)$. Let $h \in \mathcal{A}_K^*$. Using Proposition 4.2(a) with $\{a_\ell = (\ell + 1)^{-\beta} \lambda_\ell^{[-1]}\}_{\ell=0}^\infty$ (and $\beta = 0$, $\mu_n = \mu_q^*$ ), our conditions on $\lambda_\ell^{[-1]}$ imply that

$$\|\sigma_{2^n}^*(h, L^{[-1]} f)\|_{\mathbb{S}^q,p} = \| \sum_{\ell,k} h(\ell/2^n) a_\ell \widehat{\partial_\beta f}(\ell, k) Y_{\ell,k} \|_{\mathbb{S}^q,p} \leq c \|\partial_\beta f\|_{\mathbb{S}^q,p}. \tag{4.26}$$

We observe that $L^{[-1]}(\sigma_{2^n}^*(h, f)) = \sigma_{2^n}^*(h, L^{[-1]} f)$, and $\partial_\beta \sigma_{2^n}^*(h, f) = \sigma_{2^n}^*(h, \partial_\beta f)$. So, if $m \geq n$, we have

$$\begin{aligned}
&\|\sigma_{2^m}^*(h, L^{[-1]} f) - \sigma_{2^n}^*(h, L^{[-1]} f)\|_{\mathbb{S}^q,p} \\
&\quad = \|\sigma_{2^{m+1}}^*(h, \sigma_{2^m}^*(h, L^{[-1]} f) - \sigma_{2^n}^*(h, L^{[-1]} f))\|_{\mathbb{S}^q,p} \\
&\quad = \|\sigma_{2^{m+1}}^*(h, L^{[-1]}(\sigma_{2^m}^*(h, f) - \sigma_{2^n}^*(h, f)))\|_{\mathbb{S}^q,p} \\
&\quad \leq c \|\partial_\beta(\sigma_{2^m}^*(h, f) - \sigma_{2^n}^*(h, f))\|_{\mathbb{S}^q,p} \leq c \|\sigma_{2^m}^*(h, \partial_\beta f) - \sigma_{2^n}^*(h, \partial_\beta f)\|_{\mathbb{S}^q,p}.
\end{aligned}$$

Since $\partial_\beta f \in L^p(\mathbb{S}^q)$, the last sequence tends to 0 as $m, n \to \infty$. Therefore, the completeness of the $L^p$ spaces implies that the sequence $\{\sigma_{2^n}^*(h, L^{[-1]} f)\}_{n=0}^\infty$ converges to $u \in L^p(\mathbb{S}^q)$. Since $f \in \mathcal{N}_L$, $u$ is a solution of the equation $Lu = f$. Moreover, if $p = \infty$, then $u \in C(\mathbb{S}^q)$. The estimate (2.7) follows easily from (4.26).

To prove part (b), let $C_2, C_1, C$ be caps of radius $\alpha_2 > \alpha_1 > \alpha$ respectively, centered at $\mathbf{x}_0$, and $\phi \in C_0^\infty(C_2)$ such that $\phi(\mathbf{x}) = 1$ if $\mathbf{x} \in C_1$. In view of Proposition 4.4, we may choose $C_2$ to ensure that $\{\|\tau_n^*(h, f\phi)\|_{\mathbb{S}^q,p}\}_{n=0}^\infty \in \mathsf{b}_{\rho,\gamma}$. In view of Proposition 4.5, this implies, in particular, that

$$\|\tau_n^*(h, \partial_\beta(f\phi))\|_{\mathbb{S}^q,p} \leq c 2^{n(\beta - \gamma)} \|f\phi\|_{\mathbb{S}^q,p}, \qquad n = 0, 1, \ldots.$$

Hence, $\sum_{n=0}^\infty \tau_n^*(h, \partial_\beta(f\phi))$ converges in $X^p(\mathbb{S}^q)$, necessarily to $\partial_\beta(f\phi)$. Thus, $\partial_\beta(f\phi) \in X^p(\mathbb{S}^q)$.

Let $u = L^{[-1]}f$, $v$ be the solution of the equation $Lv = f\phi$. In view of Proposition 4.5 with $\mathbf{a} = \{\lambda_\nu^{[-1]}\}_{\nu=0}^\infty$,

$$\|\tau_n^*(h, v)\|_{\mathbb{S}^q,p} \le c2^{n\beta}\|\tau_n^*(h, f\phi)\|_{\mathbb{S}^q,p}.$$

Since $\{\|\tau_n^*(h, f\phi)\|_{\mathbb{S}^q,p}\}_{n=0}^\infty \in \mathsf{b}_{\rho,\gamma}$, we conclude that $\{\|\tau_n^*(h, v)\|_{\mathbb{S}^q,p}\}_{n=0}^\infty \in \mathsf{b}_{\rho,\gamma-\beta}$.

Now, using (4.18) with $h_1(x) := h(x) - h(2x)$, $\mathbf{h}_1 := \{h_1(\nu/2^n)\}$, we have for $\mathbf{x} \in C$

$$\begin{aligned}
|\tau_n^*(h, u, \mathbf{x})| &= \left|\int_{\mathbb{S}^q} \Psi_{2^n}(\mathbf{a}, h_1, \mathbf{x}\cdot\mathbf{y}) f(\mathbf{y}) d\mu_q^*(\mathbf{y})\right| \\
&\le |\tau_n^*(h, v, \mathbf{x})| + \left|\int_{\mathbb{S}^q\setminus C_1} \Psi_{2^n}(\mathbf{a}, h_1, \mathbf{x}\cdot\mathbf{y}) f(\mathbf{y})(1-\phi(\mathbf{y}))d\mu_q^*(\mathbf{y})\right| \\
&\le |\tau_n^*(h, v, \mathbf{x})| + c(\alpha, \alpha_1, \alpha_2)2^{-nQ}\|f\|_{\mathbb{S}^q,p}.
\end{aligned}$$

Since $\{\|\tau_n^*(h, v)\|_{\mathbb{S}^q,p}\}_{n=0}^\infty \in \mathsf{b}_{\rho,\gamma-\beta}$ and $Q > \gamma - \beta$, this implies that $\{\|\tau_n^*(h, u)\|_{C,p}\}_{n=0}^\infty \in \mathsf{b}_{\rho,\gamma-\beta}$. In view of Proposition 4.4, this is equivalent to the statement that $u \in B_{p,\rho,\gamma-\beta}(\mathbf{x}_0)$. This proves part (b). □

*Proof of Theorem 2.2* In view of (2.7), we have

$$\begin{aligned}
\|L^{[-1]}(f - \sigma_n^*(h, f))\|_{\mathbb{S}^q,p} &\le c\|\partial_\beta(f - \sigma_n^*(h, f))\|_{\mathbb{S}^q,p} \\
&= c\|\partial_\beta f - \sigma_n^*(h, \partial_\beta f)\|_{\mathbb{S}^q,p}.
\end{aligned}$$

The estimate (2.12) follows from (4.19).

Next, let $f \in B_{p,\rho,\gamma}(\mathbf{x}_0)$, and $u = L^{[-1]}f$. Theorem 2.1(b) implies that $u \in B_{p,\rho,\gamma-\beta}(\mathbf{x}_0)$. In view of Proposition 4.4, we may find a cap $C$, centered at $\mathbf{x}_0$, such that $\{\|\tau_k^*(h, u)\|_{C,p}\}_{k=0}^\infty \in \mathsf{b}_{\rho,\gamma-\beta}$. Then (cf. Proposition 4.4)

$$\|L^{[-1]}(f - \sigma_{2^n}^*(h, f))\|_{C,p} = \|u - \sigma_{2^n}^*(h, u)\|_{C,p} \le \sum_{k=n+1}^\infty \|\tau_k^*(h, u)\|_{C,p}.$$

Since $\{\|\tau_k^*(h, u)\|_{C,p}\}_{k=0}^\infty \in \mathsf{b}_{\rho,\gamma-\beta}$, the discrete Hardy inequality [3, Lemma 3.24, p. 27] now leads to part (b). □

*Proof of Theorem 2.3* Let $u_n := L^{[-1]}\sigma_{2^n}^*(h, f)$, $u_n^d := L^{[-1]}\sigma_{2^n}(\mu_n; h, f)$, and $a_\ell = \lambda_\ell^{[-1]}$. Then the quadrature formula (2.15) shows that for any polynomial $P \in \Pi_{2^{n-1}}^q$ and $\mathbf{x} \in \mathbb{S}^q$,

$$\begin{aligned}
u_n(\mathbf{x}) - u_n^d(\mathbf{x}) &= L^{[-1]}(\sigma_{2^n}^*(h, f - P), \mathbf{x}) - L^{[-1]}(\sigma_{2^n}(\mu_n; h, f - P), \mathbf{x}) \\
&= \int_{\mathbb{S}^q} \Psi_{2^n}(\mathbf{a}, h, \mathbf{x}\cdot\xi)(f(\xi) - P(\xi))d\mu_q^*(\xi) \\
&\quad - \int_{\mathbb{S}^q} \Psi_{2^n}(\mathbf{a}, h, \mathbf{x}\cdot\xi)(f(\xi) - P(\xi))d\mu_n(\xi). \qquad (4.27)
\end{aligned}$$

We now use Proposition 4.2, once with $\mu_q^*$ in place of $\mu_n$, and once with $\mu_n$ itself. We note that in the case when $\beta < 0$, we have to use part (b) of this proposition. We then obtain that

$$\|u_n - u_n^d\|_{\mathbb{S}^q,p} \le c\|f - P\|_{\mathbb{S}^q,p} \begin{cases} 2^{n\beta}, & \text{if } \beta \ge 0, \\ 1, & \text{if } \beta < 0. \end{cases} \tag{4.28}$$

Since $P \in \Pi_{2^{n-1}}^q$ is arbitrary, we conclude from (4.28) and Proposition 4.3(b) that

$$\|u_n - u_n^d\|_{\mathbb{S}^q,p} \le c \begin{cases} E_{2^{n-1},p}(\partial_\beta f), & \text{if } \beta \ge 0, \\ E_{2^{n-1},p}(f), & \text{if } \beta < 0. \end{cases} \tag{4.29}$$

In the case when $\beta \ge 0$, (2.16) now follows from (2.12). Now, let $\beta < 0$. Applying Proposition 4.2(b) with $\mu_q^*$ in place of $\mu_n$, $\{(\ell+1)^\beta\}_{\ell=0}^\infty$ in place of $\mathbf{a}$, we obtain that

$$\|\partial_\beta \sigma_{2^m}^*(h, f)\|_{\mathbb{S}^q,p} \le c\|f\|_{\mathbb{S}^q,p}, \qquad m \ge 0.$$

Since $\partial_\beta \sigma_{2^m}^*(h, f) \to \partial_\beta f$ in $X^p(\mathbb{S}^q)$, this leads to $\|\partial_\beta f\|_{\mathbb{S}^q,p} \le c\|f\|_{\mathbb{S}^q,p}$. Hence (cf. Proposition 4.3(a)),

$$\begin{aligned} E_{2^{n-1},p}(\partial_\beta f) &\le \|\partial_\beta f - \sigma_{2^{n-1}}^*(h, \partial_\beta f)\|_{\mathbb{S}^q,p} = \|\partial_\beta(f - \sigma_{2^{n-1}}^*(h, f))\|_{\mathbb{S}^q,p} \\ &\le c\|f - \sigma_{2^{n-1}}^*(h, f)\|_{\mathbb{S}^q,p} \le E_{2^{n-2},p}(f). \end{aligned}$$

Together with (4.29) and (2.12), this completes the proof of (2.16).

Next, let $\beta \ge 0$, $\mathbf{x}_0 \in \mathbb{S}^q$, and $C$ be a cap centered at $\mathbf{x}_0$ such that for every $\phi \in C_0^\infty(C)$, $f\phi \in B_{p,\rho,\gamma}$. Let $C'$ be another cap, centered at $\mathbf{x}_0$ and having radius less than that of $C$, and $\phi^* \in C_0^\infty(C)$ be chosen so that $\phi^*(\mathbf{x}) = 1$ if $\mathbf{x} \in C'$. In view of Theorem 2.2, we may find a cap $C''$, centered at $\mathbf{x}_0$ and having radius less than that of $C'$, such that $\{\|u - u_n\|_{C'',p}\}_{n=0}^\infty \in \mathsf{b}_{\rho,\gamma-\beta}$. Let $v_n := L^{[-1]}\sigma_{2^n}(\mu_n; h, f\phi^*)$. With $a_\ell = \lambda_\ell^{[-1]}$, we have for $\mathbf{x} \in \mathbb{S}^q$,

$$\begin{aligned} &u_n(\mathbf{x}) - v_n(\mathbf{x}) \\ &= \int_{\mathbb{S}^q} \Psi_{2^n}(\mathbf{a}, h, \mathbf{x}\cdot\xi) f(\xi)\phi^*(\xi) d\mu_q^*(\xi) - \int_{\mathbb{S}^q} \Psi_{2^n}(\mathbf{a}, h, \mathbf{x}\cdot\xi) f(\xi)\phi^*(\xi) d\mu_n(\xi) \\ &\quad + \int_{\mathbb{S}^q} \Psi_{2^n}(\mathbf{a}, h, \mathbf{x}\cdot\xi) f(\xi)(1 - \phi^*(\xi)) d\mu_q^*(\xi) \end{aligned} \tag{4.30}$$

The same argument as the one leading to (4.29) shows that

$$\begin{aligned} &\left\| \int_{\mathbb{S}^q} \Psi_{2^n}(\mathbf{a}, h, \mathbf{x}\cdot\xi) f(\xi)\phi^*(\xi) d\mu_q^*(\xi) \right. \\ &\quad \left. - \int_{\mathbb{S}^q} \Psi_{2^n}(\mathbf{a}, h, \mathbf{x}\cdot\xi) f(\xi)\phi^*(\xi) d\mu_n(\xi) \right\|_{\mathbb{S}^q,p} \\ &\le cE_{2^{n-1},p}(\partial_\beta(f\phi^*)). \end{aligned} \tag{4.31}$$

Now, let $\Delta^*$ denote the Laplace-Beltrami operator, and $g = (1-\phi^*)f$. We observe that

$$a_\ell \hat{g}(\ell, k) = \widehat{R^*(-\Delta^*)g}(\ell, k), \qquad k = 1, \ldots, d_\ell^q,\ \ell = 0, 1, \ldots.$$

Therefore,

$$\int_{\mathbb{S}^q} \Psi_{2^n}(\mathbf{a}, h, \mathbf{x}\cdot\xi) f(\xi)(1-\phi^*(\xi)) d\mu_q^*(\xi) = \int_{\mathbb{S}^q} \Phi_{2^n}(h, \mathbf{x}\cdot\xi) R^*(-\Delta^*) g(\xi) d\mu_q^*.$$

Since $g$ is supported on $\mathbb{S}^q \setminus C$, so is $R^*(-\Delta^*)g$. Hence, using (4.5), we see that for $\mathbf{x} \in C''$,

$$\left| \int_{\mathbb{S}^q} \Phi_{2^n}(h, \mathbf{x}\cdot\xi) R^*(-\Delta^*) g(\xi) d\mu_q^*(\xi) \right| \le c 2^{-nQ} \| R^*(-\Delta^*) g \|_{\mathbb{S}^q, p}.$$

Using (4.31) and (4.30), we conclude that

$$\| u_n - v_n \|_{C'', p} \le c 2^{-nQ} \| R^*(-\Delta^*) g \|_{\mathbb{S}^q, p} + c E_{2^{n-1}, p}(\partial_\beta(f\phi^*)). \quad (4.32)$$

Since $\beta \ge 0$, Proposition 4.5 shows that $\{E_{2^{n-1},p}(\partial_\beta(f\phi^*))\}_{n=0}^\infty \in \mathsf{b}_{\rho,\gamma-\beta}$. Since $\{\|u - u_n\|_{C'',p}\}_{n=0}^\infty \in \mathsf{b}_{\rho,\gamma-\beta}$ as well, (4.32) implies that $\{\|u - v_n\|_{C'',p}\}_{n=0}^\infty \in \mathsf{b}_{\rho,\gamma-\beta}$. □

## 5 Conclusions

We have studied the local smoothness properties of the solutions of pseudo-differential equations of the form $Lu = f$ on the sphere. We have discussed numerical approximation schemes for obtaining the solution, based on spectral data, as well as based on function evaluations. It is assumed that all the eigenvalues of $L$ are known in advance. In the case of methods based on function evaluations at certain points, we also assume the knowledge of appropriate quadrature formulas based on these points that are exact for high degree polynomials. The existence of the necessary quadrature formulas is known even in the case when one has no choice on where to sample the function $f$. However, practical methods for the computation of these quadrature formulas is a major topic of ongoing research.

Our methods give polynomial approximations, and are universal; i.e., neither the method nor the class of approximants depends upon the particular pseudo-differential equation in question. In the case when function evaluations are used, our method does not involve the solution of a matrix equation, as in the collocation method. The rate of convergence of our approximations on any cap depends entirely on the smoothness of the target function on a slightly larger cap, and the order of the operator in question, in spite of the approximations being globally constructed polynomials, and even though the kernels for the inverse operators are not locally supported. Our results are applicable in all $L^p$ norms, $1 \le p \le \infty$.

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