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Approximation of parabolic PDEs on spheres using spherical basis functions *

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In this paper we investigate the approximation of a class of parabolic partial differential equations on the unit spheres $S^n \subset \mathbb{R}^{n+1}$ using spherical basis functions. Error estimates in the Sobolev norm are derived.

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1. Introduction

Approximation of partial differential equations on spheres has many applications in physical geodesy, potential theory, oceanography, and meteorology [2,17,18]. Evolution equations on spherical geometry such as shallow water equations have been studied in weather forecasting services [3,23]. The geometry of the sphere is a major obstacle in constructing the approximation space for the solution of the PDEs. One way to overcome the obstacle is to construct basis functions which depend only on the geodesic distance between two points on the sphere, which are called spherical basis functions in literature [2,5,13]. Error estimates of pseudo-differential operator (which are time-independent) were studied in [2,9] but error estimates for the evolution equations remain unexplored.

In this paper we consider the following parabolic partial differential equation defined on the unit sphere $S^n \subset \mathbb{R}^{n+1}$:

$$\begin{cases} \dfrac{\partial}{\partial t} u(x,t) - \Delta u(x,t) = F(x,t) \\ u(x,0) = f(x), \quad f \in H^{2\sigma}(S^n), \end{cases} \tag{1}$$

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where Δ is the Laplace–Beltrami operator on S^n and $H^{2\sigma}(S^n)$ is the Sobolev space defined on S^n (see section 2). It is known that equation (1) describes the heat diffusion process on the surface of the sphere with external heat source $F(x, t)$.

In many applications in geophysics and global weather forecast, it is common that the function f is not known analytically everywhere but only at a finite set of scattered points.

We propose a collocation method in which the spherical basis functions are used to construct the approximate solution. The approximate solution of the partial differential equation will be of the form

$$u_X(x, t) = \sum_{i=1}^{m} c_i(t)\phi_i(x), \tag{2}$$

subject to the initial condition

$$u_X(x, 0) = I_X f(x),$$

where $\phi_i(x) = \phi(x \cdot x_i) = \Phi(x_i, x)$'s are the shifts of a spherical basis function (SBF) ϕ and $I_X f$ is the SBF interpolant of the function f. In case the basis function ϕ satisfies certain regularity conditions, we are able to obtain error estimates in certain Sobolev norms.

The paper is organized as follows: section 2 gives the necessary background on spherical harmonics and the Laplace–Beltrami operator together with the problem of interpolation on spheres using spherical basis functions. In section 3, we present the semi-discrete problem, in which the exact solution $u(x, t)$ is approximated by u_X of the form (2) and u_X is a solution of a system of ordinary differential equations. In section 4, we discretize (1) also in time variable so as to produce a completely discrete scheme for the approximate solution of our problem. Finally, some numerical experiments are presented in section 5.

2. Preliminaries

2.1. Spherical harmonics and Sobolev spaces

Spherical harmonics are polynomials which satisfy $\Delta_x Y(x) = 0$ (where Δ_x is the Laplacian operator in $\mathbb{R}^{n+1}$) and are restricted to the surface of the Euclidean sphere S^n. A more detailed discussion of spherical harmonics can be found in [10]. It is well known that Laplace–Beltrami operator Δ is linear, self-adjoint and negative definite in the spatial variables. The eigenvalues for $-\Delta$ are $\lambda_\ell = \ell(\ell + n - 1)$ for $\ell = 0, 1, 2, 3 \ldots$ and the respective eigenfunctions are the spherical harmonics $Y_\ell(x)$ of order ℓ, i.e.,

$$\Delta Y_\ell(x) = -\lambda_\ell Y_\ell(x).$$

The space of all spherical harmonics of degree ℓ on S^n, denoted by V_ℓ, has an orthonormal basis

$$\{Y_{\ell k}(x)\colon k = 1, \ldots, N(n, \ell)\},$$

where

$$N(n,0)=1 \quad \text{and} \quad N(n,\ell)=\frac{(2\ell+n-1)\Gamma(\ell+n-1)}{\Gamma(\ell+1)\Gamma(n)} \quad \text{for } \ell \geqslant 1.$$

Every function f in $L^2(S^n)$ can be expanded in terms of spherical harmonics

$$f=\sum_{\ell=0}^{\infty}\sum_{k=1}^{N(n,\ell)}\hat{f}_{\ell k}Y_{\ell k}, \quad \hat{f}_{\ell k}=\int_{S^n} f\overline{Y_{\ell k}}\,\mathrm{d}S,$$

where $\mathrm{d}S$ is the surface measure of the unit sphere. The Sobolev space $H^\sigma(S^n)$ with real parameter σ consists of all distributions f such that

$$\|f\|_\sigma^2=\sum_{\ell=0}^{\infty}\sum_{k=1}^{N(n,\ell)}(1+\lambda_\ell)^\sigma\left|\hat{f}_{\ell k}\right|^2<\infty.$$

For more details we refer to [7, section 1.7].

2.2. *Positive definite kernels on spheres*

Bizonal functions on S^n are functions that can be represented as $\phi(x\cdot y)$ for all $x, y \in S^n$, where $x\cdot y$ is the usual dot product in $\mathbb{R}^{n+1}$ and $\phi(t)$ is a continuous functions on $[-1,1]$. We shall be concerned exclusively with bizonal kernels of the type

$$\Phi(x,y)=\phi(x\cdot y)=\sum_{\ell=0}^{\infty}a_\ell P_\ell(n+1;x\cdot y), \quad a_\ell\geqslant 0,\ \sum_{\ell=0}^{\infty}a_\ell<\infty,$$

where $\{P_\ell(n+1;t)\}_{\ell=0}^{\infty}$ is the sequence of $(n+1)$-dimensional Legendre polynomials. Recall from [10] that

$$\int_{-1}^{1}P_\ell(n+1;t)P_k(n+1;t)\left(1-t^2\right)^{(n-2)/2}\mathrm{d}t=0, \quad \text{for } \ell\neq k,$$

and

$$\int_{-1}^{1}\left[P_\ell(n+1;t)\right]^2\left(1-t^2\right)^{(n-2)/2}\mathrm{d}t=\frac{|S^n|}{|S^{n-1}|N(n,\ell)},$$

where $|S^n|$ is the surface area of S^n, $|S^{n-1}|$ is the surface area of S^{n-1}.

Thanks to seminal work of Schoenberg [16], we know that such a Φ is positive definite on S^n, that is, the matrix $A:=[\Phi(x_i,x_j)]_{i,j=1}^m$ is positive semidefinite for every set of distinct points $\{x_1,\ldots,x_m\}$ on S^n for any positive integer m. When the coefficients a_ℓ are positive for every ℓ, we say that Φ is strictly positive definite, hence invertible, for every set of distinct points $\{x_1,\ldots,x_m\}$ on S^n and every m (see [24]).

Using the addition theorem (see [10]), we can express $\Phi(x,y)$ as the following:

$$\Phi(x,y)=\sum_{\ell=0}^{\infty}\hat{\phi}(\ell)\sum_{k=1}^{N(n,\ell)}Y_{\ell k}(x)\overline{Y_{\ell k}}(y), \quad \hat{\phi}(\ell)=\frac{|S^n|}{N(n,\ell)}a_\ell>0,\ \forall\ell\geqslant 0. \tag{3}$$

Upon completion, the kernel defines a reproducing kernel Hilbert space N_Φ with respect to the following inner product

$$\langle u, v\rangle_\Phi = \sum_{\ell=0}^{\infty}\sum_{k=1}^{N(n,\ell)} \frac{\hat{u}_{\ell k}\overline{\hat{v}_{\ell k}}}{\hat{\phi}(\ell)}.$$

More precisely, we define the *native space* N_Φ to be the completion of the following set:

$$N_\Phi := \left\{ f \in \mathcal{D}'\big(S^n\big)\colon\ \|f\|_\Phi^2 = \sum_{\ell=0}^{\infty}\sum_{k=1}^{N(n,\ell)} \frac{|\hat{f}_{\ell k}|^2}{\hat{\phi}(\ell)} < \infty \right\},$$

where $\mathcal{D}'(S^n)$ denotes the set of all tempered distributions defined on S^n. Note that Φ is the reproducing kernel in N_Φ in the sense that for every $f \in N_\Phi$ and for any fixed $x \in S^n$,

$$\big\langle \Phi(\cdot, x), f\big\rangle_\Phi = \sum_{\ell=0}^{\infty}\sum_{k=1}^{N(n,\ell)} \hat{\phi}(\ell)\frac{Y_{\ell k}(x)\hat{f}_{\ell k}}{\hat{\phi}(\ell)} = f(x).$$

Throughout the paper we make further assumption that $\hat{\phi}(\ell) \sim (1+\lambda_\ell)^{-\sigma}$, i.e., there are positive constants c and C and $\sigma > n/2$ such that

$$c(1+\lambda_\ell)^{-\sigma} \leqslant \hat{\phi}(\ell) \leqslant C(1+\lambda_\ell)^{-\sigma}. \tag{4}$$

We define the convolution kernel of Φ by

$$\Phi * \Phi(x, y) := \int_{S^n} \Phi(x, z)\Phi(z, y)\,\mathrm{d}S(z), \quad x, y \in S^n.$$

In terms of Fourier expansions we have

$$\Phi * \Phi(x, y) = \sum_{\ell=0}^{\infty}\big(\hat{\phi}(\ell)\big)^2 \sum_{k=1}^{N(n,\ell)} Y_{\ell k}(x)\overline{Y_{\ell k}(y)}.$$

This observation allows us to define a convolution native space to be the completion of the following set

$$N_{\Phi*\Phi} = \left\{ f \in \mathcal{D}'\big(S^n\big)\colon\ \|f\|_{\Phi*\Phi}^2 = \sum_{\ell=0}^{\infty}\sum_{k=1}^{N(n,\ell)} \frac{|\hat{f}_{\ell k}|^2}{(\hat{\phi}(\ell))^2} < \infty \right\}.$$

If the kernel Φ satisfies condition (4) then

$$N_{\Phi*\Phi} \cong H^{2\sigma}\big(S^n\big) \subset H^{\sigma}\big(S^n\big) \cong N_\Phi.$$

2.3. *Interpolation of scattered data on S^n*

Now let $X = \{x_i\colon\ i = 1, \ldots, m\}$ be a set of m distinct scattered points on S^n. Schoenberg, in [16], establishes that if the coefficients $\hat{\phi}(\ell) \geqslant 0$ for all $\ell \geqslant 0$ then the matrix $[\Phi(x_i, x_j)]$ is symmetric positive semi-definite for any configuration of X. Xu and Cheney [24] have shown that if $\hat{\phi}(\ell) > 0$ for all $\ell \geqslant 0$ then the matrix is symmetric positive definite, hence invertible. The distribution of the set X is measured by its mesh norm

$$h_X := \sup_{x \in S^n} \inf_{y \in X} d(x, y),$$

where $d(x, y) = \cos^{-1}(x \cdot y)$ is the geodesic distance on S^n. The separation distance q_X is defined as

$$q_X := \frac{1}{2} \min_{i \neq j} d(x_i, x_j).$$

We always have $\rho_X := h_X / q_X \geqslant 1$. The set X is called quasi-uniform if $1 \leqslant \rho_X < c$, where c is a constant independent of the set X. The finite-dimensional vector space is defined as

$$V_X = \operatorname{span}\bigl\{\phi_i(x)\colon\ i = 1, \ldots, m\bigr\},$$

where the spherical basis functions ϕ_i are the shifts of a strictly positive definite function ϕ on the set X, i.e.

$$\phi_i(x) := \Phi(x_i, x), \quad i = 1, \ldots, m. \tag{5}$$

Let I_X be the interpolation operator $I_X : C(S^n) \to V_X$ such that $I_X f(x_j) = f(x_j)$ for all $x_j \in X$. In practice, we have to solve the following linear system in order to find coefficients c_i's such that

$$\sum_{i=1}^{m} c_i \Phi(x_i, x_j) = f(x_j) \quad \text{for all } j = 1, \ldots, m. \tag{6}$$

The operator I_X is well defined for every function $f \in C(S^n)$ since the matrix

$$\bigl[\Phi(x_i, x_j)\bigr]_{i,j=1,\ldots,m}$$

is positive definite, hence invertible, for every configuration of the set X.

Lemma 2.1. For every $f \in N_\Phi$, we have

$$\|I_X f\|_\Phi^2 + \|f - I_X f\|_\Phi^2 = \|f\|_\Phi^2.$$

Proof. Since Φ is the reproducing kernel in the reproducing Hilbert space N_Φ, the interpolating condition $I_X f(x_j) = f(x_j)$ for all $x_j \in X$ is equivalent to

$$\bigl\langle I_X f - f, \Phi(x_j, \cdot)\bigr\rangle_\Phi = 0, \quad \forall x_j \in X.$$

Since $I_X f$ is a linear combination of $\Phi(x_j, \cdot)$'s, we have the orthogonal property

$$\langle I_X f - f, I_X f\rangle_\Phi = 0.$$

Hence the desired relation follows from the Pythagorean theorem. □

Based on [4, theorem 4.7], we have the following theorem.

Theorem 2.1. Let Φ be a positive definite kernel with $\hat{\phi}(\ell) \sim (1 + \lambda_\ell)^{-\sigma}$, and $f \in H^{2\sigma}(S^n)$. Then there exists a positive constant C independent of h_X such that

$$\|f - I_X f\|_\Phi \leqslant C h_X^\sigma \|f\|_{2\sigma}.$$

Proof.

$$\begin{aligned}
\|f - I_X f\|_\Phi^2 &= \langle f, f - I_X f\rangle_\Phi \\
&= \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(n,\ell)} \frac{\hat{f}_{\ell k}(\hat{f}_{\ell k} - \widehat{I_X f}_{\ell k})}{\hat{\phi}(\ell)} \\
&\leqslant \left(\sum_{\ell=0}^{\infty} \sum_{k=1}^{N(n,\ell)} \frac{|\hat{f}_{\ell k}|^2}{\hat{\phi}(\ell)^2} \right)^{1/2} \left(\sum_{\ell=0}^{\infty} \sum_{k=1}^{N(n,\ell)} \left(\hat{f}_{\ell k} - \widehat{I_X f}_{\ell k}\right)^2 \right)^{1/2} \\
&\leqslant \|f\|_{\Phi * \Phi} \|f - I_X f\|_{L^2(S^n)}.
\end{aligned} \tag{7}$$

Then by using [4, theorem 4.4] with $p = 2$, it follows that

$$\|f - I_X f\|_{L^2(S^n)} \leqslant C h_X^\sigma \|f - I_X f\|_\Phi. \tag{8}$$

Combining inequalities (7) and (8) and noting that $N_{\Phi * \Phi} \cong H^{2\sigma}(S^n)$ we have

$$\|f - I_X f\|_\Phi \leqslant C h_X^\sigma \|f\|_{2\sigma}. \qquad \square$$

3. Semi-discrete problem

The numerical analysis here follows a framework set out in [19], which was used to analyze the approximation of solutions of the heat equation on a bounded domain $\Omega \subset \mathbb{R}^n$ for the finite element method. However, the framework of [19] is modified significantly with the structure of the reproducing kernel Hilbert space N_Φ for a collocation method on S^n.

3.1. *The homogeneous problem*

By the method of separation of variables, see [14, section 5.7], the exact solution for the homogeneous problem:

$$\begin{cases} \dfrac{\partial}{\partial t} u(x,t) = \Delta u(x,t), \\ u(x,0) = f(x), \quad f \in L^2(S^n), \end{cases}$$

is given as the infinite series

$$u(x,t) = \sum_{\ell=0}^{\infty} \mathrm{e}^{-\lambda_\ell t} \sum_{k=1}^{N(n,\ell)} \hat{f}_{\ell k} Y_{\ell k}(x).$$

Let the approximate solution be of the following form:

$$u_X(x,t) = \sum_{i=1}^{m} c_i(t)\phi_i(x),$$

where $\phi_i(x)$ is the SBF as defined in (5). The homogeneous semi-discrete problem is formulated as the following: we require the equation (1) to be exact on the set X, i.e.,

$$\begin{cases} \dfrac{\partial}{\partial t} u_X(x_j,t) = \Delta u_X(x_j,t), & \forall x_j \in X, \\ u_X(x,0) = I_X f(x), \end{cases} \tag{9}$$

where $I_X f$ is the interpolant of f in V_X. Equation (9) can be rewritten as the following:

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{m} c_i(t)\phi_i(x_j) = \sum_{i=1}^{m} c_i(t)\Delta\phi_i(x_j), \quad \forall x_j \in X, \tag{10}$$

subject to the following initial condition:

$$\sum_{i=1}^{m} c_i(0)\phi_i(x_j) = f(x_j), \quad \forall x_j \in X.$$

If we set $A := [\phi_i(x_j)]_{i,j=1,\ldots,m}$ and $B := [\Delta\phi_i(x_j)]_{i,j=1,\ldots,m}$ then equation (10) can be written as the following system of ordinary differential equations in time:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{c}(t) = A^{-1}B\mathbf{c}(t), \tag{11}$$

where $\mathbf{c}(t) = [c_1(t),\ldots,c_m(t)]^{\mathrm{T}}$. It is known that (see, for example, [11]), in order to solve the system (11), we have to compute the distinct eigenvalues $r_1,\ldots,r_k$ of the matrix $A^{-1}B$ with multiplicities $n_1,\ldots,n_k$. For each eigenvalue r_i, we find n_i linearly independent generalized eigenvectors. Each independent solution of (11) is of the form

$$\exp\bigl(A^{-1}Bt\bigr)\mathbf{v} = \mathrm{e}^{rt}\biggl(\mathbf{v} + t\bigl(A^{-1}B - rI\bigr)\mathbf{v} + \frac{t^2}{2}\bigl(A^{-1}B - rI\bigr)^2\mathbf{v} + \cdots\biggr),$$

where r is an eigenvalue and $\mathbf{v}$ is a corresponding generalized eigenvector. If r has multiplicity n_i, then the above series reduces to the first n_i terms. The linearly independent solutions form column vectors of a matrix $E(t)$, and then the fundamental matrix $\exp(A^{-1}Bt)$ is given as

$$\exp\bigl(A^{-1}Bt\bigr) = E(t)E^{-1}(0).$$

The solution of the homogeneous semi-discrete problem is

$$u_X(x,t) = \bigl[\phi_1(x)\ldots\phi_m(x)\bigr]\exp\bigl(A^{-1}Bt\bigr)\mathbf{c}(0), \quad \text{where } \mathbf{c}(0) = A^{-1}f|_X. \tag{12}$$

We shall express the solution $u_X(x,t)$ in terms of some evolution operator. Let us consider the following operator:

$$\Delta I_X : C\big(S^n\big) \to \operatorname{span}\big\{\Delta\phi_i(x)\colon\ i = 1, \ldots, m\big\},$$
$$f \mapsto \Delta(I_X f).$$

Lemma 3.1. At the set of points X, we have

$$A\big(A^{-1}B\big)^n A^{-1}\big[f(x_j)\big]_{j=1}^m = \big[(\Delta I_X)^n f|_{x=x_j}\big]_{j=1}^m.$$

Here and thereafter, the notation $[a_j]_{j=1}^m$ stands for $[a_1, \ldots, a_m]^{\mathrm{T}}$ which is a vector in $\mathbb{R}^m$.

Proof. For $n = 1$, we have

$$AA^{-1}BA^{-1}\big[f(x_j)\big]_{j=1}^m = BA^{-1}\big[f(x_j)\big]_{j=1}^m = [\Delta I_X f|_{x=x_j}]_{j=1}^m.$$

Now assume that for $k > 1$

$$A\big(A^{-1}B\big)^k A^{-1}\big[f(x_j)\big]_{j=1}^m = \big[(\Delta I_X)^k f(x_j)\big]_{j=1}^m.$$

Then

$$\begin{aligned} A\big(A^{-1}B\big)^{k+1} A^{-1}\big[f(x_j)\big]_{j=1}^m &= A\big(A^{-1}B\big)\big(A^{-1}B\big)^k A^{-1}\big[f(x_j)\big]_{j=1}^m \\ &= BA^{-1}A\big(A^{-1}B\big)^k A^{-1}\big[f(x_j)\big]_{j=1}^m \\ &= BA^{-1}\big[(\Delta I_X)^k f\big|_{x=x_j}\big]_{j=1}^m = \big[(\Delta I_X)(\Delta I_X)^k f\big|_{x=x_j}\big]_{j=1}^m \\ &= \big[(\Delta I_X)^{k+1} f\big|_{x=x_j}\big]_{j=1}^m. \end{aligned}$$

□

Lemma 3.2. For small $t > 0$ we have

$$u_X(x,t) = I_X f + t I_X \Delta I_X f + \frac{t^2}{2} I_X(\Delta I_X)^2 f + \cdots + \frac{t^n}{n!} I_X(\Delta I_X)^n f + \cdots.$$

Proof. Using equation (12) and lemma 3.1, we have

$$\begin{aligned} \big[u_X(x_j,t)\big]_{j=1}^m &= A\exp\big(A^{-1}Bt\big)A^{-1}\big[f(x_j)\big]_{j=1}^m \\ &= A\left(I + tA^{-1}B + \cdots \frac{t^n}{n!}\big(A^{-1}B\big)^n + \cdots\right)A^{-1}\big[f(x_j)\big]_{j=1}^m \\ &= \big[f(x_j)\big]_{j=1}^m + t[\Delta I_X f|_{x=x_j}]_{j=1}^m + \cdots + \frac{t^n}{n!}\big[(\Delta I_X)^n f\big|_{x=x_j}\big]_{j=1}^m + \cdots. \end{aligned}$$

Since $u_X \in V_X$, this implies

$$u_X(x,t) = I_X f + t I_X \Delta I_X f + \cdots + \frac{t^n}{n!} I_X(\Delta I_X)^n f + \cdots.$$

□

Let us define the following evolution operator

$$E_X(t) := I + t I_X \Delta + \cdots + \frac{t^n}{n!}(I_X \Delta)^n + \cdots,$$

then $u_X(x,t) = E_X(t)I_X f(x)$. We can show that $E_X(t)$ is a stable operator in V_X in $\|\cdot\|_\Phi$ norm by the following lemma.

Lemma 3.3. For every $\psi \in V_X$,

$$\left\|E_X(t)\psi(x)\right\|_\Phi \leqslant \left\|\psi(x)\right\|_\Phi.$$

Proof. Let $\theta(x,t)$ be defined as

$$\theta(x,t) = \sum_{i=1}^{m} c_i(t)\phi_i(x).$$

We wish to solve the following PDE by a collocation method

$$\frac{\partial}{\partial t}\theta(x,t) = \Delta\theta(x,t),$$

subject to the initial condition

$$\theta(x,0) = \psi(x).$$

In our collocation method, it is required that the PDE is exact on the set of given points X, i.e.

$$\frac{\partial}{\partial t}\theta(x_j,t) = \Delta\theta(x_j,t), \quad \forall x_j \in X,$$

subject to the initial condition

$$\theta(x_j,0) = \psi(x_j), \quad \forall x_j \in X.$$

Since Φ is the reproducing kernel in the Hilbert space N_Φ,

$$\left\langle \frac{\partial}{\partial t}\theta(\cdot,t), \Phi(\cdot,x_j)\right\rangle_\Phi = \left\langle \Delta\theta(\cdot,t), \Phi(\cdot,x_j)\right\rangle_\Phi, \quad \forall x_j \in X. \tag{13}$$

Since V_X is spanned by $\Phi(x,x_j)$, for $j = 1,\ldots,m$, equation (13) implies that for every function $v \in V_X$,

$$\left\langle \frac{\partial\theta}{\partial t}, v\right\rangle_\Phi = \langle\Delta\theta, v\rangle_\Phi.$$

Since $\theta \in V_X$, we can take $v = \theta$ to obtain

$$\frac{1}{2}\frac{\partial}{\partial t}\|\theta\|_\Phi^2 = \left\langle \frac{\partial\theta}{\partial t}, \theta\right\rangle_\Phi = \langle\Delta\theta, \theta\rangle_\Phi.$$

From the definition of $\langle\cdot,\cdot\rangle_\Phi$,

$$\langle\Delta\theta, \theta\rangle_\Phi = \sum_{\ell=0}^{\infty}\sum_{k=1}^{N(n,\ell)} \frac{-\lambda_\ell|\hat{\theta}_{\ell k}|^2}{\hat{\phi}(\ell)} \leqslant 0, \quad \forall\theta \in V_X. \tag{14}$$

Thus, we obtain the result $\|\theta(x,t)\|_\Phi \leqslant \|\theta(x,0)\|_\Phi$ or in other words

$$\left\|E_X(t)\psi(x)\right\|_\Phi \leqslant \left\|\psi(x)\right\|_\Phi. \qquad \square$$

3.2. *The nonhomogeneous problem*

The approximation of the nonhomogeneous equation will be tackled via an elliptic projection from the space of the exact solution u to the finite-dimensional space V_X, which is somehow similar to the Ritz projection in the finite element method. To begin, let us define the following operator:

$$P : H^{2\sigma+2}\big(S^n\big) \to V_X$$
$$u \mapsto u_P,$$

where

$$\begin{cases} \Delta u_P(x_j) = \Delta u(x_j), & \forall x_j \in X, \\ \displaystyle\int_{S^n} u_P \,\mathrm{d}S = \int_{S^n} u \,\mathrm{d}S. \end{cases} \tag{15}$$

It is noted that Δ has zero as an eigenvalue, thus the matrix $B = [\Delta\Phi(x_i, x_j)]_{i,j=1,\dots,m}$ is not invertible. The null space of B has dimension 1. We fix the null space problem by finding $u_P = \sum_{j=1}^m \alpha_j\phi_j(x)$, where $\alpha := (\alpha_1,\dots,\alpha_m)^\mathrm{T}$ solves the following system of linear equations

$$\begin{cases} B\alpha = \big[\Delta u(x_j)\big]_{j=1}^m, \\ \displaystyle\sum_{i=1}^m \alpha_i \int_{S^n} \phi_i \,\mathrm{d}S = \int_{S^n} u \,\mathrm{d}S. \end{cases}$$

We notice that u_P is well-defined since the solution α is unique. It is also from the definition that

$$I_X\Delta P = I_X\Delta. \tag{16}$$

Lemma 3.4. Let $u \in H^{2\sigma+2}(S^n)$, and $u_P \in V_X$ be constructed from a linear combination of shifts of SBF Φ with $\hat{\phi}(\ell) \sim (1+\lambda_\ell)^{-\sigma}$. Then there is a constant C independent of h_X so that

$$\|u_P - u\|_\Phi \leqslant Ch_X^\sigma \|u\|_{2\sigma+2}.$$

Proof. Since Δu_P is the interpolation of Δu, by theorem 2.1, we have

$$\|\Delta u_P - \Delta u\|_\Phi \leqslant Ch_X^\sigma \|\Delta u\|_{2\sigma} \leqslant Ch_X^\sigma \|u\|_{2\sigma+2}. \tag{17}$$

Let $\psi = u_P - u$, then from definition (15)

$$\hat{\psi}_0 = \int_{S^n} (u_P - u)\,\mathrm{d}S = 0.$$

Hence,

$$\|\psi\|_\Phi^2 = \sum_{\ell=0}^{\infty}\sum_{k=1}^{N(n,\ell)} \frac{|\hat{\psi}_{\ell k}|^2}{\hat{\phi}(\ell)} \leqslant \left(\left|\hat{\psi}_0\right|^2 + \sum_{\ell=1}^{\infty}\sum_{k=1}^{N(n,\ell)} \frac{\lambda_\ell^2|\hat{\psi}_{\ell k}|^2}{\hat{\phi}(\ell)}\right) \leqslant \|\Delta\psi\|_\Phi^2.$$

Combining with (17), we have

$$\|u_P - u\|_\Phi \leqslant \|\Delta u_P - \Delta u\|_\Phi \leqslant Ch_X^\sigma \|u\|_{2\sigma+2}. \qquad \square$$

The collocation semi-discrete equation (9) now takes the following form

$$\frac{\partial}{\partial t}u_X(x_j,t) - \Delta u_X(x_j,t) = F(x_j,t), \quad \forall x_j \in X, \tag{18}$$

subject to the initial condition

$$u_X(x,0) = I_X f(x).$$

Theorem 3.1. Let $f, u_t \in H^{2\sigma+2}(S^n)$ and u, u_X be the solution for (1) and (18), respectively. The approximate solution u_X is constructed as a linear combination of shifts of a spherical basis function $\Phi(x,y) = \phi(x\cdot y)$ which satisfies $\hat{\phi}(\ell) \sim (1+\lambda_\ell)^{-\sigma}$. Then there is a positive constant C, independent of h_X, so that the following error estimate holds:

$$\left\|u(T) - u_X(T)\right\|_\Phi \leqslant Ch_X^\sigma\left(\|f\|_{2\sigma} + \|f\|_{2\sigma+2} + \int_0^T \|u_t\|_{2\sigma+2}\,\mathrm{d}s\right).$$

Proof. Let $\theta := u_X - u_P$, and let $\gamma := u_P - u$. Note that $\theta \in V_X$. When being restricted on the set X, using the relation $\Delta u_P|_X = \Delta u|_X$ we have the following equations:

$$\begin{aligned}\left(\frac{\partial}{\partial t}\theta - \Delta\theta\right)\bigg|_X &= \left(\frac{\partial}{\partial t}u_X - \Delta u_X\right)\bigg|_X - \left(\frac{\partial}{\partial t}u_P - \Delta u_P\right)\bigg|_X \\ &= F|_X - \left(\frac{\partial u_P}{\partial t} - \Delta u\right)\bigg|_X \\ &= F|_X - \left(\frac{\partial u}{\partial t} - \Delta u\right)\bigg|_X + \left(\frac{\partial u}{\partial t} - \frac{\partial u_P}{\partial t}\right)\bigg|_X \\ &= \frac{\partial}{\partial t}(u - u_P)\bigg|_X,\end{aligned}$$

or in terms of a PDE in the finite-dimensional space V_X,

$$\frac{\partial\theta}{\partial t} - \Delta\theta = -I_X\frac{\partial\gamma}{\partial t}. \tag{19}$$

By Duhamel's principle, see [14, section 3.11], we have

$$\theta(T) = E_X(T)\theta(0) - \int_0^T E_X(T-s)I_X\frac{\partial\gamma}{\partial t}\,\mathrm{d}s.$$

Since $\|E_X(T)v\|_\Phi \leqslant \|v\|_\Phi$ for all $v \in V_X$ (by lemma 3.3), we have

$$\|\theta(T)\|_\Phi \leqslant \|\theta(0)\|_\Phi + \int_0^T \left\| I_X \frac{\partial \gamma}{\partial t}(s) \right\|_\Phi \mathrm{d}s.$$

Here,

$$\begin{aligned}\|\theta(0)\|_\Phi = \|I_X f - Pf\|_\Phi &\leqslant \|I_X f - f\|_\Phi + \|Pf - f\|_\Phi \\ &\leqslant Ch^\sigma\big(\|f\|_{2\sigma} + \|f\|_{2\sigma+2}\big).\end{aligned}$$

We can use lemma 3.4 to obtain

$$\|\gamma_t\|_\Phi = \left\| \frac{\partial}{\partial t}(u - u_P) \right\|_\Phi \leqslant Ch_X^\sigma \|u_t\|_{2\sigma+2}.$$

Using lemma 2.1, we obtain

$$\|I_X \gamma_t\|_\Phi \leqslant \|\gamma_t\|_\Phi \leqslant Ch_X^\sigma \|u_t\|_{2\sigma+2}.$$

We know from lemma 3.4 that

$$\begin{aligned}\|\gamma(T)\|_\Phi = \|u(T) - u_P(T)\|_\Phi &\leqslant Ch_X^\sigma \|u(T)\|_{2\sigma+2} \\ &\leqslant Ch_X^\sigma \left(\left\| f + \int_0^T u_t(s)\,\mathrm{d}s \right\|_{2\sigma+2} \right) \\ &\leqslant Ch_X^\sigma \left(\|f\|_{2\sigma+2} + \int_0^T \|u_t\|_{2\sigma+2}\,\mathrm{d}s \right).\end{aligned}$$

Therefore, after adjusting the constant C, we obtain

$$\begin{aligned}\|u - u_X\|_\Phi &\leqslant \|\theta(T)\|_\Phi + \|\gamma(T)\|_\Phi \\ &\leqslant Ch_X^\sigma \left(\|f\|_{2\sigma} + \|f\|_{2\sigma+2} + \int_0^T \|u_t\|_{2\sigma+2}\,\mathrm{d}s \right).\end{aligned}$$ □

4. Time discretization

4.1. Backward Euler method

Let us discretize the time derivative using backward Euler method as

$$\frac{u(x,t) - u(x,t-\tau)}{\tau} + \mathrm{o}(1) - \Delta u(x,t) = F(x,t).$$

The collocation equation for u_X is

$$u_X(x_j,t) - u_X(x_j,t-\tau) - \tau\Delta u_X(x_j,t) = \tau F(x_j,t), \quad \forall x_j \in X. \tag{20}$$

Let us define $t_N := N\tau$, $U_N(x) := u_X(x,t_N)$ and introduce the notation

$$\overline{\partial}_t U_N := \frac{U_N - U_{N-1}}{\tau}.$$

The collocation equation (20) can be rewritten as

$$\overline{\partial}_t U_N(x_j) - \Delta U_N(x_j) = F(x_j, t_N), \quad \forall x_j \in X, \tag{21}$$

subject to the initial condition

$$U_0 = I_X f.$$

If we write $U_N = \sum_{i=1}^m c_{N,i}\phi_i(x)$ then in terms of matrices A and B, defined in section 3, we have

$$(A - \tau B)\mathbf{c}_N = A\mathbf{c}_{N-1} + \tau\big[F(x_j, N\tau)\big]_{j=1}^m, \tag{22}$$

with the initial condition

$$A\mathbf{c}_0 = \big[f(x_j)\big]_{j=1}^m.$$

We now estimate the difference between U_N and the exact solution u at the time t_N.

Theorem 4.1. Let us assume that $u_{tt} \in H^\sigma(S^n)$ and $u_t, f \in H^{2\sigma+2}(S^n)$ and let U_N be the solution of (21). The approximate solution U_N is constructed from shifts of a spherical basis function Φ with $\hat{\phi}(\ell) \sim (1+\lambda_\ell)^{-\sigma}$. Then there are positive constants C_1 and C_2 so that we have the following error estimate:

$$\big\|U_N - u(t_N)\big\|_\Phi \leqslant C_1 h_X^\sigma \Gamma(f, u_t) + C_2\tau\int_0^{t_N} \|u_{tt}\|_\sigma \,\mathrm{d}s,$$

where

$$\Gamma(f, u_t) := \|f\|_{2\sigma} + \|f\|_{2\sigma+2} + \int_0^{t_N} \big\|u_t(s)\big\|_{2\sigma+2}\,\mathrm{d}s + \tau\sum_{j=1}^N \big\|u_t(t_j)\big\|_{2\sigma}.$$

Proof.

$$U_N - u(t_N) = U_N - Pu(t_N) + Pu(t_N) - u(t_N) =: \theta_N + \gamma_N.$$

We already know

$$\begin{aligned}\big\|Pu(t_N) - u(t_N)\big\|_\Phi = \|\gamma_N\|_\Phi &\leqslant Ch_X^\sigma\big\|u(t_N)\big\|_{2\sigma+2}\\ &\leqslant Ch_X^\sigma\bigg(\|f\|_{2\sigma+2} + \int_0^{t_N}\big\|u_t(s)\big\|_{2\sigma+2}\,\mathrm{d}s\bigg).\end{aligned}$$

Similar to (19), we have

$$\overline{\partial}_t\theta_N(x_j) - \Delta\theta_N(x_j) = -\omega_N(x_j), \quad \forall x_j \in X, \tag{23}$$

where

$$\omega_N = \overline{\partial}_t Pu(t_N) - I_X u_t(t_N).$$

We can rewrite equation (23) as

$$(1-\tau\Delta)\theta_N(x_j) = \theta_{N-1}(x_j) - \tau\omega_N(x_j), \quad \forall x_j \in X. \tag{24}$$

In terms of the inner product $\langle\cdot,\cdot\rangle_\Phi$ in the reproducing kernel Hilbert space N_Φ,

$$\bigl\langle \theta_N - \tau\Delta\theta_N, \Phi(x_j,\cdot)\bigr\rangle_\Phi = \bigl\langle \theta_{N-1} - \tau\omega_N, \Phi(x_j,\cdot)\bigr\rangle_\Phi, \quad \forall x_j \in X. \tag{25}$$

Since V_X is spanned by $\Phi(x_j,\cdot)$'s, $j = 1,\dots,m$, this means for every $v \in V_X$,

$$\langle \theta_N - \tau\Delta\theta_N, v\rangle_\Phi = \langle \theta_{N-1} - \tau\omega_N, v\rangle_\Phi. \tag{26}$$

By taking $v = \theta_N$, we have

$$\begin{aligned} \langle \theta_N - \tau\Delta\theta_N, \theta_N\rangle_\Phi &= \langle \theta_{N-1} - \tau\omega_N, \theta_N\rangle_\Phi, \\ \|\theta_N\|_\Phi^2 - \tau\langle\Delta\theta_N, \theta_N\rangle_\Phi &= \langle\theta_{N-1}, \theta_N\rangle_\Phi - \tau\langle\omega_N, \theta_N\rangle_\Phi. \end{aligned}$$

Since $\langle\Delta\theta_N, \theta_N\rangle_\Phi \leqslant 0$ (cf. inequality (14)), we can conclude

$$\begin{aligned} \|\theta_N\|_\Phi^2 &\leqslant \langle\theta_{N-1}, \theta_N\rangle_\Phi + \tau\bigl|\langle\omega_N, \theta_N\rangle_\Phi\bigr| \\ &\leqslant \|\theta_{N-1}\|_\Phi\|\theta_N\|_\Phi + \tau\|\omega_N\|_\Phi\|\theta_N\|_\Phi. \end{aligned}$$

Simplifying $\|\theta_N\|_\Phi$ on both sides, we obtain

$$\|\theta_N\|_\Phi \leqslant \|\theta_{N-1}\|_\Phi + \tau\|\omega_N\|_\Phi.$$

By repeated application,

$$\|\theta_N\|_\Phi \leqslant \|\theta_0\|_\Phi + \tau\sum_{j=1}^N \|\omega_j\|_\Phi.$$

Here, as before,

$$\|\theta_0\|_\Phi = \|I_X f - Pf\|_\Phi \leqslant Ch_X^\sigma\bigl(\|f\|_{2\sigma} + \|f\|_{2\sigma+2}\bigr).$$

Now for every $1 \leqslant j \leqslant N$,

$$\begin{aligned} \omega_j &= \overline{\partial}_t Pu(t_j) - \overline{\partial}_t u(t_j) + \bigl(\overline{\partial}_t u(t_j) - I_X u_t(t_j)\bigr) \\ &=: \omega_{j,1} + \omega_{j,2}. \end{aligned}$$

We note that

$$\omega_{j,1} = (P-I)\tau^{-1}\int_{t_{j-1}}^{t_j} u_t \,\mathrm{d}s = \tau^{-1}\int_{t_{j-1}}^{t_j} (P-I)u_t \,\mathrm{d}s,$$

whence

$$\tau\sum_{j=1}^N \|\omega_{j,1}\|_\Phi \leqslant \sum_{j=1}^N Ch_X^\sigma \int_{t_{j-1}}^{t_j} \bigl\|u_t(s)\bigr\|_{2\sigma+2}\,\mathrm{d}s = Ch_X^\sigma\int_0^{t_N} \bigl\|u_t(s)\bigr\|_{2\sigma+2}\,\mathrm{d}s.$$

Further,

$$\omega_{j,2} = \frac{u(t_j) - u(t_{j-1})}{\tau} - u_t(t_j) + u_t(t_j) - I_X u_t(t_j)$$
$$= -\frac{1}{\tau}\int_{t_{j-1}}^{t_j} (s - t_{j-1}) u_{tt}(s)\,\mathrm{d}s + u_t(t_j) - I_X u_t(t_j),$$

so that

$$\tau \sum_{j=1}^{N} \|\omega_{j,2}\|_\Phi \leqslant \sum_{j=1}^{N} \left\| \int_{t_{j-1}}^{t_j} (s - t_{j-1}) u_{tt}(s)\,\mathrm{d}s \right\|_\Phi + \tau \sum_{j=1}^{N} \left\| u_t(t_j) - I_X u_t(t_j) \right\|_\Phi$$
$$\leqslant \tau \int_0^{t_N} \|u_{tt}\|_\Phi \,\mathrm{d}s + C\tau h_X^\sigma \sum_{j=1}^{N} \left\| u_t(t_j) \right\|_{2\sigma}.$$

Therefore, by setting $C_1 := C$ and noting $\|\cdot\|_\Phi \sim \|\cdot\|_\sigma$, we obtain a constant C_2 so that

$$\tau \sum_{j=1}^{N} \|\omega_j\|_\Phi \leqslant \tau \sum_{j=1}^{N} \|\omega_{j,1}\|_\Phi + \tau \sum_{j=1}^{N} \|\omega_{j,2}\|_\Phi$$
$$\leqslant C_1 h_X^\sigma \left(\int_0^{t_N} \left\| u_t(s) \right\|_{2\sigma+2} \mathrm{d}s + \tau \sum_{j=1}^{N} \left\| u_t(t_j) \right\|_{2\sigma} \right)$$
$$+ C_2 \tau \int_0^{t_N} \left\| u_{tt}(s) \right\|_\sigma \mathrm{d}s.$$

Thus

$$\left\| u(T) - U_N(T) \right\|_\Phi \leqslant \|\gamma_N\|_\Phi + \|\theta_N\|_\Phi \leqslant \|\gamma_N\|_\Phi + \|\theta_0\|_\Phi + \tau \sum_{j=1}^{N} \|\omega_j\|_\Phi$$
$$\leqslant C_1 h_X^\sigma \Gamma(f, u_t) + C_2 \tau \int_0^{t_N} \left\| u_{tt}(s) \right\|_\sigma \mathrm{d}s,$$

where

$$\Gamma(f, u_t) := \|f\|_{2\sigma} + \|f\|_{2\sigma+2} + \int_0^{t_N} \left\| u_t(s) \right\|_{2\sigma+2} \mathrm{d}s + \tau \sum_{j=1}^{N} \left\| u_t(t_j) \right\|_{2\sigma}. \qquad \square$$

4.2. Crank–Nicolson method

We now turn to the Crank–Nicolson method in which the semi-discrete equation is discretized in a symmetric fashion around the point $t_{N-1/2} := (N - 1/2)\tau$, which will produce a second order in time accurate method. More precisely, U_N in V_X can be defined recursively by

$$\overline{\partial}_t U_N(x_j) - \frac{\Delta(U_N(x_j) + U_{N-1}(x_j))}{2} = F(x_j, t_{N-1/2}), \quad \forall x_j \in X, \tag{27}$$

given that

$$U_0 = I_X f.$$

In matrix form

$$\left(A - \frac{1}{2}\tau B\right)\mathbf{c}_N = \left(A + \frac{1}{2}\tau B\right)\mathbf{c}_{N-1} + \tau\big[F(x_j, t_{N-1/2})\big]_{j=1}^m,$$

given that

$$A\mathbf{c}_0 = \big[f(x_j)\big]_{j=1}^m.$$

Theorem 4.2. Let U_N and u be the solutions of (27) and (1), respectively. We assume that $f, u_t \in H^{2\sigma+2}(S^n)$ and $u_{ttt}, \Delta u_{tt} \in H^\sigma(S^n)$. The approximate solution U_N is constructed from shifts of a spherical basis function Φ with $\hat{\phi}(\ell) \sim (1+\lambda_\ell)^{-\sigma}$. Then there are positive constants C_1 and C_2, independent of h_X, so that

$$\big\|U_N - u(t_N)\big\|_\Phi \leqslant C_1 h_X^\sigma \Gamma(f, u_t) + C_2\tau^2\left(\int_0^{t_N} \|u_{ttt}\|_\sigma + \|\Delta u_{tt}\|_\sigma \,\mathrm{d}s\right),$$

where

$$\Gamma(f, u_t) := \|f\|_{2\sigma} + \|f\|_{2\sigma+2} + \int_0^{t_N} \big\|u_t(s)\big\|_{2\sigma+2}\,\mathrm{d}s + \tau\sum_{j=1}^N \big\|u_t(t_{j-1/2})\big\|_{2\sigma}.$$

Let

$$U_N - u(t_N) = U_N - Pu(t_N) + Pu(t_N) - u(t_N) =: \theta_N + \gamma_N.$$

With the above notation we have

$$\overline{\partial}_t\theta_N(x_j) - \frac{\Delta(\theta_N(x_j) + \theta_{N-1}(x_j))}{2} = -\eta_N(x_j), \quad \forall x_j \in X,$$

where now

$$\begin{aligned}
\eta_N &= \overline{\partial}_t Pu(t_N) - \partial_t I_X u(t_{N-1/2}) + I_X\Delta\left(u(t_{N-1/2}) - \frac{u(t_N) + u(t_{N-1})}{2}\right)\\
&= (P - I)\overline{\partial}_t u(t_N) + \left(\overline{\partial}_t u(t_N) - I_X u_t(t_{N-1/2})\right)\\
&\quad + I_X\Delta\left(u(t_{N-1/2}) - \frac{u(t_N) + u(t_{N-1})}{2}\right)\\
&=: \eta_{N,1} + \eta_{N,2} + \eta_{N,3}.
\end{aligned}$$

Applying arguments similar to (25) and (26) we arriving at

$$\left\langle \theta_N - \theta_{N-1} - \frac{\tau}{2}\Delta(\theta_N + \theta_{N-1}), \chi\right\rangle_\Phi = -\tau\langle\eta_N, \chi\rangle_\Phi, \quad \forall\chi \in V_X.$$

By taking $\chi = \theta_N + \theta_{N-1}$ and note that $\langle \Delta(\theta_N + \theta_{N-1}), \theta_N + \theta_{N-1}\rangle_\Phi \leqslant 0$ (cf. inequality (17)), we have

$$\|\theta_N\|_\Phi^2 - \|\theta_{N-1}\|_\Phi^2 \leqslant -\tau\big\langle \eta_N, (\theta_N + \theta_{N-1})\big\rangle_\Phi \leqslant \tau\|\eta_N\|_\Phi\big(\|\theta_N\|_\Phi + \|\theta_{N-1}\|_\Phi\big).$$

Simplifying the common factor $(\|\theta_N\|_\Phi + \|\theta_{N-1}\|_\Phi)$ on both sides of the inequality, we obtain

$$\|\theta_N\|_\Phi \leqslant \|\theta_{N-1}\|_\Phi + \tau\|\eta_N\|_\Phi.$$

After repeated application this yields

$$\|\theta_N\|_\Phi \leqslant \|\theta_0\|_\Phi + \tau\sum_{j=1}^{N}\big(\|\eta_{j,1}\|_\Phi + \|\eta_{j,2}\|_\Phi + \|\eta_{j,3}\|_\Phi\big).$$

The term $\|\theta_0\|_\Phi$ can be estimated as before. For the latter sum, we have

$$\|\eta_{j,1}\|_\Phi = \big\|(P - I)\overline{\partial}_t u(t_j)\big\|_\Phi \leqslant C\tau^{-1}h_X^\sigma\int_{t_{j-1}}^{t_j}\|u_t\|_{2\sigma+2}\,\mathrm{d}s.$$

Further,

$$\begin{aligned}
\|\eta_{j,2}\|_\Phi &= \big\|\overline{\partial}_t u(t_j) - I_X u_t(t_{j-1/2})\big\|_\Phi \\
&\leqslant \big\|\overline{\partial}_t u(t_j) - u_t(t_{j-1/2})\big\|_\Phi + \big\|u_t(t_{j-1/2}) - I_X u_t(t_{j-1/2})\big\|_\Phi \\
&= \frac{1}{2\tau}\left\|\int_{t_{j-1}}^{t_{j-1/2}}(s - t_{j-1})^2 u_{ttt}(s)\,\mathrm{d}s + \int_{t_{j-1/2}}^{t_j}(s - t_j)^2 u_{ttt}(s)\,\mathrm{d}s\right\|_\Phi \\
&\quad + \big\|u_t(t_{j-1/2}) - I_X u_t(t_{j-1/2})\big\|_\Phi \\
&\leqslant \tau\int_{t_{j-1}}^{t_j}\|u_{ttt}\|_\Phi\,\mathrm{d}s + Ch_X^\sigma\big\|u_t(t_{j-1/2})\big\|_{2\sigma}.
\end{aligned}$$

Let

$$\begin{aligned}
\psi &:= u(t_{j-1/2}) - \frac{u(t_j) + u(t_{j-1})}{2} \\
&= \frac{1}{2}\int_{t_{j-1}}^{t_{j-1/2}}(t_{j-1} - s)u_{tt}(s)\,\mathrm{d}s + \frac{1}{2}\int_{t_{j-1/2}}^{t_j}(s - t_j)u_{tt}(s)\,\mathrm{d}s.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\|\eta_{j,3}\|_\Phi &= \left\|I_X\Delta\left(u(t_{j-1/2}) - \frac{1}{2}\big(u(t_j) + u(t_{j-1})\big)\right)\right\|_\Phi \\
&= \|I_X\Delta\psi\|_\Phi \leqslant \|\Delta\psi\|_\Phi \quad \text{(see lemma 2.1)} \\
&\leqslant C_2\tau\int_{t_{j-1}}^{t_j}\|\Delta u_{tt}\|_\sigma\,\mathrm{d}s, \quad \text{since } \|\cdot\|_\Phi \sim \|\cdot\|_\sigma.
\end{aligned}$$

Altogether, with $C_1 := C$, we have

$$
\begin{aligned}
&\tau \sum_{j=1}^{N} \big(\|\eta_{j,1}\|_\Phi + \|\eta_{j,2}\|_\Phi + \|\eta_{j,3}\|_\Phi\big) \\
&\quad \leqslant C_1 h_X^\sigma \biggl(\int_0^{t_N} \|u_t\|_{2\sigma+2}\,\mathrm{d}s + \tau \sum_{j=1}^{N} \big\| u_t(t_{j-1/2}) \big\|_{2\sigma} \biggr) \\
&\qquad + C_2 \tau^2 \int_0^{t_N} \big(\big\| u_{ttt}(s) \big\|_\sigma + \big\| \Delta u_{tt}(s) \big\|_\sigma \big)\,\mathrm{d}s.
\end{aligned}
$$

Thus

$$
\|\theta_N\|_\Phi + \|\gamma_N\|_\Phi \leqslant C_1 h_X^\sigma \Gamma(f, u_t) + C_2 \tau^2 \biggl(\int_0^{t_N} \|u_{ttt}\|_\sigma + \|\Delta u_{tt}\|_\sigma \,\mathrm{d}s \biggr),
$$

where

$$
\Gamma(f, u_t) := \|f\|_{2\sigma} + \|f\|_{2\sigma+2} + \int_0^{t_N} \big\| u_t(s) \big\|_{2\sigma+2}\,\mathrm{d}s + \tau \sum_{j=1}^{N} \big\| u_t(t_{j-1/2}) \big\|_{2\sigma}.
$$

5. Numerical experiments on S^2

Let us consider the function

$$
G(z) = 1 - 2\ln\biggl(1 + \sqrt{\frac{1-z}{2}}\biggr).
$$

We can expand $G(z)$ as a series of Legendre polynomials (cf. [8]):

$$
G(z) = \sum_{\ell=1}^{\infty} \frac{1}{\ell(\ell+1)} P_\ell(z).
$$

The following PDE describes the heat diffusion process from the north pole onto the surface of the unit sphere:

$$
\begin{cases}
\dfrac{\partial}{\partial t} u(x,t) = \Delta u(x,t), & x \in S^2, \\
u(x,0) = G(x \cdot p), & \text{where } p = (0,0,1)^{\mathrm{T}}.
\end{cases}
\tag{28}
$$

Since the initial condition $u(x,0)$ is a zonal function which depends only on the geodesic distance from any given point on the sphere to the north pole, the solution $u(x,t)$ also depends only on the geodesic distance to the north pole. The problem (28) is reduced to

$$
\frac{\partial u}{\partial t} = \frac{\partial}{\partial z}\big(1 - z^2\big)\frac{\partial u}{\partial z},
$$

subject to the following initial condition:

$$u(z,0) = G(z), \quad z \in [-1,1].$$

We know that the Legendre polynomials are eigenfunctions of the operator

$$\frac{\partial}{\partial z}\left(1 - z^2\right)\frac{\partial}{\partial z}.$$

Thus, by the method of separation of variables, the exact solution of (28) is given as

$$u(z,t) = \sum_{\ell=1}^{\infty} \frac{e^{-\ell(\ell+1)t}}{\ell(\ell+1)} P_\ell(z).$$

We can approximate $u(z,t)$ by the truncated series:

$$u_L(z,t) = \sum_{\ell=1}^{L} \frac{e^{-\ell(\ell+1)t}}{\ell(\ell+1)} P_\ell(z).$$

The error is estimated by using the fact that $\|P_\ell(z)\|_\infty = 1$ (see [10])

$$\begin{aligned}\|u - u_L\|_\infty &= \left\| \sum_{\ell=L+1}^{\infty} \frac{e^{-\ell(\ell+1)t}}{\ell(\ell+1)} P_\ell(z) \right\|_\infty \\ &\leqslant e^{-L(L+1)t} \int_L^\infty \frac{dx}{x(x+1)} \leqslant e^{-L(L+1)\tau} \ln\left(1 + \frac{1}{L}\right).\end{aligned}$$

For time-step $\tau = 0.00125$, in order to obtain the accuracy of order 10^{-16} it is required that $L \geqslant 160$.

The spherical basis functions used to construct the approximate solution are derived from a class of locally supported radial basis function proposed by Wendland [21]. These functions $\psi(x)$ are rotation invariant and are thus function of $|x|$ only. So the corresponding convolution kernel $\psi(x-y), x, y \in S^n$, is a function of $|x-y| = \sqrt{2 - 2x \cdot y}$. We may therefore define a function

$$\Phi(x,y) = \phi(x \cdot y) := \psi(x-y), \quad x, y \in S^n.$$

Note that $\Phi(x,y)$ inherits the property of positive definiteness from ψ, and $\hat{\phi}(\ell) \sim (1+\lambda_\ell)^{-\sigma}$ for some $\sigma > 0$ (see [13, section 4]). For our numerical study, we use the function $\psi(r) = (1-r)_+^4(4r+1)$.

The set of points which are used in constructing the SBFs is generated according to an algorithm in [15]. These points are generated uniformly, in the sense that each point is a center of a cell on the unit sphere of area $4\pi/m$.

The iterative equation (22) becomes

$$\left(I - \tau A^{-1} B\right)\mathbf{c}_N = \mathbf{c}_{N-1},$$

with the initial equation

$$\mathbf{c}_0 = A^{-1} f|_X.$$

Table 1
Backward Euler method with different sets of points and time-steps.

m	h_X	q_X	$E_\infty(\tau = 0.01)$	$E_\infty(\tau = 0.005)$	$E_\infty(\tau = 0.0025)$
200	0.1942	0.1130	0.0224	0.0225	0.0225
400	0.1288	0.0731	0.0137	0.0138	0.0139
600	0.1122	0.0675	0.0088	0.0089	0.0090
800	0.0950	0.0577	0.0060	0.0061	0.0062
1000	0.0849	0.0516	0.0044	0.0045	0.0046
1200	0.0789	0.0476	0.0034	0.0036	0.0036

Since A is positive definite and B has nonpositive eigenvalues, it can be shown that all the eigenvalues of the matrix $(I - \tau A^{-1}B)$ are in the interval (0, 1] (see the appendix). Hence the numerical algorithm is stable.

The following table show the numerical errors between the iterated solution U_N obtained by backward Euler method and u_{160}. Here, $N = 1.5/\tau$ and

$$E_\infty(\tau) := \max_{x\in S^2} |U_N - u_L|.$$

Appendix

Lemma A.1 (cf. [22, chapter 1, section 31]). Let A be a symmetric positive definite matrix and B be a symmetric positive semi-definite (negative semi-definite). Then all of the eigenvalues of AB are non-negative (nonpositive).

Proof. Since A is symmetric positive definite, there is an invertible matrix P such that $A = P^{\mathrm{T}}P$. Let $C = PBP^{\mathrm{T}}$, then C and AB have the same set of eigenvalues since $(P^{\mathrm{T}})^{-1}ABP^{\mathrm{T}} = PBP^{\mathrm{T}} = C$. The matrix C is symmetric since $C^{\mathrm{T}} = PB^{\mathrm{T}}P^{\mathrm{T}} = PBP^{\mathrm{T}} = C$ since B is symmetric. Now since B is positive semi-definite,

$$\left(P^{\mathrm{T}}\mathbf{x}\right)^{\mathrm{T}}B\left(P^{\mathrm{T}}\mathbf{x}\right) \geqslant 0 \quad \text{for all } \mathbf{x} \in \mathbb{R}^m.$$

Hence C is symmetric semi-positive definite. Hence all of the eigenvalues of C are non-negative, so are the eigenvalues of the matrix AB. □

Lemma A.2. Let A be a symmetric positive definite matrix and B be a symmetric negative semi-definite. Then for any $\varepsilon > 0$, all of the eigenvalues of $(I - \varepsilon A^{-1}B)^{-1}$ are in the interval (0, 1].

Proof. Let μ be an eigenvalue of $I - \varepsilon A^{-1}B$, then $1/\mu$ is an eigenvalue of $(I - \varepsilon A^{-1}B)^{-1}$. It is observed that $\mu = 1 - \varepsilon\delta$ where δ is an eigenvalue of $A^{-1}B$. By lemma A.1, $\delta \leqslant 0$, and therefore, $\mu \in [1, \infty)$. Thus, $1/\mu \in (0, 1]$. □

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