Orthogonal Projections

\[ P : \mathcal{H} \mapsto \mathcal{H} \quad S = P(\mathcal{H}) \]

\( P \) orthogonal projection on \( S \) iff \( \forall x, y \in \mathcal{H} \& \quad \forall \alpha, \beta \in \mathbb{C} \)

(a) \( P \) linear: \( P(\alpha x + \beta y) = \alpha P(x) + \beta P(y) \)

(b) \( P^2 = P \) : \( P(y) = y \quad \forall y \in S \)

(c) \( \langle P(x), y \rangle = \langle x, P(y) \rangle \)

Lemma (a) \( x - P(x) \perp S \ \forall x \in \mathcal{H} \)

(b) \( \exists! y_0 \in S : \inf_{y \in S} \|x - y\| = \|x - y_0\| \& \quad P(x) = y_0 \)

Proof (a) \( y \in S \) \( \langle x - P(x), y \rangle = \langle x - P(x), P(y) \rangle \)

\[ = \langle P(x - P(x)), y \rangle = \ldots = 0 \]

\( P(x - P(x)) = P(x) - P(P(x)) = P(x) - P(x) = 0 \)

\( \in S \)

(c) \( \|P(x)\|^2 = \langle P(x), P(x) \rangle \)

(b) \( y_0 := P(x) \) \( y \in S : y \neq y_0 \)

\[ \|x - y\|^2 = \| (x - y_0) + (y_0 - y) \|^2 \]

\[ = x - P(x) \perp \in S \]

by Pythagoras Theorem

\[ \|x - y\|^2 = \|x - y_0\|^2 + \|y_0 - y\|^2 \]

\[ > 0 \quad \text{since} \quad y \neq y_0 \]

both distance & uniqueness follow from here

(\text{c}) \quad \|P(x)\| \leq \|x\| \quad \forall x \in \mathcal{H} \)

(d) \( S \) is closed

\( \forall x \in \mathcal{H} \)

\[ \|x - y\| \geq \|x - y_0\| \]

by Cauchy-Schwartz

\[ \leq \|P(x)\| \cdot \|x\| \]