The Wrong Trigonometry

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1 What's wrong with trigonometry?

Trigonometry begins with the study of triangles. A triangle has three side lengths, three vertex angles, and an area. Classical trigonometry studies these seven quantities and the relations between them. It then applies this understanding to more complicated figures such as quadrilaterals and other polygons, along with three dimensional boxes, pyramids and wedges. Then it solves numerous problems in surveying, navigation, engineering, construction, physics, chemistry and other branches of mathematics.

Surely understanding a triangle cannot be hard. But each year millions of students around the world are turned off further study in mathematics because of problems learning classical trigonometry. Somehow the subject is a lot more complicated than you would at first guess. Why is this? Is it necessarily so? Are there any alternatives?

Let's describe the subject of classical trigonometry in a bit more detail. In keeping with tradition, precise definitions will be avoided, because they are invariably too subtle. Even so, you'll perhaps agree that classical trigonometry is difficult, and that it's not surprising that students don't grasp the material well.

Then we'll reveal a subversive secret—*there's a much simpler way.*

2 Basic Concepts

The basic concepts of classical planar trigonometry are distance, area and angle. Informally distance is what you measure with a ruler. More precisely it is given by the formula

$$|A_1, A_2| = \sqrt{(a_2 - a_1)^2 + (b_2 - b_1)^2}$$

for points $A_1 = [a_1, b_1]$ and $A_2 = [a_2, b_2]$. Informally area is what you measure with a weighing scale–by weighing a planar segment and dividing by its density. More precisely area is defined for a triangle by the rule one-half base times height.

Informally angle is what you measure with a protractor.To define it more precisely turns out to be surprisingly tricky.

One way is to define it in terms of arclengths of circular arcs, but this requires a prior understanding of calculus. Another common approach is to try to define an angle θ as $\arctan(y/x)$ but this requires occasional fiddling with an extra π, and a prior understanding of the arctan function. Well, $\arctan x$ is a suitably defined inverse function of $\tan\theta$, which is itself the ratio $\sin\theta/\cos\theta$, and $\sin\theta$ and $\cos\theta$ are defined in terms of ratios or projections involving the angle θ. So we are back where we started, with the angle θ.

This logical circularity is rarely acknowledged, as educators generally reveal only part of this circle of ideas at any one time. Let us proceed forward nonetheless.

3 Trigonometric relationships

Figure 1 shows a triangle with side lengths a, b and c, with angles A, B and C, and with area Δ. The lengths are measured using some standard unit, such as centimeters or inches. The angles are usually measured either in degrees (from 0° to 360°) or in radians (from 0 to 2π). The area is measured in units of length squared, such as square centimeters or square inches.

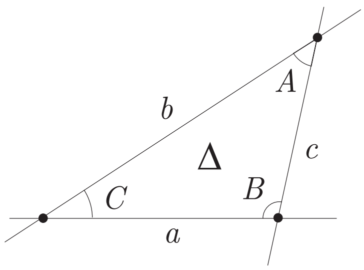


Figure 1: A basic triangle

The classical formulas relating the sides and the angles are the **Sine law**

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

the **Cosine law**

$$c^2 = a^2 + b^2 - 2ab\cos C$$

and the **Law of tangents**

$$\frac{a-b}{a+b} = \frac{\tan\frac{A-B}{2}}{\tan\frac{A+B}{2}}.$$

These involve the three basic trigonometric functions $\sin\theta$, $\cos\theta$ and $\tan\theta$, called respectively sine, cosine and tangent. There are others, such as half-angle formulas, but these are the main ones. There are also relations linking the area Δ to the sides and angles, such as

$$\Delta = \frac{ab\sin C}{2}$$

and **Heron's**, or **Archimedes'**, **formula**

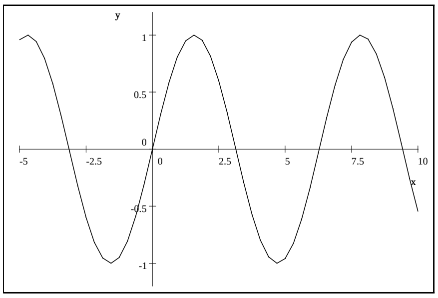
$$\Delta = \sqrt{s\,(s-a)\,(s-b)\,(s-c)}$$

where $s = (a+b+c)/2$.

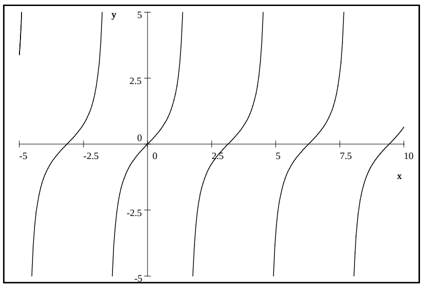
4 Graphs of the trigonometric functions

It should be clear that classical trigonometry relies heavily on an understanding of the trigonometric functions $\cos\theta, \sin\theta$ and $\tan\theta$, along with their reciprocals $\sec\theta, \csc\theta$ and $\cot\theta$. But what exactly is the sine function, for example? How does one calculate a quantity like $\sin 37°$?

Here is the graph of the sine function $\sin\theta$, which is a 2π-periodic function of the variable θ measured in radians.



The cosine function $\cos x$ is similiar. But the tangent function $\tan x$ looks quite different.



Students who wish to learn classical trigonometry must become familiar with these graphs, their various properties and special values.

5 Calculating the trigonometric functions

To find accurate values of these functions, the most straightforward approach is via a *power series expansion*. This is like a polynomial with an infinite number of terms, so to interpret its values requires care. The power series for $\sin x, \cos x$ and $\tan x$ are

$$\begin{aligned}\sin x &= x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \frac{1}{362\,880}x^9 - \frac{1}{39\,916\,800}x^{11} + \cdots \\ \cos x &= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \frac{1}{40\,320}x^8 - \frac{1}{3628\,800}x^{10} + \cdots \\ \tan x &= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9 + \frac{1382}{155\,925}x^{11} + \cdots\end{aligned}$$

The series for $\sin x$ and $\cos x$ are not as complicated as the formulas might suggest, with compact ways of writing the n-th coefficient, discovered by Newton. However, the series for $\tan x$ *really is* difficult—there is no known way to compactly write down its general coefficient. These subtleties are largely hidden from view with our powerful calculators and computers.

To calculate $\sin 37°$, your calculator does something like this. It first converts $37°$ into radians by the formula

$$\frac{37\pi}{180} = 0.645\,771\,824$$

It then substitutes the value of $x = 0.645\,771\,824$ into, say, the first six terms of the power series for $\sin x$ given above, yielding the expression

$$\sin 37° = 0.645\,771\,824 - \frac{1}{6}(0.645\,771\,824)^3 + \cdots - \frac{1}{39\,916\,800}(0.645\,771\,824)^{11}$$

It then evaluates this using ordinary arithmetic, yielding

$$\sin 37° = 0.601\,815\,024.$$

Your calculator might differ at the last digit.

The secant, cosecant and cotangent functions also have power series. Like the tangent function, these series have no compact forms.

6 Inverse trigonometric functions

Unfortunately, this is only part of the story. A trigonometric formula generally does not yield an angle, but only an expression for some trigonometric function of that angle. For example, you might be able to compute $\cos C = 0.527\,468\,3$ from the Cosine Law, but then you will need to use the inverse function $\arccos x$ to recover C. A number of subtle points arise, having to do with possible ambiguities in defining inverse functions.

Ultimately your calculator will have to resort to power series expansions, or something equivalent to them. For example the usual inverse functions for $\sin\theta$, $\cos\theta$ and $\tan\theta$ have the following power series

$$\begin{aligned}
\arcsin x &= x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \frac{35}{1152}x^9 + \frac{63}{2816}x^{11} + \cdots \\
\arccos x &= \frac{1}{2}\pi - x - \frac{1}{6}x^3 - \frac{3}{40}x^5 - \frac{5}{112}x^7 - \frac{35}{1152}x^9 - \frac{63}{2816}x^{11} - \cdots \\
\arctan x &= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 - \frac{1}{11}x^{11} + \frac{1}{13}x^{13} - \cdots .
\end{aligned}$$

Curiously in this case $\arctan x$ has the simpler expansion.

7 Relationships among the trig functions

Having understood the definitions of the trigonometric functions, how to compute them, and how they appear in the formulas for triangles is not enough. The trigonometric functions satisfy relations between themselves. There are certainly hundreds of these, however a dozen or two should get you through most applications.

Some of the relations are reasonably simple, like

$$\cos^2 x + \sin^2 x = 1$$

and

$$\tan x = \frac{\sin x}{\cos x}.$$

Others, like the double angle formulas

$$\begin{aligned}
\cos 2x &= \cos^2 x - \sin^2 x \\
\sin 2x &= 2\cos x \sin x \\
\tan 2x &= \frac{2\tan x}{1-\tan^2 x}
\end{aligned}$$

are not too bad. Others, like the formulas

$$\begin{aligned}
\cos(x+y) &= \cos x \cos y - \sin x \sin y \\
\sin(x+y) &= \cos x \sin y + \sin x \cos y \\
\tan(x+y) &= \frac{\tan x + \tan y}{1 - \tan x \tan y}
\end{aligned}$$

are a bit more difficult to remember. Many more crop up in high school courses.

Despite the large amount of theory necessary to sustain classical trigonometry, students are constantly given examples from only a handful of basic scenarios. The two triangles with angles

$90°/45°/45°$ and $90°/60°/30°$ appear over and over again in this subject, for the simple reason that they are basically the only ones that don't require calculators. In real life these triangles are nowhere near as common as they are in trigonometry classes. In real applications calculating angles is quite painful without a calculator or tables.

8 A new trigonometry

Millions of students have struggled with classical trigonometry over the centuries. Despite being about something very simple—namely a triangle—the theory involves all kinds of *complicated concepts and formulas,* and relies on *infinite processes* for its calculations. To properly understand these, you really need to first learn calculus. But trigonometry is always a *preliminary to calculus.* So that by the time you get around to calculus in university, you and your professors will already assume you understand trigonometry. A bit circular, isn't it?

The notions of distance and angle are so familiar that it is easy to believe that they are fundamental and self-evident. If you allow the possibility that there might be *other concepts* which replace distance, angle and area, then it becomes possible that the same subject might be a lot simpler and more pleasant. This is exactly what happens with **rational trigonometry**, as developed in the recent book *'Divine Proportions: Rational Trigonometry to Universal Geometry'* [Wildberger].

Once you realize that distance and angle are neither self-evident nor fundamental, a new world of mathematics opens up before you. Rational trigonometry uses the quadratic and more elementary concepts of *quadrance*, *spread* and *quadrea*, and suddenly a lot of the complication evaporates! The trigonometric functions become incidental to trigonometry and geometry. The fundamental laws are simpler and can be solved without calculators or tables. Round-off errors can be eliminated, yielding more accurate answers to a host of practical geometric problems.

The trig functions $\sin\theta$ and $\cos\theta$ still have a role to play in the study of circular or harmonic motion, but there the knowledge needed is rather minimal. Indeed for the study of circular motion the trigonometric functions are best understood in terms of the (complex) exponential function.

So once you learn rational trigonometry, you realize that classical trigonometry is *wrong*, and the traditional confusion of students is quite justified. Educators will find that with the *right approach*—rational trigonometry—the doors of mathematics swing open much more easily.

References

[Wildberger] N J Wildberger, *Divine Proportions: Rational Trigonometry to Universal Geometry*, Wild Egg (http://wildegg.com), Sydney 2005.