

THE IMPOSSIBILITY OF TRISECTING AN ANGLE WITH STRAIGHTEDGE AND COMPASS: AN APPROACH USING RATIONAL TRIGONOMETRY

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A classical problem in geometry is to trisect an angle using only a straightedge and compass. That is, given an angle θ , give a procedure using only straightedge and compass that will construct the angle $\theta/3$ in a finite number of steps. This problem, which dates to around 400 B.C., fascinated mathematicians and amateurs over the centuries and many “solutions” have been proposed. Unfortunately, all of these “solutions” are flawed as it can be shown that the trisection of a general angle is impossible.

Today, the proof that straightedge-and-compass trisection is impossible is usually done using algebraic methods. The following fact, which can be found in any good abstract algebra textbook, is needed.

Theorem A: If a real number α is constructible using only straightedge and compass, then α satisfies a polynomial f that (i) has rational coefficients, (ii) is irreducible over the rational numbers, and (iii) has degree 2^k for some nonnegative integer k .

The following theorem and its proof are quite standard. (“Constructible” means “constructible using only straightedge and compass”).

Theorem B: An angle of 20° is not constructible.

Proof B1: We proceed as follows:

1. An angle θ is constructible if and only if $\cos\theta$ is constructible.
2. The trigonometric identity $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ implies that when $\alpha = \cos 20^\circ$, $4\alpha^3 - 3\alpha = \cos 60^\circ = 1/2$ or $8\alpha^3 - 6\alpha - 1 = 0$.
3. The polynomial $f(x) = 8x^3 - 6x - 1$ is irreducible over \mathbb{Q} by the Rational Roots Theorem. (Specifically, we need to check that none of the possible rational roots of $f(x) = 0$, namely $x = \pm 1, \pm 1/2, \pm 1/4, \pm 1/8$, actually works.)
4. It now follows from Theorem A that $\alpha = \cos 20^\circ$, and hence 20° , is not constructible.

We can give an alternative proof of Theorem B using rational trigonometry. What is interesting about this approach is that, instead of the Rational Roots Theorem, we can

make use of Eisenstein's Irreducibility Criterion. Recall that for integers a and b , $a \mid b$ means a is a factor of b .

Eisenstein's Criterion: Let $f(x) = a_n x^n + \cdots + a_1 x + a_0$ be a polynomial with integer coefficients. If there exists a prime number p such that (i) $p \mid a_i$ for $i = 0, 1, \dots, n-1$, (ii) $p \nmid a_n$, and (iii) $p^2 \nmid a_0$, then f is irreducible over \mathbb{Q} .

Let $s = s(\theta)$ denote the spread of the angle θ .

Proof B2: We proceed as follows:

1. For any real number α , if α is constructible then so are α^2 and $\sqrt{\alpha}$. It follows that an angle θ is constructible if and only if $s(\theta)$ is constructible.
2. If $s = s(\theta)$, then $s(3\theta) = s(3 - 4s)^2$. (This is the 3rd spread polynomial.) Thus, when $\alpha = s(20^\circ)$, we have $\alpha(3 - 4\alpha)^2 = s(60^\circ) = 3/4$ or $64\alpha^3 - 96\alpha^2 + 36\alpha - 3 = 0$.
3. The polynomial $f(x) = 64x^3 - 96x^2 + 36x - 3$ is irreducible over \mathbb{Q} by Eisenstein's Criterion with $p = 3$.
4. It now follows from Theorem A that $\alpha = s(20^\circ)$, and hence 20° , is not constructible.

While proofs B1 and B2 are clearly analogous, note that Eisenstein's Criterion is considerably easier to use than the Rational Roots Theorem.

References

1. Gallian, J. A., *Contemporary Abstract Algebra*, Houghton Mifflin, Boston, MA, 2006.
2. Wildberger, N. J., *Divine Proportions: Rational Trigonometry to Universal Geometry*, Wild Egg, Sydney, AU, 2006.