A Comparison
of
Rational and Classical Trigonometry

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Introduction. The problems listed below originate from Norman Wildberger's paper, Survivor: The Trigonometry Challenge. (See http://web.maths.unsw.edu.au/~norman/papers/Survivor.pdf) In the paper, three men are stranded on an island without calculators and two of the men challenge one another to solve four trigonometric problems. One man solves the problems using the classical (traditional) approach, and the other uses rational trigonometry. The purpose of the paper is to compare rational and classical trigonometry, and make the case that the rational method is easier, faster, more accurate, and requires less calculations when calculators are not used.

The purpose of this article is provide an objective comparison between the classical and rational methods. It not only compares the two methods for solving the same four problems, but discusses some of the other issues that Wildberger raises in his book, *Divine Proportions: Rational Trigonometry to Universal Geometry*, and in other papers he has authored. The papers and several chapters of his book are available from his website. (See http://web.maths.unsw.edu.au/~norman/Rational1.htm)

There are several things worth noting about the comparison shown below. First, the reader will observe that some problems are using a different classical formula than the one presented in the paper. It is reasonable that the classical method be allowed to choose the appropriate classical formula for solving a given problem. These formulas are not new and are commonly found in traditional textbooks and on the Internet.

Second, notice in some cases rational trigonometry only gives an appearance of a rational result by not evaluating the square root function. (See problem 4.) When the function is evaluated, it generally produces an irrational number. Since rational trigonometry achieves a rational-looking answer by not evaluating the square root function, it is only fair that the classical method be allowed to produce a rational-looking answer, too, by not evaluating the sine function. Angles historically have been expressed as a sine or chord (twice the sine), so expressing an angle as a sine is within the classical framework. As demonstrated, the classical approach is capable of producing rational answers in the same way as the rational method.

Third, what is not revealed in the paper is that rational trigonometry is not applicable to most problems involving circles or circular rotation. The paper also doesn't mention that the solutions produced by rational trigonometry cannot be used to solve science and engineering problems that rely on a linear system of measurements (meters, miles per hour, radians, degrees, etc.) unless the nonlinear units (quadrances and spreads) are converted to/from linear units (lengths and angles). For example, rational trigonometry may need to use the sine function for converting a linear input angle (radian or degree) to a nonlinear spread, and it may need to use the arcsine function for converting a nonlinear spread to a linear output angle.

Since it is unfair to compare two methods without ensuring both methods produce solutions that can be applied to everyday problems, the arcsine function was added as an optional last step to both the classical and rational methods when the problem statement deemed it necessary. (Although these four problems did not require a sine function be added prior to the first step, other problems that express a triangle in terms of an angle may require the function be added to both the classical and rational methods.)

The sine and arcsine functions only have a significant effect on the number of calculations when calculators are not used. If calculators were permitted, these functions could be evaluated by the touch of a button, so their impact is negligible. Although it is unreasonable to believe that calculators would not be used in real life, we will stick with the conditions as set forth and not allow their use when comparing the two methods, even though the comparison may not be very realistic.

Also, if the comparison included other problems involving circular rotation, linear measurements, the sum or difference of two spreads, etc., the reader would immediately see rational trigonometry in a much different light. However, this comparison will solve only the four problems that Wildberger chose in his paper.

All four problems are based on the same general triangle with three given vertices. We begin by performing some pre-calculations. To make it easier to compare the two methods, the choice of some variables were changed so that an uppercase variable indicates a square of its lowercase variable. Readers may view the calculations and draw their own conclusions as to which approach is easier.

Comparison Table

Classical	Rational
Distance squared	Quadrance
Sine squared	Spread
Cosine squared	Cross
Law of sines	Spread law
Law of cosines	Cross law

Pre-calculations.

<u>Given</u>: A triangle with three vertices: $A_1 = (4, 1)$, $A_2 = (1, 2)$, and $A_3 = (2, 4)$.

Unlike the rational method, the classical method does not need to calculate the sine and cosine for the second angle A_2 to solve the four problems.

Classical Approach	Rational Approach
Compute the lengths (distances) for each side: $d_1 = \lvert A_2, A_3 \rvert = \sqrt{(1-2)^2 + (2-4)^2} = \sqrt{5}$ $d_2 = \lvert A_1, A_3 \rvert = \sqrt{(4-2)^2 + (1-4)^2} = \sqrt{13}$ $d_3 = \lvert A_1, A_2 \rvert = \sqrt{(4-1)^2 + (1-2)^2} = \sqrt{10}$	Compute the quadrances for each side: $D_1 = \lvert A_2, A_3 \rvert^2 = (1-2)^2 + (2-4)^2 = 5$ $D_2 = \lvert A_1, A_3 \rvert^2 = (4-2)^2 + (1-4)^2 = 13$ $D_3 = \lvert A_2, A_3 \rvert^2 = (4-1)^2 + (1-2)^2 = 10$
Compute the cosine for two angles using the law of cosines, and the sine using the Pythagorean theorem: $c_1 = \cos(A_1) = \frac{13+10-5}{2\sqrt{10}\sqrt{13}} = \frac{9}{\sqrt{130}}$ $s_1 = \sin(A_1) = \sqrt{1-\cos^2(A_1)} = \sqrt{1-c_1^{\,2}} = \sqrt{1-\left(\frac{9}{\sqrt{130}}\right)^2} = \sqrt{1-\frac{81}{130}} = \sqrt{\frac{49}{130}}$	Compute the spreads for three angles using the cross formula. Note, the cross formula uses both the law of cosines and the Pythagorean theorem: $C_1 = \cos^2(A_1) = \frac{(13+10-5)^2}{4\cdot 10\cdot 13} = \frac{81}{130}$ $S_1 = \sin^2(A_1) = 1-\cos^2(A_1) = 1-C_1 = 1-\frac{81}{130} = \frac{49}{130}$
	$C_2 = \cos^2(A_2) = \frac{(-13+10+5)^2}{4\cdot 5\cdot 10} = \frac{1}{50}$ $S_2 = \sin^2(A_2) = 1-\cos^2(A_2) = 1-C_2 = 1-\frac{1}{50} = \frac{49}{50}$
$c_3 = \cos(A_3) = \frac{13-10+5}{2\sqrt{5}\sqrt{13}} = \frac{4}{\sqrt{65}}$ $s_3 = \sin(A_3) = \sqrt{1-\cos^2(A_3)} = \sqrt{1-c_3^{\,2}} = \sqrt{1-\left(\frac{4}{\sqrt{65}}\right)^2} = \sqrt{1-\frac{16}{65}} = \sqrt{\frac{49}{65}}$	$C_3 = \cos^2(A_3) = \frac{(13-10+5)^2}{4\cdot 5\cdot 13} = \frac{16}{65}$ $S_3 = \sin^2(A_3) = 1-\cos^2(A_3) = 1-C_3 = 1-\frac{16}{65} = \frac{49}{65}$

Having completed the preliminary calculations, we can now move on to solving the first problem.

Problem 1. Find the area of the triangle.

Since the triangle was originally stated in terms of three vertices, the classical method, unlike the rational method, can use a formula that does not require the use of the square root function.

Classical Approach	Rational Approach
$area = \left\lvert \frac{x_1(y_3 - y_2) + x_2(y_1 - y_3) + x_3(y_2 - y_1)}{2} \right\rvert$ $area = \left\lvert \frac{4(4-2) + 1(1-4) + 2(2-1)}{2} \right\rvert$ $area = \left\lvert \frac{8-3+2}{2} \right\rvert = \frac{7}{2} = 3.5$	$area = \sqrt{\frac{(D_1 + D_2 + D_3)^2 - 2(D_1^2 + D_2^2 + D_3^2)}{16}}$ $area = \sqrt{\frac{(5+13+10)^2 - 2(5^2 + 13^2 + 10^2)}{16}}$ $area = \sqrt{\frac{196}{16}} = \frac{14}{4} = 3.5$

Below is an alternative solution using a different version of Heron's formula that does use the square root function.

Classical Approach	Rational Approach
$area = \frac{1}{4}\sqrt{2d_1^2 d_2^2 + 2d_2^2 d_3^2 + 2d_1^2 d_3^2 - d_1^4 - d_2^4 - d_3^4}$ $area = \frac{1}{4}\sqrt{2 \cdot 5 \cdot 13 + 2 \cdot 13 \cdot 10 + 2 \cdot 5 \cdot 10 - 5^2 - 13^2 - 10^2}$ $area = \frac{1}{4}\sqrt{130 + 260 + 100 - 25 - 169 - 100} = \frac{\sqrt{196}}{4} = \frac{14}{4} = 3.5$	$area = \sqrt{\frac{(D_1 + D_2 + D_3)^2 - 2(D_1^2 + D_2^2 + D_3^2)}{16}}$ $area = \sqrt{\frac{(5+13+10)^2 - 2(5^2 + 13^2 + 10^2)}{16}}$ $area = \sqrt{\frac{28^2 - 2(294)}{16}} = \sqrt{\frac{196}{16}} = \frac{14}{4} = 3.5$

Problem 2. What is the length, or quadrance, of the altitude from A_1 to the opposite side A_2A_3?

Wildberger's paper showed the classical method calculating an angle so that its sine could be found using a power series. This was unnecessary. Recall, that the value of the sine for A_3 was previously calculated. (See the pre-calculations above.) If we eliminate the unnecessary power series calculation, then the classical solution is decidedly different.

Classical Approach	Rational Approach
$h = d_2 \sin(A_3) = \sqrt{13} \cdot \sqrt{\frac{49}{65}} = \sqrt{\frac{49}{5}}$	$H = D_2 S_3 = 13 \cdot \frac{49}{65} = \frac{49}{5}$

Problem 3. Let's consider the median from A_2 to the side A_1A_3. What's its length, or quadrance? And what's the angle, or spread, between it and the side A_1A_3?

Classical Approach	Rational Approach
$p^2 = \frac{2d_1^2 + 2d_3^2 - d_2^2}{4}$ $p^2 = \frac{2\cdot(\sqrt{5})^2 + 2\cdot(\sqrt{10})^2 - (\sqrt{13})^2}{4} = \frac{2\cdot 5 + 2\cdot 10 - 13}{4} = \frac{17}{4}$ $p = \sqrt{\frac{17}{4}}$ $\sin^2(\alpha) = \left(\frac{d_3 \sin(A_1)}{p}\right)^2 = \frac{d_3^2 \sin^2(A_1)}{p^2}$ (law of sines) $\sin^2(\alpha) = \frac{(\sqrt{10})^2 \cdot \left(\sqrt{\frac{49}{130}}\right)^2}{\frac{17}{4}} = \frac{10\cdot\frac{49}{130}}{\frac{17}{4}} = \left(\frac{49}{17}\right)\left(\frac{4}{17}\right) = \frac{196}{221}$ $\sin(\alpha) = \sqrt{\frac{196}{221}}$	$\left(P - D_3 - \frac{D_2}{4}\right)^2 = 4\cdot D_3 \cdot \frac{D_2}{4}\cdot(1 - S_1)$ where $P = p^2$ $\left(P - 10 - \frac{13}{4}\right)^2 = 4\cdot 10\cdot\frac{13}{4}\cdot\left(1 - \frac{49}{130}\right)$ $P^2 - \frac{53}{2}P + \frac{1513}{16} = 0$ Eq 1. $\left(P - D_1 - \frac{D_2}{4}\right)^2 = 4\cdot D_1\cdot\frac{D_2}{4}\cdot(1 - S_3)$ $\left(P - 5 - \frac{13}{4}\right)^2 = 4\cdot 5\cdot\frac{13}{4}\cdot\left(1 - \frac{49}{65}\right)$ $P^2 - \frac{33}{2}P + \frac{833}{16} = 0$ Eq 2. Subtracting Eq 1 from Eq 2 $\frac{33}{2}P - \frac{833}{16} - \frac{53}{2}P + \frac{1513}{16} = 0$ $\frac{20}{2}P = \frac{85}{2}$ $P = \frac{17}{4}$ $\sin^2(\alpha) = \frac{D_3 S_1}{P}$ (spread law) $\sin^2(\alpha) = \frac{10\cdot\frac{49}{130}}{\frac{17}{4}} == \frac{49}{13}\cdot\frac{4}{17} = \frac{196}{221}$
If a radian solution is required, then both methods need to perform these additional calculations:	
$\alpha = \arcsin\left(\sqrt{\frac{196}{221}}\right)$	$\sin(\alpha) = \sqrt{\frac{196}{221}}$ $\alpha = \arcsin\left(\sqrt{\frac{196}{221}}\right)$

Problem 4. Let's consider the angle bisector at A_3. What is its length, or quadrance, and what is the angle, or spread, which it makes with A_1A_2?

Let b be the angle bisector at A_3. Let C be the point where the bisector b meets line A_1A_2. Let P_1 be the quadrance of the line A_1C, and let P_2 be the quadrance of the line A_2C. Let B be the quadrance of the bisector. Let α be the angle A_3CA_1.

Classical Approach

$$b = 2d_1d_2\left(\frac{\cos\left(\frac{A_3}{2}\right)}{d_1+d_2}\right)$$ where $\cos\left(\frac{A_3}{2}\right)=\sqrt{\frac{1+\cos(A_3)}{2}}$

therefore

$$b^2 = 4(d_1d_2)^2\left(\frac{\frac{1+\cos(A_3)}{2}}{(d_1+d_2)^2}\right) = \frac{2(d_1d_2)^2(1+\cos(A_3))}{d_1^2+2d_1d_2+d_2^2}$$

$$b^2 = \frac{2\cdot 65\left(1+\frac{4}{\sqrt{65}}\right)}{5+13+2\sqrt{65}} = \frac{65\left(\frac{4+\sqrt{65}}{\sqrt{65}}\right)}{9+\sqrt{65}} = \frac{65(4+\sqrt{65})}{\sqrt{65}(9+\sqrt{65})}$$

$$b^2 = \frac{(260+65\sqrt{65})}{(65+9\sqrt{65})} = \frac{21125-1885\sqrt{65}}{1040} = \frac{325-29\sqrt{65}}{16}$$

$$b=\sqrt{\frac{325-29\sqrt{65}}{16}}$$

$$\sin^2(\alpha) = \left(\frac{d_2\sin(A_1)}{b}\right)^2 = \frac{d_2^2\sin^2(A_1)}{b^2}$$ (law of sines)

$$\sin^2(\alpha)=\frac{13\left(\frac{49}{130}\right)}{\frac{325-29\sqrt{65}}{16}} = \left(\frac{49}{10}\right)\left(\frac{16}{325-29\sqrt{65}}\right)=\frac{392}{1625-145\sqrt{65}}$$

$$\sin^2(\alpha)=\frac{637000+56840\sqrt{65}}{1274000}=\frac{1}{2}+\frac{29\sqrt{65}}{650}$$

$$\sin(\alpha)=\sqrt{\frac{1}{2}+\frac{29\sqrt{65}}{650}}$$

Rational Approach

$$\frac{\sin^2\left(\frac{A_3}{2}\right)}{P_1}=\frac{S_1}{B}$$ and $$\frac{\sin^2\left(\frac{A_3}{2}\right)}{P_2}=\frac{S_2}{B}$$

therefore

$$\frac{P_2}{P_1}=\frac{S_1}{S_2}=\frac{\frac{49}{130}}{\frac{49}{50}}=\frac{5}{13}$$ Eq 1.

$$(P_1+P_2-10)^2=4P_1P_2$$

$$\frac{(P_1+P_2-10)^2}{P_1^2}=\frac{4P_1P_2}{P_1^2}$$

$$\left(1+\frac{P_2}{P_1}-\frac{10}{P_1}\right)^2=\frac{4P_2}{P_1}$$

$$\left(1+\frac{5}{13}-\frac{10}{P_1}\right)^2=\frac{4\cdot 5}{13}$$

$$\left(\frac{18P_1}{13}-\frac{10}{1}\right)^2=P_1^2\left(\frac{20}{13}\right)$$

$$\frac{(18P_1-130)^2}{169}=P_1^2\left(\frac{20}{13}\right)$$

$$(18P_1-130)^2=260P_1^2$$

$$16P_1^2-1170P_1+4225=0$$

$$P_1=\frac{585\pm 65\sqrt{65}}{16}$$

substituting P_1 into Eq 1.

$$P_2=\frac{5(585\pm 65\sqrt{65})}{16\cdot 13}=\frac{5(585\pm 65\sqrt{65})}{208}$$

use the cross laws on the bisected angle

$$(B-P_1-D_2)^2=4P_1D_2(1-S_1)$$ Eq 2

$$(B-P_2-D_1)^2=4P_2D_1(1-S_2)$$ Eq 3

or

$$\left(B-\left(\frac{585-65\sqrt{65}}{16}\right)-13\right)^2=4\left(\frac{585-65\sqrt{65}}{16}\right)\cdot 13\left(1-\frac{49}{130}\right)$$ Eq 2a

$$\left(B-5\left(\frac{585-65\sqrt{65}}{16\cdot 13}\right)-5\right)^2=20\left(\frac{585-65\sqrt{65}}{16\cdot 13}\right)\cdot 5\left(1-\frac{49}{50}\right)$$ Eq 3a

or

$$B^2+B\left(\frac{65}{8}\sqrt{65}-\frac{793}{8}\right)+\frac{300105}{128}-\frac{34697}{128}\sqrt{65}=0$$ Eq 2b

$$B^2+B\left(\frac{25}{8}\sqrt{65}-\frac{308}{8}\right)+\frac{66105}{128}-\frac{7545}{128}\sqrt{65}=0$$ Eq 3b

take the difference of Eq 2b and Eq 3b

$$B(61-5\sqrt{65})-\frac{14625}{8}+\frac{1697}{8}\sqrt{65}=0$$

$$B=\frac{325-29\sqrt{65}}{16}$$

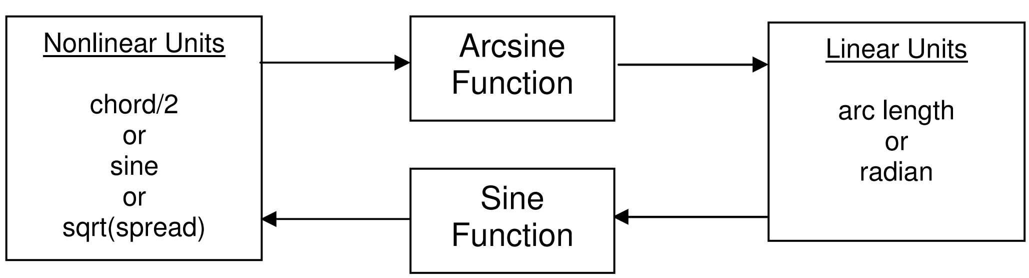
	$\frac{\sin^2(\alpha)}{D_2} = \frac{\sin^2(A_1)}{B}$ (spread law) $\frac{\sin^2(\alpha)}{13} = \frac{49}{130B}$ $\sin^2(\alpha) = \frac{49}{10}\left(\frac{325}{16} - \frac{29}{16}\sqrt{65}\right)^{-1} = \frac{1}{2} + \frac{29\sqrt{65}}{650}$
If a radian solution is required, then both methods need to perform these additional calculations:	
$\alpha = \arcsin\left(\sqrt{\frac{1}{2} + \frac{29\sqrt{65}}{650}}\right)$	$\sin(\alpha) = \sqrt{\frac{1}{2} + \frac{29\sqrt{65}}{650}}$ $\alpha = \arcsin\left(\sqrt{\frac{1}{2} + \frac{29\sqrt{65}}{650}}\right)$

Note: In Wildberger's paper, the classical approach found angle A_2CA_1, and the rational approach found A_3CA_1.

Teaching Rational Trigonometry. Aside from the claim that rational trigonometry is easier and produces rational results that are more accurate than classical trigonometry, Wildberger also believes that rational trigonometry is more intuitive for students and it resolves the circular logic students are taught. The circular logic begins when students are told they need to learn calculus before they can understand the trigonometric functions.

It is debatable that students will find quadrances and spreads to be more intuitive than lengths and angles. It's normal for the student to use distance to measure the separation between two points, and so it's intuitive to use this same concept for measuring the separation between two lines by using a chord (twice the sine). Unlike quadrances and spreads, distances and chords can be easily illustrated geometrically, and most students already have a foundation in geometry and linear measurements prior to taking trigonometry.

Contrary to what most educators and students believe (or were taught), the trigonometric functions can be understood without first learning calculus. Contrasting Ptolemy's method for producing trigonometric tables, a geometrically derived algorithm exists that can easily produce the sine for any given angle between 0 and pi/2. A similar geometrically derived algorithm exists for the arcsine function. These algorithms provide the means to illustrate to the student how a nonlinear chord (or sine) is converted to a linear arc length (radian angle), and back again. Unfortunately, few educators use this approach, relying instead on power series that are derived by calculus, which the student has no knowledge.



Finally, Wildberger submits that students be first taught rational trigonometry, then at some later time be taught parts of classical trigonometry to fill in the *missing* trigonometric theory (e.g., circles, circular rotation, etc.). The impact this will have on the teaching of science courses that depend on linear units of measure has yet to be fully realized. It may introduce the same circular logic in these courses that rational trigonometry is trying to eliminate in mathematics. That is, a topic is introduced, but students can't fully understand or apply it because they haven't yet learned the *missing* theory. For some, it may seem they are unlearning what they had previously learned in elementary school, namely that measuring the separation of lines (angle) is additive, and relearning that the separation of lines (spread) is neither additive nor proportional, and a whole series of calculations using the Triple Spread and the Quadratic formulas are required before two spreads can be added or subtracted.

Although Wildberger may very well be correct in stating the way trigonometry is taught is wrong, it's a mistake to say classical trigonometry is the cause or that rational trigonometry is a better alternative. Educators simply must change the way they teach trigonometry, not replace it with a nonlinear theory that is incompatible with our linear system of measurement, has a limited application (e.g., mainly triangles), usually involves more calculations, may be less intuitive, and yet still requires the student to learn all or portions of the classical theory.