Set Theory: Should You Believe?

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"I protest against the use of infinite magnitude as something completed, which is never permissible in mathematics. Infinity is merely a way of speaking, the true meaning being a limit which certain ratios approach indefinitely close, while others are permitted to increase without restriction." (Gauss)

"I don't know what predominates in Cantor's theory - philosophy or theology, but I am sure that there is no mathematics there." (Kronecker)

"...classical logic was abstracted from the mathematics of finite sets and their subsets...Forgetful of this limited origin, one afterwards mistook that logic for something above and prior to all mathematics, and finally applied it, without justification, to the mathematics of infinite sets. This is the Fall and original sin of [Cantor's] set theory ..." (Weyl)

Modern mathematics as religion

Modern mathematics *doesn't make complete sense*. The unfortunate consequences include difficulty in deciding what to teach and how to teach it, many papers that are logically flawed, the challenge of recruiting young people to the subject, and an unfortunate teetering on the brink of irrelevance.

If mathematics made complete sense it would be a lot easier to teach, and a lot easier to learn. Using flawed and ambiguous concepts, hiding confusions and circular reasoning, pulling theorems out of thin air to be justified 'later' (i.e. never) and relying on appeals to authority don't help young people, they make things more difficult for them.

If mathematics made complete sense there would be higher standards of rigour, with fewer but better books and papers published. That might make it easier for ordinary researchers to be confident of a small but meaningful contribution. If mathematics made complete sense then the physicists wouldn't have to thrash around quite so wildly for the right mathematical theories for quantum field theory and string theory. Mathematics that makes complete sense tends to parallel the real world and be highly relevant to it, while mathematics that doesn't make complete sense rarely ever hits the nail right on the head, although it can still be very useful.

So where exactly are the logical problems? The troubles stem from the *consistent refusal by the Academy to get serious about the foundational aspects of the subject*, and are augmented by the twentieth centuries' *whole hearted and largely uncritical embrace of Set Theory*.

Most of the problems with the foundational aspects arise from mathematicians' erroneous belief that they properly understand the content of public school and high school mathematics, and that further clarification and codification is largely unnecessary. Most (but not all) of the difficulties of Set Theory arise from the insistence that there exist 'infinite sets', and that it is the job of mathematics to study them and use them.

In perpetuating these notions, modern mathematics takes on many of the aspects of a *religion*. It has its essential creed—namely Set Theory, and its unquestioned assumptions, namely that mathematics is based on 'Axioms', in particular the Zermelo-Fraenkel 'Axioms of Set Theory'. It has its anointed priesthood, the *logicians*, who specialize in studying the *foundations of mathematics*, a supposedly deep and difficult subject that requires years of devotion to master. Other mathematicians learn to invoke the official mantras when questioned by outsiders, but have only a hazy view about how the elementary aspects of the subject hang together logically.

Training of the young is like that in secret societies—immersion in the cult involves intensive undergraduate memorization of the standard thoughts before they are properly understood, so that comprehension often follows belief instead of the other (more healthy) way around. A long and often painful graduate school apprenticeship keeps the cadet busy jumping through the many required hoops, discourages critical thought about the foundations of the subject, but then gradually yields to the gentle acceptance and support of the brotherhood. The ever-present demons of inadequacy, failure and banishment are however never far from view, ensuring that most stay on the well-trodden path.

The large international conferences let the fellowship gather together and congratulate themselves on the uniformity and sanity of their world view, though to the rare outsider that sneaks into such events the proceedings no doubt seem characterized by jargon, mutual incomprehensibility and irrelevance to the outside world. The official doctrine is that all views and opinions are valued if they contain truth, and that ultimately only elegance and utility decide what gets studied. The reality is less ennobling—the usual hierarchical structures reward allegiance, conformity and technical mastery of the doctrines, elevate the interests of the powerful, and discourage dissent.

There is no evil intent or ugly conspiracy here—the practice is held in place by a mixture of well-meaning effort, inertia and self-interest. We humans have a fondness for believing what those around us do, and a willingness to mold our intellectual constructs to support those hypotheses which justify our habits and make us feel good.

The problem with foundations

The reason that mathematics doesn't make complete sense is quite easy to explain when we look at it from the educational side. Mathematicians, like everyone else, begin learning mathematics before kindergarten, with counting and basic shapes. Throughout the public and high school years (K-12) they are exposed to a mishmash of subjects and approaches, which in the better schools or with the better teachers involves learning about numbers, fractions, arithmetic, points, lines, triangles, circles, decimals, percentages, congruences, sets, functions, algebra, polynomials, parabolas, ellipses, hyperbolas, trigonometry, rates of change, probabilities, logarithms, exponentials, quadrilaterals, areas, volumes, vectors and perhaps some calculus. The treatment is non-rigorous, inconsistent and even sloppy. The aim is to get the average student through the material with a few procedures under their belts, not to provide a proper logical framework for those who might have an interest in a scientific or mathematical career.

In the first year of university the student encounters calculus more seriously and some linear algebra, perhaps with some discrete mathematics thrown in. Sometime in their second or third year, a dramatic change happens in the training of aspiring pure mathematicians. They start being introduced to the idea of *rigorous thinking* and *proofs*, and gradually become aware that they are not at the peak of intellectual achievement, but just at the foothills of a very onerous climb. Group theory, differential equations, fields, rings, topological spaces, measure theory, operators, complex analysis, special functions, manifolds, Hilbert spaces, posets and lattices—it all piles up quickly. They learn to think about mathematics less as a jumble of facts to be memorized and algorithms to be mastered, but as a coherent logical structure. Assignment problems increasingly require serious thinking, and soon all but the very best are brain-tired and confused.

Do you suppose the curriculum at this point has time or inclination to return to the material they learnt in public school and high school, and finally organize it properly? When we start to get really picky about logical correctness, doesn't it make sense to go back and ensure that all those subjects that up to now have only been taught in a loose and cavalier fashion get a proper rigorous treatment? Isn't this the appropriate time to finally learn what a *number* in fact is, why exactly the *laws of arithmetic hold*, what the *correct definitions of a line and a circle are*, what we mean by a *vector*, a *function*, an *area* and all the rest? You might think so, but there are two very good reasons why this is nowhere done.

The first reason is that even the professors mostly don't know! They too have gone through a similar indoctrination, and never had to *prove* that multiplication is associative, for example, or learnt what is the right order of topics in trigonometry. Of course they know how to solve all the problems in elementary school texts, but this is quite different from being able to correct all the logical defects contained there, and give a complete and proper exposition of the material.

The modern mathematician walks around with her head full of the tight logical relationships of the specialized theories she researches, with only a rudimentary understanding of the logical foundations underpinning the entire subject. But the worst part is, she is largely unaware of this inadequacy in her training. She and her colleagues *really do* believe they profoundly understand elementary mathematics. But a few well-chosen questions reveal that this is not so. Ask them just *what a fraction is*, or how to properly *define an angle*, or whether a *polynomial* is really a *function* or not, and see what kind of non-uniform rambling emerges! The more elementary the question, the more likely the answer involves a lot of philosophizing and bluster. The issue of the correct approach to the definition of a fraction is a particularly crucial one to public school education.

Mathematicians like to reassure themselves that foundational questions are resolved by some mumbo-jumbo about 'Axioms' (more on that later) but in reality successful mathematics requires familiarity with a large collection of 'elementary' concepts and underlying linguistic and notational conventions. These are often unwritten, but are part of the training of young people in the subject. For example, an entire essay could be written on the use, implicit and explicit, of *ordering* and *brackets* in mathematical statements and equations. Codifying this kind of *implicit syntax* is a job professional mathematicians are not particularly interested in.

The second reason is that any attempt to lay out elementary mathematics properly would be resisted by both students and educators as not going forward, but backwards. Who wants to spend time worrying about the correct approach to polynomials when *Measure theory* and the *Residue calculus* beckon instead? The consequence is that a large amount of elementary mathematics is never properly taught anywhere.

But there are two foundational topics that are introduced in the early undergraduate years: *infinite set theory* and *real numbers*. Historically these are very controversial topics, fraught with logical difficulties which embroiled mathematicians for decades. The presentation these days is matter of fact—'an infinite set is a collection of mathematical objects which isn't finite' and 'a real number is an equivalence class of Cauchy sequences of rational numbers'.

Or some such nonsense. Set theory as presented to young people *simply doesn't make sense*, and the resultant approach to real numbers is in fact *a joke*! You heard it correctly—and I will try to explain shortly. The point here is that these logically dubious topics are slipped into the curriculum in an off-hand way when students are already overworked and awed by all the other material before them. There is not the time to ruminate and discuss the uncertainties of generations gone by. With a slick enough presentation, the whole thing goes down just like any other of the subjects they are struggling to learn. From then on till their retirement years, mathematicians have a busy schedule ahead of them, ensuring that few get around to critically examining the subject matter of their student days.

Infinite sets

I think we can agree that (finite) set theory is understandable. There are many examples of (finite) sets, we know how to manipulate them effectively, and the theory is useful and powerful (although not as useful and powerful as it should be, but that's a different story).

So what about an '*infinite set*'? Well, to begin with, you should say precisely what the term means. Okay, if *you* don't, at least *someone* should. Putting an adjective in front of a noun does not in itself make a mathematical concept. Cantor declared that an '*infinite set*' is *a set which is not finite.* Surely that is unsatisfactory, as Cantor no doubt suspected himself. It's like declaring that an '*all-seeing Leprechaun*' is a Leprechaun which can see everything. Or an '*unstoppable mouse*' is a mouse which cannot be stopped. These grammatical constructions do not create concepts, except perhaps in a literary or poetic sense. It is not clear that there *are* any sets that are not finite, just as it is not clear that there *are* any Leprechauns which can see everything, or that there *are* mice that cannot be stopped. Certainly in science there is no reason to suppose that 'infinite sets' exist. Are there an infinite number of quarks or electrons in the universe? If physicists had to hazard a guess, I am confident the majority would say: No. But even if there *were* an infinite number of electrons, it is unreasonable to suppose that you can get an infinite number of them all together as a single 'data object'.

The dubious nature of Cantor's definition was spectacularly demonstrated by the contradictions in 'infinite set theory' discovered by Russell and others around the turn of the twentieth century. Allowing any old 'infinite set' à la Cantor allows you to consider the 'infinite set' of 'all infinite sets', and this leads to a self-referential contradiction. How about the 'infinite sets' of 'all finite sets', or 'all finite groups', or perhaps 'all topological spaces which are homeomorphic to the sphere'? The paradoxes showed that unless you are very particular about the exact meaning of the concept of 'infinite set', the theory collapses. Russell and Whitehead spent decades trying to formulate a clear and sufficiently comprehensive framework for the subject.

Let me remind you that mathematical theories are not in the habit of collapsing. We do not routinely say, "Did you hear that Pseudo-convex cohomology theory collapsed last week? What a shame! Such nice people too."

So did analysts retreat from Cantor's theory in embarrassment? Only for a few years, till Hilbert rallied the troops with his battle-cry "No one shall expel us from the paradise Cantor has created for us!" To which Wittgenstein responded "If one person can see it as a paradise for mathematicians, why should not another see it as a joke?"

Do modern texts on set theory bend over backwards to say precisely *what is and what is not an infinite set*? Check it out for yourself—I cannot say that I have found much evidence of such an attitude, and I have looked. Do those students learning 'infinite set theory' for the first time wade through *The Principia*? Of course not, that would be too much work for them and their teachers, and would dull that pleasant sense of superiority they feel from having finally 'understood the infinite'.

The bulwark against such criticisms, we are told, is having the appropriate collection of 'Axioms'! It turns out, completely against the insights and deepest intuitions of the greatest mathematicians over thousands of years, that it all comes down to *what you believe*. Fortunately what we as good modern mathematicians believe has now been encoded and deeply entrenched in the '*Axioms of Zermelo–Fraenkel*'. Although there was quite a bit of squabbling about this in the early decades of the last century, nowadays there are only a few skeptics. We mostly attend the same church, dutifully repeat the same incantations, and insure our students do the same.

Let us have a look at these 'Axioms', these bastions of modern mathematics. In what follows, X and Y are 'sets'.

1. Axiom of Extensionality: If X and Y have the same elements, then $X = Y$.
2. Axiom of the Unordered Pair: For any a and b there exists a set $\{a, b\}$ that contains exactly a and b.
3. Axiom of Subsets: If ϕ is a property (with parameter p), then for any X and p there exists a set $Y = \{u \in X : \phi(u, p)\}$ that contains all those u in X that have the property ϕ.
4. Axiom of Union: For any X there exists a set $Y = \cup X$, the union of all elements of X.
5. Axiom of the Power Set: For any X there exists a set $Y = P(X)$, the set of all subsets of X.
6. Axiom of Infinity: There exists an infinite set.
7. Axiom of Replacement: If F is a function, then for any X there exists a set $Y = F[X] = \{F(x) : x \in X\}$.
8. Axiom of Foundation: Every nonempty set has a minimal element, that is one which does not contain another in the set.
9. Axiom of Choice: Every family of nonempty sets has a choice function, namely a function which assigns to each of the sets one of its elements.

All completely clear? This *sorry list of assertions* is, according to the majority of mathematicians, the proper foundation for set theory and modern mathematics! Incredible!

The 'Axioms' are first of all unintelligible unless you are already a trained mathematician. Perhaps you disagree? Then I suggest an experiment—inflict this list on a random sample of educated non-mathematicians and see if they buy—or even understand—any of it. However even to a mathematician it should be obvious that these statements are awash with difficulties. What is a *property*? What is a *parameter*? What is a *function*? What is a *family of sets*? Where is the explanation of what all the symbols mean, if indeed they have any meaning? How many further assumptions are hidden behind the syntax and logical conventions assumed by these postulates?

And Axiom 6: *There is an infinite set!?* How in heavens did this one sneak in here? One of the whole points of Russell's critique is that one must be extremely careful about what the words 'infinite set' denote. One might as well declare that: *There is an all-seeing Leprechaun*! or *There is an unstoppable mouse*!

Just to get you thinking about whether in fact *you* understand the *'*Axioms*'*, *consider* the set

$$A = \{a\}\,.$$

As we do. Please stop reading for a moment, and just *consider* this set.

Thanks for considering it. Ah, but someone has a question! Yes? You would like to know what a is? Very well, I will tell you. I am not sure if I am legally obligated to (am I?), but I will tell you anyway—a is itself a set, also a very simple one, with just two elements, called a_1 and a_2. Thus

$$a = \{a_1, a_2\}\,.$$

Can we move on now? Wait, someone insists on knowing: what are a_1 and a_2? They are also sets, also with two elements each, so that

$$\begin{aligned} a_1 &= \{a_{11}, a_{12}\} \\ a_2 &= \{a_{21}, a_{22}\}\,. \end{aligned}$$

And, before you ask, each of the elements a_{11}, a_{12}, a_{21} and a_{22} is itself a set, with also exactly two elements. Does the pattern continue? Suppose it *does*, would that make A legitimate? But suppose it *doesn't*, and that I refuse to reveal a pattern, perhaps because non exists. In modern mathematics we are allowed to consider patterns that do not have any pattern to them. In such a case does A still exist? Does it exist if I invoke some appropriate new 'Axiom'?

The Zermelo-Fraenkel 'Axioms' are but the merry beginning of a zoo of possible starting points for mathematics, according to modern practitioners. The 'Axiom of Choice' has numerous variants. There is the 'Axiom of Countable Choice'. 'The Axiom of Dependent Choice'. There is the 'Axiom that all subsets of $\mathbb{R}$ are Lebesgue measurable' (which contradicts the 'Axiom of Choice'). Not to mention all the higher possible axioms concerned with large cardinals. You can mix and match as you please.

I have been a working mathematician for more than twenty years, and none of this resembles in any way, shape, or form the subject as I have come to experience it. In my studies of Lie theory, hypergroups and geometry, there has never been a point at which I have pondered—should I assume *this postulate* about the mathematical world, or *that postulate*? Of course one makes decisions all the time about which *definitions* to focus on, but the nature of the mathematical world that I investigate appears to me to be absolutely fixed. Either G_2 has an eleven dimensional irreducible representation or it doesn't (in fact it doesn't). My religious/ philosophical/Axiomatic position has nothing to do with it. So I am confident that a view of mathematics as swimming ambiguously on a sea of potential Axiomatic systems strongly misrepresents the practical reality of the subject.

Does mathematics require axioms?

Occasionally logicians inquire as to whether the current 'Axioms' need to be changed further, or augmented. The more fundamental question—*whether mathematics requires any Axioms*—is not up for discussion. That would be like trying to get the high priests on the island of Okineyab to consider not whether the Divine Ompah's Holy Phoenix has twelve or thirteen colours in her tail (a fascinating question on which entire tomes have been written), but rather whether the Divine Ompah exists at all. Ask *that* question, and icy stares are what you have to expect, then it's off to the dungeons, mate, for a bit of retraining.

Mathematics does not require 'Axioms'. The job of a pure mathematician is not to build some elaborate castle in the sky, and to proclaim that it stands up on the strength of some arbitrarily chosen assumptions. The job is to *investigate the mathematical reality of the world in which we live.* For this, no assumptions are necessary. Careful observation is necessary, clear definitions are necessary, and correct use of language and logic are necessary. But at no point does one need to start invoking the existence of objects or procedures that we cannot see, specify, or implement.

The difficulty with the current reliance on 'Axioms' arises from a grammatical confusion, along with the perceived need to have some (any) way to continue certain ambiguous practices that analysts historically have liked to make. People use the term 'Axiom' when often they really mean *definition.* Thus the 'axioms' of group theory are in fact just definitions. We say exactly what we mean by a group, that's all. There are no assumptions anywhere. At no point do we or should we say, 'Now that we have defined an abstract group, let's assume they exist'. Either we can demonstrate they exist by constructing some, or the theory becomes vacuous. Similarly there is no need for 'Axioms of Field Theory', or 'Axioms of Set theory', or 'Axioms' for any other branch of mathematics—or for mathematics itself!

Euclid may have called certain of his initial statements Axioms, but he had something else in mind. Euclid had a lot of geometrical facts which he wanted to organize as best as he could into a logical framework. Many decisions had to be made as to a convenient order of presentation. He rightfully decided that simpler and more basic facts should appear before complicated and difficult ones. So he contrived to organize things in a linear way, with most Propositions following from previous ones by logical reasoning alone, with the exception of *certain initial statements* that were taken to be self-evident. To Euclid, *an Axiom was a fact that was sufficiently obvious to not require a proof.* This is a quite different meaning to the use of the term today. Those formalists who claim that they are following in Euclid's illustrious footsteps by casting mathematics as a game played with symbols which are not given meaning are misrepresenting the situation.

We have politely swallowed the standard gobble dee gook of modern set theory from our student days—around the same time that we agreed that there most certainly *are* a whole host of 'uncomputable real numbers', even if you or I will never get to meet one, and yes, there no doubt *is* a non-measurable function, despite the fact that no one can tell us what it is, and yes, there surely *are* non-separable Hilbert spaces, only we can't specify them all that well, and it surely *is* possible to dissect a solid unit ball into five pieces, and rearrange them to form a solid ball of radius two.

And yes, all right, the Continuum hypothesis doesn't really need to be true or false, but is allowed to hover in some no-man's land, falling one way or the other depending on *what you believe.* Cohen's proof of the independence of the Continuum hypothesis from the 'Axioms' should have been the long overdue wake-up call. In ordinary mathematics, statements are either true, false, or they don't make sense. If you have an elaborate theory of 'hierarchies upon hierarchies of infinite sets', in which you cannot *even in principle* decide whether there is anything between the first and second 'infinity' on your list, *then it's time to admit that you are no longer doing mathematics.*

Whenever discussions about the foundations of mathematics arise, we pay lip service to the 'Axioms' of Zermelo-Fraenkel, but do we every use them? Hardly ever. With the notable exception of the 'Axiom of Choice', I bet that fewer than 5% of mathematicians have ever employed even one of these 'Axioms' explicitly in their published work. The average mathematician probably can't even remember the 'Axioms'. I think I am typical—in two weeks time I'll have retired them to their usual spot in some distant ballpark of my memory, mostly beyond recall.

In practise, working mathematicians are quite aware of the lurking contradictions with 'infinite set theory'. We have learnt to keep the demons at bay, not by relying on 'Axioms' but rather by developing conventions and intuition that allow us to seemingly avoid the most obvious traps. Whenever it smells like there may be an 'infinite set' around that is problematic, we quickly use the term 'class'. For example: A topology is an 'equivalence class of atlases'. Of course most of us could not spell out exactly what does and what does not constitute a 'class', and we learn to not bring up such questions in company.

There is also the useful term 'category'. Consider the 'category of all finite groups'. Given any set a whatsoever, I can create a one element set $A = \{a\}$ whose single element is a. Then I can define A to be a group, by defining $a \cdot a = a$. Thus for every set a, there is a group with one element which determines a. So if you believe that the 'set of all sets' doesn't make good sense, then how can the 'category of all finite groups' be any better? Do category theorists begin their lectures to the rest of us with a quick primer as to what the term 'category' might precisely mean? Does the audience get nervous not knowing? Back in the good old nineteenth century they probably did, but nowadays those who attend research seminars regularly are quite used to taking for granted abstractions that they feel incapable of understanding.

Another good example arises from the usual definition of a *function.* Although the official doctrine is that a function is prescribed by a domain (a set) and a codomain (a set) as well as a rule that tells us what do with an element of the former to get an element of the latter, we know that in practice the domain and codomain can be dispensed with in shady circumstances, or the term can be replaced by the somewhat more flexible 'functor', particularly in category theory. To illustrate—when we define the *fundamental group* $\pi(X)$ of a topological space X, we instinctively know that it is better not to write

$$\pi : \text{Top} \rightarrow \text{Group}$$

because chances are Top and Group are not 'properly defined infinite sets'. We just employ the everyday understanding of a function, namely that it suffices to say what *kind* of an object it inputs, what *kind* of an object it outputs, and what it *does* precisely to an input to get an output. No need to have all the possible inputs and outputs arranged in front of us neatly as two sets. This kind of understanding can be usefully extended to many more mundane situations. Do you really think you need to have all the natural numbers together in a set to define the function $f(n) = n^2 + 1$ on natural numbers? Of course not—the *rule* itself, together with the specification of the *kinds* of objects it inputs and outputs is enough. As computer scientists already know.

Why real numbers are a joke

According to the status quo, the continuum is properly modelled by the 'real numbers'. What is a real number? Let's start with an easier question: What is a rational number? Here comes set theory to our aid. It is, according to some accounts, nothing but an *equivalence class of ordered pairs of integers*. Thus when my six year old daughter uses the fraction 2/3, what she is *really* doing is using the 'equivalence class'

$$2/3 = \{[2,3], [4,6], [-22,-33], [14,21], [86,129], \cdots\}.$$

Good grief. But let's carry on. A *Cauchy sequence* of rational numbers is a sequence

$$\lambda = [r_1, r_2, r_3, \cdots]$$

where each of the r_i is a rational number (of the kind just mentioned) with the property that for all $\varepsilon > 0$ there exists a natural number N such that if n and m are bigger than N, then

$$|r_n - r_m| \leq \varepsilon.$$

But here is a *very important point*: we are *not* obliged, in modern mathematics, to actually have a *rule or algorithm* that specifies the sequence $r_1, r_2, r_3, \cdots$. In other words, 'arbitrary' sequences are allowed, as long as they have the Cauchy convergence property. This removes the obligation to specify concretely the objects which you are talking about. Sequences generated by algorithms can be specified by those algorithms, but what possibly could it mean to discuss a 'sequence' which is not generated by such a finite rule? Such an object would contain an 'infinite amount' of information, and there are no concrete examples of such things in the known universe. This is metaphysics masquerading as mathematics.

To get you used to the modern magic of Cauchy sequences, here is one I just made up:

$$\mu = [2/3, 2/3, 2/3, 2/3, 2/3, 2/3, \cdots] \,.$$

Anyone want to guess what the limit is? Oh, you want some more information first? The initial billion terms are all 2/3. Now would you like to guess? No, you want more information. All right, the billion and first term is 475. Now would you like to guess? No, you want more information. Fine, the next trillion terms are all 2/3. You are getting tired of asking for more information, so you want me to tell you the pattern once and for all? Ha Ha! Modern mathematics doesn't require it! There doesn't *need* to be a pattern, and in this case, there isn't, because I say so. You are getting tired of this game, so you guess 2/3? Good effort, but sadly you are wrong. The actual answer is -17. That's right, after the first trillion and billion and one terms, the entries start doing *really* wild and crazy things, which I don't need to describe to you, and then 'eventually' they start heading towards -17, but how they do so and at what rate is not known by anyone. Isn't modern religion fun?

So now what is a real number? It is an equivalence class of Cauchy sequences! That's right, not just one, not just two, but an entire equivalence class of them. We can't even list the elements of such a 'class', since each and every one of them contains an 'uncountable' number of Cauchy sequences. So of course we have already absorbed the 'infinite set theory' to make sense of these statements, and we still ought to 'explain' the equivalence relation. Let's forego that, and just present a representative example. Here is a real number, where I have saved considerable space by not presenting rational numbers in their full glory:

$$\{[2/3, -14, 1/3, 2/3, \cdots], [4/9, 4, -4/17, 2458, \cdots], \; [78, 2/29, 3, 4, 5/3, \cdots], \; \cdots\} \,.$$

Like to guess what real number this is? You're right! It is $5\pi + e$. However did you know?

Now that you are comfortable with the definition of real numbers, perhaps you would like to know how to do arithmetic with them? How to add them, and multiply them? And perhaps you might want to check that once you have defined these operations, they obey the properties you would like, such as associativity etc. Well, all I can say is—good luck. If you write this all down coherently, you will certainly be the first to have done so. On top of the manifold ugliness and complexity of the situation, you will be continually dogged by the difficulty that in all these sequences there does *not have to be a pattern*—they are allowed to be completely 'arbitrary'. That means you are unable to say when two given real numbers are the same, or when a particular arithmetical statement involving real numbers is correct. Even a simple statement like $1 + 1 = 2$ will cause you consternation, since you have to phrase everything in terms of unending Cauchy sequences, and in the absence of solid conventions for specifying infinite sequences, you will wrestle with the question of whether the Cauchy sequence $[1, 1, 1, \cdots]$ really does represent 1, or perhaps just appears to from this end of things.

Perhaps you would like to consult the usual 'Constructing the Real Numbers' section in your favourite calculus text instead. Have a look, and see what passes for logical thinking in modern mathematics education. Then to really sink your spirits, open up a 'rigorous' analysis text, and thumb through to the critical section where they *explain the continuum—exactly what a real number is and how one operates with them.* This is the heart of the matter—the bedrock on which modern analysis is built. And in all such books, waffling and ambiguity is what you find, unless the subject is passed over altogether. Some of them are honest about it. Others cleverly confuse the issue by allowing talk about 'sets' of rational numbers without any mention of how you actually *specify* them. It is in this gap that the logical difficulties lurk. A set of rational numbers is essentially a sequence of zeros and ones, and such a sequence is specified properly when you have a finite function or computer program which generates it. Otherwise 'it' is not accessible in a finite universe.

This critical issue of describing the points on the continuum *should* have a strong connection with notions of *computability*, but it turns out, according to the standard dogma, that computable real numbers are just a 'measure zero slice' of 'all real numbers'. Despite the fact that neither you nor anyone else has been able to write down a single 'non-computable real number' and the undeniable fact that they never play the slightest role in any actual scientific, engineering or applied mathematical calculation.

Even the 'computable real numbers' are quite misunderstood. Most mathematicians reading this paper suffer from the impression that the 'computable real numbers' are *countable*, and that they are *not complete.* As I mention in my recent book, this is quite wrong. Think clearly about the subject for a few days, and you will see that the computable real numbers are *not countable*, and are *complete.* Think for a few more days, and you will be able to see how to make these statements without any reference to 'infinite sets', and that this suffices for Cantor's proof that not all irrational numbers are algebraic.

When it comes to foundational issues, *modern analysis is off in la-la land.*

But what about the natural numbers?

Okay, you say, perhaps you have a bit of a point here, but surely you are going too far in denouncing infinite sets altogether. After all, there is one infinite set that we can be *absolutely sure of*, one that is so familiar, so cut and dried, it is beyond reproach. What about—*the set* $\mathbb{N}$ *of all natural numbers* you ask?? Have a look, here it is in its glorious entirety:

$$\mathbb{N} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, \cdots\}.$$

Well, perhaps not in its entirety, but we all know what those three little dots represent, don't we? All the rest of those numbers, squeezed in between the 18 and the right bracket!

The ancient Greeks believed that the natural numbers are not finite, but that didn't mean they agreed that you could *put them all together to form a well-*

defined mathematical object. A finite set we can describe explicitly and specify completely—we can list all its elements so there is no possible ambiguity. But the question is—are we allowed to state that *all* of the natural numbers are collectible into one big set?

Some will argue that a mathematician can do whatever she likes, as long as a logical contradiction doesn't result. But things are not so simple. Are we allowed to introduce *all-seeing Leprechauns* into mathematics as long as they seem to behave themselves and not cause contradictions? A far better approach to create beautiful and useful mathematics is to ensure that all basic concepts are *entirely clear and straightforward right from the start.* The onus is on us to demonstrate that our notions make sense, instead of challenging someone else to find a contradiction.

We'll now see that the concept of the 'set of natural numbers' is neither clear nor straightforward, but immersed in complexity. The difficulties start when we leave the familiar and comfortable domain of microscopic natural numbers, and start pushing on through the sequence in an effort to write down bigger and bigger numbers. Pretty soon expressing numbers in decimal form like

$$a = 23,518,800,234,444,511,009$$

gets uneconomical, and it is easier to use exponential notation. Iteration allows us to write a tower of three tens:

$$b = 10^{10^{10}}.$$

Let's keep on going, and write down the number

$$c = 10^{10^{\cdot^{\cdot^{\cdot^{10}}}}} \}10^{10^{10}}$$

where the tower of exponents on the left has altogether b tens. My guess is that c is already bigger than any number ever used (meaningfully) in mathematics or science, but I could be wrong. In any case, it's still early days in our exploration of $\mathbb{N}$, as we've only been at it for five minutes. How about

$$d = 10^{10^{\cdot^{\cdot^{\cdot^{10}}}}} \}10^{10^{\cdot^{\cdot^{\cdot^{10}}}}} \}\cdots 10^{10^{\cdot^{\cdot^{\cdot^{10}}}}} \}10^{10^{10}}$$

where the number of brackets is c? Please think about this number for a few minutes. This should not be too much of a burden on you, since you routinely bandy the set $\mathbb{N}$ of *all* natural numbers about.

Next we could introduce e, then f, then some suitable iterate of iterates, say a_1, then b_1, then eventually a_2 and so on, and so on, constrained only by the limits of our imaginations, and the amount of writing paper at our disposal. Assuming our imaginations are not a problem, there is the issue of space, for as we keep going and keep going, we are going to start running out of memory space to write down our increasingly large numbers. First they will fill a page, then a book, then our hard drives. Of course we can make our computers bigger and our coding more efficient, start dismantling stars and spreading our memory banks across galaxies. But... *the universe is almost certainly finite*. Eventually, you and I may have vaporized and rearranged all the stars, furniture and other creatures in our quest to write down yet bigger numbers, and now we are starting to run out of particles with which to extend our galactic hard drive. Suppose you reduce me to atoms in the interests of science, and perhaps your outer extremities too. At some point, you are going to write down a number so vast that it requires all the particles of the universe (except for some minimal amount of what's left of you). May I humbly suggest you call this number w, in honour of the last person you vaporized to create it?

Now here is a dilemma. Once you have written down and marvelled at w in all its glory, where are you going to find $w + 1$? From this end of things—the working end—the endless sequence of natural numbers does not appear either natural nor endless. And where is the infinite set $\mathbb{N}$?

The answer is—nowhere. It doesn't exist. It is a convenient metaphysical fiction that allows mathematicians to be sloppy in formulating various questions and arguments. It allows us to avoid issues of specification and replace concrete understandings with woolly abstractions. What seems to be a happy and well behaved sequence when viewed from the beginning is more like an enormous fractal when viewed from the other end.

Unlike a, the numbers b, c and d are dramatic anomalies in the zoo of natural numbers, because they can be written down using so little space. Their complexity, or informational content, is much smaller than they are themselves. Most numbers are not like this at all. To emphasize this point, let's make a crude calculation to bound the number of possible numbers we could write down by treating the entire universe as an enormous hard-drive, packed row upon row with elementary particles to encode some gigantic number. Suppose that in one dimension the universe is at most $10^{10^{10}}$ metres wide, that there are perhaps 10^{10^2} dimensions (to make room for future versions of string theory), that the smallest possible particle size is 10^{-10^3} metres, and that there are say 10^{10} different particles that we could place at any one point in the universe. So the number of possible configurations of particles filling up all of the universe is at most

$$10^{10^{\left(\left(10^{10^{10}} \times 10^{10^3}\right)^{10^{10^2}}\right)}}.$$

Although this is a respectable number, it pales to insignificance when compared to c. Conclusion: The vast majority of numbers less than c cannot be written down in our universe. These numbers are *completely inaccessible to us, and always will be*. But c can be written down in one line. Numbers 'close' to c in the sense of having expressions that are not all that different from that of c form little 'islands of simplicity' in a sea of overwhelming complexity.

It follows that long before you get to w, you are going to *reach numbers whose prime factorizations are impossible*, since some of the factors, if they existed, would require more room to write down than w. For example $c + 23$ is almost surely such a number—I claim it *has no prime factorization*. Neither you nor I nor anyone ever living in this universe will ever be able to factor this number, since most of its 'prime factors' are almost surely so huge as to be inexpressible, which means they *don't exist*.

Perhaps you believe that even though you cannot write down numbers bigger than w, you can still *abstractly contemplate them*! This is a metaphysical claim. What does *a number bigger than* w mean, if there is nothing that it counts, and it can't even be written down? Believing you can 'visualize' an all-seeing Leprechaun or an unstoppable mouse in your mind, by some melange of images, descriptive phrases and vague feelings, does not mean they exist. By all means, write plays and poems about all those numbers beyond w, but don't imagine you are doing mathematics. Twentieth century physicists have learnt to disregard 'concepts' which are not measurable or observable in some form or another, and we mathematicians ought to be equally skeptical.

Elementary mathematics needs to be understood in the *right way*, and the entire subject needs to be rebuilt so that it makes complete sense right from the beginning, without any use of dubious philosophical assumptions about infinite sets or procedures. Show me *one fact* about the real world (i.e. applied maths, physics, chemistry, biology, economics etc.) that *truly requires* mathematics involving 'infinite sets'! Mathematics was always really about, and always will be about, finite collections, patterns and algorithms. All those theories, arguments and daydreams involving 'infinite sets' need to be recast into a precise finite framework or relegated to philosophy. Sure it's more work, just as developing Schwartz's theory of distributions is more work than talking about the delta function as 'a function with total integral one that is zero everywhere except at one point where it is infinite'. But Schwartz's clarification inevitably led to important new applications and insights.

If such an approach had been taken in the twentieth century, then (at the very least) two important consequences would have ensued. First of all, mathematicians would by now have arrived at a reasonable consensus of how to formulate *elementary and high school mathematics in the right way*. The benefits to mathematics education would have been profound. We would have strong positions and reasoned arguments from which to encourage educators to adopt certain approaches and avoid others, and students would have a much more sensible, uniform and digestible subject.

The second benefit would have been that our ties to computer science would be much stronger than they currently are. If we are ever going to get serious about understanding the continuum—and I strongly feel we should—then we must address the critical issue of how to specify and handle the computational procedures that determine points (i.e. decimal expansions). There is *no avoiding the development of an appropriate theory of algorithms*. How sad that mathematics lost the interesting and important subdiscipline of computer science largely because we preferred convenience to precision!

But let's not cry overlong about missed opportunities. Instead, let's get out of our dreamy feather beds, smell the coffee, and make complete sense of mathematics.