One Dimensional Metrical Geometry

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Abstract

One dimensional metrical geometry may be developed in either an affine or projective setting over a general field using only algebraic ideas and quadratic forms. Some basic results of universal geometry are already present in this situation, such as the Triple quad formula, the Triple spread formulas and the Spread polynomials, which are universal analogs of the Chebyshev polynomials of the first kind. Chromogeometry appears here, and the related metrical and algebraic properties of the projective line are brought to the fore.

1 Introduction

This paper introduces the metrical geometry of *one dimensional space*, a surprisingly rich but largely unexplored subject. This theory is a natural chapter of *universal geometry*, introduced in [5], where all fields not of characteristic two are treated uniformly.

Over the 'real numbers' it is usual to consider the two models of one dimensional geometry to be the *line* and the *circle*. Either way one must struggle with the proper definition of the 'continuum'. In this paper I wish to promote instead the idea that the two fundamental one dimensional objects are the *affine line* and the *projective line*, and that a purely algebraic approach to metrical structure simplifies the logical coherence of the subject.

The affine line has a unique metrical structure (not a metric in the usual sense!) derived from the notion of *quadrance* between two points. This is the square of the distance in the usual 'real number' sense, but its algebraic nature allows it to be defined over a general field.

The projective line has a family of metrical structures determined by *forms,* with *projective quadrance* playing the role of quadrance. Projective quadrance between projective points is equivalent to *spread* between lines in the plane, and many of the results of this paper are basic for *rational trigonometry*, as described in [5].

The formulas developed here also anticipate results of planar Euclidean geometry, including Heron's or Archimedes' formula on the area of a triangle and Brahmagupta's formula for the area of a cyclic quadrilateral. They extend to affine and projective metrical geometries associated to arbitrary quadratic forms in higher dimensions, as described in [7], and so become important for elliptic and hyperbolic geometries.

Of particular interest in the one dimensional situation are three forms called *blue*, *red* and *green,* as these lead to *chromogeometry,* an entirely new framework for our understanding of planar metrical geometry, as described in [6]. The blue form corresponds to Euclidean geometry, while the red and green forms are hyperbolic, or relativistic, analogs. Each yields an *algebraic structure* to the projective line, and these three geometries interact in a pleasant way even in the one dimensional situation.

The metrical and algebraic properties of the projective line introduced here should be of interest to algebraic geometers, as the natural extension of these ideas to higher dimensions allows *metrical analysis* of projective curves and varieties—still completely in the algebraic context.

2 The affine line

2.1 Quadrance and Triple quad formula

Fix a field, whose elements are called **numbers**. A **point** $A \equiv [x]$ is a number enclosed in square brackets.

The **quadrance** $Q(A_1, A_2)$ between the points $A_1 \equiv [x_1]$ and $A_2 \equiv [x_2]$ is the number

$$Q(A_1, A_2) \equiv (x_2 - x_1)^2 .$$

Clearly

$$Q(A_1, A_2) = Q(A_2, A_1),$$

and $Q(A_1, A_2)$ is zero precisely when $A_1 = A_2$.

The next result is fundamental, with implications throughout universal geometry.

Theorem 1 (Triple quad formula) *For points A_1, A_2 and A_3, define $Q_1 \equiv Q(A_2, A_3)$, $Q_2 \equiv Q(A_1, A_3)$ and $Q_3 \equiv Q(A_1, A_2)$. Then*

$$(Q_1 + Q_2 + Q_3)^2 = 2\left(Q_1^2 + Q_2^2 + Q_3^2\right).$$

Proof. First verify the polynomial identity

$$(Q_1 + Q_2 + Q_3)^2 - 2\left(Q_1^2 + Q_2^2 + Q_3^2\right) = 4Q_1Q_2 - (Q_1 + Q_2 - Q_3)^2 .$$

Suppose that $A_1 \equiv [x_1]$, $A_2 \equiv [x_2]$ and $A_3 \equiv [x_3]$. Then

$$\begin{aligned} Q_1 + Q_2 - Q_3 &= (x_3 - x_2)^2 + (x_3 - x_1)^2 - (x_2 - x_1)^2 \\ &= 2(x_3 - x_2)(x_3 - x_1). \end{aligned}$$

So

$$(Q_1 + Q_2 - Q_3)^2 = 4Q_1Q_2.$$

■

Motivated by the Triple quad formula, define **Archimedes' function** $A(a,b,c)$ for numbers a,b and c by

$$A(a,b,c) \equiv (a+b+c)^2 - 2\left(a^2+b^2+c^2\right).$$

A set $\{a,b,c\}$ is a **quad triple** precisely when $A(a,b,c)=0$. Note that $A(a,b,c)$ is a symmetric function of a,b and c, and that

$$\begin{aligned} A(a,b,c) &= 4ab - (a+b-c)^2 \\ &= 2(ab+bc+ca) - \left(a^2+b^2+c^2\right) \\ &= 4(ab+bc+ca) - (a+b+c)^2 \\ &= \begin{vmatrix} 2a & a+b-c \\ a+b-c & 2b \end{vmatrix} \\ &= -\begin{vmatrix} 0 & a & b & 1 \\ a & 0 & c & 1 \\ b & c & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}. \end{aligned}$$

Theorem 2 (Heron's identity) *Suppose that* $D_1 = d_1^2$, $D_2 = d_2^2$ *and* $D_3 = d_3^2$ *for some numbers* d_1, d_2 *and* d_3. *Then*

$$A(D_1,D_2,D_3) = (d_1+d_2+d_3)(-d_1+d_2+d_3)(d_1-d_2+d_3)(d_1+d_2-d_3).$$

Proof. Observe that

$$\begin{aligned} &(d_1+d_2+d_3)(-d_1+d_2+d_3)(d_1-d_2+d_3)(d_1+d_2-d_3) \\ &= \left((d_1+d_2)^2 - d_3^2\right)\left(d_3^2 - (d_1-d_2)^2\right) \\ &= \left((d_1+d_2)^2 + (d_1-d_2)^2\right)D_3 - (d_1+d_2)^2(d_1-d_2)^2 - D_3^2 \\ &= 2(D_1+D_2)D_3 - (D_1-D_2)^2 - D_3^2 \\ &= (D_1+D_2+D_3)^2 - 2\left(D_1^2+D_2^2+D_3^2\right). \end{aligned}$$

■

By the classical *Heron's formula*, known by Archimedes, the expression $A\left(d_1^2, d_2^2, d_3^2\right)$ is sixteen times the square of the area of a triangle with side lengths d_1, d_2 and d_3.

2.2 Quadruple quad formula

The next two results extend the Triple quad formula to four points.

Theorem 3 (Two quad triples) *Suppose that $\{a,b,x\}$ and $\{c,d,x\}$ are both quad triples. Then*

$$\left((a+b+c+d)^2-2\left(a^2+b^2+c^2+d^2\right)\right)^2=64abcd.$$

Furthermore if $a+b\neq c+d$ then

$$x=\frac{(a-b)^2-(c-d)^2}{2\,(a+b-c-d)}.$$

Proof. Suppose that $\{a,b,x\}$ and $\{c,d,x\}$ are quad triples, so that

$$(x-a-b)^2=4ab \qquad (1)$$
$$(x-c-d)^2=4cd. \qquad (2)$$

Take the difference between these two equations to yield

$$2x\,(a+b-c-d)+(c+d)^2-(a+b)^2=4cd-4ab.$$

If $a+b\neq c+d$ then

$$x=\frac{(a-b)^2-(c-d)^2}{2\,(a+b-c-d)}.$$

Substitute this into (1) or (2) to get

$$\left((a-b)^2-(c-d)^2-2\,(a+b-c-d)\,(a+b)\right)^2-16ab\,(a+b-c-d)^2=0. \qquad (3)$$

The left hand side of this equation can be rearranged to get

$$\left((a+b+c+d)^2-2\left(a^2+b^2+c^2+d^2\right)\right)^2-64abcd.$$

On the other hand if $a+b=c+d$ then by (1) we see that $4ab=4cd$, so that $(a-b)^2=(c-d)^2$, and then (3) reduces also to zero. ■

The **Quadruple quad function** $Q\,(a,b,c,d)$ is defined by

$$Q\,(a,b,c,d)\equiv\left((a+b+c+d)^2-2\left(a^2+b^2+c^2+d^2\right)\right)^2-64abcd.$$

It is interesting that the expression

$$(a+b+c+d)^2-2\left(a^2+b^2+c^2+d^2\right)$$

appears in Descartes' circle theorem, a result motivated by a question of Princess Elisabeth of Bohemia around 1640, and rediscovered by both Beecroft and Soddy (see [3], [1]).

Theorem 4 (Quadruple quad formula) *For points A_1, A_2, A_3 and A_4, define the quadrances $Q_{ij} \equiv Q(A_i, A_j)$ for all $i, j = 1, 2, 3$ and 4. Then*

$$Q(Q_{12}, Q_{23}, Q_{34}, Q_{14}) = 0.$$

Furthermore

$$Q_{13} = \frac{(Q_{12} - Q_{23})^2 - (Q_{34} - Q_{14})^2}{2(Q_{12} + Q_{23} - Q_{34} - Q_{14})}$$
$$Q_{24} = \frac{(Q_{23} - Q_{34})^2 - (Q_{12} - Q_{14})^2}{2(Q_{23} + Q_{34} - Q_{12} - Q_{14})}$$

provided the denominators are not zero.

Proof. Both $\{Q_{12}, Q_{23}, Q_{13}\}$ and $\{Q_{14}, Q_{13}, Q_{34}\}$ are quad triples, so the Two quad triples theorem shows that $Q(Q_{12}, Q_{23}, Q_{34}, Q_{14}) = 0$ and gives the stated formula for Q_{13}. The result for Q_{24} is similar. ■

Theorem 5 (Brahmagupta's identity) *Suppose that $D_{12} \equiv d_{12}^2$, $D_{23} \equiv d_{23}^2$, $D_{34} \equiv d_{34}^2$ and $D_{14} \equiv d_{14}^2$ for some numbers d_{12}, d_{23}, d_{34} and d_{14}. Then*

$$\begin{aligned} &Q(D_{12}, D_{23}, D_{34}, D_{14}) \\ &= (d_{12} - d_{14} + d_{23} + d_{34})(d_{12} + d_{14} + d_{23} - d_{34})(d_{14} - d_{12} + d_{23} + d_{34}) \\ &\times (d_{12} + d_{14} - d_{23} + d_{34})(d_{12} + d_{14} + d_{23} + d_{34})(d_{12} - d_{14} - d_{23} + d_{34}) \\ &\times (d_{12} - d_{14} + d_{23} - d_{34})(d_{23} - d_{14} - d_{12} + d_{34}). \end{aligned}$$

Proof. Make the substitutions $D_{ij} = d_{ij}^2$ for all i and j to turn the expression

$$\left((D_{12} + D_{23} + D_{34} + D_{14})^2 - 2\left(D_{12}^2 + D_{23}^2 + D_{34}^2 + D_{14}^2\right)\right)^2 - 64 D_{12} D_{23} D_{34} D_{14}$$

into a difference of squares. This is then the product of the expression

$$\begin{aligned} &\left(d_{14}^2 + d_{34}^2 + d_{12}^2 + d_{23}^2\right)^2 - 2\left(d_{14}^4 + d_{34}^4 + d_{12}^4 + d_{23}^4\right) + 8 d_{14} d_{34} d_{12} d_{23} \\ &= (-d_{12} + d_{14} + d_{23} + d_{34})(d_{12} - d_{14} + d_{23} + d_{34}) \\ &\times (d_{12} + d_{14} - d_{23} + d_{34})(d_{12} + d_{14} + d_{23} - d_{34}) \end{aligned}$$

and the expression

$$\begin{aligned} &\left(d_{14}^2 + d_{34}^2 + d_{12}^2 + d_{23}^2\right)^2 - 2\left(d_{14}^4 + d_{34}^4 + d_{12}^4 + d_{23}^4\right) - 8 d_{14} d_{34} d_{12} d_{23} \\ &= (d_{12} + d_{14} + d_{23} + d_{34})(d_{12} - d_{14} - d_{23} + d_{34}) \\ &\times (d_{12} - d_{14} + d_{23} - d_{34})(d_{23} - d_{14} - d_{12} + d_{34}). \end{aligned}$$

■

By a classical theorem of Brahmagupta, the first of these two expressions corresponds in the decimal number system to sixteen times the square of the area of a convex cyclic quadrilateral with side lengths d_1, d_2, d_3 and d_4. The second corresponds to an analogous result for a non-convex cyclic quadrilateral with these side lengths, as discussed in [4].

2.3 Higher quad formulas

Generalizations of the Triple quad formula and Quadruple quad formula exist for more than four points. Writing down pleasant expressions for these seems an interesting challenge in pure algebra.

2.4 Isometries of the affine line

We adopt the convention that the image of the point A under the function, or map, σ is denoted $A\sigma$, and that if σ_1 and σ_2 are two maps, $\sigma_1\sigma_2$ denotes the composite map given by

$$A(\sigma_1\sigma_2) = (A\sigma_1)\,\sigma_2.$$

This allows us to write $A(\sigma_1\sigma_2)$ simply as $A\sigma_1\sigma_2$.

An **isometry of the affine line** is a function σ which inputs and outputs points and preserves quadrance, in the sense that for any points A and B,

$$Q(A,B) = Q(A\sigma, B\sigma).$$

Theorem 6 *An isometry σ of the affine line has exactly one of the two forms:*

$$[x]\,\sigma = [x+\alpha]$$

or

$$[x]\,\sigma = [\alpha - x]$$

for some number α.

Proof. Let $O = [0]$ and $I = [1]$. Suppose σ is an isometry with $O\sigma = A = [\alpha]$ and $I\sigma = B = [\beta]$. Then we must have

$$Q(A,B) = (\beta-\alpha)^2 = Q(O,I) = 1$$

and it follows that $\beta = \alpha \pm 1$. Suppose that $\beta = \alpha + 1$. In this case if $[x]\,\sigma = [y]$, then

$$\begin{aligned} x^2 &= (y-\alpha)^2 \\ (x-1)^2 &= (y-\beta)^2 = (y-\alpha-1)^2. \end{aligned}$$

It follows that $y = x + \alpha$. Now suppose that $\beta = \alpha - 1$. In this case if $[x]\,\sigma = [y]$, then

$$\begin{aligned} x^2 &= (y-\alpha)^2 \\ (x-1)^2 &= (y-\beta)^2 = (y-\alpha+1)^2. \end{aligned}$$

It follows that $y = \alpha - x$. Each of these maps is easily checked to be an isometry. ■

Note in particular that an isometry is invertible.

3 The projective line

3.1 Projective quadrance

A **projective point** is a proportion $a \equiv [x : y]$ where x and y are not both zero, and where $[x_1 : y_1] = [x_2 : y_2]$ precisely when

$$x_1 y_2 - x_2 y_1 = 0.$$

This is equivalent to the condition $[x : y] = [\lambda x : \lambda y]$ for any non-zero λ. A projective point will often be called a **p-point**.

A **form** is a proportion $F \equiv (d : e : f)$ where d, e and f are not all zero. Again we agree that

$$(d : e : f) = (\lambda d : \lambda e : \lambda f)$$

for any non-zero λ. The form $F \equiv (d : e : f)$ is **degenerate** precisely when the **discriminant**

$$df - e^2$$

is zero. Everything in this section depends on the choice of a non-degenerate form $F \equiv (d : e : f)$, which we consider arbitrary but fixed.

A p-point $a \equiv [x : y]$ is **null** precisely when

$$dx^2 + 2exy + fy^2 = 0.$$

The p-points $a_1 \equiv [x_1 : y_1]$ and $a_2 \equiv [x_2 : y_2]$ are **perpendicular** precisely when

$$dx_1 x_2 + ex_1 y_2 + ex_2 y_1 + fy_1 y_2 = 0.$$

The **projective quadrance**, or more simply the **p-quadrance**, between non-null p-points $a_1 \equiv [x_1 : y_1]$ and $a_2 \equiv [x_2 : y_2]$ is the number

$$q = q(a_1, a_2) \equiv \frac{\left(df - e^2\right)(x_1 y_2 - x_2 y_1)^2}{\left(dx_1^2 + 2ex_1 y_1 + fy_1^2\right)\left(dx_2^2 + 2ex_2 y_2 + fy_2^2\right)}.$$

The presence of the discriminant $df - e^2$ ensures that the formulas that follow are independent of the form F. Note that

$$q(a_1, a_2) = q(a_2, a_1)$$

and that $q(a_1, a_2)$ is zero precisely when $a_1 = a_2$.

Theorem 7 (Perpendicular p-points) *Two non-null p-points a_1 and a_2 are perpendicular precisely when $q(a_1, a_2) = 1$.*

Proof. The **generalized Fibonacci's identity**

$$\begin{aligned}&\left(df - e^2\right)(x_1 y_2 - x_2 y_1)^2 + (dx_1 x_2 + ex_1 y_2 + ex_2 y_1 + fy_1 y_2)^2 \\ &= \left(dx_1^2 + 2ex_1 y_1 + fy_1^2\right)\left(dx_2^2 + 2ex_2 y_2 + fy_2^2\right)\end{aligned}$$

shows that if $a_1 \equiv [x_1 : y_1]$ and $a_2 \equiv [x_2 : y_2]$, then

$$1 - q(a_1, a_2) = \frac{(dx_1x_2 + ex_1y_2 + ex_2y_1 + fy_1y_2)^2}{(dx_1^2 + 2ex_1y_1 + fy_1^2)(dx_2^2 + 2ex_2y_2 + fy_2^2)}.$$

This is zero precisely when a_1 and a_2 are perpendicular. ■

The next result is fundamental for metrical projective geometry. It is the analog of the Triple quad formula, and differs from it only by a cubic term. In planar rational trigonometry this result is called the *Triple spread formula*.

Theorem 8 (Projective triple quad formula) *For p-points a_1, a_2 and a_3, define the p-quadrances $q_1 \equiv q(a_2, a_3)$, $q_2 \equiv q(a_1, a_3)$ and $q_3 \equiv q(a_1, a_2)$. Then*

$$(q_1 + q_2 + q_3)^2 = 2\left(q_1^2 + q_2^2 + q_3^2\right) + 4q_1q_2q_3.$$

Proof. If $a_1 \equiv [x_1 : y_1]$, $a_2 \equiv [x_2 : y_2]$, and $a_3 \equiv [x_3 : y_3]$ then

$$q_1 = \frac{\left(df - e^2\right)(x_2y_3 - x_3y_2)^2}{(dx_2^2 + 2ex_2y_2 + fy_2^2)(dx_3^2 + 2ex_3y_3 + fy_3^2)}$$
$$q_2 = \frac{\left(df - e^2\right)(x_1y_3 - x_3y_1)^2}{(dx_1^2 + 2ex_1y_1 + fy_1^2)(dx_3^2 + 2ex_3y_3 + fy_3^2)}$$
$$q_3 = \frac{\left(df - e^2\right)(x_1y_2 - x_2y_1)^2}{(dx_1^2 + 2ex_1y_1 + fy_1^2)(dx_2^2 + 2ex_2y_2 + fy_2^2)}.$$

The following is an algebraic identity:

$$(x_2y_3 - x_3y_2)^2\left(dx_1^2 + 2ex_1y_1 + fy_1^2\right) + (x_1y_3 - x_3y_1)^2\left(dx_2^2 + 2ex_2y_2 + fy_2^2\right)$$
$$-(x_1y_2 - x_2y_1)^2\left(dx_3^2 + 2ex_3y_3 + fy_3^2\right)$$
$$= 2(x_2y_3 - x_3y_2)(x_1y_3 - x_3y_1)(dx_1x_2 + ex_1y_2 + ex_2y_1 + fy_1y_2)$$

Square both sides and rearrange using the Perpendicular p-points theorem, to get

$$(q_1 + q_2 - q_3)^2 = 4q_1q_2(1 - q_3).$$

This can be rewritten as the symmetrical equation

$$(q_1 + q_2 + q_3)^2 - 2\left(q_1^2 + q_2^2 + q_3^2\right) = 4q_1q_2q_3.$$

■

Motivated by the Projective triple quad formula, define the **Triple spread function** $S(a, b, c)$ for numbers a, b and c by

$$S(a, b, c) \equiv (a + b + c)^2 - 2\left(a^2 + b^2 + c^2\right) - 4abc.$$

The reason for the terminology is that in two dimensional geometry the projective quadrance between p-points becomes the *spread* between lines. Note that

$S(a,b,c)$ is a symmetric function of a,b and c, and that

$$\begin{aligned}
S(a,b,c) &= A(a,b,c) - 4abc \\
&= 2(ac+bc+ab) - \left(a^2+b^2+c^2\right) - 4abc \\
&= 4(ab+bc+ca) - (a+b+c)^2 - 4abc \\
&= 4(1-a)(1-b)(1-c) - (a+b+c-2)^2 \\
&= 4(1-a)bc - (a-b-c)^2 \\
&= -\begin{vmatrix} 0 & a & b & 1 \\ a & 0 & c & 1 \\ b & c & 0 & 1 \\ 1 & 1 & 1 & 2 \end{vmatrix} \\
&= 4bc(1-b)(1-c) - (a-b-c+2bc)^2. \qquad (4)
\end{aligned}$$

The last of these equations is useful to solve $S(a,b,c)=0$ for a if b and c are known.

A set $\{a,b,c\}$ is a **spread triple** precisely when $S(a,b,c)=0$.

3.2 Projective quadruple quad formula

The next theorems extend the Projective triple quad formula to four p-points.

Theorem 9 (Two spread triples) *Suppose that $\{a,b,x\}$ and $\{c,d,x\}$ are both spread triples. Then*

$$\begin{aligned}
&\left(\begin{array}{c} (a+b+c+d)^2 - 2\left(a^2+b^2+c^2+d^2\right) \\ -4(abc+abd+acd+bcd) + 8abcd \end{array} \right)^2 \\
&= 64abcd(1-a)(1-b)(1-c)(1-d).
\end{aligned}$$

Furthermore if $a+b-2ab \neq c+d-2cd$ then

$$x = \frac{(a-b)^2 - (c-d)^2}{2(a+b-c-d-2ab+2cd)}.$$

Proof. Suppose that $\{a,b,x\}$ and $\{c,d,x\}$ are both spread triples. Then (4) gives

$$(x-a-b+2ab)^2 = 4ab(1-a)(1-b) \qquad (5)$$

$$(x-c-d+2cd)^2 = 4cd(1-c)(1-d). \qquad (6)$$

Take the difference between these two equations. If $a+b-2ab \neq c+d-2cd$ then you may solve for x to get

$$x = \frac{(a-b)^2 - (c-d)^2}{2(a+b-c-d-2ab+2cd)}.$$

Substitute back into (5) or (6) to get

$$\begin{aligned}&\Big((a-b)^2-(c-d)^2-2\,(a+b-c-d-2ab+2cd)\,(a+b-2ab)\Big)^2\\&=16ab\,(1-a)\,(1-b)\,(a+b-c-d-2ab+2cd)^2\,.\end{aligned}\tag{7}$$

This condition can be rewritten more symmetrically as

$$\begin{aligned}&\left(\begin{array}{c}(a+b+c+d)^2-2\left(a^2+b^2+c^2+d^2\right)\\-4\,(abc+abd+acd+bcd)+8abcd\end{array}\right)^2\\&=64abcd\,(1-a)\,(1-b)\,(1-c)\,(1-d)\,.\end{aligned}$$

If

$$a+b-2ab=c+d-2cd$$

then by (5) also

$$4ab\,(1-a)\,(1-b)=4cd\,(1-c)\,(1-d)\,.$$

Hence the identity

$$(a+b-2ab)^2-4ab\,(1-a)\,(1-b)=(a-b)^2$$

implies that $(a-b)^2=(c-d)^2$, and then (7) is automatic. ■

The **Quadruple spread function** $R\,(a,b,c,d)$ is defined for numbers a,b,c and d by

$$\begin{aligned}R\,(a,b,c,d)\equiv&\left(\begin{array}{c}(a+b+c+d)^2-2\left(a^2+b^2+c^2+d^2\right)\\-4\,(abc+abd+acd+bcd)+8abcd\end{array}\right)^2\\&-64abcd\,(1-a)\,(1-b)\,(1-c)\,(1-d)\,.\end{aligned}$$

This is a symmetric function of a,b,c and d.

Theorem 10 (Projective quadruple quad formula) *Suppose* a_1,a_2,a_3 *and* a_4 *are non-null p-points with p-quadrances* $q_{ij}\equiv q\,(a_i,a_j)$ *for all* $i,j=1,2,3$ *and* 4*. Then*

$$R\,(q_{12},q_{23},q_{34},q_{14})=0.$$

Furthermore

$$q_{13}=\frac{(q_{12}-q_{23})^2-(q_{34}-q_{14})^2}{2\,(q_{12}+q_{23}-q_{34}-q_{14}-2q_{12}q_{23}+2q_{34}q_{14})}$$

$$q_{24}=\frac{(q_{23}-q_{34})^2-(q_{12}-q_{14})^2}{2\,(q_{23}+q_{34}-q_{12}-q_{14}-2q_{23}q_{34}+2q_{12}q_{14})}$$

provided the denominators are non-zero.

Proof. Since $\{q_{12},q_{23},q_{13}\}$ and $\{q_{14},q_{34},q_{13}\}$ are both spread triples, and $\{q_{23},q_{34},q_{24}\}$ and $\{q_{12},q_{14},q_{24}\}$ are also both spread triples, the formulas follow from the Two spread triples theorem. ■

3.3 An example

Suppose $a_1 \equiv [1:0]$, $a_2 \equiv [2:3]$, $a_3 \equiv [4:-1]$ and $a_4 \equiv [3:5]$ over the rational number field, and that we choose the Euclidean form $F \equiv (1:0:1)$, corresponding to x^2+y^2. Then the projective quadrance is given by

$$q([x_1:y_1],[x_2:y_2]) = \frac{(x_1y_2-x_2y_1)^2}{(x_1^2+y_1^2)(x_2^2+y_2^2)}.$$

You may then calculate that with $q_{ij} \equiv q(a_i,a_j)$,

$$q_{12} = 9/13 \qquad q_{23} = 196/221 \qquad q_{34} = 529/578 \qquad q_{14} = 25/34$$

and that

$$R\left(\frac{9}{13},\frac{196}{221},\frac{529}{578},\frac{25}{34}\right) = 0.$$

Furthermore

$$\frac{(q_{12}-q_{23})^2-(q_{34}-q_{14})^2}{2(q_{12}+q_{23}-q_{34}-q_{14}-2q_{12}q_{23}+2q_{34}q_{14})} = \frac{1}{17} = q_{13}$$
$$\frac{(q_{23}-q_{34})^2-(q_{12}-q_{14})^2}{2(q_{23}+q_{34}-q_{12}-q_{14}-2q_{23}q_{34}+2q_{12}q_{14})} = \frac{1}{442} = q_{24}.$$

3.4 Higher quadruple spread formulas

The same remarks made earlier about higher quad formulas apply to higher spread formulas.

4 The Spread polynomials

The Projective triple quad formula, or Triple spread formula

$$(a+b+c)^2 = 2\left(a^2+b^2+c^2\right)+4abc$$

yields an important sequence of polynomials with integer coefficients, called *spread polynomials*. These are universal analogs of the Chebyshev polynomials of the first kind, and make sense over any field, not of characteristic two. It is interesting that these appear already in one dimensional geometry.

To motivate the spread polynomials, note that in the Triple spread formula if $a \equiv b \equiv s$ then c is either 0 or $4s(1-s)$. If $a \equiv 4s(1-s)$ and $b \equiv s$ then c is either s or $s(3-4s)^2$. Continuing, there is a sequence of polynomials $S_n(s)$ for $n = 0,1,2,\cdots$ with the property that $S_{n-1}(s)$, s and $S_n(s)$ always satisfy the Triple spread formula.

As in [5], the **spread polynomial** $S_n(s)$ is defined recursively by $S_0(s) \equiv 0$ and $S_1(s) \equiv s$, together with the rule

$$S_n(s) \equiv 2(1-2s)\,S_{n-1}(s) - S_{n-2}(s) + 2s.$$

The coefficient of s^n in $S_n(s)$ is a power of four, so in any field not of characteristic two the degree of the polynomial $S_n(s)$ is n. Over the decimal number field

$$S_n(s) = \frac{1 - T_n(1-2s)}{2}$$

where T_n is the n-th Chebyshev polynomial of the first kind. The affine transformation which maps the square $[-1,1]\times[-1,1]$ to the square $[0,1]\times[0,1]$ takes the graph of the n-th Chebyshev polynomial to the graph of the n-th spread polynomial.

The first few spread polynomials are

$$\begin{aligned}
&S_0(s) = 0\\
&S_1(s) = s\\
&S_2(s) = 4s - 4s^2 = 4s(1-s)\\
&S_3(s) = 9s - 24s^2 + 16s^3 = s(4s-3)^2\\
&S_4(s) = 16s - 80s^2 + 128s^3 - 64s^4 = 16s(1-s)(2s-1)^2\\
&S_5(s) = 25s - 200s^2 + 560s^3 - 640s^4 + 256s^5 = s\left(16s^2 - 20s + 5\right)^2.
\end{aligned}$$

Note that $S_2(s)$ is the logistic map. As shown in [5], $S_n \circ S_m = S_{nm}$ for $n, m \geq 1$, and the spread polynomials have interesting orthogonality properties over finite fields.

S. Goh [2] observed that there is a sequence of 'spread-cyclotomic' polynomials $\phi_k(s)$ of degree $\phi(k)$ with integer coefficients such that for any $n = 1, 2, 3, \cdots$

$$S_n(s) = \prod_{k|n} \phi_k(s).$$

This factorization shows that arithmetically the spread polynomials have quite different properties than the closely related Chebyshev polynomials.

5 Chromogeometry

There are three forms which are particularly important and useful, and which interact in a surprising way. This becomes particularly clear in two dimensional geometry, but there are already hints here in the one dimensional situation.

The **blue form** is $F_b \equiv (1:0:1)$, the **red form** is $F_r \equiv (1:0:-1)$ and the **green form** is $F_g \equiv (0:1:0)$. Two p-points $a_1 \equiv [x_1 : y_1]$ and $a_2 \equiv [x_2 : y_2]$

are then **blue, red and green perpendicular** precisely when

$$x_1x_2 + y_1y_2 = 0$$
$$x_1x_2 - y_1y_2 = 0$$
$$x_1y_2 + x_2y_1 = 0$$

respectively. Recall that $a_1 = a_2$ precisely when

$$x_1y_2 - x_2y_1 = 0$$

and these four equations exhaust the bilinear equations in the coordinates of a_1 and a_2 which employ coefficients of ± 1.

The **blue, red and green perpendiculars** of the p-point $a \equiv [x : y]$ are respectively the p-points

$$a^b \equiv [-y : x] \qquad a^r \equiv [y : x] \qquad a^g \equiv [x : -y] \,.$$

We let q^b, q^r and q^g denote the blue, red and green p-quadrances associated to the blue, red and green forms respectively. Then

$$\begin{aligned} q^b\left([x_1 : y_1], [x_2 : y_2]\right) &= \frac{(x_1y_2 - x_2y_1)^2}{(x_1^2 + y_1^2)(x_2^2 + y_2^2)} \\ q^r\left([x_1 : y_1], [x_2 : y_2]\right) &= -\frac{(x_1y_2 - x_2y_1)^2}{(x_1^2 - y_1^2)(x_2^2 - y_2^2)} \\ q^g\left([x_1 : y_1], [x_2 : y_2]\right) &= -\frac{(x_1y_2 - x_2y_1)^2}{4x_1y_1x_2y_2}. \end{aligned}$$

Theorem 11 *For any p-points a_1 and a_2*

$$\frac{1}{q^b(a_1, a_2)} + \frac{1}{q^r(a_1, a_2)} + \frac{1}{q^g(a_1, a_2)} = 2.$$

Proof. This follows from the previous formulas and the identity

$$\left(x_1^2 + y_1^2\right)\left(x_2^2 + y_2^2\right) - \left(x_1^2 - y_1^2\right)\left(x_2^2 - y_2^2\right) - 4x_1y_1x_2y_2 = 2\left(x_1y_2 - x_2y_1\right)^2.$$

■

Theorem 12 *For any p-point a,*

$$q^b\left(a^r, a^g\right) = q^r\left(a^g, a^b\right) = q^g\left(a^b, a^r\right) = 1.$$

Proof. For $a \equiv [x : y]$

$$\begin{aligned} q^b\left(a^r, a^g\right) &= q^b\left([y : x], [x : -y]\right) = 1 \\ q^r\left(a^g, a^b\right) &= q^r\left([x : -y], [-y : x]\right) = 1 \\ q^g\left(a^b, a^r\right) &= q^g\left([-y : x], [y : x]\right) = 1. \end{aligned}$$

■

Theorem 13 *For any two p-points* $a_1 = [x_1 : y_1]$ *and* $a_2 = [x_2 : y_2]$ *and any colour* c *(either* b, r *or* g*)*

$$q^c(a_1, a_2) = q^c\left(a_1^b, a_2^b\right) = q^c(a_1^r, a_2^r) = q^c(a_1^g, a_2^g).$$

Proof. A computation using the above formulas. ∎

Theorem 14 *For any p-points* a_1 *and* a_2,

$$\begin{aligned} q^b(a_1^r, a_2^g) &= q^b(a_1^g, a_2^r) \\ q^r\left(a_1^g, a_2^b\right) &= q^r\left(a_1^b, a_2^g\right) \\ q^g\left(a_1^b, a_2^r\right) &= q^g\left(a_1^r, a_2^b\right). \end{aligned}$$

Proof. A computation. ∎

6 Projective isometries: blue, red and green

Let q denote one of the blue, red or green p-quadrances. A projective **isometry** is a map σ that inputs and outputs projective points and satisfies

$$q(a, b) = q(a\sigma, b\sigma)$$

for any projective points a and b. Note that the action of the map σ on the p-point a is denote $a\sigma$.

One way of describing a map on projective points is to use a **projective matrix**

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where by definition

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{bmatrix}$$

for any non-zero number λ. Such a matrix defines the map

$$[x : y]\,\sigma = [x : y]\begin{bmatrix} a & b \\ c & d \end{bmatrix} = [ax + cy : bx + dy]$$

and we write

$$\sigma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

6.1 Blue isometries

Theorem 15 *An isometry of the blue p-quadrance is either*

$$\sigma^b_{[a:b]} = \begin{bmatrix} a & b \\ b & -a \end{bmatrix} \qquad or \qquad \rho^b_{[a:b]} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

for some blue non-null projective point $[a : b]$.

Proof. Suppose that σ sends $i_1 \equiv [1:0]$ to $[a:b]$ and $i_2 \equiv [0:1]$ to $[c:d]$. Then since $q_b(i_1, i_2) = 1$, we must have

$$ac + bd = 0.$$

So $[c:d] = [b:-a]$. Now given an arbitrary projective point $u \equiv [x:y]$ with $u\sigma \equiv v \equiv [w:z]$,

$$\begin{aligned} q_b(u, i_1) &= 1 - \frac{x^2}{x^2+y^2} = \frac{y^2}{x^2+y^2} \\ &= q(v, [a:b]) = \frac{(az-bw)^2}{(a^2+b^2)(w^2+z^2)} \end{aligned}$$

and

$$\begin{aligned} q_b(u, i_2) &= 1 - \frac{y^2}{x^2+y^2} = \frac{x^2}{x^2+y^2} \\ &= q(v, [b:-a]) = \frac{(bz+aw)^2}{(a^2+b^2)(w^2+z^2)}. \end{aligned}$$

This gives the following quadratic equations for w and z :

$$\begin{aligned} \frac{y^2}{x^2+y^2} &= \frac{(az-bw)^2}{(a^2+b^2)(w^2+z^2)} \\ \frac{x^2}{x^2+y^2} &= \frac{(bz+aw)^2}{(a^2+b^2)(w^2+z^2)}. \end{aligned}$$

The two possible solutions for $[w:z]$ are

$$[ax+by : bx-ay] \qquad \text{and} \qquad [ax-by : bx+ay]$$

which correspond to $u\sigma^b_{[a:b]}$ and $u\rho^b_{[a:b]}$ respectively. ■

Theorem 16 *For any blue non-null projective points* $[a:b]$ *and* $[c:d]$

$$\sigma^b_{[a:b]}\sigma^b_{[c:d]} = \rho^b_{[ac+bd:ad-bc]} \qquad \rho^b_{[a:b]}\rho^b_{[c:d]} = \rho^b_{[ac-bd:ad+bc]}$$

$$\rho^b_{[a:b]}\sigma^b_{[c:d]} = \sigma^b_{[ac+bd:ad-bc]} \qquad \sigma^b_{[a:b]}\rho^b_{[c:d]} = \sigma^b_{[ac-bd:ad+bc]}.$$

Proof. This is a straightforward verification. Note that the Fibonacci identities

$$(ac+bd)^2 + (ad-bc)^2 = \left(a^2+b^2\right)\left(c^2+d^2\right) = (ac-bd)^2 + (ad+bc)^2$$

show that the resultant isometries are also associated to non-null points. ■

Define the group G^b of all isometries of the blue projective line, and distinguish the subgroup G^b_e of **blue rotations** $\rho^b_{[a:b]}$. These latter are naturally in bijection with the non-null projective points.

The coset G_o^b consists of **blue reflections** $\sigma^b_{[a:b]}$ which are also naturally in bijection with the non-null points. The non-commutative group G^b naturally acts on the blue non-null projective points, as well as on the blue null projective points. Both actions are transitive.

The group structure of the blue rotations can be transferred to give a **multiplication** of blue non-null projective points, by the rule

$$[a:b] \times_b [c:d] = [ac-bd : ad+bc] \,.$$

This multiplication is associative, commutative, has identity $[1:0]$, and the inverse of $[a:b]$ is $[a:-b]$. Of course this rule anticipates multiplication of unit complex numbers, but there are subtle differences. For any p-point $[a:b]$ with $a^2+b^2=1$ and $a \neq -1$ you can compute that

$$\begin{aligned}[a+1:b] \times_b [a+1:b] &= \left[(a+1)^2 - b^2 : 2\,(a+1)\,b\right] \\ &= \left[2\left(a^2+a\right) : 2\,(a+1)\,b\right] = [a:b]\end{aligned}$$

so that such a p-point has a distinguished algebraic 'square root' without requiring a field extension.

6.2 Red Isometries

Theorem 17 *An isometry of the red p-quadrance is either*

$$\sigma^r_{[a:b]} = \begin{bmatrix} a & b \\ -b & -a \end{bmatrix} \qquad \text{or} \qquad \rho^r_{[a:b]} = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

for some red non-null projective point $[a:b]$.

Proof. Suppose that σ sends $i_1 \equiv [1:0]$ to $[a:b]$ and $i_2 \equiv [0:1]$ to $[c:d]$. Now given an arbitrary p-point $u \equiv [x:y]$ with $u\sigma \equiv [w:z]$, a similar analysis to the previous proof gives the two quadratic equations

$$\begin{aligned}\frac{y^2}{x^2-y^2} &= \frac{(az-bw)^2}{(a^2-b^2)\,(w^2-z^2)} \\ \frac{x^2}{x^2-y^2} &= \frac{(bz-aw)^2}{(a^2-b^2)\,(w^2-z^2)}.\end{aligned}$$

The two possible solutions for $[w:z]$ are

$$[ax-by : bx-ay] \qquad \text{and} \qquad [ax+by : bx+ay]$$

which correspond to $u\sigma_{[a:b]}$ and $u\rho_{[a:b]}$ respectively. ■

Theorem 18 *For any red non-null projective points* $[a:b]$ *and* $[c:d]$

$$\sigma^r_{[a:b]}\sigma^r_{[c:d]} = \rho^r_{[ac-bd:ad-bc]} \qquad \rho^r_{[a:b]}\rho^r_{[c:d]} = \rho^r_{[ac+bd:ad+bc]}$$

$$\rho^r_{[a:b]}\sigma^r_{[c:d]} = \sigma^r_{[ac-bd:ad-bc]} \qquad \rho^r_{[c:d]}\sigma^r_{[a:b]} = \sigma^r_{[ac+bd:ad+bc]}.$$

Proof. A straightforward verification. Note that the hyperbolic versions of Fibonacci's identities:

$$(ac-bd)^2-(ad-bc)^2=\left(a^2-b^2\right)\left(c^2-d^2\right)=(ac+bd)^2-(ad+bc)^2$$

show that the resultant isometries are also associated to red non-null points. ■

We now define the group G^r of all isometries of the red projective line, and distinguish the subgroup $G_e^r=G_e^r$ of red rotations $\rho_{[a:b]}^r$. These latter are naturally in bijection with the non-null projective points.

The coset $G_o^r=G_o^r$ consists of red reflections $\sigma_{[a:b]}^r$ which are also naturally in bijection with the non-null points. The non-commutative group G^r naturally acts on the red non-null projective points, as well as on the red null projective points. Both actions are transitive.

The group structure of the red rotations can be transferred to give a **multiplication** of red non-null projective points, by the rule

$$[a:b]\times_r[c:d]=[ac+bd:ad+bc]\,.$$

This multiplication is associative, commutative, has identity $[1:0]$, and the inverse of $[a:b]$ is $[a:-b]$.

6.3 Green isometries

Theorem 19 *An isometry of the green p-quadrance has one of the two forms*

$$\sigma_{[a:b]}^g=\begin{bmatrix}0&a\\b&0\end{bmatrix}\qquad or\qquad \rho_{[a:b]}^g=\begin{bmatrix}a&0\\0&b\end{bmatrix}$$

for some green non-null projective point $[a:b]$.

Proof. Suppose that σ sends $j_1\equiv[1:1]$ to $[a:b]$ and $j_2\equiv[1:-1]$ to $[c:d]$. Then since $q_g(j_1,j_2)=1$, we must have

$$-\frac{(ad-bc)^2}{4abcd}=1$$

which implies that

$$ad+bc=0.$$

So $[c:d]=[a:-b]$. Now given an arbitrary p-point $u\equiv[x:y]$ with $u\sigma\equiv v\equiv[w:z]$,

$$\begin{aligned}q_g(u,j_1)&=-\frac{(x-y)^2}{4xy}\\&=q_g(v,[a:b])=-\frac{(za-wb)^2}{4wzab}\end{aligned}$$

and

$$q_g(u, j_2) = \frac{(x+y)^2}{4xy}$$
$$= q_g(v, [a:-b]) = \frac{(wb+za)^2}{4wzab}.$$

So

$$-\frac{(x-y)^2}{4xy} = -\frac{(za-wb)^2}{4wzab}$$
$$\frac{(x+y)^2}{4xy} = \frac{(wb+za)^2}{4wzab}$$

The two possible solutions for $[w:z]$ are

$$[ay:bx] \qquad \text{and} \qquad [ax:by]$$

which correspond to $u\sigma^g_{[a:b]}$ and $u\rho^g_{[a:b]}$ respectively. ■

Theorem 20 *For any non-null projective points* $[a:b]$ *and* $[c:d]$

$$\sigma^g_{[a:b]}\sigma^g_{[c:d]} = \rho^g_{[ad:bc]} \qquad \rho^g_{[a:b]}\rho^g_{[c:d]} = \rho^g_{[ac:bd]}$$

$$\rho^g_{[a:b]}\sigma^g_{[c:d]} = \sigma^g_{[ac:bd]} \qquad \sigma^g_{[a:b]}\rho^g_{[c:d]} = \sigma^g_{[ad:bc]}.$$

Proof. These are immediate. Note that the resultant isometries are also associated to non-null points. ■

We now define the group G^g of all isometries of the green quadrance, and distinguish the subgroup G^g_e of green rotations $\rho^g_{[a:b]}$. These latter are naturally in bijection with the non-null projective points.

The coset G^g_o consists of green reflections $\sigma^g_{[a:b]}$ which are also naturally in bijection with the non-null points. The group G^g naturally acts on the green non-null projective points, as well as on the green null projective points. Both actions are transitive.

The group structure of the green rotations can be transferred to give a **multiplication** of green non-null projective points, by the rule

$$[a:b] \times_g [c:d] = [ac:bd].$$

This multiplication is associative, commutative, has identity $[1:1]$, and the inverse of $[a:b]$ is $[b:a]$. This is a familiar algebraic object: it is just the multiplicative group of non-zero fractions.

In particular if we wish to take powers in this group, then say $[2:3]^2 = [4:9]$ and $[2:3]^3 = [8:27]$ and so on. But each of these powers are then equally spaced with respect to green p-quadrance. As an application we may easily prove the following result about values of spread polynomials.

Theorem 21 *If* $s = -\frac{(y-x)^2}{4xy}$ *then* $S_n(s) = -\frac{(y^n-x^n)^2}{4x^ny^n}$.

Proof. The number $s = -\frac{(y-x)^2}{4xy}$ is $q_g(1,1,x,y)$ while $r = -\frac{(y^n-x^n)^2}{4x^ny^n}$ is $q_g(1,1,x^n,y^n)$. But since in the green multiplication $[x:y]^n = [x^n:y^n]$ we must have $r = S_n(s)$. ■

References

[1] H. S. M. Coxeter, 'The Descartes circle theorem', 189–192, in *The Changing Shape of Geometry*, ed. C. Pritchard, Cambridge University Press, 2003.

[2] S. Goh, *Chebyshev Polynomials and Spread Polynomials*, Honours Thesis, School of Mathematics and Statistics, UNSW, 2005.

[3] D. Pedoe, 'On a theorem in geometry', *The American Mathematical Monthly,* **74** (1967), 627–640.

[4] D. A. Robbins, 'Areas of polygons inscribed in a circle', *The American Mathematical Monthly,* **102** Number 6 (1995), 523–530.

[5] N. J. Wildberger, *Divine Proportions: Rational Trigonometry to Universal Geometry*, Wild Egg Books (http://wildegg.com), Sydney, 2005.

[6] N. J. Wildberger, 'Chromogeometry', preprint 2007.

[7] N. J. Wildberger, 'Affine and projective universal geometry', arXiv:math.MG/0612499v1 18Dec 2006.