The Mutation Game, Coxeter Graphs, and Partially Ordered Multisets

N. J. Wildberger*

Introduction

The Mutation Game

Mars is populated by Martians and anti-Martians, who live in a network of cities connected by one-way roads. There may be many roads, or none at all, between any two cities. Martians and anti-Martians annihilate each other in any given city, so that each city contains only Martians or only anti-Martians or is empty.

Occasionally a city $A$ mutates; its inhabitants all change their ‘type’ (that is Martians turn into anti-Martians and vice versa) while any city $B$ with at least one road leading to $A$ clones its own population sufficiently to send a copy of its population down every road leading to $A$ while leaving its own population intact. These arriving armies at $A$ cancel or add to give the new population at $A$. Thus a mutation at $A$ changes only the population of $A$.

Here is an example of three cities $A$, $B$, $C$ with populations 3, 4 and 10 respectively.

If city $C$ mutates the populations become 3, 4 and $-3$ respectively, where $-3$ means 3 anti-Martians. A subsequent mutation at city $B$ results in populations 3, $-1$, $-3$

---

*University of New South Wales, Sydney 2052, Australia. email: n.wildberger@unsw.edu.au
respectively. A population of just one Martian in just one city will be called a **simple root**. Any population obtainable from a simple root by mutations we will call a **root** population. If $X$ is the directed multigraph representing cities as nodes and roads as directed edges, then we let $R(X)$ denote the set of all possible root populations, considered as a set of functions on the vertices of $X$.

One of the key features of our approach is that we work in the context of directed multigraphs. It is useful to point out at this stage that the case of simple (undirected) graphs can be viewed as a special case by the convention that an undirected edge is equivalent to two directed edges, one in each direction.

There is a wealth of depth and subtlety in this situation, as well as connections with much of modern mathematics, including Lie theory, finite group theory, Von Neumann algebras, hypergroups, algebraic geometry, quadratic forms, spectral theory of graphs, singularities of curves, lattices and coding theory, affine Lie algebras, representation theory, quivers, and combinatorics - and no doubt much more this author is unaware of. To get some sense of the mutation game, the reader is invited to spend a few days pondering the following.

### The Mutation Problem

Show that if $X$ is a simple graph then any root population consists entirely of Martians or entirely of anti-Martians.

### Coxeter graphs

A Coxeter graph is an undirected graph with all its edges labelled with either an integer $k \geq 3$, or $\infty$. By convention edges labelled with 3 are unmarked. If $S$ denotes the set of vertices of a Coxeter graph $\Gamma$, set

$$m(s, s') = \begin{cases} 1 & \text{if } s = s' \\ 2 & \text{if there is no edge between } s, s' \\ \ell & \text{if there is an edge labelled } \ell \text{ between } s, s' \end{cases}$$

Now associate to $\Gamma$ the group $W$ generated by $S$ with relations

$$(ss')^{m(s,s')} = 1$$

for all $s, s' \in S$. For example if $\Gamma$ is $A_{n-1}$

![Diagram of Coxeter graph](image)

with $n - 1$ vertices, then $W$ is $S_n$, the permutation group on $n$ objects. The $j^{th}$ vertex corresponds to the transposition $(j, j + 1) \in S_n$. 
Theorem 0.1. Let \( \Gamma \) be a (connected) Coxeter graph. Then \( W \) is finite iff \( \Gamma \) is one of the following:

\[
\begin{align*}
A_n & \quad \ldots \quad \ (n \geq 1 \text{ vertices}) \\
B_n & \quad 4 \ldots \quad \ (n \geq 2 \text{ vertices}) \\
D_n & \quad \ldots \quad \quad \ (n \geq 4 \text{ vertices}) \\
E_6 & \\
E_7 & \\
E_8 & \\
F_4 & \quad 4 \\
G_2 & \quad 6 \\
H_3 & \quad 5 \\
H_4 & \quad 5 \\
I_2(m) & \quad m \quad \quad \quad \quad \ (m = 5 \text{ or } 7 \leq m < \infty)
\end{align*}
\]

A related result is the following. Let \( b_{s,s'} = -\cos \pi/m(s, s') \), and let \( B_\Gamma \) be the symmetric bilinear form on \( \mathbb{R}^n \ (n = |S|) \) defined by the matrix \( b_{s,s'} \).

Theorem 0.2. The group \( W \) is finite iff \( B_\Gamma \) is positive and nondegenerate.

We may realize \( W \) when it is finite by a group of reflections in \( \mathbb{R}^n \) by defining

\[
\sigma_s(x) = x - 2B_\Gamma(e_s, x)e_s
\]

where \( e_s \) is the basis vector associated to \( s \in S \). It is then true that any finite group of reflections in Euclidean space is obtained this way. Furthermore \( W \) is said to satisfy the crystallographic condition if there is a lattice \( \mathbb{Z}^n \subseteq \mathbb{R}^n \) which is invariant under \( W \). The crystallographic Coxeter groups are then the groups of type \( A, B, D, E, F \) and \( G \), while the groups of type \( H \) and \( I_2(m), \ m = S \text{ or } m \geq 7 \) do not satisfy this condition. The crystallographic Coxeter groups turn out to be the same as the Weyl groups associated to simple Lie algebras and the classification of the latter is closely related to the above and involves Dynkin diagrams.
Before explaining this connection, let us point out that when the Coxeter graph is of type $ADE$, all edges have label 3, the elements $b_{s,s'}$ are either 0, 1 or $-\frac{1}{2}$, so the form $B_\Gamma$ is essentially integral (by multiplying it by 2). However for the other cases $B_\Gamma$ has terms like $-\frac{\sqrt{2}}{2}$, $-\frac{1+\sqrt{5}}{4}$ so is not integral.

**Root systems and Dynkin diagrams**

Let $E$ be a Euclidean space, by which we mean a finite dimensional real vector space with a Euclidean inner product $(,)$. A **root system** $R \subset E$ is a subset such that

1. $R$ is finite, $0 \notin R$ and span $(R) = E$
2. $\forall \alpha \in R$ the reflection $s_\alpha$ defined by
   \[ xs_\alpha = x - 2\frac{(x, \alpha)\alpha}{(\alpha, \alpha)} \]
   sends $R$ to $R$
3. $\forall \alpha, \beta \in R$, $\beta s_\alpha - \beta$ is an integer multiple of $\alpha$.

The root system is called **reduced** if $R \cup \mathbb{R}\alpha = \{\alpha, -\alpha\}$ for all $\alpha \in R$. The associated **Weyl group** is the subgroup $W$ of $GL(E)$ generated by the reflections $s_\alpha, \alpha \in R$. The rank of $R$ is the dimension of $E$.

Here are the rank 2 reduced root systems:
Let $R$ be a reduced root system. Then it is always possible to find a basis $S \subset R$ of $E$ with the property that every element of $R$ is a linear combination of elements of $S$ with either all coefficients non-negative or all coefficients non-positive. Associated to such a choice $S$ of simple roots is the Cartan matrix $(n(\alpha, \beta))_{\alpha, \beta \in S}$ where

$$n(\alpha, \beta) = 2 \frac{(\alpha, \beta)}{(\beta, \beta)} \in \mathbb{Z},$$

and the Dynkin diagram constructed as follows. Draw a vertex for each $s \in S$. Connect $i$ and $j$ with

- i) a simple edge if $n(i, j) = n(j, i) = -1$, $i \overset{\bullet}{\longrightarrow} j$
- ii) a directed double edge from $j$ to $i$ if $2n(i, j) = n(j, i) = -2$, $i \overset{	ext{double}}{\longrightarrow} j$
- iii) a directed triple edge from $j$ to $i$ if $3n(i, j) = n(j, i) = -3$, $i \overset{	ext{triple}}{\longrightarrow} j$

These are the only possibilities. The Dynkin diagram of a root system determines the root system and is essentially unique. If $R$ is irreducible, by which we mean there is no non-trivial decomposition $R = R_1 \cup R_2$ with $R_1 \subset E_1, R_2 \subset E_2$ root systems with $E = E_1 \times E_2$, the Dynkin diagram is connected.

Of the rank 2 root systems $A_1 \times A_1$ is reducible, the others have Dynkin diagrams

- \[ \overset{\bullet}{\longrightarrow}, \quad \overset{\longrightarrow}{\longrightarrow} \text{ and } \overset{\longrightarrow}{\longrightarrow}. \]

**Theorem 0.3.** A reduced irreducible root system $R$ corresponds to exactly one of the following Dynkin diagrams, where in each case the index refers to the number of vertices.

- $A_n$
- $B_n$
- $C_n$
- $D_n$
- $E_6$
- $E_7$
- $E_8$
- $F_4$
- $G_2$
This classification arose historically through the connection with simple Lie algebras. Each such Lie algebra has associated to it a root system and the above theorem is a key ingredient in showing that simple Lie algebras are classified by the same list of Dynkin diagrams.

Note that by replacing Dynkin type bonds \( \bullet \leftrightarrow \) and \( \bullet \leftrightarrow \) with the corresponding Coxeter type bonds \( 4, 6 \) we get a subset of the Coxeter graphs of Theorem ?? in fact exactly the crystallographic Coxeter graphs. This is a reflection of the fact that Weyl groups and crystallographic Coxeter groups coincide.

**Directed Multigraphs**

A directed multigraph \( \mathcal{X} \) is a set of vertices \( \{x_1, \cdots, x_n\} \) and a multiset of directed edges, or ordered pairs \((x_i, x_j)\). We refer to an edge \((x_i, x_j) = e\) as an edge from \(x_i\) to \(x_j\), with \(b(e) = x_i\) and \(f(e) = x_j\). An edge of the form \((x_i, x_i)\) will be called a **loop**. If there are \(m\) edges from \(x_i\) to \(x_j\) and \(n\) edges from \(x_j\) to \(x_i\) then we will abuse notation slightly by saying there is an edge of type \((m, n)\) from \(x_i\) to \(x_j\). Such a situation will be represented by

\[
\begin{align*}
  &\xymatrix{ x_i & \cdots & \cdots & x_j \\
  & m \ar@{-}[l] & n \ar@{-}[l] \\
}
\quad \text{or} \quad
\begin{align*}
  &\xymatrix{ x_i & \cdots & \cdots & x_j \\
  & m \ar@{-}[l] & n \ar@{-}[l] \\
}
\end{align*}
\]

The special cases of edges of type \((1,1)\), \((1,2)\) or \((2,1)\) will be abbreviated as follows:

\[
\begin{align*}
  &\xymatrix{ x & \ar@{-}[l] & y \ar@{-}[l] } \iff \xymatrix{ x & \ar@{-}[l] & y \\
  &\xymatrix{ x & \ar@{-}[l] & y \ar@{-}[l] } \\
  &\xymatrix{ x & \ar@{-}[l] & y \ar@{-}[l] } \iff \xymatrix{ x & \ar@{-}[l] & y }
\end{align*}
\]

In this way we will regard undirected graphs as special cases of directed multigraphs.

Associated to \( \mathcal{X} \) is the adjacency matrix \( A \), an \( n \times n \) matrix indexed by the vertices of \( \mathcal{X} \) with the property that

\[
A(x_i, x_j) = \# \text{ edges from } x_i \text{ to } x_j.
\]

Identifying \( A \) with a linear transformation from \( \mathbb{R}^n \) to \( \mathbb{R}^n \) (acting say on column vectors),

\[
\| A \| = \max\{ \| A \xi \| : \xi \in \mathbb{R}^n, \| \xi \| \leq 1 \} = \| X \|
\]
is the operator norm of $A$, and by definition the norm of $X$. If the eigenvalues of $A$ are $\lambda_1, \cdots, \lambda_n \in \mathbb{C}$ possibly with multiplicity, then

$$sp(A) = \max\{|\lambda_i|; i = 1, \cdots, n\} = sp(X)$$

is the spectral radius of $A$, and by definition the spectral radius of $X$. For symmetric matrices $\|A\| = sp(A)$ but in general $sp(A) \leq \|A\|$.

For undirected graphs, the classification of those $X$ for which $\|X\| = 2$ or $\|X\| < 2$ are well known (see [?]). We are here interested in the more general question of classifying those directed multigraphs $X$ for which $sp(X) = 2$ and $sp(X) < 2$, especially in the case when $X$ has no loops and is bi-directed, meaning for all $i, j$ $A(x_i, x_j) > 0$ implies $A(x_j, x_i) > 0$.

A square $n \times n$ matrix $A$ is irreducible iff for each $i, j \in \{1, \cdots, n\}$ there exists an integer $p$ (depending on $i$ and $j$) such that $A^p(i, j) > 0$. If $A$ has non-negative entries, a Perron-Frobenious vector for $A$ is an eigenvector $\xi \in \mathbb{R}^n$ of $A$ with non-negative entries. If $A, B$ are matrices, we define $A \leq B$ iff $B - A$ has non-negative entries. Here is the fundamental tool one needs in this subject.

**Theorem 0.4 (Perron-Frobenius).** Let $A$ be an irreducible non-negative $n \times n$ matrix. Then

i) a Perron-Frobenius vector $\xi$ exists and is unique up to multiplication by a positive scalar.

ii) $\xi$ corresponds to a simple eigenvalue $\lambda$ which is the spectral radius of $A$.

iii) if $A'$ is a second non-negative $n \times n$ matrix with $A' \leq A$ and $A' \neq A$, then $sp(A') < sp(A)$.

**Theorem 0.5.** Let $X$ be a bi-directed multigraph with no loops. Then $sp(X) = 2$ iff $X$ is in List A below, and $sp(X) < 2$ iff $X$ is in List B below.

**List A**

a) ![Diagram a](image1)

b) ![Diagram b](image2)

c) ![Diagram c](image3)

d) ![Diagram d](image4)

e) ![Diagram e](image5)

**List B**

f) ![Diagram f](image6)
List B

a)  
1 1 1 1 1 1 1

b)  
1 2 2 2 2 1 1 1

c)  
1 2 3 2 1 2 2

d)  
2 3 4 3 2 1 2 2

e)  
2 3 4 3 2 1 2 2

f)  
1 1 1 1 1 1 1 1

or  
1 2 2 2 2 2 2 2
Remark. List B has some duplications - the same graph appears twice under the last 4 entries with different functions appearing. This will be explained below and is needed for the proof and subsequent developments - not in the statement of the Theorem.

Proof. By assumptions of the theorem, the adjacency matrix $A$ is irreducible so the Perron Frobenius Theorem applies to it. Each graph in List A is given with a Perron-Frobenius vector as a function on the vertices. Such a vector $\xi$ can be checked to be associated to the eigenvalue 2 by noting that

$$2\xi(y) = \sum_x A(x, y)\xi(x) = \sum_{x \text{ a neighbour of } y} \xi(x) \times (\# \text{ edges from } x \text{ to } y).$$

Thus each graph in List A has spectral radius 2. Each graph in List B is a proper subgraph of a graph in List A, obtained by removing some vertex and all edges connected to it. It follows from the Perron-Frobenius Theorem that the spectral radius of each graph in List B is strictly less than 2. List B also contains the restrictions of the Perron-Frobenius vectors of those List A graphs which contain the given List B graph. In the last 4 cases there are 2 possible such List A graphs for each List B graph.

Now suppose that $\text{sp}(X) \leq 2$ and $X$ is not in List A. By Perron-Frobenius it cannot contain any of the List A graphs as proper subgraphs, for otherwise $\text{sp}(X) > 2$. Not containing a) means $X$ has no cycles and so is a tree, b) means $X$ has no vertices of degree 4 or more, c) means at most one vertex of degree 3, d) means a degree 3 vertex must have at least one branch of length 1, 3) means a degree 3 vertex must have at most one branch of length 3 or more and f) means a degree 3 vertex with branches of length 1 and 2 has its largest branch of length at most 5.

If $X$ has all edges of type (1,1) then it follows that $X$ either has no vertices of degree 3 or more, in which case it is a) of List B, or it has one vertex of degree 3
with two branches of length 1 in which case it is b) of List B, or it has one vertex of degree 3 with branches of length 1 and 2 and the other of lengths 2,3 or 4, giving c) d) or e) of List B.

\( X \) can have at most one edge not of type (1,1), for otherwise we could find a path between the two edges giving a subgraph of type either g) i) or k) of List A. So suppose \( X \) has exactly one edge not of type (1,1), say joining \( x \) and \( y \). Then h) and j) of List A show that \( X \) contains no vertex of degree 3 or more. Furthermore p) and q) of List A show that the edge between \( x \) and \( y \) must be of type (1,3) or (3,1) or less, and if it is of type (1,3) or (3,1) then n) and p) of List A show it must be i) of List B. If neither \( x \) and \( y \) are endpoints then l) and m) of List A show that \( X \) must be h) of List B. If \( x \) or \( y \) are endpoints the only remaining possibilities are f) and g) in List B.

In conclusion, \( X \) must be one of the graphs of List B, and this concludes the proof. \( \square \)

**Corollary.** If \( X \) is a bidirected multigraph with no loops and \( sp(X) > 2 \) then \( X \) contains a List A graph as a subgraph.

**Proof.** The proof of the Theorem shows that \( sp(X) \leq 2 \) and \( X \) not in List A implies \( X \) is in List B. But in fact the same argument shows that \( sp(X) > 2 \) and \( X \) not containing a List A graph implies \( X \) is in List B, which is impossible by the Theorem. \( \square \)

**Mutation-reflections on directed multigraphs**

Let \( X \) be a directed multigraph, perhaps not connected. Let \( P(X) \) denote the set of integral valued functions on the vertices of \( X \), with \( P_+(X) \) and \( P_-(X) \) respectively the subsets of non-negative and non-positive functions. For each vertex \( x \), we define a map \( s_x : P(X) \to P(X) \) by

\[
(p s_x)(y) = \begin{cases} 
  p(y) & \text{if } y \neq x \\
  -p(x) + \sum_{z \neq x} A(z, x) p(z) & \text{if } y = x
\end{cases}
\]

which we call the mutation, or reflection at \( x \).

**Theorem 0.6.** If \( x \) and \( y \) are vertices of \( X \) then

i) \( s_x^2 = \text{identity} \)

ii) \( s_x s_y = s_y s_x \) if \( x \) and \( y \) are not neighbours

iii) \( s_x s_y \) has finite order greater than 2 iff the edge type of \((x, y)\) is a) (1,1) b) (1,2) or (2,1) or c) (1,3) or (3,1) in which case \( s_x s_y \) has order a) 3 b) 4 or c) 6 respectively.
Proof. i) Since \( s_x \) leaves all vertices except \( x \) fixed, we need only compute that for a population \( p \) in \( P(X) \)

\[
(ps_x^2)(x) = -ps_x(x) + \sum_{z \neq x} A(z, x)ps_x(z)
\]

\[
= -(-p(x) + \sum_{z \neq x} A(z, x)p(z)) + \sum_{z \neq x} A(z, x)p(z)
\]

\[
= p(x)
\]

since \( ps_x(z) = p(z) \) for \( z \neq x \).

ii) Obvious.

iii) Suppose the edge type of \((x, y)\) is \((m, n)\). Let \( p \in P(X) \) and suppose that \( p(x) = a, p(y) = b \) and \( \sum_{b(e) \neq y} f(e) = x \).

Then

\[
(ps_x s_y)(z) = \begin{cases} p(z) & \text{if } z \neq x, z \neq y \\ A + nb - a & \text{if } z = x \\ B + m(A + nb - a) - b & \text{if } z = y. \end{cases}
\]

If we write \( a', b' \) for the values of \( ps_x s_y \) at \( x \) and \( y \) respectively, then

\[
\begin{vmatrix} a' \\ b' \end{vmatrix} = \begin{vmatrix} a \\ b \end{vmatrix} + \begin{vmatrix} A \\ B \end{vmatrix} = M \begin{vmatrix} a \\ b \end{vmatrix} + \begin{vmatrix} c \\ d \end{vmatrix}
\]

where

\[
M = \begin{vmatrix} -1 & n \\ -m & mn - 1 \end{vmatrix}
\]

has integer entries and determinant one, and is not the identity. Iterating, we get

\[
\begin{vmatrix} a^{(k)} \\ b^{(k)} \end{vmatrix} = M^k \begin{vmatrix} a \\ b \end{vmatrix} + (M^{k-1} + \cdots + M + I) \begin{vmatrix} c \\ d \end{vmatrix}
\]

\[
= M^k \begin{vmatrix} a \\ b \end{vmatrix} + \frac{M^k - I}{M - I} \begin{vmatrix} c \\ d \end{vmatrix}.
\]

Thus \( s_x s_y \) has finite order \( k \) iff \( M^k = I \). If the eigenvalues of \( M \) are \( \lambda \) and \( \lambda^{-1} \) then \( M^k = I \) implies \( \lambda + \lambda^{-1} = 2 \cos \frac{2\pi \ell}{k} \) for some \( \ell \). Since also \( \lambda + \lambda^{-1} = \text{tr}M = mn - 2 \) we get

\[
\cos \frac{2\pi \ell}{k} = \frac{nm}{2} - 1.
\]

This has solutions in the list \((n, m) = (t, 0), (0, t), (1, 1), (1, 2), (2, 1), (1, 3), (3, 1), (2, 2), (1, 4), (4, 1)\). The first two result in \( M \) of the form

\[
\begin{vmatrix} -1 & 0 \\ t & -1 \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} -1 & t \\ 0 & -1 \end{vmatrix}
\]

which do
not have finite order, and the last three give \( M \) to be one of
\[
\begin{bmatrix}
-1 & 2 \\
-2 & 3 \\
-1 & 3 \\
-4 & 3 \\
\end{bmatrix},
\]
each of which has 1 as a double eigenvalue, is not diagonalizable, and so not of finite order.

If \((n,m) = (1, 1)\) then \( M = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \) has order 3, if \((n,m) = (1, 2)\) or \((2, 1)\) then \( M = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix} \) or \( \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} \) each of order 4 and if \((n,m) = (1, 3)\) or \((3, 1)\) then \( M = \begin{bmatrix} -1 & 1 \\ -3 & 2 \end{bmatrix} \) or \( \begin{bmatrix} -1 & 3 \\ -1 & 2 \end{bmatrix} \) each of order 6. \(\square\)

We will say that \( X \) is of **finite mutation type** (or finite reflection type) iff all neighbouring vertices have edge types \((1,1), (1,2), (2,1), (1,3)\) or \((3,1)\). In any case we let \( W \) be the group of \( \mathbb{Z} \)-linear transformations of \( \mathcal{P}(X) \) generated by the \( s_x, x \) a vertex of \( X \).

Elements of \( \mathcal{P}(X) \) will be called **populations** on \( X \). A **simple root** is a population which has all values 0 except at a single vertex, say \( x \), where it has value 1 - it will be denoted by \( \delta_x \). Note that \( \delta_x s_x = -\delta_x \). A **root** population is one obtainable from some simple root by a sequence of mutations. We let \( \mathcal{R}(X) \) denote the set of all root populations, or **roots**.

Let \( \mathcal{R}_+(X) = \mathcal{R}(X) \cap \mathcal{P}_+(X) \) and \( \mathcal{R}_-(X) = \mathcal{R}(X) \cap \mathcal{P}_-(X) \) denote the **positive roots** and **negative roots** respectively. Clearly \( \mathcal{R}_-(X) = -\mathcal{R}_+(X) \).

**Proposition 0.1.** Let \( X \) be a bidirected multigraph with no loops, and suppose that \( sp(X) \geq 2 \). Then \( \mathcal{R}(X) \) is an infinite set.

**Proof.** The assumption implies that \( X \) contains a List A graph as a subgraph, so we need only show that \( |\mathcal{R}(X)| = \infty \) for \( X \) in List A. We do this by exhibiting explicitly for each List A graph a simple root \( \delta_x \) and a sequence of mutations \( s_{y_1} s_{y_2} \cdots s_{y_k} \) such that \( \delta_x s_{y_1} s_{y_2} \cdots s_{y_k} = S_x + \alpha_0 \) where \( \alpha_0 \) is the Perron-Frobenius population also given in List A; or a slight variant. In each case this shows, since \( \alpha_0 s_y = \alpha_0 \) for all \( y \), that we may increase populations arbitrarily so that \( \mathcal{R}(X) \) is an infinite set. To simplify matters, we introduce the notation

\[
s_{y_1} s_{y_2} \cdots s_{y_k} = s(y_1, y_2, \ldots, y_k).
\]

The sequences of mutations we give here will have applications later in the theory, we denote the list as List A2, and introduce simultaneously a labelling of each vertex of List A.
List A2

a) 
\[ \delta_1 s(2, 3, \cdots, n, *) = \delta_* + \alpha_0 \]

b) 
\[ \delta_2 s(2, 3, 0, *, 2) = \delta_* + \alpha_0 \]

c) 
\[ \delta_2 s(0, 1, 3, \cdots, n - 3, n - 2, n - 1, *, n - 2, n - 3, \cdots, 3, 2) = \delta_2 + \alpha_0 \]

d) 
\[ \delta_3 s(2, 4, 0, 1, 5, *, 3, 2, 4, 0, 3) = \delta_3 + \alpha_0 \]

e) 
\[ \delta_3 s(2, 4, 0, 1, 5, *, 6, 3, 2, 4, 1, 5, 3, 2, 4, 0, 3) = \delta_3 + \alpha_0 \]
\[
\delta_3 s(2, 4, 0, 1, 5, 6, 7, *, 3, 2, 4, 5, 6, 7, 3, 0, 4, 5, 6, 3, 2, 4, 1, 5, 3, 2, 4, 0, 3) = \delta_3 + \alpha_0
\]

\[
d_s s(1, 2, \cdots, n) = \delta_n + \alpha_0
\]

\[
\delta_{n-1} s(n-1, \cdots, 2, 1) = \delta_{n-1} + \alpha_0
\]

\[
\delta_1 s(2, 3, \cdots, n-1, n, *, n-1, \cdots, 3, 2, 1) = \delta_1 + \alpha_0.
\]

\[
\delta_1 s(2, 3, \cdots, n, *) = \delta_* + \alpha_0
\]

\[
\delta_* s(n, \cdots, 3, 2, 1) = \delta_1 + \alpha_0
\]

\[
\delta_n s(n-1, \cdots, 2, 1, 2, \cdots, n-1, *) = \delta_n + \alpha_0
\]

\[
\delta_* s(n-1, \cdots, 2, 1, 2, \cdots, n-1, *) = \delta_n + \alpha_0
\]
\[ \delta_1 s(2, 3, \ldots, n, *, n, \ldots, 3, 2, 1) = \delta_1 + \alpha_0 \]

\[ \delta_2 s(1, 3, *, 4, 2, 1, 3, 2) = \delta_2 + \alpha_0 \]

\[ \delta_2 s(1, 3, 4, *, 2, 3, 4, 2, 1, 3, 1) = \delta_2 + \alpha_0 \]

\[ \delta_1 s(*, 2, 1) = \delta_1 + \alpha_0 \]

\[ \delta_2 s(*, 1, 2) = \delta_2 + \alpha_0 \]

\[ \delta_1 s(*, 1) = \delta_1 + \alpha_0 \]

\[ \delta_1 s(*, 1) = \delta_1 + \alpha_0. \]
Simple graphs and the canonical bilinear form

Let $X$ be a simple undirected graph. We recall that our conventions for labelling edges means that $X$ is a special kind of directed multigraph, indeed with no loops and bidirected. In this case we will refer to the $s_x$ as reflections, not mutations.

Define a symmetric bilinear $\mathbb{Z}$-valued form on $P(X)$ by linearly extending

$$(\delta_x, \delta_y) = \begin{cases} 
2 & \text{if } x = y \\
-1 & \text{if } x \text{ and } y \text{ are neighbours} \\
0 & \text{otherwise.}
\end{cases}$$

Let $Q(p) = Q_x(p) = (p, p)$ be the associated quadratic form. If $p, q \in P(X)$ are regarded as column vectors then $(p, q) = p^T(2I - A)q$ where $A$ is the adjacency matrix of $X$. This form is non-degenerate iff $2I - A$ is nonsingular iff $A$ does not have eigenvalue 2. If $X$ is in List A, the Perron-Frobenius population, call it $\alpha_0$, satisfies $Q(\alpha_0) = 0$, and if $X$ is in List B then $Q$ is non-degenerate. Furthermore $Q$ is positive-definite iff all eigenvalues of $2I - A$ are positive iff $X$ is in List B. If $X$ is neither in List A or List B then $A$ must have eigenvalues both greater than and less than 2, so that $Q$ is indefinite.

Proposition 0.2. Let $x$ be any vertex of the simple undirected graph $X$. Then for any $p, q \in P(X)$,

$$(ps_x, qs_x) = (p, q).$$

Proof. By linearity it suffices to prove this for $p = \delta_y$, $q = \delta_z$. If $y = z$ then the statement is clear if $x = y$ or if $x$ is not a neighbour of $y$. If $y = z$ and $x$ is a neighbour of $y$ then

$$\delta_y s_x, \delta_y s_x) = (\delta_y + \delta_x, \delta_y + \delta_x) = 2 - 1 - 1 + 2 = (\delta_y, \delta_y).$$

If $y \neq z$ and $y$ and $z$ are not neighbours, then the only case of interest is if $x$ is a joint neighbour of $y$ and $z$, in which case

$$(\delta_y s_x, \delta_y s_x) = (\delta_y + \delta_x, \delta_z + \delta_x) = 0 - 1 + 2 = (\delta_y, \delta_z).$$

If $y$ and $z$ are neighbours and $x$ is a joint neighbour,

$$(\delta_y s_x, \delta_z s_x) = (\delta_y + \delta_x, \delta_z + \delta_x) = -1 - 1 + 2 = (\delta_y, \delta_z).$$

If $y$ and $z$ are neighbours and $x = y$ then

$$(\delta_y s_x, \delta_z s_x) = (-\delta_y, \delta_z + \delta_y) = 1 - 2 = (\delta_y, \delta_z).$$

These are essentially all the cases. $\square$

Corollary. If $X$ is a simple undirected graph in List B then $W$ is a finite group and $R(X)$ is a finite set.
Proof. This follows from the discreteness of \( W \), the fact that \( Q \) is positive definite and that \( W \) preserves \( Q \), and the fact that \( R(X) \) is at most a finite union of \( W \) orbits.

In fact it is easy to see that for \( X \) a (connected) undirected graph, any two simple roots \( \delta_x \) and \( \delta_y \) are in the same \( W \) orbit. Just take a path from \( x \) to \( y \), reflect successively along this path from \( x \) to \( y \) to get the full ‘chain’ from \( x \) to \( y \), and then reflect again from \( x \) to \( y \) along the chain. For example, in \( E_7 \)

\[
\delta_0 s_3 s_4 s_5 s_6 s_0 s_3 s_4 s_5 = \delta_6.
\]

**Quadratic forms on general multigraphs.**

Let \( X \) be a directed multigraph. We will now inquire into the existence of a \( W \)-invariant bilinear or quadratic form on \( P(X) \).

The general quadratic form on \( P(X) \) may be described as follows. Let the vertices of \( X \) be labelled \( x_i \) and let \( p \in P(X) \) with \( p(x_i) = p_i \). Then

\[
Q(p) = \sum_i \alpha_i p_i^2 - \frac{1}{2} \sum_{i \neq j} \beta_{ij} p_i p_j
\]

for some \( \alpha_i \) and \( \beta_{ij} \). Suppose such a \( Q \) is **\( W \)-invariant**, that is

\[
Q(p) = Q(ps_x)
\]

for all mutations \( s_x \) and all populations \( p \).

**Lemma 0.1.** If \( (x_i, x_j) \) and \( (x_j, x_i) \) are not edges of \( X \) then \( \beta_{ij} = 0 \).

**Proof.** Let \( p = \delta_{x_i} + \delta_{x_j} \) and use \( Q(p) = Q(ps_{x_i}) \). \( \square \)

**Lemma 0.2.** Suppose \( (x_i, x_j) \) is an edge of type \((m, n)\). Then \( \beta_{ij} = p_i n = p_j m = \beta_{ji} \).

**Proof.** The edge \( (x_i, x_j) \) looks like

\[
x_i \quad m \quad n \quad x_j
\]
Let \( p = p_i \delta_{x_i} + p_j \delta_{x_j} \). Then

\[
Q(p) = \alpha_i p_i^2 + \alpha_j p_j^2 - \beta_{ij} p_i p_j = Q(ps_{x_i})
\]

\[
= Q(p_i (-p_i + np_j) + p_j \delta_{x_j})
\]

\[
= \alpha_i (-p_i + np_j)^2 + \alpha_j p_j^2 - \beta_{ij} (-p_i + np_j) p_j
\]

\[
= \alpha_i p_i^2 + (\alpha_j + \alpha_i n^2 - \beta_{ij} n) p_j^2 - (2\alpha_i n - \beta_{ij}) p_i p_j
\]

In order for this equality to hold for all \( p_i \) and \( p_j \) we need

\[
\alpha_i n^2 - \beta_{ij} n = 0
\]

and

\[
\beta_{ij} = 2\alpha_i n - \beta_{ij},
\]

that is, we need \( \beta_{ij} = 2\alpha_i n \). Symmetrically we get \( \beta_{ij} = \alpha_j m \).

\[\square\]

A \( W \)-invariant \( Q \) is thus determined by the weights \( \alpha_i \), which must necessarily satisfy the equalities

\[
\alpha_i n = \alpha_j m
\]

if \((x_i, x_j)\) is an edge of type \((m, n)\). Let us declare such a set of weights to be balanced. Notice that if \( n \) or \( m \) is zero, then at least one of the weights \( \alpha_i \) or \( \alpha_j \) is also zero, so that \( Q \) cannot be non-degenerate. The presence of such zeroes complicates the analysis.

Define a directed multigraph \( X \) to be balanced iff for any cycle \( x_1, x_2, \ldots, x_k, x_1 \) the product of the numbers of edges traversed in a forward direction equals the product traversed in the opposite direction. That is if the edges have types as indicated,
then \(m_1m_2 \cdots m_k = n_1n_2 \cdots n_k\).

**Lemma 0.3.** Let \(X\) be a bidirected multigraph with no loops. Then \(X\) has a balanced set of weights iff \(X\) is balanced.

**Proof.** If \(\alpha_i\) is a balanced set of weights on \(X\) then on the cycle \(x_1, x_2, \ldots, x_k, x_1\) pictured above we must have the equalities \(\alpha_1n_1 = \alpha_2m_1, \alpha_2n_2 = \alpha_3m_2, \ldots \alpha_kn_k = \alpha_1m_k\). Taking the product gives

\[
\alpha_1\alpha_2 \cdots \alpha_k n_1 \cdots n_k = \alpha_1 \cdots \alpha_k m_1 \cdots m_k.
\]

If one of the \(\alpha_i\) is zero they all are, which we don’t allow, so that \(n_1 \cdots n_k = m_1 \cdots m_k\).

Conversely given a balanced \(X\) we may assign a weight of say \(\alpha_1 = 1\) to \(x_1\) and then all other weights are determined by connectivity and the bidirected nature of \(X\). Consistency of choices is just equivalent to the above product equality around any cycle.

Note that the weight function is unique up to a constant.

**Theorem 0.7.** Let \(X\) be a balanced bidirected multigraph with no loops. Then \(P(X)\) has a \(W\)-invariant quadratic form, unique up to a constant.

**Proof.** Let \(X\) be a balanced bidirected multigraph with no loops, and let \(\{\alpha_i\}\) be a balanced set of weights on the vertices. For any \(i, j\) let us agree that the edge type of \((x_i, x_j)\) be \((m_{ij}, n_{ij})\) even if these are both zero.

Set

\[
\beta_{ij} = \alpha_jm_{ij} = \alpha_in_{ij} = \beta_{ji}
\]

which is well-defined by our assumption, and let

\[
Q(p) = \sum_i \alpha_i p_i^2 - \frac{1}{2} \sum_{i \neq j} \beta_{ij} p_ip_j.
\]

Fix a vertex \(x_k\). Applying \(s_k\) to \(p\) changes \(p_k\) to \(-p_k + \sum_{\ell \neq k} n_{k\ell}p_\ell\) and leaves the other \(p_i\) unchanged.

Thus

\[
Q(ps_k) = \sum_{i \neq k} \alpha_i p_i^2 + \alpha_k(-p_k + \sum_{\ell \neq k} n_{k\ell}p_\ell)^2 - \frac{1}{2} \sum_{i \neq j} \beta_{ij} p_ip_j
\]

\[ - \frac{1}{2} \sum_{j \neq k} \beta_{kj}(-p_k + \sum_{\ell \neq k} n_{k\ell}p_\ell)p_j \]

\[ - \frac{1}{2} \sum_{i \neq k} \beta_{ik}(-p_k + \sum_{\ell \neq k} n_{k\ell}p_\ell).
\]
In this expression, the coefficient of $p_i^2, i \neq k$, is

$$\alpha_i + \alpha_k n_{ki}^2 - \frac{1}{2} \beta_{ki} n_{ki} - \frac{1}{2} \beta_{ik} n_{ki} = \alpha_i + \beta_{ki} n_{ki} - \beta_{ki} n_{ki}$$

$$= \alpha_i.$$

The coefficient of $p_k^2$ is $\alpha_k$. The coefficient of $p_ip_j$ for $i \neq k, j \neq k, i \neq j$ is

$$2\alpha_k n_{ki} n_{kj} - \frac{1}{2} (\beta_{ij} + \beta_{ji}) - \frac{1}{2} (\beta_{kj} n_{ki} + \beta_{ik} n_{kj})$$

$$= 2\alpha_k n_{ki} n_{kj} - \beta_{ij} - \beta_{ik} n_{ki} - \beta_{kj} n_{ki}$$

$$= -\beta_{ij} + 2\alpha_k n_{ki} n_{kj} - \alpha_k n_{ki} n_{kj} - \alpha_k n_{kj} n_{ki}$$

$$= -\beta_{ij},$$

which is also the coefficient of $p_i p_j$ in $Q(p)$. Finally, the coefficient of $p_i p_k$ for $i \neq k$ is

$$-2\alpha_k n_{ki} + \frac{1}{2} \beta_{ki} + \frac{1}{2} \beta_{ik}$$

$$= -2\alpha_k n_{ki} + \beta_{ki}$$

$$= -\alpha_k n_{ki}$$

$$= -\beta_{ki}$$

which is also in agreement with $Q(p)$. Thus $Q(p) = Q(p_{sk})$ and we are done. \qed

**Corollary.** If $X$ is a bidirected multigraph which is a tree, then $\mathcal{P}(X)$ has a $W$-invariant quadratic form, unique up to a constant.

**Example.** Here is a balanced bidirected multigraph $X$ with weights as indicated

![Diagram of a balanced bidirected multigraph](image)

\begin{align*}
\alpha_1 &= 3 & \beta_{12} &= 12 \\
\alpha_2 &= 4 & \beta_{23} &= 36 \\
\alpha_3 &= 18 & \beta_{34} &= 180 \\
\alpha_4 &= 60 & \beta_{41} &= 60
\end{align*}

The $W$-invariant quadratic form is

$$Q(p) = 3p_1^2 - 12p_1p_2 + 4p_2^2 - 36p_2p_3 + 18p_3^2 - 180p_3p_4 + 60p_4^2 - 60p_4p_1.$$  

**Theorem 0.8.** Let $X$ be a bidirected multigraph with no loops. Then $\mathcal{P}(X)$ has a $W$-invariant bilinear form iff $X$ is balanced. In this case the bilinear from is symmetric and unique up to a constant.

**Proof.** If $\mathcal{P}(X)$ has a $W$-invariant bilinear form then it has a $W$-invariant quadratic form so it follows from our earlier discussion that $X$ is balanced. Suppose that $X$ is
balanced with \( \{\alpha_i\} \) a balanced set of weights on the vertices and \( Q \) a \( W \)-invariant quadratic form. Define the symmetric bilinear form \( (\cdot,\cdot) \) by
\[
(p,q) = (q,p) = \frac{1}{4}(Q(p+q) - Q(p-q))
\]
which is clearly \( W \)-invariant. We need only show this is the only choice of \( W \)-invariant bilinear form with associated quadratic form \( Q \).

To this end, let \( (\cdot,\cdot) \) be an arbitrary such form. If \( (x,y) \) is an edge of type \((m,n)\) then
\[
\langle \delta_x, \delta_y \rangle = \langle \delta_x s_x, \delta_y s_x \rangle = \langle -\delta_x, \delta_y + n\delta_x \rangle = \langle \delta_x s_y, \delta_y s_y \rangle = \langle \delta_x + m\delta_y, -\delta_y \rangle
\]
from which we deduce that
\[
\langle \delta_x, \delta_y \rangle = -\frac{1}{2}n\alpha_x = -\frac{1}{2}m\alpha_y = \langle \delta_y, \delta_x \rangle.
\]
Thus \( (\cdot,\cdot) \) is symmetric, which means it must coincide with \( (\cdot,\cdot) \).

\textbf{Theorem 0.9.} Let \( X \) be a balanced bidirected multigraph with no loops, with \( W \)-invariant quadratic form \( Q \). Then \( W \) is finite iff \( R(X) \) is finite iff \( Q \) is positive definite iff \( \text{sp}(X) < 2 \) iff \( X \) is in List B.

\textit{Proof.} We already know that \( \text{sp}(X) < 2 \) iff \( X \) is in List B. If \( X \) is in List B and \( X \) is undirected then we know \( Q \) is positive definite, \( R(X) \) is finite and \( W \) is finite. The remaining graphs in List B have the following associated quadratic forms \( Q \), as represented by a matrix.
\[
\begin{pmatrix}
1 & -1 \\
-1 & 2 & -1 \\
-1 & 2
\end{pmatrix}
\]
Minors: 1, 1, 1, \cdots
g) \[
\begin{array}{cccc}
1 & 2 & 3 & \cdots & n-1 & n \\
\end{array}
\]

Minors: 4, 4, 4, \cdots

h) \[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\end{array}
\]

Minors: 2, 3, 4, 4

i) \[
\begin{array}{cccc}
\end{array}
\]

Minors: 2, 3

The minors of these matrices are shown and can be calculated easily - an induction is needed for f) and g). The positive definiteness of \(Q\) in each case follows. Now since \(W\) is discrete and preserves \(Q\) it follows that \(W\) is finite and then so is \(R(X)\).

On the other hand if \(X\) is not in List B then it contains a List A subgraph, so \(sp(X) \geq 2\). But then Proposition shows \(R(X)\) is infinite and so \(W\) is infinite too, and \(Q\) must be indefinite. This concludes the proof.

Let \(X\) be balanced with weight set \(\{\alpha_i\}\) and

\[
Q(p) = \sum \alpha_i p_i^2 - \frac{1}{2} \sum_{i \neq j} \beta_{ij} p_i p_j
\]

as before, where \(\beta_{ij} = \alpha_j m_{ij} = \alpha_i n_{ij} = \beta_{ij}\) for \((x_i, x_j)\) an edge of type \((m_{ij}, n_{ij})\).

**Lemma 0.4.** The associated bilinear form satisfies

\[
(\delta_{x_i}, \delta_{x_j}) = \begin{cases} 
\alpha_i & \text{if } x_i = x_j \\
-\frac{1}{2} \beta_{ij} & \text{if } x_i \neq x_j 
\end{cases}
\]
Proof. i) If \( x_i = x_j \) then

\[
(\delta_{x_i}, \delta_{x_i}) = \frac{1}{4}(Q(\delta_{x_i} + \delta_{x_i}) - Q(\delta_{x_i} - \delta_{x_i})) = \alpha_i
\]

ii) while if \( x_i \neq x_j \) then

\[
(\delta_{x_i}, \delta_{x_j}) = \frac{1}{4}(Q(\delta_{x_i} + \delta_{x_j}) - Q(\delta_{x_i} - \delta_{x_j}))
\]

\[
= -\frac{1}{2}\beta_{ij}.
\]

\[\square\]

Remark. In the list A and B graphs, the \( \beta_{ij} \) are even if the \( \alpha_i \) are chosen to be even. Clearly we can choose the \( \alpha_i \) so that all the \( \alpha_i \) and \( \beta_{ij} \) are integers, which we henceforth assume.

Explicit Posets of Positive Roots

Let \( X \) be a bi-directed multigraph with no loops, and \( R_+(X) \) the set of positive roots of \( X \). Declare \( p \leq q \) iff \( p(x) \leq q(x) \) for all vertices \( x \), and \( p \leq q \) iff \( p \leq q \) and \( q \) is obtained from \( q \) by a series of mutations that increase populations. Let \( R_{++}(X) \) denote the set \( R_+(X) \) with the partial order \( \leq \); with the partial order \( \leq \) we refer to \( R_+(X) \).

We now list explicitly all the posets so obtained, with the infinite series being indicated by a representative sampling - the pattern is in all cases obvious [hopefully]. It is practical to depict the reverse Hasse diagrams of posets. Another useful convention is to indicate the difference between adjacent roots, necessarily a multiple of a simple root, or vertex, by a multiple edge.
\( R_+(A_4) = R_{++}(A_4) \)

\( R_{++}(B_2) \)

\( R_{++}(B_3) \)
$B_4$

1000
1200
1220
1222
1221

0100 0010 0001
1100 0110 0011
1110 0111
1210 1111
1211

$R_{++}(B_4)$

$C_2$

10
11
21

01
$R_{++}(C_2)$

$C_3$

100
110
111
210
211
221

010 001
011
$R_{++}(C_3)$
$C_4$

$D_4$

$R_+(D_4) = R_{++}(D_4)$
$D_5$

\[
\begin{array}{cccc}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 \\
1 & 2 & 1 & 0 \\
1 & 2 & 1 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 \\
0 & 2 & 1 & 0 \\
0 & 2 & 1 & 1 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 \\
1 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 \\
1 & 2 & 1 & 0 \\
1 & 2 & 1 & 1 \\
1 & 2 & 2 & 1 \\
\end{array}
\]

$R_+(D_5) = R_{++}(D_5)$
$E_6$

$R_+ (E_6) = R_{++} (E_6)$
The Dual Mutation Game

Venus is populated by Venetians and anti-Venetians, who, like the inhabitants of Mars, live in a network of cities connected by one way roads. The only difference is what happens when a city on Venus mutates. In this case, the city clones itself sufficiently to send an army down every road leading away from the city while its original population all change their type.

Here is an example of three cities $A$, $B$, $C$ with populations $12, 5$ and $1$ respectively.

A dual mutation at $A$ results in populations of $-12, 17, 13$ respectively. A subsequent dual mutation at $B$ results in populations of $22, -17$ and $47$ respectively. For reasons that will become clear shortly, we prefer to think of dual mutations as acting on an identical but separate space of dual populations where $X$ is the underlying directed multigraph of cities and roads on Venus. We will denote the singleton dual population at $x$ by $\epsilon_x$, and refer to it as the fundamental weight at the vertex $x$. Any dual population obtainable from a fundamental weight by a series of dual mutations will be called a weight and the set of all weights of $X$ will be denoted $Q(X)$.

More precisely for a vertex $x$ and a dual population $q$, we define the dual mutation/reflection $r_x$ at $x$ by

$$(qr_x)(y) = \begin{cases} 
-q(y) & \text{if } y = x \\
q(y) + A(x, y)q(x) & \text{if } y \neq x
\end{cases}$$

**Theorem 0.10.** If $x$ and $y$ are vertices of $X$ then

i) $r_x^2 = \text{identity}$

ii) $r_xr_y = r_yr_x$ if $x$ and $y$ are not neighbours

iii) $r_xr_y$ has finite order greater than 2 iff the edge type of $(x, y)$ is a) $(1,1)$ b) $(1,2)$ or $(2,1)$ or c) $(1,3)$ or $(3,1)$ in which case $r_xr_y$ has order a) $3$ b) $4$ or c) $6$ respectively.
Proof. i) Clearly \( q^2_r(x) = q(x) \) for any dual population \( q \). For \( y \neq x \)

\[
q^2_r(y) = q_r(x) + A(x, y)q_r(x) = q(y) + A(x, y)q(x) + A(x, y)(-q(x)) = q(y).
\]

ii) Obvious.

iii) Suppose the edge type of \( (x, y) \) is \((m, n)\). Let \( q \in Q(X) \) and suppose that \( q(x) = a, q(y) = b \) and \( z \) another vertex of \( X \). Then if \( q(z) = c \),

\[
(q_r r_y)(z) = \begin{cases}
  nb + ma - a & \text{if } z = x \\
  -b - ma & \text{if } z = y \\
  q(z) + (A + Bm)a + Bb & \text{if } z \neq x, y \text{ with } A(x, z) = A, A(y, z) = B
\end{cases}
\]

If we write \( a', b' \) for the values of \( q_r r_y \) at \( x \) and \( y \) respectively, then

\[
\begin{vmatrix}
  a' \\
  b'
\end{vmatrix}
= \begin{vmatrix}
  mn - 1 & n \\
  -m & -1
\end{vmatrix}
\begin{vmatrix}
  a \\
  b
\end{vmatrix} = M \begin{vmatrix}
  a \\
  b
\end{vmatrix}
\]

while the value at \( z \) of \( q_r r_y \) is

\[
c' = c + (A + Bm, B) \begin{vmatrix}
  a \\
  b
\end{vmatrix}.
\]

After \( k \) iterations,

\[
\begin{vmatrix}
  a^{(k)} \\
  b^{(k)}
\end{vmatrix} = M^k \begin{vmatrix}
  a \\
  b
\end{vmatrix}
\]

while

\[
c^{(k)} = c + (A + Bm, B)(I + M + \cdots + M^{k-1}) \begin{vmatrix}
  a \\
  b
\end{vmatrix}.
\]

After this the analysis and conclusion follows the proof of Theorem (0.10). \( \square \)

We now show that \( s_x \) and \( r_x \) are adjoint operators. Define \( \tau : \mathcal{P}(X) \to Q(X) \) by the rule \( p\tau = q \) where

\[
q(x) = p\tau(x) = (ps_x - p)(x).
\]

Thus \( q(x) \) measures the change in \( p \) at \( x \) resulting from a mutation at \( x \).

**Proposition 0.3.** For any \( p \in \mathcal{P}(X) \) and any vertex \( x \) of \( X \),

\[
ps_x \tau = p\tau r_x.
\]
Proof. We show that $p_s x \tau(y) = p \tau r_x(y)$ for any vertex $y$.

i) $y = x$

\[
\begin{align*}
p_s x \tau(x) &= (p_s x s_x - p_s x)(x) \\
&= 2p(x) - \sum_{z \neq x} A(z, x)p(z) \\
&= -p \tau(x) \\
&= p \tau r_x(x).
\end{align*}
\]

ii) $y \neq x$

\[
\begin{align*}
p_s x \tau(y) &= (p_s x s_y - p_s x)(y) \\
&= \sum_z A(z, y)p_s x(z) - 2p_s x(y) \\
&= \sum_z A(z, y)p(z) + A(x, y)(-p(x) + \sum_{z \neq x} A(z, x)p(z)) - 2p(y)
\end{align*}
\]

while

\[
\begin{align*}
p \tau r_x(y) &= p T(y) + A(x, y)p \tau(x) \\
&= (p s_y - p)(y) + A(x, y)(p s_x - p)(x) \\
&= -2p(y) + \sum_z A(z, y)p(z) + A(x, y)(-2p(x) + \sum_z A(z, x)p(z)) \\
&= p s_x \tau(y).
\end{align*}
\]

The formula $p \tau(x) = -2p(x) + \sum_z A(z, x)p(z)$ means that as an operator $\tau$ can be represented by the matrix $A - 2I$. It is then invertible iff $A$ does not have eigenvalue 2. In particular $\tau$ is invertible if $X$ is in List B, and $\tau$ is not invertible if $X$ is in List A. In the latter case, the Perron Frobenius vector $\alpha_0$ is the kernel of $\tau$. The inverse operator, or matrix, $(A - 2I)^{-1}$, contains significant information when it exists.

There are a number of advantages to working with dual mutations. One is that restrictions to subgraphs work better. If $Y$ is a subgraph of $X$, obtained by taking a subset of the vertices of $X$ and all edges of $X$ between them, then for $y$ a vertex of $Y$, $r_y$ can be considered as operating on populations of $X$ or of $Y$. If $p \in \mathcal{P}(X)$,

\[
\left. p r_y \right|_Y = p \left|_Y r_y.
\]

The corresponding statement for mutations is false.

Another advantage is that dual mutations reveal increases clearly. We say a mutation $s_x$ increases a population $p$ if $p s_x(x) > p(x)$, and that it increments $p$ if $p s_x(x) = p(x) + 1$. If $q = p \tau$ is the dual population, then $s_x$ increases $p$ iff $q(x) > 0$ and increments $p$ iff $q(x) = 1$.

We say that a sequence $s(x_1, \cdots, x_k)$ of mutations is increasing on a population $p$ if $s(x_l)$ is increasing on $p s(x_1, \cdots, x_{l-1})$ for all $l = 1, \cdots, k$. This is clearly equivalent
to each dual mutation acting only on vertices where the dual mutation is positive. For a population \( p \), let \( I(p) = \{ x | p\tau(x) > 0 \} \) be the increasing set for \( p \), and \( I_1(p) = \{ x | p\tau(x) = 1 \} \) the incrementing set for \( p \). Suppose \( I(p) \) is a subset of the vertices of \( X \) with the property that no two elements are neighbours - an isolated subset. Then if \( x_1, x_2, \cdots, x_k \) is an ordering of \( I(p) \), the population

\[
p_s(x_1, \cdots, x_k) = p_s(I(p))
\]

is actually independent of the given ordering since all the \( s_{x_i} \) commute.

As an application, suppose \( X \) is bipartite-its vertices can be coloured red and black so that all edges join vertices of different colours. If \( x \) is any vertex, \( \delta_x \tau \) is a dual population with value \(-2\) at \( x \) and value \( m \) at any vertex \( y \) with \( m \) edges from \( x \) to \( y \). In particular, \( I(p) \) consists of vertices all of the same colour. Then \( p_s(I(p)) = p_2 \) also has this property, since on the dual side each \( r_y, y \in I(p) \) changes the sign of \( p\tau = q \) at \( y \) and only increases vertices of a different colour to itself. It follows that the sequence \( p_1, p_2, p_3, \cdots \) where \( p_k = p_{k-1}s(I(p_{k-1})) \) is an increasing sequence of populations until \( I(p_k) = \phi \) for some \( k \). We call it the canonical increasing sequence from \( \delta_x = p \). A natural, and interesting question is: For which graphs \( X \) (bipartite say) and which vertices \( x \) does this sequence terminate?

An \( X \)-frame \( F \) is increasing [incremental] on a population \( p \in \mathcal{P}(X) \) iff for any total ordering \( x_1, \cdots, x_k \) of \( F \) each mutation in the sequence \( p s_{x_1}, s_{x_2} \cdots s_{x_k} \) acts as an increase [increment]. An increasing [incremental] \( X \)-frame \( F \) on \( p \) is fully increasing [fully incremental] iff any \( p' \in \mathcal{P}(X) \) obtainable from \( p \) by increasing [incremental] mutations can be obtained from some total ordering of \( F \) applied to \( p \) and truncated at some stage.

We now introduce a partial order on \( \mathcal{P}(X) \) by \( p \leq p' \) iff 1) \( p(x) \leq p'(x) \) for all vertices \( x \) and 2) \( p' \) can be obtained from \( p \) by a sequence of increasing mutations. We will be most interested in the restriction of \( \leq \) to \( \mathcal{R}(X) \).

For a vertex \( x \) of \( X \), let

\[
\mathcal{R}(X, x) = \{ p \in \mathcal{R}(x) | \delta_x \leq p \}.
\]

**Proposition 0.4.** Let \( F \) be a fully increasing \( X \)-frame on \( \delta_x \) for some vertex \( x \) of \( X \). Then as posets

\[
J(F) \simeq \mathcal{R}(X, x).
\]

**Proof.** Let \( T : J(F) \rightarrow \mathcal{R}(X, x) \) be defined by \( T(I) = \delta_x s_{y_1} - s_{y_k} \) for some total ordering \( y_1, \cdots, y_k \) of \( I \). This is well defined, respects the partial orders on both sets because \( F \) is increasing and is onto since \( F \) is fully increasing.

To show that \( T \) is \( | \cdot | \), suppose \( T(I_1) = T(I_2) \) with \( I_1 \neq I_2 \). Let \( I = I_1 \cap I_2 \) and choose \( x \in I \), say with \( x \notin I \) but with \( x \) directly above \( I \), that is \( y \leq x \) for all \( y \in I \) and \( y \leq z \leq x \) for some \( y \in I \) implies \( z = x \) or \( z \in I \). Since the occurrence \( x \) is not in \( I_2 \) it means that the vertex \( x \) occurs later in \( I_2 \), in fact it must occur after an
occurrence of some neighbouring vertex say $z$. That means when $x$ does occur in $I_2$, the correspondence mutation increases $x$ more than does the occurrence of $x$ in $I$, directly after $I$. It follows that $\delta_w s(I) s_x$.

**An example.** Let us illustrate explicitly the correspondence between mutations and dual mutations for the graph $X$ labelled as shown.

![Graph X](image)

We consider the sequence of mutations $s_w s_z s_x s_y$ acting on the population $\delta_y$.

![Mutations on X](image)
Reflections

Suppose that $X$ is a balanced bidirected multigraph with no loops, and that $(\ , \ )$ is a $W$-invariant bilinear form on $\mathcal{P}(X)$, assumed here non-degenerate.

**Lemma 0.5.** For any vertex $x$ of $X$ and $p \in \mathcal{P}(X)$,

$$ ps_x = p - \frac{2(p, \delta_x)}{(\delta_x, \delta_x)} \delta_x. $$

**Proof.** Let $\{\alpha_i\}$ be a balanced set of weights on the vertices of $X$ and

$$ \mathcal{Q}(p) = \sum_i \alpha_i p_i^2 - \frac{1}{2} \sum_{i \neq j} \beta_{ij} p_i p_j $$

the corresponding $W$-invariant quadratic form.
Then \((\delta_x, \delta_x) = \alpha_x\) and
\[
(p, \delta_x) = p(x)\alpha_x - \frac{1}{2} \sum_{y \neq x} \beta_{yx} p(y).
\]
But \(\beta_{yx} = \alpha_x m_{yx}\) where \(m_{yx} = A(y, x)\). Thus
\[
(p, \delta_x) = \alpha_0(p(x) = \frac{1}{2} \sum_{y \neq x} m_{yx} p(y))
\]
and
\[
\frac{p - 2(p, \delta_x)}{(\delta_x, \delta_x)} \delta_x = p - (2p(x)\delta_x - \sum_{y \neq x} m_{yx} p(y)\delta_x) = ps_x.
\]

Note that this identifies \(s_x\) as a legitimate reflection with respect to \((\ , \ )\) on \(P(X)\).

Let \(P_Q(X)\) denote the space of \(Q\)-valued functions on the vertices of \(X\). The previous Lemma motivates us to define, for any \(q \in P(X)\), the operator
\[
ps_q = p - \frac{2(p, q)}{(q, q)} q
\]
acting on \(P_Q(X)\), which we call the reflection at \(p\).

**Proposition 0.5.** Let \(p, q \in P(X)\). Then \(s^{-1}_q = s_q\) and
\[
s_{qs_p} = s_p^{-1}s_q s_p = s_p s_q s_p.
\]

**Proof.** For any \(r \in P_Q(X)\),
\[
r s_{qs_p} = r - \frac{2(r, qs_p)}{(qs_p, qs_p)} qs_p.
\]
Here \((qs_p, qs_p)\) refers to the \(Q\)-linear extension of \((\ , \ )\) to \(P_Q(X)\). One checks easily that \((qs_p, q, s_p) = (q, q)\). In fact for any \(q, q' \in P(X)\)
\[
(qs_p, q's_p) = (q - 2\frac{(q, p)}{(p, p)} p, q' - 2\frac{(q', p)}{(p, p)} p)
\]
\[
= (q, q') - 2\frac{(q, p)(p, p)}{(p, p)} - 2\frac{(q', p)(p, q)}{(p, p)} + 4\frac{(q, p)(q', p)}{(p, p)}
\]
\[
= (q, q').
\]

36
Thus,
\[ rs_{qs} = r - 2 \frac{(rs_p^{-1}, q)}{(q, q)} qs_p. \]
\[ = (rs_p^{-1} - 2 \frac{(rs_p^{-1}, q)}{(q, q)} q)s_p \]
\[ = rs_p^{-1}s_q s_p. \]

Finally,
\[ ps_q^2 = ps_q - 2 \frac{(ps_q, q)}{(q, q)} q \]
\[ = p - 2 \frac{(p, q)}{(q, q)} q - 2 \frac{(p, q)}{(q, q)} q + 4 \frac{(p, q)}{(q, q)} (q, q) q \]
\[ = p \]

so \( s_q^{-1} = s_q. \)

\textbf{Corollary.} For \( p \in \mathcal{R}(X), \mathcal{P}(X)s_p = \mathcal{P}(X). \)

\textit{Proof.} Clearly \( s_x \) takes \( \mathcal{P}(X) \) to \( \mathcal{P}(X) \) for any vertex \( x \). Since any root \( p \in \mathcal{R}(X) \) is obtained from a simple root by reflections, we may write \( p = \delta_x s_{y_1} s_{y_2} \cdots s_{y_k} \) for some vertices \( x, y_1, \cdots, y_k \). But then
\[ s_p = s_{y_k} \cdots s_{y_1} s_x s_{y_1} \cdots s_{y_k} \]
so \( s_p \) takes \( \mathcal{P}(X) \) to \( \mathcal{P}(X). \)

Let \( T = \{ s_p \mid p \in \mathcal{R}(X) \} \subseteq W. \) We call \( T \) the set of \textit{reflections} in \( W. \) The above shows that \( T \) consists exactly of the conjugates of \( S = \{ s_x \mid x \text{ a vertex of } X \} \) in \( W. \)

\textbf{Proposition 0.6.} The map \( p \mapsto s_p \) establishes a bijection between \( \mathcal{R}(X) \) and \( T. \)

\textit{Proof.} If \( s_p = s_q \) then
\[ ps_p = -p = ps_q = p - 2 \frac{(p, q)}{(q, q)} q \]
so that \( 2p = 2 \frac{(p, q)}{(q, q)} q. \) Thus \( p \) and \( q \) are multiples, i.e. \( (q, q)p = (p, q)q. \) But by symmetry, \( (p, q)p = (q, p)p. \) Thus \( (q, q)(p, p) = (p, q)^2. \) Suppose \( p \) is derived from the simple root \( \delta_x, \) and \( q \) is derived from the simple root \( \delta_y. \) But then \( \delta_x w \) is a multiple of \( \delta_y \) for some \( w \) and conversely \( \delta_y v \) is a multiple of \( \delta_x \) for some \( v \) in \( W. \) But since \( W \) acts invertibly on \( \mathcal{P}(X), \) these multiples must be 1, and \( p = q. \)
We now define an important map \( \psi \) from sequences in \( T \) to sequences in \( T \). Given \((s_1, s_2, \ldots, s_k)\) in \( T \), define

\[
\begin{align*}
t_1 &= s_1 \\
t_2 &= s_1 s_2 s_1 = s_1^{-1} s_2 s_1 \\
t_3 &= s_1 s_2 s_3 s_2 s_1 = (s_2 s_1)^{-1} s_3 (s_2 s_1) \\
& \vdots \\
t_k &= s_1 s_2 \cdots s_k s_{k-1} \cdots s_1 = (s_{k-1} \cdots s_1)^{-1} s_k (s_{k-1} \cdots s_1).
\end{align*}
\]

Note then that \((t_1, \ldots, t_k) = \psi(s_1, \ldots, s_k)\) is a sequence in \( T \) and that

\[
\begin{align*}
t_1 &= s_1 \\
t_1 t_2 &= s_2 s_1 \\
t_1 t_2 t_3 &= s_3 s_2 s_1 \\
& \vdots \\
t_1 \cdots t_k &= s_k \cdots s_1.
\end{align*}
\]

This makes it clear that \( \psi^2 = \text{identity} \). We call \( \psi \) the complementation map.

**Lemma 0.6.** Suppose that \( s_i, s_{i+1} \) commute. Then \( t_i, t_{i+1} \) commute.

**Proof.**

\[
\begin{align*}
t_k &= (s_{k-1} \cdots s_1)^{-1} s_k (s_{k-1} \cdots s_1) \\
t_{k+1} &= (s_k \cdots s_1)^{-1} s_{k+1} (s_k \cdots s_1)
\end{align*}
\]

So

\[
\begin{align*}
t_k t_{k+1} &= (s_{k-1} \cdots s_1)^{-1} s_k (s_{k-1} \cdots s_1) (s_k \cdots s_1)^{-1} s_{k+1} (s_k \cdots s_1) \\
&= (s_{k-1} \cdots s_1)^{-1} s_k s_{k+1} (s_{k-1} \cdots s_1)
\end{align*}
\]

\[
\begin{align*}
t_{k+1} t_k &= (s_k \cdots s_1)^{-1} s_{k+1} (s_k \cdots s_1) (s_{k-1} \cdots s_1)^{-1} s_k (s_{k-1} \cdots s_1) \\
&= (s_{k-1} \cdots s_1)^{-1} s_k^{-1} s_{k+1} s_k^2 (s_{k-1} \cdots s_1)
\end{align*}
\]

Now if \( s_k \) and \( s_{k+1} \) commute then \( t_k \) and \( t_{k+1} \) commute. \( \square \)

**Proposition 0.7.** If \( s_i \) is reflection in \( \alpha_i \) for \( i = 1, \ldots, k \) then \( t_i \) is reflection in \( \alpha_i s_{i-1} \cdots s_1 = \alpha_i t_1 \cdots t_{i-1} \) for \( i = 1, \ldots, k \).

**Proof.** Follows immediately from Proposition and our definition of \( \psi \). \( \square \)

Since \( T \) and \( R_+(X) \) may be naturally identified, we may use the previous Proposition to define \( \psi \) on sequences of positive roots. Thus more generally if \( a = (p_1^{a_1}, p_2^{a_2}, \ldots, p_k^{a_k}) \) is a multi sequence of roots (with \( p_i \) occurring with multiplicity \( a_i \)), then define

\[
\psi(a) = (p_1^{a_1} p_2^{a_2} s_{p_1}, p_3^{a_3} s_{p_2} s_{p_1}, \ldots, p_k^{a_k} s_{p_{k-1}} \cdots s_{p_1})
\]
where by convention $p_i^{a_i}s_{p_j} = (p_izs_{p_j})^{a_i}$. Note that this is just linearity of the $s_{p_j}$; after all $p_i^{a_i}$ is just a notational shorthand for $a_ip_i$.

We should also warn the reader that even though all the $p_i$ may be positive roots, there is no guarantee that each element of $\psi(a)$ is a positive root - in general it will be only a multiple of a root.

0.1 The complementation map and permutations on Mars

This section is an aside on the meaning of the map $\psi$ and is not needed elsewhere in the paper.

Differences in physiology and psychology between Martians and Earthlings manifest themselves in striking disparities when it comes to mathematics. The dependence on a linear notational system with the subsequent heavy emphasis on labelling objects with integers found on Earth is absent from Mars, where various unordered labelling systems are in common use, even amongst children. Unfortunately this renders Martian approaches to permutation theory awkward to describe. We will do our best.

Consider say a set of objects like \[\{\triangle, +, \bigcirc, \times\}\] and a set of positions like \[\{\text{B, L, M, R, F}\}\], where these are short for Left, Middle, Right, Back, and Forwards not respectively. Here is an assignment of objects to positions, or equivalently positions to objects [on Mars there is complete symmetry between the notions], and a series of permutations of these objects/positions with the corresponding notation that Martians and anti Martians use to describe them. [Martians follow objects; anti Martians follow positions.]
4. \( \left( \begin{array}{cccc} \triangle & \times & + & \bigcirc & \square \\ \square & \triangle & \bigcirc & + & \times \end{array} \right) = T_1 \)

3. \( \left( \begin{array}{cccc} L & M & R & B & F \\ F & L & B & R & M \end{array} \right) = \sigma_1 \)

2. \( \left( \begin{array}{c} \bigcirc \\ \bigcirc & \triangle & + & \bigcirc & + & \bigcirc & + \end{array} \right) = T_2 \)

5. \( \left( \begin{array}{cccc} L & M & R & B & F \\ B & L & R & F & M \end{array} \right) = \sigma_2 \)

1. \( \left( \begin{array}{cccc} \bigcirc \\ \bigcirc & \triangle & + & \bigcirc & + \end{array} \right) = T_3 \)

7. \( \left( \begin{array}{cccc} L & M & B & F & R \\ B & R & F & L & M \end{array} \right) = \sigma_3 \)

Explanations

1. The original assignment of objects to positions or positions to objects. We say object □’s position is L or equivalently position L’s object is □. Such an assignment we denoted mathematically by the array

\[ \alpha = \left[ \begin{array}{ccccc} \square & \bigtriangleup & \bigcirc & + & \bigtimes \\ L & M & R & B & F \end{array} \right] \]

For a Martian this would confusing since the order of the symbols is unimportant in the sense that only the pairs \( \{(\square L), (\bigtriangleup M), (\bigcirc R), (\bigtimes B), (\bigtimes F)\} \) have meaning.

2. After the first permutation we get the new assignment

\[ \left[ \begin{array}{ccccc} \bigtriangleup & \times & + & \bigcirc & \square \\ L & M & R & B & F \end{array} \right] \]
The first permutation is given by the sequence of instructions ‘position L’s object goes to position F’ etc. This is the anti-Martian approach which we call a **position permutation**.

It can equivalently be given by the sequence ‘object □’s position goes to object △’ etc. This is the Martian approach which we call a **object permutation**.

Thus the position permutation \( \begin{pmatrix} L & M & R & B & F \\ F & L & B & R & M \end{pmatrix} \) is read ‘top to bottom’

while the object permutation \( \begin{pmatrix} △ & × & + & □ \\ □ & △ & ○ & + & × \end{pmatrix} \) is read ‘bottom to top’.

As before these arrays are to be considered as shorthands for the two sets

\[
\left\{ \begin{pmatrix} L \\ F \end{pmatrix}, \begin{pmatrix} B \\ R \end{pmatrix}, \begin{pmatrix} M \\ L \end{pmatrix}, \begin{pmatrix} R \\ B \end{pmatrix}, \begin{pmatrix} F \\ M \end{pmatrix} \right\}
\]

\[
\left\{ \begin{pmatrix} + \\ □ \\ × \\ ○ \\ △ \\ × \end{pmatrix} \right\}.
\]

Let us agree that a pair \( \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \) can be vertically concatenated, or multiplied, by a pair \( \begin{pmatrix} p_3 \\ p_4 \end{pmatrix} \) to get \( δ_{p_2=p_3} \begin{pmatrix} p_1 \\ p_4 \end{pmatrix} \). This is written in either anti-Martian or Martian style as

\[
\begin{pmatrix} L \\ M \end{pmatrix} \begin{pmatrix} □ \\ △ \\ × \end{pmatrix}, \begin{pmatrix} B \\ R \end{pmatrix} \begin{pmatrix} ○ \\ △ \\ △ \end{pmatrix}, \begin{pmatrix} M \\ L \end{pmatrix} \begin{pmatrix} ○ \\ △ \end{pmatrix}, \begin{pmatrix} R \\ B \end{pmatrix} \begin{pmatrix} × \\ △ \end{pmatrix}, \begin{pmatrix} F \\ M \end{pmatrix} \begin{pmatrix} × \\ △ \end{pmatrix}
\]

Position permutations act on assignments on the bottom; object permutations act on the top. Thus

\[
\begin{pmatrix} □ & △ & ○ & + & × \\ L & M & R & B & F \end{pmatrix} \begin{pmatrix} △ & × & + & □ \\ □ & △ & ○ & + & × \end{pmatrix}
\]

and

\[
\begin{pmatrix} □ & △ & ○ & + & × \\ F & L & B & R & M \end{pmatrix} \begin{pmatrix} △ & × & + & □ \\ □ & △ & ○ & + & × \end{pmatrix}
\]
and both results are the same. Notice also that inverses are obtained by simply flipping an array about its mid line, and that associativity is obvious.

The sequence $3 \ 5 \ 7$ of position permutations applied to $\alpha$ yields the same result as the sequence of object permutations $4 \ 6 \ 8$. Thus

$$
\begin{align*}
\alpha & = \tau_3 \\
\sigma_1 & = \tau_2, \quad \alpha & = \tau_2, \quad \sigma_1 & = \tau_1 \\
\alpha_2 & = \tau_1, \quad \sigma_1 & = \tau_1, \quad \alpha & = \tau_1 \\
\alpha_3 & = \alpha \\
\end{align*}
$$

A bit of algebra, Martian-style, then shows that

$$
\begin{align*}
\alpha & = \sigma_1 \alpha \sigma_1^{-1} \\
\tau_3 & = \sigma_2 \tau_2 \sigma_2^{-1} \\
\alpha & = \tau_1 \alpha \tau_1^{-1} \\
\end{align*}
$$

Thus in general we get:

**Theorem 0.11.** Let $\alpha$ be an initial assignment. If the effect of the sequence $\sigma_1, \sigma_2, \cdots, \sigma_k$ of position permutations is at each step the same as the effect of the sequence $\tau_1, \tau_2, \cdots, \tau_k$ of object permutations, then

$$
\begin{align*}
\alpha & = \sigma_1 \alpha \sigma_1^{-1} \\
\tau_3 & = \sigma_2 \tau_2 \sigma_2^{-1} \\
\alpha & = \tau_1 \alpha \tau_1^{-1} \\
\end{align*}
$$

We here recognise a general form of the complementation map $\psi$. If objects and positions are identified (confused?) with the set $\{1, \cdots, n\}$ and $\alpha$ is the identity assignment, we get a correspondence between a sequence $\sigma_i$ and its complementary sequence $\tau_i$. However the fact that $\tau_i$’s are read bottom to top means that the proper definition of $\psi$ is
\tau_1^{-1} = \sigma_1 \\
(\tau_1 \tau_2)^{-1} = \sigma_1 \sigma_2 \\
(\tau_1 \tau_2 \tau_3)^{-1} = \sigma_1 \sigma_2 \sigma_3 \\
\vdots \\
\tau_1 = \sigma_1^{-1} \\
\tau_2 = \sigma_1 \sigma_2 \sigma_3^{-1} \\
\tau_3 = \sigma_1 \sigma_2 \sigma_3^{-1} (\sigma_1 \sigma_2)^{-1} \\
\vdots \\
This happens to coincide with our earlier definition for reflections where each \( s_i^2 = 1 \).

**Posets, Lattices, and Generalisations**

The mutation game associates to a multigraph \( X \) an interesting family of partially ordered sets. In this section we review some necessary preliminaries, establish notation and discuss generalisations arising from the emphasis on multi-sets which we adopt.

Let \((P, \leq)\) be a partially ordered set, by which we mean that \( \leq \) satisfies i) \( x \leq x \) ii) \( x \leq y \) and \( y \leq x \) implies \( x = y \) iii) \( x \leq y \) and \( y \leq z \) implies \( x \leq z \). We say \( y \) covers \( x \) iff \( x < y \) and \( x \leq z < y \) implies \( z = x \), and represent \( P \) by a reverse Hasse diagram with elements of \( P \) as vertices and a downward line from \( x \) to \( y \) iff \( y \) covers \( x \).

An ideal of \( P \) is a subset \( I \) such that \( y \in I, x \leq y \) imply \( x \in I \). A filter of \( P \) is a subset \( J \) such that \( y \in J, y \leq x \) imply \( x \in J \). The set \( L(P) \) of all ideals of \( P \) is itself a poset under inclusion.

A lattice is a poset \( P \) in which \( x \lor y \), the supremum or join of \( x \) and \( y \), and \( x \land y \), the infimum, or meet of \( x \) and \( y \), are defined for all \( x, y \). A lattice is distributive iff for all \( x, y, z \)

\[ x \land (y \lor z) = (x \land y) \lor (x \land z) \]

and modular iff for all \( x, y, z \)

\[ x \geq z \Rightarrow x \land (y \lor z) = (x \land y) \lor z. \]

The ‘diamond’ lattice

```
            •
           / \     \ 
          /   \     \ 
         /     \     
```

is modular but not distributive. The ‘pentagon’ lattice

```
            •
           / | \     \ 
          /   |   \     
         /     |     
```

43
is neither modular nor distributive. [In fact any distributive lattice is modular].

The standard example of a distributive lattice is a lattice of subsets of some set under inclusion. The converse is included in this important result.

**Theorem 0.12.** Let \( L \) be a lattice. Then \( L \) is distributive iff one of the following equivalent conditions hold.

i) \( L \cong L(P) \) for some poset \( P \);

ii) \( L \) does not contain the diamond or pentagon lattices as sub-lattices.

In case \( L \) is distributive, the poset \( P \) may be identified in a natural way as the set of join-irreducible of \( L \); those \( x \in L \) such that \( x \neq 0 \) (in case \( L \) has a minimal element 0, which ours do) and such that \( x = a \lor b \) implies \( x = a \) or \( x = b \). Denote the set of join-irreducibles of \( L \) by \( J(L) \). In our situation join-irreducibles can be easily identified by the fact that they cover only one element. Thus \( L \cong L(JI(L)) \).

A **multi-set** is a collection of objects, some of which are possibly indistinguishable from others. We denote such a collection by square brackets, such as

\[
X = [x, y, z, x, x, y].
\]

This multi-set, or **m-set**, has 6 elements, so we write \(|X| = 6\). We say there are 3 **occurrences** of \( x \) in \( X \), or that the **multiplicity** of \( x \) in \( X \) is \( m_X(x) = 3 \). It will also be useful to allow ourselves the convenient shorthand \( X = 3x + 2y + z \).

A **partial order** on a multi-set \( X \) is a collection of arrows from the elements of \( X \) to the elements of \( X \) which has the property that by following errors we can never return to our starting point. This makes \( X \) into a **partially ordered m-set**, or **pomset**.

Here is a partial order on the example above

\[
\begin{align*}
x & \rightarrow y \\
z & \leftarrow x \\
\end{align*}
\]
An arrow from $y$ to $x$ means $x < y$. The associated reverse Hasse diagram for this example is

\[
\begin{array}{c}
x \\
y \\
z \\
x \\
z
\end{array}
\]

We allow also the possibility of more than one arrow from $y$ to $x$; this makes a pomset just a special type of multi-graph. If furthermore several occurrences of an element $x$ are treated uniformly by a partial order, we will sometimes only indicate the element with its multiplicity in a reverse Hasse diagram, expressed as an exponent or by circling. Hopefully the following makes this clear.

\[
\begin{array}{c}
x \\
y \\
z \\
x \\
z \\
u \\
w \\
v
\end{array} \Leftrightarrow
\begin{array}{c}
x \\
z^2 \\
u \\
w \\
v
\end{array} \Leftrightarrow
\begin{array}{c}
x \\
z \\
u \\
w \\
v
\end{array}
\]

We adopt the convention that for small values of $k$, $z^{k+1}$ is represented by $z$ surrounded by $k$ circles.

**Mutation Frames**

Let $X$ be a bidirected multigraph with no loops. We will be associating posets, called **frames**, to sequences of vertices or mutations of $X$.

Given a sequence $u = (x_1, \cdots, x_k)$ of vertices of $X$, we refer to the $x_i$ as the **occurrences** in the sequence, so that there are exactly $k$ occurrences even if the $x_i$’s repeat. Define two sequences $u = (x_1, x_2, \cdots, x_k)$, $u' = (x'_1, x'_2, \cdots, x'_k)$ to be **equivalent** if one is obtained from the other by a finite number of switches of adjacent elements in the sequence whose corresponding vertices are not neighbours in $X$. Such switches we will call **free** switches.

**Example.**
The sequence \( u = (1234503243) \) is equivalent to \( u' = (1234053243) \) but not to \( u'' = (1234530243) \).

An equivalence class of sequences of vertices of \( X \) will be called an \textbf{\( X \)-string}, and denoted \([u] = [x_1, x_2, \ldots, x_k]\).

Given a sequence \( u = (x_1, x_2, \ldots, x_k) \) we define a partial order on the occurrences in the sequence by defining \( x_i < x_j \) iff 1) \( i < j \) and 2) \( x_i \) and \( x_j \) are neighbouring vertices in \( X \). This is unchanged by free switches, so the resulting poset depends only on the \( X \)-string \([u] = [x_1, \ldots, x_k]\) and will be called the \textbf{framing poset}, or \textbf{frame}, of that \( X \)-string and denoted \( F[u] \). A frame obtained from the graph \( X \) will be called an \textbf{\( X \)-frame}.

\textbf{Example}. The reverse Hasse diagram for the frame of the \( X \)-string \([1234503243]\) in the previous example is

![Diagram](attachment:image.png)

We leave the proof of the following to the reader.

\textbf{Lemma 0.7}. \textit{The set of all sequences belonging to an} \( X \)-\textit{string is just the set of all total orderings of the corresponding} \( X \)-\textit{frame}.

Let \( F \) be an \( X \)-frame, say \( F = F[u] \) for some \( X \)-string \([u] = [x_1, \ldots, x_k]\) and let \( p \) be a population on \( X \) with \( pt = q \) the corresponding dual population. We say \( F \) is \textbf{increasing} \textbf{[incremental]} on \( p \) if \( s(x_1, \ldots, x_k) \) is an increasing sequence of mutations on \( p \). This is well defined since switching the order of two commuting increasing [incremental] mutations still gives two increasing [incremental] mutations.

\( F \) is increasing on \( p \) iff \( qr_{x_1} \cdots r_{x_{i-1}}(x_i) > 0 \) for all \( i = 1, \ldots, k \). \( F \) is incremental on \( p \) iff \( qr_{x_1} \cdots r_{x_{i-1}}(x_i) = 1 \) for all \( i = 1, \ldots, k \).
For \( p \in \mathcal{P}(X) \) and a sequence \([u] = (x_1, \cdots, x_k)\), we may associate the augmentation sequence \( A([u]) \)
\[
c([u]) = (p s_{x_1} - p, p s_{x_1} s_{x_2} - p s_{x_1}, \cdots, p s_{x_1} \cdots s_{x_k} - p s_{x_1} \cdots s_{x_k - 1}) \\
= (q(x_1) \delta_{x_1}, q r_{x_1}(x_2) \delta_{x_2}, \cdots, q r_{x_1} \cdots r_{x_k - 1}(x_k) \delta_{x_k}).
\]
For notational convenience, we adopt the convention that this sequence will also be written in multiplicative form, that is, as,
\[
c([u]) = (x_1^{q(x_1)}, x_2^{q r_{x_1}(x_2)}, \cdots, x_k^{q r_{x_1} \cdots r_{x_k - 1}(x_k)}) = (x_1^{a_1}, \cdots, x_k^{a_k}).
\]
Such a sequence of vertices of \( X \) with multiplicities will be called a generalised sequence. The partial order on sequences may be extended to generalised sequences by defining \( x_i^{a_i} < x_j^{a_j} \) iff 1) \( i < j \) and 2) \( x_i \) and \( x_j \) are neighbouring vertices in \( X \). This is unchanged by free switches, a switch of \( x_i^{a_i} \) and \( x_j^{a_j} \) if they are consecutive terms in the sequence with \( x_i \) and \( x_j \) not neighbours in \( X \). So we call an equivalence class under \( X \)-switches a generalised \( X \)-string, and the resulting poset a generalised \( X \)-frame.

**Proposition 0.8.** Let \( X \) be a simple undirected graph and let \( c = (x_1^{a_1}, \cdots, x_k^{a_k}) \) be the change sequence for some \( p \in \mathcal{P}(X) \) and sequence of mutations \( s(x_1, \cdots, x_k) \). Let \( x^a \) and \( x^b \) be two successive occurrences in \( c \) of a vertex \( X \) and let \( y_1^{\gamma_1}, \cdots, y_{\ell}^{\gamma_{\ell}} \) be the occurrences of neighbouring vertices in \( X \) between \( x^a \) and \( x^b \). Then
\[
a + b = \gamma_1 + \cdots + \gamma_{\ell}.
\]

*Proof.* Suppose first that there are no intermediary occurrences of neighbours between \( x^a \) and \( x^b \). Then since \( s_x^2 = \text{identity} \), \( b = -a \). Now if neighbours \( y_1, \cdots, y_{\ell} \) have occurred with multiplicities \( \gamma_1, \cdots, \gamma_{\ell} \), the total population of those vertices has increased by a sum of \( \gamma_1 + \cdots + \gamma_{\ell} \). The next occurrence of \( s_x \) results in a change not of \( b = -a \), but of \( b = -a + \gamma_1 + \cdots + \gamma_{\ell} \) since each \( y_i \) is connected to \( x \) by a single edge. \( \square \)

There is an obvious generalisation of this to more general graphs. Let \( X \) be an arbitrary directed multigraph.

**Theorem 0.13.** If \( c = (x_1^{a_1}, \cdots, x_k^{a_k}) \) is a change sequence and \( x^a, x^b \) two successive occurrences in \( c \) of a vertex \( x \) with \( y_1^{\gamma_1}, \cdots, y_{\ell}^{\gamma_{\ell}} \) the occurrences of neighbouring vertices in \( X \) between \( x^a \) and \( x^b \), then \( a + b = \gamma_1 m_1 + \cdots + \gamma_{\ell} m_{\ell} \) where \( m_i = \# \text{ edges from } y_i \text{ to } x \).

An \( X \)-switch in the mutation sequence \( u = (x_1, \cdots, x_k) \) results in a corresponding \( X \)-switch in the change sequence \( c = (x_1^{a_1}, \cdots, x_k^{a_k}) \) and vice versa. The generalised \( X \)-frame \( F([x_1^{a_1}, \cdots, x_k^{a_k}]) \) depends thus only on \( p \tau = q \) and \( \tilde{F} = f([x_1, \cdots, x_k]) \). We write
\[
F([x_1^{a_1}, \cdots, x_k^{a_k}]) = F(q)
\]
\[47\]
and call it the **mutation frame** associated to $q$ and $F$.

Let us clarify these ideas with an example. Suppose $q$ is the dual population on $E_6$ given by

![Diagram](image)

and consider the sequence of dual mutations $r(2, 3, 4, 5, 2, 3)$ acting on $q$. This results in the sequence

![Diagram](image)

The corresponding change sequence is $(2^2, 3^3, 4^3, 5^3, 2, 3)$, and the associated mutation frame is

![Diagram](image)
We will initially be interested in the case when \( q = \epsilon \) a fundamental weight, and \( F = F_{(x_1, \ldots, x_k)} \) acts in an increasing fashion on \( \delta_x \).

In the case of \( X \) bipartite, which of course includes the case when \( X \) is a tree, such an \( X \)-string can be fashioned very simply. Suppose that \( y_i \) are the odd vertices of \( X \) and \( z_i \) are the even vertices. Then let

\[
    s_O = \prod_{y_i} s_{y_i}, \quad s_E = \prod_{z_i} s_{z_i}
\]

be the products, in any order, of the odd and even mutations respectively, and similarly

\[
    r_O = \prod_{y_i} r_{y_i}, \quad r_E = \prod_{z_i} r_{z_i}
\]

the products of odd and even dual mutations respectively. Note that since non adjacent mutations commute, these elements are well-defined. Let \( y = (y_1, y_2, \cdots) \) and \( z = (z_1, z_2, \cdots) \) be the corresponding sequences, in some order, and consider

\[
    F_O = F_{[y,z,y,z,\cdots]}, \quad F_E = F_{[z,y,z,y,\cdots]}.
\]

These \( X \)-frames will be called the **canonical** odd and even \( X \)-frames or respectively. Now if \( \delta_x = q \) is even, \( qr(F_O) \) gives the **canonical** mutation frame at \( x \), and if \( \delta_x = q \) is odd, \( qr(F_E) \) is the canonical mutation frame at \( x \).

Let us denote by \( M(X, x) \) the canonical mutation frame for the bipartite multigraph \( X \) starting with a fundamental weight at \( x \).

We now list some examples of such \( M(X, x) \).

\[ A_3 \]

\[ D_4 \]
In the following, we list the dual populations at each successive stage. Note the remarkable properties (symmetry, spindle-shaped, Sperner) of these lattices.
$E_7$
$M(E_7, 1)$
Root Spindles

For $X$ a bidirected multigraph with no loops, $\mathcal{R}(X)$ carries at least three interesting partial orders. Define

1. $p \leq q \iff p(x) \leq q(x)$ for all vertices $x$.

2. $p \preceq q \iff p(x) \leq q(x)$ and $q$ is obtained from $p$ by a sequence of increasing mutations.

3. $p \preceq_1 q \iff p(x) \leq q(x)$ and $q$ is obtained from $p$ by a sequence of incremental mutations.

Clearly $p \preceq_1 q \Rightarrow p \preceq q \Rightarrow p \leq q$. These poset structures on $\mathcal{R}(X)$ allow us to identify certain substructures which are of particular interest. For a vertex $x$ of $X$, define

$$R_+(X, x) = \{ p \in \mathcal{R}_+(X) | \delta_x \leq p, \ p(x) = 1 \}$$

$$R_{++}(X, x) = \{ p \in \mathcal{R}_+(X) | \delta_x \preceq p, \ p(x) = 1 \}$$

$$R_{+++}(X, x) = \{ p \in \mathcal{R}_+(X) | \delta_x \preceq_1 p, \ p(x) = 1 \}$$

We call these respectively the (positive) root spindle above $x$, the increasing root spindle above $x$ and the incremental root spindle above $x$. They are posets with respect to $\leq$, $\preceq$, and $\preceq_1$ respectively.

For $x$ a vertex of $X$, let $X \setminus x$ denote the subgraph of $X$ obtained by removing $x$ and all edges containing $x$. This graph is generally not connected.

Define

$$D_+(X, x) = \bigcup_{y \in N(x)} R_+(X \setminus x, y)$$

$$D_{++}(X, x) = \bigcup_{y \in N(x)} R_{++}(X \setminus x, y)$$

$$D_{+++}(X, x) = \bigcup_{y \in N(x)} R_{+++}(X \setminus x, y),$$

posets with respect to $\leq$, $\preceq$, $\preceq_1$ respectively.

**Observation** Let $X$ be a simple undirected graph in List B (in other words, on $A\overline{D}-E$ graph) and $x$ a vertex of $X$. Then

i) $R_+(X, x) \simeq R_{++}(X, x) \simeq R_{+++}(X, x)$

ii) $D_+(X, x) \simeq D_{++}(X, x) \simeq D_{+++}(X, x)$

iii) $R_+(X, x) \simeq J(D_+(X, x))$
The graphs in the Observation are all trees and it is readily seen that since any subgraph which is connected is also of this form, the case when \( x \) is an endpoint of \( X \) is the crucial situation.

It follows that the root spindles of the \( A-D-E \) graphs exhibit a remarkable inductive or ‘cascading’ structure which links them all together in the following pattern.

\[
\begin{align*}
A_1 & \quad \uparrow \\
A_2 & \quad \uparrow \\
A_3 & \quad \uparrow \\
A_4 & \quad D_4 \\
A_5 & \quad D_5 \\
A_6 & \quad D_6 \\
A_7 & \quad D_7 \\
A_8 & \quad D_8 \\
A_9 & \quad D_9 \\
\vdots & \\
\end{align*}
\]

From this diagram, the poset structures of the various root spindles can be built up, once one understands the isomorphism in part iii) of the Observation. For this we need to connect root spindles to mutation frames, and the key tool is the complementation map \( \psi \). It then will be seen that the inductive cascading shown above is but a special case of a much more general phenomenon.
Let us illustrate this cascading phenomenon explicitly. Here is $R_+(E_6, 5)$ where $E_6$ is labelled

Here is $D_+(E_6, 5) = R_+(D_5, 4)$ and $M(D_5, 4)$
The elements of $R_+$ are positive roots of $D_5$. The elements of $M(D_5, 4)$ are also positive roots of $D_5$, they are in fact simple roots. The crucial fact is that

$$R_+(D_5, 4) = \psi(M(D_5, 4)).$$

In other words, if we choose a total ordering of $M(D_5, 4)$, say

$$a = (4, 3, 2, 1, 0, 3, 2, 4, 3, 0)$$

and apply $\psi$ to this sequence, we get

$$\psi(a) = (\delta_4, \delta_3s_4, \delta_2s_3s_4, \delta_1s_2s_3s_4, \cdots, \delta_0s_3s_4s_2s_3s_0s_1s_2s_3s_4)$$

which turns out to be the sequence

$$(0001, 0011, 0111, 0121, 1111, 1121, 1111, 1000, 1010, 0000)$$

which is a total ordering of $R_+(D_5, 4)$.

This may be described more directly. For an element $\alpha$ in $M(D_5, 4)$, let $I(\alpha) = \{\beta \in M(D_5, 4) | \beta < \alpha\}$, and let $\beta_1, \beta_2, \cdots, \beta_e$ be a total ordering of $I(\alpha)$. Then

$$\psi(\alpha) = \alpha \cdot s_{\beta_e} \cdots s_{\beta_1}.$$

Thus for the squared 4, we see

$$\psi(4) = \delta_4 \cdots s_3 \cdots s_0s_3s_4 = \frac{1}{0121}.$$

To find the inverse of $\psi$ we need recall that if $\psi(s_1, \cdots, s_k) = (t_1, \cdots, t_k)$ then $t_1t_2\cdots t_3 = s_es_{e-1} \cdots s_1$ for all $e = 1, \cdots, k$. Of course $\psi^2 = \text{identity}$, so the inverse
of \( \psi \) is itself. Thus

\[
\psi^{-1}\left(\begin{array}{c} 1 \\ 0121 \end{array}\right) = \psi\left(\begin{array}{c} 1 \\ 0121 \end{array}\right) = \begin{array}{ccccccc} 1 & 0111 & 1 & s & 0 & s & 0 \\ 0121 & 0111 & 0011 & 0011 & 0001 \end{array}
\]

but this formula involves non-simple reflections. Using the identity and assuming \( \psi^{-1}(\gamma) \) has been found for all \( \gamma \in I\left(\begin{array}{c} 1 \\ 0121 \end{array}\right) \), we may see that if \( I\left(\begin{array}{c} 1 \\ 0121 \end{array}\right) \) has ordering \( s_1, s_2, \cdots s_e \), then

\[
\psi\left(\begin{array}{c} 1 \\ 0121 \end{array}\right) = \begin{array}{ccccccc} 1 & 0121 & \cdot & s_{\psi(s_1)}s_{\psi(s_2)} & \cdots & s_{\psi(s_e)}. \end{array}
\]

In our case this becomes

\[
\psi^{-1}\left(\begin{array}{c} 1 \\ 0121 \end{array}\right) = \begin{array}{ccccccc} 1 & 0121 & \cdot & s_4s_3s_2s_1s_3 = 1 & 0001 & .
\end{array}
\]

These observations generalise.

**Observation** Let \( X \) be a simple undirected graph in List B (an \( A-D-E \) graph) and \( x \) an extreme vertex of \( X \). Then

i) \( R_+(X,x) \cong M(X,x) \) and the isomorphism is given by the complementation map \( \psi \).

ii) If \( \alpha \in M(X,x) \) and \( \beta_1, \cdots, \beta_e \) a total ordering of the ideal \( I(\alpha) = \{ \beta < \alpha \} \) in \( M(X,x) \) then \( \psi(\alpha) = \alpha \cdot s_{\beta_e} \cdots s_{\beta_1} \).

iii) If \( \gamma \in R_+(X,x) \) and \( \delta_1, \cdots, \delta_e \) a total ordering of the ideal \( I(\gamma) = \{ \delta < \gamma \} \) in \( R_+(X,x) \) then

\[
\psi(\gamma) = \gamma s_{\psi(\delta_1)} \cdots s_{\psi(\delta_e)}
\]
\[ R_+(E_6, 1) \]
$R_+(E_6, 0)$
\( R_+ (E_7, 6) \)
**B₂**

a) \[ \alpha_0 = \begin{bmatrix} 1 & 1 & 2 \\ * & 1 & 2 \end{bmatrix} \]

b) \[ \alpha_0 = \begin{bmatrix} 1 & 1 & 1 \\ * & 1 & 2 \end{bmatrix} \]

c) \[ \alpha_0 = \begin{bmatrix} 1 & 2 & * \\ 1 & 2 & \end{bmatrix} \]

d) \[ \alpha_0 = \begin{bmatrix} 1 & 2 & * \\ 1 & 2 & \end{bmatrix} \]

Not Perron-Frobenius extension of \( B₂ \)

**C₂**

a) \[ \alpha_0 = \begin{bmatrix} 2 & 2 & * \\ 1 & 2 & \end{bmatrix} \]

b) \[ \alpha_0 = \begin{bmatrix} 2 & 2 & * \\ 1 & 2 & \end{bmatrix} \]

c) \[ \alpha_0 = \begin{bmatrix} 001 \\ 011 \\ 211 \\ 221 \end{bmatrix} \]

Not Perron-Frobenius extension.
Only a) and c) are extensions of $C_2$ with the property that the restrictions of the Perron-Frobenius population $\alpha_0$ to $C_2$ is a root and $\alpha_0(*) = 1$. Call such extensions Perron-Frobenius extensions.
b) \[ \begin{array}{c}
1 & 2 & 3 & * \\
\end{array} \]
\[ \alpha_0 = \begin{array}{c}
1 & 1 & 1 & 1 \\
\end{array} \]

This is not a Perron-Frobenius extension of \( C_3 \) since \( \alpha_0 \bigg|_{C_3} \) is not a root.

c) \[ \begin{array}{c}
1 & 2 & 3 & * \\
\end{array} \]
\[ \alpha_0 = \begin{array}{c}
2 & 2 & 2 & 1 \\
\end{array} \]
$D_4$

\[ \alpha_0 = \]

\[ \begin{array}{ccc}
1 & 2 & 3 \\
* & 1 & 2
\end{array} \]
$\alpha_0 =$

Diagram of $D_5$ with nodes labeled by $0, 1, 2, 3, 4$ and edges labeled by 0, 1, 0, 0.