On the complexification of the classical geometries and exceptional numbers

April 1995

0. Introduction

The classical groups $\mathrm{O}(n;\mathbf{R})$, $\mathrm{Sp}(2n;\mathbf{R})$, and $\mathrm{GL}(n;\mathbf{C})$ appear as the isometry groups of special geometries on a real vector space. In fact, the orthogonal group $\mathrm{O}(n;\mathbf{R})$ represents the linear isomorphisms of a real vector space V (of dimension n) leaving invariant a positive definite and symmetric bilinear form g on V. The symplectic group $\mathrm{Sp}(2n;\mathbf{R})$ represents the isometry group of a real vector space (of dimension $2n$) leaving invariant a nondegenerate skew-symmetric bilinear form ω on V. Finally, $\mathrm{GL}(n;\mathbf{C})$ represents the linear isomorphisms of a real vector space V (of dimension $2n$) leaving invariant a complex structure on V, i.e., an endomorphism $J\colon V\to V$ satisfying $J^2=-1$. These three geometries $\mathrm{O}(2n;\mathbf{R})$, $\mathrm{Sp}(2n;\mathbf{R})$, and $\mathrm{GL}(n;\mathbf{C})$ in $\mathrm{GL}(2n;\mathbf{R})$ intersect (even pairwise) in the unitary group $\mathrm{U}(n;\mathbf{C})$.

Considering now the relative versions of these geometries on a real manifold (of dimension $2n$) leads to the notions of a Riemannian manifold, an almost-symplectic manifold, and an almost-complex manifold. A symplectic manifold however is an almost-symplectic manifold (X,ω), i.e., X is a manifold and ω is a nondegenerate 2-form on X, so that the 2-form is closed. Similarly an almost-complex manifold (X,J) is called complex, if the torsion tensor $N(J)$ vanishes. These three geometries "intersect" in the notion of a Kähler manifold.

In view of the importance of the complexification of the real Lie groups—for instance in the structure theory and representation theory of semi-simple real Lie groups—we consider here the question on the underlying geometrical structures on a complex vector space (of dimension $2n$) giving rise to isometry groups which are the Lie theoretic complexifications of the classical groups. While the procedure is quite clear for the orthogonal group $\mathrm{O}(n;\mathbf{C})=\mathrm{O}(n;\mathbf{R})^c$ and the symplectic group $\mathrm{Sp}(2n;\mathbf{C})=\mathrm{Sp}(2n;\mathbf{R})^c$, it turns out that the so-called exceptional numbers $\mathbf{E}$ (cf. [Wildberger]) are quite useful to describe the complexifications of $\mathrm{GL}(n;\mathbf{C})\subseteq\mathrm{GL}(2n;\mathbf{R})$ and accordingly of $\mathrm{U}(n;\mathbf{C})$. In fact one finds that

$$\mathrm{GL}(n;\mathbf{E})\cong\mathrm{GL}(n;\mathbf{C})^c=\mathrm{GL}(n;\mathbf{C})\times\mathrm{GL}(n;\mathbf{C})$$

and

$$\mathrm{U}(n;\mathbf{E})\cong\mathrm{U}(n;\mathbf{C})^c=\mathrm{GL}(n;\mathbf{C}).$$

After clarifying this task of so to say elementary linear algebra over the exceptional numbers, we come to the relative notions of these groups over a given complex manifold. The corresponding notions are called a euclidean, symplectic or exceptional structure on a complex manifold (of dimension $2n$). They "intersect" in the notion of a Kähler manifold. The question of existence of such structures are briefly discussed. The main point however is the final example. Consider a semi-simple complex Lie algebra L and let $a\in L$ be a semi-simple element. Denote further by $G=\mathrm{Int}(L)\subseteq\mathrm{GL}(L)$ the adjoint group of L. It is shown that the adjoint orbit $G(a)\subseteq L$ carries the structure of a Kähler manifold.

1. Elementary theory of exceptional vector spaces

(1.1) Euclidean structures. Consider a complex vector space V of dimension n. A *euclidean structure* on V is a symmetric and nondegenerate bilinear form $g\colon V\times V\to\mathbf{C}$.

Proposition 1. *If g is a euclidean structure on V, there exists an orthonormal basis $(v_1, \dots, v_n)$ on V, i.e., $g(v_\mu, v_\nu) = \delta_{\mu\nu}$.*

Proof. Since $g \neq 0$ we can choose a vector $v \in V$ so that $g(v, v) \neq 0$. By rescaling v—observe that $z \mapsto z^2$ is surjective on $\mathbf{C}$—we may assume that $g(v, v) = 1$. The orthogonal complement $W := v^\perp = \{w \in V \mid g(v, w) = 0\}$ is a euclidean subspace of V, i.e., the restriction of g to $W \times W$ remains nondegenerate. In fact, if $w \in W$ and suppose that $W \subseteq w^\perp$, then, since $v \in w^\perp$, one finds that $V \subseteq w^\perp$ implying $w = 0$, since g is nondegenerate on V. Now by induction on the dimension the result follows.

Remark. Recall that the rescaling argument does not apply in the real, i.e., in the classical case. In fact, in this case, a nondegenerate bilinear form has an additional invariant: its index, that is the dimension of a maximal subspace of V on which g is positive definite.

Let $V = \mathbf{C}^n$ and define the *standard euclidean structure* by

$$g(z, w) = \sum_{\nu=1}^{n} z_\nu w_\nu .$$

Then any euclidean vector space is equivalent to $(\mathbf{C}^n, g)$. In particular, the complex orthogonal group

$$\mathrm{O}(n; \mathbf{C}) = \{A \in \mathrm{GL}(n; \mathbf{C}) \mid A^t A = 1\}$$

is—up to isomorphy—the isometry group of such a structure.

(1.2) Exceptional structures. Consider a complex vector space V of finite dimension and an endomorphism $J: V \to V$ satisfying $J^2 = -1$. Since the minimal polynomial of J divides $X^2 + 1 = (X - i)(X + i)$, J is necessarily semi-simple, i.e., V splits into the $\pm i$-eigenspaces,

$$V = \mathrm{Eig}(J, i) + \mathrm{Eig}(J, -i).$$

We call J an *exceptional structure* on V, if the multiplicities of the $+i$- and $-i$-eigenspace of J are equal,

$$\dim \mathrm{Eig}(J, i) = \dim \mathrm{Eig}(J, -i).$$

Proposition 2. *Let J be an exceptional structure on a finite dimensional complex vector space V. Then V is even dimensional, say of dimension $2n$, and there exists a basis $(v_1, \dots, v_n, w_1, \dots, w_n)$ of V such that the matrix of J with respect to this basis is*

$$I := \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}.$$

Proof. Let $(\tilde v_1, \dots, \tilde v_n, \tilde w_1, \dots, \tilde w_n)$ be a basis such that the corresponding matrix of J is $\left(\begin{smallmatrix} i1_n & 0 \\ 0 & -i1_n \end{smallmatrix}\right)$. Then the basis $v_\nu := \tilde v_\nu + \tilde w_\nu$, $w_\nu := \tilde v_\nu - \tilde w_\nu$ satisfies the assertion.

Recall the basic construction of the exceptional numbers (see [Wildberger]). Here $\mathbf{E} = \mathbf{C} \times \mathbf{C}$ together with its vectorspace structure over $\mathbf{C}$ and its multiplicative structure resulting from $j^2 := -1$, where $j := (0, 1) \in \mathbf{E}$. The null numbers were defined as the set of noninvertible elements $\alpha \in \mathbf{E}$, i.e., $\alpha = z\, e_+$ with $z \in \mathbf{C}$ or $\alpha = z\, e_-$ with $z \in \mathbf{C}$. Here

$$e_+ := \frac{1}{2}(1 - ij), \quad e_- := \frac{1}{2}(1 + ij).$$

The $\mathbf{C}$-algebra $\mathbf{E}$ is isomorphic to the direct product (of $\mathbf{C}$-algebras) of $\mathbf{C}$ with itself via

$$\mathbf{C}^2 \to \mathbf{E}, \quad (\alpha^+, \alpha^-) \mapsto \alpha^+ e_+ + \alpha^- e_-,$$

since $e_+^2 = e_+$, $e_-^2 = e_-$, and $e_+e_- = 0$, as one verifies immediately. Observe furthermore that multiplication by $j \in \mathbf{E}$ gives $\mathbf{E}$ its canonical exceptional structure, since

$$je_+ = ie_+, \quad je_- = -ie_-.$$

Now we can identify our complex vector space (of dimension $2n$) with the $\mathbf{E}$-modul $\mathbf{E}^n$ in the obvious way. Set for $z, w \in \mathbf{C}$ and $v \in V$

$$(z + jw) \cdot v := zv + wJ(v).$$

Then it follows immediately that $V \cong \mathbf{E}^n$.

Remark. Recall that in the real case a complex structure on a real vector space is defined only by an endomorphism $J: V \to V$ with $J^2 = -1$ satisfying no additional properties on its eigenvalue multiplicities. Denoting by V^c the complexification of V, i.e., $V^c = V \otimes_{\mathbf{R}} \mathbf{C}$, and by $J^c: V^c \to V^c$ the complexification of J, i.e., its unique complex linear extension from V to V^c, then the eigenvalue multiplicities of J^c are automatically equal. In fact, the complex conjugation $\mathrm{conj}: z \mapsto \bar{z}$ on $\mathbf{C}$ gives rise to an $\mathbf{R}$-linear map $\mathrm{bar} := 1 \otimes \mathrm{conj}: V^c \to V^c$, which is an isomorphism. Since $\mathrm{bar}\big(\mathrm{Eig}(J^c, i)\big) = \mathrm{Eig}(J^c, -i)$, the multiplicities of $+i$ and $-i$ coincide.

Let us briefly look at the isometries of $\mathbf{C}^{2n}$ concerning the *standard exceptional structure* $J: \mathbf{C}^{2n} \to \mathbf{C}^{2n}$ given by the matrix

$$I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

with respect to the canonical basis of $\mathbf{C}^{2n}$,

$$\mathrm{GL}(n; \mathbf{E}) := \{A \in \mathrm{GL}(2n; \mathbf{C}) \mid AI = IA\}.$$

Looking at $A \in \mathrm{GL}(n; \mathbf{E})$ as an $\mathbf{E}$-linear endomorphism F of $\mathbf{E}^n$ and representing it with respect to the null basis (e_+, e_-) in each factor $\mathbf{E}$ shows that the matrix of F has to commute with $\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$, i.e., it has bock form $\left(\begin{smallmatrix} A_+ & 0 \\ 0 & A_- \end{smallmatrix}\right)$, with $A_+, A_- \in \mathrm{GL}(n; \mathbf{C})$. This shows that

$$\mathrm{GL}(n; \mathbf{E}) \cong \mathrm{GL}(n; \mathbf{C}) \times \mathrm{GL}(n; \mathbf{C}).$$

(1.3) Symplectic structures. Consider a complex vector space V of finite dimension. A *symplectic structure* on V is a nondegenerate and skew-symmetric bilinear form $\omega: V \times V \to \mathbf{C}$.

Proposition 3. *Let ω be a symplectic structure on V. Then V is necessarily even dimensional—say of dimension $2n$—and there exists a symplectic basis $(v_1, \ldots, w_n)$ of V, i.e., $\omega(v_\mu, v_\nu) = \omega(w_\mu, w_\nu) = 0$ and $\omega(v_\mu, w_\nu) = \delta_{\mu\nu}$.*

Proof. Since $\omega \neq 0$ there exist vectors $v, w \in V$ so that $\omega(v, w) \neq 0$. One can assume that $\omega(v, w) = 1$. Let $H := \mathrm{span}(v, w)$. Now considering the symplectic complement $H^\perp = \{x \in V \mid \omega(x, v) = \omega(x, w) = 0\}$, we see that $V = H + H^\perp$ is direct sum decomposition of symplectic vector spaces. In fact, H is obviously a symplectic subspace (i.e., $\omega | H \times H$ is nondegenerate) implying $H \cap H^\perp = 0$ and therefore $V = H + H^\perp$ as vector spaces for dimension reasons, and finally $H^\perp$ is also symplectic. Indeed, if $x \in H^\perp$ and $H^\perp \subseteq x^\perp$, then, since $H \subseteq x^\perp$, we find that $V \subseteq x^\perp$ implying $x = 0$. Now by induction the proposition follows.

The *standard symplectic structure* on $\mathbf{C}^{2n}$ is given by

$$\omega\big((x_1, y_1), (x_2, y_2)\big) = g(x_1, y_2) - g(x_2, y_1),$$

where g is the standard euclidean structure on $\mathbf{C}^n$. The complex symmetric group is

$$\mathrm{Sp}(2n; \mathbf{C}) = \{A \in \mathrm{GL}(2n; \mathbf{C}) \mid A^t I A = I\},$$

and the proposition shows that the isometry group of a complex symplectic vector space of dimension $2n$ is isomorphic to $\mathrm{Sp}(2n, \mathbf{C})$.

(1.4) The unitary group over the exceptional numbers. Now we want to compute the pairwise intersection of the complex groups $\mathrm{O}(2n;\mathbf{C})$, $\mathrm{GL}(n;\mathbf{E})$, and $\mathrm{Sp}(2n;\mathbf{C})$ in $\mathrm{GL}(2n;\mathbf{C})$. To that end consider the free $\mathbf{E}$-modul $\mathbf{E}^n$. Denote by $\bar{\ }\colon \mathbf{E} \to \mathbf{E}$ the complex linear conjugation $z + jw \mapsto z - jw$ on $\mathbf{E}$. Observe that $\bar{e}_+ = e_-$ and $\bar{e}_- = e_+$. The *canonical hermitean form* $\langle \cdot, \cdot \rangle \colon \mathbf{E}^n \times \mathbf{E}^n \to \mathbf{E}$ is defined by

$$\langle \alpha, \beta \rangle := \sum_{\nu=1}^{n} \bar{\alpha}_\nu \beta_\mu.$$

Decomposition in complex and fictitious part yields the equation

$$\langle \alpha, \beta \rangle = g(\alpha, \beta) + j\omega(\alpha, \beta),$$

where g and ω are the standard euclidean and symplectic structures on $\mathbf{E}^n = \mathbf{C}^{2n}$. A hermitean form h is by definition a sesquilinear form on $V = \mathbf{E}^n$, i.e., a $\mathbf{C}$-bilinear form satisfying $h(v, \alpha w) = \alpha h(v, w)$ and $h(w, v) = \overline{h(v, w)}$ for all $\alpha \in \mathbf{E}$ and $v, w \in V$. According to the standard $\mathbf{C}$-basis $(e_1, \ldots, e_n, je_1, \ldots, je_n)$ of $\mathbf{C}^{2n} = \mathbf{E}^n$ the matrices of g, J, and ω are

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

respectively. On the other hand, the matrices according to the null basis $\big((e_+)_1, \ldots, (e_+)_n, (e_-)_1, \ldots, (e_-)_n\big)$ is

$$\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \quad \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \quad \begin{pmatrix} 0 & \frac{i}{2} \\ -\frac{i}{2} & 0 \end{pmatrix}.$$

Define now by $\mathrm{U}(n;\mathbf{E})$ the complex-linear transformations of $\mathbf{C}^{2n} = \mathbf{E}^n$ respecting the standard hermitean form on $\mathbf{E}^n$, i.e.,

$$\mathrm{U}(n;\mathbf{E}) = \{A \in \mathrm{GL}(n;\mathbf{E}) \mid \bar{A}^t A = 1\}.$$

In fact, such a transformation is necessary $\mathbf{E}$-linear, i.e, the matrix A commutes with $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) = I$. Since $A \in \mathrm{O}(2n;\mathbf{C})$, we have $A^t = A^{-1}$ and since $A \in \mathrm{Sp}(2n;\mathbf{C})$, we find $A^t I A = I$, i.e., $IA = (A^t)^{-1} I = AI$. We have now seen that $\mathrm{U}(n;\mathbf{E}) \subseteq \mathrm{Sp}(2n;\mathbf{C}) \cap \mathrm{GL}(n;\mathbf{E}) \cap \mathrm{O}(2n;\mathbf{C})$. On the other hand the equations

$$\langle \cdot, \cdot \rangle = g + j\omega, \quad g(\alpha, \beta) = \omega(\alpha, J\beta)$$

show that $\mathrm{Sp}(2n;\mathbf{C}) \cap \mathrm{GL}(n;\mathbf{E}) = \mathrm{O}(2n;\mathbf{C}) \cap \mathrm{GL}(n;\mathbf{E}) = \mathrm{Sp}(2n;\mathbf{C}) \cap \mathrm{O}(n;\mathbf{C}) \subseteq \mathrm{U}(n;\mathbf{E})$. The three complex geometries on $\mathbf{C}^{2n}$ intersect in the exceptional unitary group $\mathrm{U}(n;\mathbf{E})$.

To identify $\mathrm{U}(n;\mathbf{E})$ let us finally recall what it means that a matrix $X = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ in $\mathrm{GL}(2n;\mathbf{C})$ is symplectic, i.e., $X^t I X = X$:

(i) $A^t C = C^t A$,
(ii) $B^t D = D^t B$,
(iii) $A^t D - C^t B = 1$.

Representing the linear transformation $X\colon \mathbf{E}^n \to \mathbf{E}^n$ with respect to the null basis yields the same result, since the representing matrix of ω is $-\frac{i}{2} I$, as we have noted above. In this basis X has the description $\left(\begin{smallmatrix} A_+ & 0 \\ 0 & A_- \end{smallmatrix}\right)$, since $X \in \mathrm{GL}(n;\mathbf{E})$. Thus the three conditions come down to $A_+^t A_- = 1$, showing that

$$\mathrm{U}(n;\mathbf{E}) \cong \mathrm{GL}(n;\mathbf{C}).$$

In general a $\mathbf{C}$-bilinear map $h\colon V \times V \to \mathbf{E}$ on a free $\mathbf{E}$-modul V of rank n is called a *Kählerian structure*, if it is a nondegenerate hermitean form on V.

Proposition 4. *Let V be a free $\mathbf{E}$-modul of rank n and let h be a Kählerian structure on V. Then there exists an $\mathbf{E}$-basis $(v_1, \ldots, v_n)$ so that $h(v_\mu, v_\nu) = \delta_{\mu\nu}$.*

Proof. Since $h \neq 0$ (and the polarization formula) we find a vector $v \in V$ so that $h(v,v) \neq 0$. Now $h(v,v) = \overline{h(v,v)}$, so $h(v,v) \in \mathbf{C} \subseteq \mathbf{E}$. Thus we can rescale $v \in V$ so that $h(v,v) = 1$. Now the hermitean complement $v^\perp = \{w \mid h(w,v) = 0\}$ is again a free $\mathbf{E}$-modul. In fact it is at least a $\mathbf{C}$-vector space of dimension $2n-2$ and it is J-invariant, where J is the exceptional structure on V. But $J|v^\perp$ is again an exceptional structure since $J(v+jv) = j(v+Jv)$ and $J(v-jv) = -j(v-Jv)$ and thus the multiplicities of i and $-i$ of $J|v^\perp$ coincide. Furthermore the restriction of h on $v^\perp$ is again a Kählerian structure which is now left to the reader. So $v^\perp$ is a free $\mathbf{E}$-modul of rank $n-1$ with Kählerian structure $h|v^\perp$ and the induction hypothesis applies.

(1.5) Calibrated exceptional structures. Let (V,ω) be a (complex) symplectic vector space. A *calibrated exceptional structure* is given by an exceptional structure J so that $J \in \mathrm{Sp}(V,\omega)$, i.e., $\omega(Jv,Jw) = \omega(v,w)$ for all $v,w \in V$. The corresponding euclidean structure is defined by $g\colon V \times V \to \mathbf{C}$,

$$g(v,w) := \omega(v,Jw),$$

and the corresponding Kählerian structure $h\colon V \times V \to \mathbf{E}$ is given by

$$h(v,w) := g(v,w) + j\omega(v,w).$$

The triple (V,J,h) is then a Kählerian vector space. In fact, it is straight forward to see that g is nondegenerate and moreover g is also J-invariant, i.e., $g(Jv,Jw) = g(v,w)$ for all $v,w \in V$.

Recall that in the real case a complex structure on a (real) symplectic vector space (V,ω) is called calibrated if the corresponding bilinear form g is in addition positive definite. However, there isn't such an assumption in the complex case.

(1.6) A fundamental example. Consider now a (finite-dimensional) complex Lie algebra L (with Lie bracket $[\cdot,\cdot]$). The infinitesimal adjoint action of L is given by the Lie homomorphism $\mathrm{ad}\colon L \to \mathbf{gl}(L)$, $\mathrm{ad}(a)(b) = [a,b]$. For a given $a \in L$ the orbit $L(a) = \{\mathrm{ad}a(b) \mid b \in L\}$ is in one to one correspondence with the vector space $V_a := L/C(a)$, where $C(a) = \ker \mathrm{ad}a$ is the centralizer of a. In fact, $C(a)$ is nothing else than the isotropy algebra of the L-action in the point a. Consider next the dual vector space L^* of L together with its (infinitesimal) coadjoint action of L, i.e., $\mathrm{ad}^*\colon L \to \mathbf{gl}(L^*)$,

$$\langle \mathrm{ad}^*(a)(\alpha), b\rangle := \langle \alpha, \mathrm{ad}(-a)(b)\rangle = \langle \alpha, [-a,b]\rangle,$$

for $\alpha \in L^*$ and $a,b \in L$. The coadjoint orbit through α is then identified with the vector space $L/I(\alpha)$, where

$$I(\alpha) = \{a \in L \mid \langle \alpha, [a,b]\rangle = 0 \quad \forall b \in L\},$$

since $I(\alpha)$ is obviously the isotropy algebra of L in α. Therefore the skew-symmetric form $\omega_\alpha\colon L \times L \to \mathbf{C}$,

$$\omega_\alpha(a,b) := \langle \alpha, [a,b]\rangle$$

induces a symplectic structure on the vector space $L/I(\alpha)$. It is called the *Kirillov-Kostant structure*.

Next assume that the Lie algebra L is semi-simple, i.e., the symmetric bilinear form $B\colon L \times L \to \mathbf{C}$,

$$B(a,b) := \mathrm{tr}\big(\mathrm{ad}(a)\mathrm{ad}(b)\big)$$

is nondegenerate, i.e., it defines a euclidean structure on L. B is the *Killing form* on L. Thus it defines an isomorphism $\Phi\colon L \to L^*$ by

$$\langle \Phi(a), b\rangle = B(a,b),$$

which is in addition equivariant with respect to the infinitesimal adjoint action of L on L and the infinitesimal coadjoint action of L on L^*,

$$\Phi\big(\mathrm{ad}(a)(b)\big) = \mathrm{ad}^*(a)\big(\Phi(b)\big).$$

This follows from the relation

$$B([a,b],c) = B(a,[b,c])$$

for the Killing form B, coming from the Jacobi identity for $[\cdot,\cdot]$. In particular, the isotropy algebras corresponds and we have a canonical symplectic structure on $V_a = L/C(a)$, induced from $\omega_a\colon L \times L \to \mathbf{C}$,

$$\omega_a(b,c) = B(a,[b,c]).$$

We are now going to define an exceptional structure on V_a calibrated with respect to ω_a, if a is a semi-simple element (i.e., $\mathrm{ad}a\colon L \to L$ is semi-simple). Accordingly this will give V_a the structure of a Kählerian complex vector space in the sense of section 1.4. To this end decompose L into the eigenspaces of the semi-simple endomorphism $\mathrm{ad}a$. Thus, if we denote by $L_\lambda = \mathrm{Eig}(\mathrm{ad}a, \lambda)$ and observe that $L_0 = C(a)$, we obtain

$$V = C(a) + \sum_{\lambda\in\Phi} L_\lambda,$$

where $\Phi \subseteq \mathbf{C}$ denotes the non-zero eigenvalues of $\mathrm{ad}a$. A fundamental observation is the relation

$$[L_\lambda, L_\mu] \subseteq L_{\lambda+\mu} \tag{$*$}$$

for $\lambda, \mu \in \mathbf{C}$, since $\delta := \mathrm{ad}a$ is a derivation, i.e., $\delta([b,c]) = [\delta b, c] + [b, \delta c]$, coming from the Jacobi identity of $[\cdot,\cdot]$. This implies that L_λ is orthogonal to L_μ if $\lambda+\mu \neq 0$, since for $a \in L_\lambda$, $b \in L_\mu$ we have that $\mathrm{ad}a\,\mathrm{ad}b(L_\nu) \subseteq L_{\lambda+\mu+\nu}$ and therefore $B(a,b) = \mathrm{tr}(\mathrm{ad}a\,\mathrm{ad}b) = 0$. Now using that B is nondegenerate, we find that $\lambda \in \Phi$ implies $-\lambda \in \Phi$ and moreover $\dim L_\lambda = \dim L_{-\lambda}$, since B induces a nondegenerate pairing on $L_\lambda \times L_{-\lambda} \to \mathbf{C}$. Thus the nonzero eigenvalues come in pairs $\pm\lambda$ and choosing a decomposition of Φ into a positive and negative part,

$$\Phi = \Phi^+ \dot\cup \Phi^-,$$

i.e., $\lambda \in \Phi^+$ if and only $-\lambda \in \Phi^-$, gives now the decomposition

$$V = C(a) + N^+ + N^-,$$

where

$$N^+ = \sum_{\lambda\in\Phi^+} L_\lambda, \quad N^- = \sum_{\lambda\in\Phi^-} L_\lambda.$$

Defining now $J|N^+ := i\,\mathrm{id}_{N^+}$ and $J|N^- := -i\,\mathrm{id}_{N^-}$ induces an exceptional structure $J = J_a\colon V_a \to V_a$, since $\dim N^+ = \dim N^-$. (We identify V_a with $N^+ + N^-$.) Finally, this exceptional structure is calibrated with respect to the Kirillov-Kostant structure ω_a as is seen by the following argument. If $b,c \in N^+$, then $[b,c] \notin C(a)$ by relation $(*)$; so $\omega_a(b,c) = B(a,[b,c]) = 0 = \omega_a(ib,ic) = \omega_a(Jb,Jc)$. The same argument applies for the case $b,c \in N^-$. If $b \in N^+$ and $c \in N^-$, then $\omega_a(Jb,Jc) = \omega_a(ib,-ic) = \omega_a(b,c)$. Thus we conclude that

$$\omega_a(Jb,Jc) = \omega_a(b,c)$$

for all $b,c \in N^+ + N^-$ showing that J is a calibrated exceptional structure on V_a. The associated euclidean structure $g_a\colon V_a \times V_a \to \mathbf{C}$ is therefore defined by

$$g_a(b,c) = \omega_a(b,Jc),$$

and the associated Kählerian structure $h_a\colon V_a \times V_a \to \mathbf{E}$ by $h_a = g_a + j\omega_a$ as usual.

Since $B\colon L_\lambda \times L_{-\lambda} \to \mathbf{C}$ is a nondegenerate pairing, we can find a basis $(E_1^\lambda \ldots, E_{r_\lambda}^\lambda)$ of L_λ ($r_\lambda := \dim L_\lambda$) and $(E_{-1}^\lambda, \ldots, E_{-r_\lambda}^\lambda)$ of $L_{-\lambda}$ so that $B(E_\rho^\lambda, E_{-\sigma}^\lambda) = \delta_{\rho\sigma}$ for $1 \leq \rho,\sigma \leq r_\lambda$, and $\lambda \in \Phi^+$. Let $H_\rho^\lambda := [E_\rho^\lambda, E_{-\rho}^\lambda] \in C(a)$ and define the complex numbers

$$\gamma_\rho^\lambda := B(a, H_\rho^\lambda).$$

Denoting by $\Gamma = \mathrm{diag}(\gamma_\rho^\lambda)$, we see that the matrix describing the Kirillov-Kostant structure ω_a on V_a with respect to $(E_\rho^\lambda, E_{-\rho}^\lambda)$ is

$$\begin{pmatrix} 0 & \Gamma \\ -\Gamma & 0 \end{pmatrix}.$$

Of course, the matrix describing the exceptional structure J_a corresponding to this basis is

$$\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix},$$

implying that the matrix of g_a is

$$\begin{pmatrix} 0 & -i\Gamma \\ -i\Gamma & 0 \end{pmatrix}.$$

Remark. Note that the centralizer $C(a)\subseteq L$ is itself a euclidean subspace with respect to the Killing structure on L. In fact, since $C(a) = L_0$, the Killing form B induces a nondegenerate pairing of $C(a)$ with itself. In particular, the quotient $V_a = L/C(a)$ inherits a euclidean structure B_a using the natural identification of V_a with $C(a)^\perp$. Now $C(a)^\perp = N^+ + N^- \subseteq L$. However, by the choice of the basis $(E_\rho^\lambda, E_\rho^\lambda)$, the matrix of the euclidean structure B_a is given by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This shows that the Killing form is in general *not* the complex part of the Kählerian structure on V_a described above. In fact, this is the case if and only if $\Gamma = i$, i.e., $\gamma_\rho^\lambda = i$ for all $\lambda \in \Phi^+$ and $\rho = 1, \ldots, r_\lambda$.

Remark. Observe that the Kählerian structure on V_a described above is not canonical in the sense that we have chosen a decomposition of the non-zero roots Φ of a into positive and negative ones. However, in the real case, when L is a real semi-simple Lie algebra, it is not difficult to see that for a semi-simple $a \in L$ the roots of $\mathrm{ad}a$ are all on the real or imaginary axes of $\mathbf{C}$. This gives then a canonical choice of positive and negative by declaring a root λ to be positive, if $\lambda \in \mathbf{R}_+$ or $\lambda \in i\mathbf{R}_+$. On the other hand, the induced euclidean structure g_a on V_a is only positive definite (i.e., the complex structure J_a is calibrated with respect to ω_a) if and only if *all* roots are on the imaginary axes, i.e., the Lie algebra L is compact. Thus, in the case of a compact semi-simple Lie algebra, the adjoint orbit V_a carries a canonical Kähler structure, for any $a \in L$ (since every element is semi-simple).

2. G-structures on complex manifolds

(2.1) Definition of G-structures. Consider a complex manifold X. We denote by $\pi: FX \to X$ the bundle of linear frames on X. Thus a point p in the fibre $\pi^{-1}(x)$ over $x \in X$ is a basis of the tangent space TX_x of X in x. It may be seen as the linear isomorphism from the standard complex vector space $\mathbf{C}^n$ to TX_x carrying the canonical basis $(e_1, \ldots, e_n)$ of $\mathbf{C}^n$ into the basis given by p,

$$p: \mathbf{C}^n \to TX_x.$$

The Lie group $\mathrm{GL}(n, \mathbf{C})$ acts naturally on FX by

$$g.p(z) := p(g^{-1}z),$$

for $p \in FX$, $g \in \mathrm{GL}(n; \mathbf{C})$ and $z \in \mathbf{C}^n$. In fact, FX is a principle $\mathrm{GL}(n; \mathbf{C})$-bundle, i.e., there exists a covering (U_α) of X and equivariant holomorphic diffeomorphisms $\pi \times \varphi_\alpha: \pi^{-1}(U_\alpha) \to U_\alpha \times \mathrm{GL}(n; \mathbf{C})$, where $\mathrm{GL}(n; \mathbf{C})$ acts only on the right factor,

$$\varphi_\alpha(g.p) = g.\varphi_\alpha(p).$$

Therefore, for $x \in U_\alpha \cap U_\beta$ the matrix $\varphi_\beta^{-1}(p)\varphi_\alpha(p) \in \mathrm{GL}(n;\mathbf{C})$ does not depend on $p \in \pi^{-1}(x)$, defining the transition functions $\varphi_{\alpha\beta}: U_\alpha \cap U_\beta \to \mathrm{GL}(n;\mathbf{C})$, $\varphi_{\alpha\beta}(x) = \varphi_\beta^{-1}(p)\varphi_\alpha(p)$ (for some arbitrary $p \in \pi^{-1}(x)$). Of course, the $\{\varphi_{\alpha\beta}\}$ characterize the principle bundle and in the case at hand they are nothing else than the transition functions of the tangent bundle itself. (FX is the so-called associated principle bundle for the vector bundle TX, see e.g. [Steenrod].)

Now for any G-principle bundle $P \to X$ (G a complex Lie group), one defines a reduction of P to a closed subgroup $H \subseteq G$ to be a homomorphism $\rho: Q \to P$ of an H-principle bundle Q to P, i.e., $\pi_P \circ \rho = \pi_Q$ and ρ has to be H-equivariant. A G-principle bundle is thus reducible to the closed subgroup H if and only if there exists a bundle covering (U_α) of X so that the corresponding transition functions $(\varphi_{\alpha\beta})$ take their values (not only in G, but) in H.

Definition. Let G be a closed subgroup of $\mathrm{GL}(n;\mathbf{C})$ and X a complex manifold of dimension n. A *G-structure* on X is a reduction of the bundle of linear frames to G.

If $\rho: Q \to FX$ is the reduction homomorphism, we call sometimes the submanifold $B_G := \rho(Q) \subseteq FX$ the G-structure. In particular for $G = \mathrm{O}(n;\mathbf{C})$, B_G is called an *almost-euclidean structure* (or *Riemannian structure*), for $G = \mathrm{Sp}(2n;\mathbf{C}) \subseteq \mathrm{GL}(2n;\mathbf{C})$ an *almost-symplectic structure* and for $G = \mathrm{GL}(n;\mathbf{E}) \subseteq \mathrm{GL}(2n;\mathbf{C})$ an *almost-exceptional structure* on X.

In general, if $P \to X$ is a G-principle bundle and Y is a complex G-manifold, meaning that G acts holomorphically by biholomorphisms, one can build the associated fibre bundle

$$P \times_G Y,$$

which is the quotient of $P \times Y$ by the diagonal action of G. If in particular P is the principle $\mathrm{GL}(n;\mathbf{C})$-bundle of linear frames over X and $G \subseteq \mathrm{GL}(n;\mathbf{C})$ is a closed subgroup, then the coset space $Y = \mathrm{GL}(n;\mathbf{C})/G$ is in a natural way a $\mathrm{GL}(n;\mathbf{C})$-space, and thus we can build the associated fibre bundle

$$F := FX \times_{\mathrm{GL}(n;\mathbf{C})} \mathrm{GL}(n;\mathbf{C})/G.$$

Now a reduction of P to G is the same as a holomorphic section of X in this fibre bundle. If $\mathrm{GL}(n;\mathbf{C})/G$ is contractible, then there exists always a continuous section of F. This is the reason (in this language) why on a real manifold there always exists a Riemannian structure, since $\mathrm{O}(n;\mathbf{R}) \subseteq \mathrm{GL}(n;\mathbf{R})$ is maximal compact, i.e., $\mathrm{GL}(n;\mathbf{R})/\mathrm{O}(n;\mathbf{R})$ is diffeomorphic to a cell. In a sense the obstruction for the existence of a G-structure is only in the topology of $\mathrm{GL}(n;\mathbf{R})/G$ in the real case. Similarly, any symplectic manifold carries an almost-complex structure since $\mathrm{U}(n;\mathbf{C}) \subseteq \mathrm{Sp}(2n;\mathbf{R})$ is again a maximal compact subgroup.

Now, in the complex analytic case, we have, in addition to the topological obstruction of the homogeneous space $\mathrm{GL}(n;\mathbf{C})/G$, an analytical obstruction. In fact, the existence of a continuous section does not at all imply the existence of a holomorphic section (in contrast to the differentiable case). However, if X is a Stein manifold, a fundamental example of Grauert (Grauert's Oka principle) says, that F has a continuous section if and only if F has a holomorphic section. Thus over a Stein manifold, we conclude again that the obstruction for the existence of a G-structure is in a sense only in the topology of $\mathrm{GL}(n;\mathbf{C})/G$ (meaning the existence of a section of the topological fibre bundle $F \to X$, forgetting about the complex analytic structure of X).

Finally, observe that all the homogeneous spaces $\mathrm{GL}(n;\mathbf{C})/\mathrm{O}(n;\mathbf{C})$, $\mathrm{GL}(2n;\mathbf{C})/\mathrm{GL}(n;\mathbf{E})$, $\mathrm{GL}(2n;\mathbf{C})/\mathrm{Sp}(n;\mathbf{C})$ and $\mathrm{Sp}(2n;\mathbf{C})/\mathrm{U}(n;\mathbf{E})$, parametrizing the euclidean structures on $\mathbf{C}^n$, the exceptional structures on $\mathbf{C}^{2n}$, the symplectic structures on $\mathbf{C}^{2n}$, and the calibrated exceptional structures on $\mathbf{C}^{2n}$ with respect to the standard symplectic structure are *not* contractible. This shows that in general these structures do *not* exist on a complex manifold, even in the case when X is Stein.

A fundamental question in the theory of G-structures is: When is a given G-structure locally flat? That means that it is locally equivalent to the standard G-structure on $\mathbf{C}^n$ (or $\mathbf{R}^n$ in the real case)? If $G = \mathrm{O}(n;\mathbf{C})$, we call a locally flat G-structure a *euclidean structure*, if $G = \mathrm{GL}(n;\mathbf{E}) \subseteq \mathrm{GL}(2n;\mathbf{C})$ an *exceptional structure*, and if $G = \mathrm{Sp}(2n;\mathbf{C}) \subseteq \mathrm{GL}(2n;\mathbf{C})$ we call a locally flat structure a *symplectic structure* on X. A necessary condition in the case $G = \mathrm{O}(n;\mathbf{C})$ is that the sectional curvature tensor R—defined as in the real case—vanishes. In the case $G = \mathrm{GL}(n;\mathbf{E})$ a necessary condition is that the torsion tensor of J, i.e., $N(J): TX \times TX \to TX$,

$$N(J)(\xi,\eta) := J([\xi,\eta] - [J\xi, J\eta]) - ([J\xi,\eta] + [\xi, J\eta])$$

vanishes. Finally, in the case $G = \mathrm{Sp}(2n; \mathbf{C})$ it is necessary that the almost-symplectic form ω is closed. In the real case, these necessary conditions are also known to be sufficient. Thus we define an almost-euclidean structure g on a complex manifold to be *euclidean*, if the associated Riemannn curvature tensor vanishes; we define an almost-exceptional structure J on a $2n$-dimensional complex manifold to be *exceptional*, if the associated torsion tensor vanishes, and we define an almost-symplectic structure ω on a $2n$-dimensional complex manifold to be *symplectic*, if it is closed.

If (X, ω) is a symplectic manifold, an exceptional structure J on X is called *calibrated*, if it is pointwise calibrated with respect to ω. Then the associated Kählerian structure is said to define a *Kähler structure* on X. Now we think that the given definitions of *euclidean*, *exceptional* and *symplectic structure* agree, i.e., we hope that the answer of the following question is in the affirmative.

Question. Are the conditions $R = 0$, $N(J) = 0$ and $d\omega = 0$ equivalent to a locally flat structure also in the complex case?

(2.2) A fundamental example. Consider now a complex Lie group G and let $\mathbf{g}$ be its Lie algebra. Then G operates on $\mathbf{g}$ via its adjoint action and on $\mathbf{g}^*$ via its coadjoint action. The tangent space of the coadjoint orbit $G(\alpha) \subseteq \mathbf{g}^*$ for some $\alpha \in \mathbf{g}^*$ is canonically identified with $\mathbf{g}/\mathbf{g}_\alpha$, where $\mathbf{g}_\alpha = \{a \in \mathbf{g} \mid \langle \alpha, [a, b] \rangle = 0 \text{ for all } b \in \mathbf{g}\}$. It carries therefore the Kirillov-Kostant structure discussed earlier. Now it is easy to see that the corresponding 2-form ω on $X = G(\alpha)$ is G-invariant and moreover closed. In fact, this follows from Cartan's formula for $d\omega$, i.e.,

$$\begin{aligned} d\omega(X, Y, Z) =& X\omega(Y, Z) - Y\omega(X, Z) + Z\omega(Y, Z) \\ & - \omega([X, Y], Z) + \omega([X, Z], Y) - \omega([Y, Z], X). \end{aligned}$$

Thus for any complex Lie group the coadjoint orbit carries a natural structure of a symplectic complex manifold.

Consider now a semi-simple complex Lie group G and let $a \in \mathbf{g}$ be a semi-simple element. Then the roots of $\mathrm{ad}a$ are the same as the roots of $\mathrm{ad}\big(\mathrm{Ad}g(a)\big)$ for any $g \in G$. In fact, if one identifies G via $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathbf{g}) \subseteq \mathrm{GL}(\mathbf{g})$ with $\mathrm{Ad}(G)$ and $\mathbf{g}$ with $\mathrm{ad}\mathbf{g} \subseteq \mathbf{gl}(\mathbf{g})$, then Ad is the conjugation action on $\mathbf{g}$. Thus choosing once a decomposition of the non-zero roots Φ of a into negative and positive ones,

$$\Phi = \Phi^+ \,\dot\cup\, \Phi^-,$$

gives an almost-exceptional structure J on the adjoint orbit $G(a) \subseteq \mathbf{g}$, which is clearly G-invariant by construction. To check that the torsion tensor vanishes, i.e.,

$$J([X, Y] - [JX, JY]) - ([JX, Y] + [X, JY]) = 0,$$

we may assume that $X = \mathrm{ad}a(b)$ and $Y = \mathrm{ad}a(c)$ for $b, c \in N^+ + N^-$ and since the infinitesimal action of $\mathbf{g}$ on the vector fields of $G(a)$ is (up to a minus sign) a Lie homomorphism, one has to check the above relation simply for $X = b$ and $Y = c$ in $N^+ + N^-$. Now, if $b \in N^+$ and $c \in N^-$ then the equation is obviously fulfilled since $J|N^+ = i\,\mathrm{id}$ and $J|N^- = -i\,\mathrm{id}$. However, to satisfy the relation also in the cases $b, c \in N^+$ and $b, c \in N^-$, we need that the decomposition has the property

$$\lambda, \mu \in \Phi^+ \Longrightarrow \lambda + \mu \notin \Phi^-.$$

In this case the relation is true also in these cases. An equivalent formulation is of course, that N^+ and N^- are Lie subalgebras of $\mathbf{g}$ (which are necessarily nilpotent then). We have proved now:

Theorem. *Let $G(a)$ be a semi-simple adjoint orbit of a semi-simple complex Lie group G. Let $\Phi \subseteq \mathbf{C}$ be the non-zero eigenvalues of $\mathrm{ad}(a)$. Then $\lambda \in \Phi$ if and only if $-\lambda \in \Phi$ and any choice of a decomposition of Φ into negative and positive eigenvalues, $\Phi = \Phi^+ \cup \Phi^-$, meaning that $\lambda \in \Phi^+$ if and only if $\lambda \in \Phi^-$, satisfying $\Phi^+ + \Phi^+ \subseteq \mathbf{C} \setminus \Phi^-$ induces a Kähler structure on X.*

As a particular case consider $G = \mathrm{SL}(n+1; \mathbf{C})$ and $a = \left(\begin{smallmatrix} -n & 0 \\ 0 & 1_n \end{smallmatrix}\right) \in \mathbf{sl}(n+1; \mathbf{C})$. Then the adjoint orbit of G is naturally identified with $\mathrm{SL}(n+1; \mathbf{C})/\mathrm{GL}(n; \mathbf{C})$. In view of the above discussion we may identify

this as $\mathrm{SU}(n+1;\mathbf{E})/\mathrm{U}(n;\mathbf{E})$ and call it the *exceptional projective space* $\mathbf{P}^n(\mathbf{E})$. The exceptional structure comes from the natural decomposition of $\mathbf{sl}(n+1;\mathbf{C})$ according to a,

$$\mathbf{sl}(n+1;\mathbf{C}) = \mathbf{gl}(n;\mathbf{C}) + N^+ + N^-,$$

where

$$\mathbf{gl}(n;\mathbf{C}) \cong \left\{ \begin{pmatrix} -\mathrm{tr}(b) & 0 \\ 0 & b \end{pmatrix} \mid b \in \mathbf{gl}(n;\mathbf{C}) \right\}$$

and

$$N^+ = \left\{ \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix} \mid v \in \mathbf{C}^n \right\}, \quad N^- = \left\{ \begin{pmatrix} 0 & 0 \\ v & 0 \end{pmatrix} \mid v \in \mathbf{C}^n \right\}.$$

Since $\mathrm{SU}(n+1;\mathbf{E})$ acts transitively on $\{v \in \mathbf{E}^{n+1} \mid \langle v, v\rangle = 1\}$, we conclude that we may identify $\mathbf{P}^n(\mathbf{E})$ with the quotient of $(\mathbf{E}^{n+1})^* := \{v \in \mathbf{E}^{n+1} \mid \langle v, v\rangle \neq 0\}$ by the natural diagonal action of $\mathbf{E}^*$, $\alpha.v = \alpha v$, i.e.,

$$\mathbf{P}^n(\mathbf{E}) = (\mathbf{E}^{n+1})^*/\mathbf{E}^*.$$

Adress: Frank Loose, Mathematisches Institut, D – 40225 Düsseldorf, e-mail: loose@cs.uni-duesseldorf.de