RATIONAL TRIGONOMETRY: COMPUTATIONAL VIEWPOINT

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Main idea behind rational trigonometry. Traditional trigonometry uses non-algebraic functions such as $\sin(x)$ or $\cos(x)$ to "solve" triangles, i.e., to use the values of some of its parameters (sides a_1, a_2, a_3, angles A_1, A_2, A_3) for finding the values of others parameters. The use of non-algebraic functions complicates mathematical analysis, makes computations more difficult, and makes trigonometry more difficult to learn.

To avoid these difficulties, N. J. Wildberger proposed to describe each side a_i by its square $s_i \stackrel{\text{def}}{=} a_i^2$ (which he calls *quadrance*), and each angle A_i by its "spread" $Q_i \stackrel{\text{def}}{=} \sin^2(A_i)$. In these terms, all the formulas become algebraic: e.g., the sine law takes the form

$$\frac{s_1}{Q_1} = \frac{s_2}{Q_2} = \frac{s_3}{Q_3};$$

see, e.g., (Wildberger 2005) and (Henle 2007). These formulas are called *rational trigonometry*.

Natural alternatives to rational trigonometry. One can easily see that to avoid non-algebraic functions, it is sufficient to keep a_i and/or consider $S_i \stackrel{\text{def}}{=} \sin(A_i)$ or $C_i \stackrel{\text{def}}{=} \cos(A_i)$. Indeed, e.g., in terms of s_i and S_i, the sine law takes the algebraic form

$$\frac{s_1}{S_1^2} = \frac{s_2}{S_2^2} = \frac{s_3}{S_3^2};$$

and in terms of a_i and C_i, the sine law takes the algebraic form

$$\frac{a_1^2}{1 - C_1^2} = \frac{a_2^2}{1 - C_2^2} = \frac{a_3^2}{1 - C_3^2}.$$

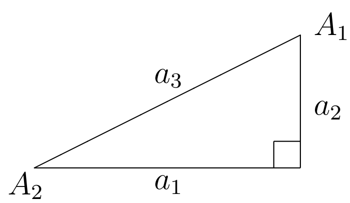
Is rational trigonometry better than its alternatives? From the purely *theoretical* viewpoint (of making all the formulas algebraic), all the above alternatives are as good as the original rational trigonometry. However, one can notice that the formulas of rational trigonometry are somewhat easier that the corresponding formulas of the alternative trigonometries. So, a natural hypothesis is that from the *computational* viewpoint, rational trigonometry is probably better. This is exactly what we prove in this paper.

Computational viewpoint: a brief description. As we have mentioned, the main idea behind rational trigonometry is that algebraic relations are "better" than non-algebraic ones, and that, e.g., computing a square root is easier than computing a sine.

In real computers, however, only arithmetic operations – addition, subtraction, multiplication, and division – are directly hardware supported (and thus faster to compute). All non-arithmetic standard functions, be it $\sqrt{x}$ or $\sin(x)$, take much longer time to compute – and this time is approximately the same for all such functions, whether they are algebraic (like $\sqrt{x}$) or non-algebraic (like $\sin(x)$).

So, from the computational viewpoint, what matters is the number of calls to all non-arithmetic standard functions.

Case of right triangles. Let us show that for right triangles, in terms of the number of such calls, rational trigonometry (s, Q) is indeed better than its alternatives (a, Q), (s, S), (s, C), (a, S), (a, C) – and than the traditional trigonometry (a, A). For a right triangle



there are 5 parameters a_1, a_2, a_3, A_1, A_2. If we know any 2 of them (except for A_1 and A_2), we can determine the rest. Modulo symmetry $a_1 \leftrightarrow a_2$, $A_1 \leftrightarrow A_2$, we thus have 5 possible problems:

1. we know a_1 and a_2;

2. we know a_1 and a_3;

3. we know a_1 and A_1;
4. we know a_1 and A_2;
5. we know a_3 and A_1.

In rational trigonometry and in alternative trigonometries, we assume that we know the corresponding characteristics of sides and angles, and we must compute the remaining ones. For example, in the 1st problem under rational trigonometry, we know s_1 and s_2, and we want to compute s_3, Q_1, and Q_2.

In the traditional trigonometry (a, A), in all 5 problems, a non-arithmetic operation is needed: $\sqrt{x}$ to compute $a_3 = \sqrt{a_1^2 + a_2^2}$ and $a_2 = \sqrt{a_3^2 - a_1^2}$ in the first 2 problems, and trig functions to compute a_2 in other problems.

In (a, S) and in (a, C), we still need $\sqrt{x}$ to compute $a_3 = \sqrt{a_1^2 + a_2^2}$ and $a_2 = \sqrt{a_3^2 - a_1^2}$. In (s, S), once we know S_1, we need $\sqrt{x}$ to compute $S_2 = \sqrt{1 - S_1^2}$. Similarly, in (s, C), once we know C_1, we need $\sqrt{x}$ to compute $C_2 = \sqrt{1 - C_1^2}$.

In contrast, in (s, Q), all 5 problems can be solved by using only arithmetic operations:

1. compute $s_3 = s_1 + s_2$, $Q_1 = s_1/s_3$, and $Q_2 = 1 - Q_1$;
2. compute $s_2 = s_3 - s_1$, $Q_1 = s_1/s_3$, and $Q_2 = 1 - Q_1$;
3. compute $s_3 = s_1/Q_1$, $s_2 = s_3 - s_1$, and $Q_2 = 1 - Q_1$;
4. compute $Q_1 = 1 - Q_2$, $s_3 = s_1/Q_1$, and $s_2 = s_3 - s_1$;
5. compute $s_1 = s_3 \cdot Q_1$, $s_2 = s_3 - s_1$, and $Q_2 = 1 - Q_1$.

Thus, for right triangles, rational trigonometry is indeed computationally faster.

Case of general triangles. For general triangles, with A_3 possibly different from $\pi/2 = 90°$, modulo symmetry, we have 5 possible problems:

1. we know a_1, a_2, and a_3;
2. we know a_1, a_2, and A_3;
3. we know a_1, a_2, and A_1;
4. we know a_1, A_1, and A_2;
5. we know a_1, A_2, and A_3.

All these problems can be solved by using the sine law, the cosine law

$$a_3^2 = a_1^2 + a_2^2 - 2a_1 \cdot a_2 \cdot \cos(A_3),$$

and the fact that $A_1 + A_2 + A_3 = \pi$.

In (s, Q), the sine law is purely arithmetic; the cosine law takes the form

$$s_3 = s_1 + s_2 - 2\sqrt{s_1 \cdot s_2 \cdot (1 - Q_3)},$$

with one non-arithmetic operation $\sqrt{x}$. The formula

$$A_3 = \pi - (A_1 + A_2)$$

leads to

$$\sin^2(A_3) = (\sin(A_1) \cdot \cos(A_2) + \sin(A_2) \cdot \cos(A_1))^2,$$

i.e., to

$$Q_3 = Q_1 \cdot (1 - Q_2) + Q_2 \cdot (1 - Q_1) + 2\sqrt{Q_1 \cdot (1 - Q_2) \cdot Q_2 \cdot (1 - Q_1)},$$

also with one non-arithmetic operation $\sqrt{x}$. So, all 5 problems can be solved by using at most one non-arithmetic operation $\sqrt{x}$:

1. use the cosine law to compute $1 - Q_3 = \dfrac{(s_1 + s_2 - s_3)^2}{4s_1 \cdot s_2}$ and similarly Q_1 and Q_2; 0 non-arithmetic operations;
2. use the cosine law to compute s_3 (one call to $\sqrt{x}$), then act as in Problem 1; 1 non-arithmetic operation;

3. use the sine law to compute $Q_2 = Q_1 \cdot (s_2/s_1)$, find Q_3 (one call to $\sqrt{x}$), then use the sine law to compute s_3; 1 non-arithmetic operation;

4. find Q_3 (one call to $\sqrt{x}$), then use the sine law to compute s_2 and s_3; 1 non-arithmetic operation;

5. find Q_1 (one call to $\sqrt{x}$), then use the sine law to compute s_2 and s_3; 1 non-arithmetic operation.

One can check that in the traditional trigonometry (a, A) and in all alternatives (a, Q), (s, S), (s, C), (a, S), (a, C), for one of the above 5 problems, at least two non-arithmetic operations are needed. Specifically:

- In (a, S), for the 2nd problem, the only law we can apply is the cosine law; this application requires two calls to $\sqrt{x}$: first to compute $\cos(A_3)$ as $\sqrt{1 - S_3^2}$ and the second to compute $a_3 = \sqrt{a_1^2 + a_2^2 - 2a_1 \cdot a_2 \cdot \cos(A_3)}$.

- In (a, A), for this same 1st problem, we need an arccosine function to find A_3 and then one more arcsine or arccosine to find one angle A_1 or A_2 (once two angles are known, we can find the third one as $\pi - A_3 - A_1$.

- In (a, C) and (s, C), for the 3rd problem, the only law we can apply is the sine law; this application requires two calls to $\sqrt{x}$: first to compute $S_1 = \sin(A_1)$ as $\sqrt{1 - C_1^2}$ and then, after computing S_2, to compute $C_2 = \sqrt{1 - S_2^2}$.

- Finally, in (s, S), for the 2nd problem, we need one square root to compute the cosine C_3, and then at least one more square root to find S_1 after applying the sine law $s_1/S_1^2 = s_3/S_3^2$.

Other alternatives? Instead of using a_i or $s_i = a_i^2$, we could use an arbitrary function $f(s_i)$. Similarly, instead of A_i or $Q_i = \sin^2(A_i)$, we could use an arbitrary function $F(Q_i)$. We have shown that for s_i and Q_i, we need 0 non-arithmetic operations for right triangles and at most 1 for general triangles. A natural question is: for what other functions $f(x)$ and $F(x)$ is this property true?

For the right triangle, the Pythagoras theorem leads to $s_3 = s_1 + s_2$. We want to compute $f(s_3)$ from $f(s_1)$ and $f(s_2)$ without using any non-arithmetic operations, i.e., by applying a rational function. So, the desired function $f(x)$ must satisfy the property $f(s_1+s_2) = R(f(s_1), f(s_2))$ for some rational function $R(x,y)$. As proven in (Aczel 2006), such functions are either rationally equivalent to $f(x) = x$ – i.e., have the form $f(x) = \dfrac{a \cdot x + b}{c \cdot x + d}$ – or to $f(x) = \exp(k \cdot x)$. One can check that for $f(x) = \exp(k \cdot x)$, other formulas cannot be arithmetic, so, in essence, we have to consider $f(x) = x$ and the quadrance $f(s_i) = s_i = a_i^2$.

Similarly, one can prove that we have to consider a function $F(x) = x$, i.e., in effect, the spread $F(Q_i) = Q_i = \sin^2(A_i)$. Strictly speaking, we can consider functions are known to be either rationally equivalent to $F(x) = x$. Let us give some trig-relevant examples:

- For $F(x) = 1 - x$, we get $\cos^2(A_i)$.
- For $F(x) = x/(1 - x)$, we get $\tan^2(A_i)$.
- For $F(x) = (1 - x)/x$, we get $\cot^2(A_i)$.
- For $F(x) = 1/x$, we get $\operatorname{cosec}^2(A_i) = 1/\sin^2(A_i)$.
- For $F(x) = 1/(1 - x)$, we get $\sec^2(A_i) = 1/\cos^2(A_i)$.

Conclusions. Traditional trigonometry uses non-algebraic functions like $\sin(x)$ to "solve triangles", i.e., to find some parameters of the triangle from the others. In (Wildberger 2005), N. J. Wildberger showed that if we characterize each side a_i by its "quadrance" $s_i \stackrel{\text{def}}{=} a_i^2$ and each angle A_i by its "spread" $Q_i \stackrel{\text{def}}{=} \sin^2(A_i)$, then all the formulas for solving triangles become algebraic. Formulas using s_i and Q_i are called *rational trigonometry*.

The above "algebraic property" holds not only for the rational trigonometry (i.e., for the combination of s_i and Q_i), but also for other combinations: e.g., for a_i and $S_i \stackrel{\text{def}}{=} \sin(A_i)$. Which combination should be select?

One of the main original objectives behind the algebraic property is the desired reduction of computation time. In view of this fact, in this paper, we analyze which combination leads to the smallest computation time. In modern computers, any non-arithmetic operation is much slower than an arithmetic one, so it is reasonable to use the number of non-arithmetic operations as a measure of computation time. We have shown that for s_i and Q_i, we need 0 non-arithmetic operations to solve right triangles and at most 1 non-arithmetic operation to solve general triangles. We have also shown that every combination which with this property is (rationally equivalent to) s_i and Q_i. Thus, rational trigonometry is indeed the computationally fastest way of solving triangles.

Open questions. In the above text, we only considered triangles. What happens if we similarly analyze more complex geometric constructions? Constructions in spherical, hyperbolic (Lobachevsky), or pseudo-Euclidean space?

In some cases, the traditional approach is clearly computationally faster. For example, if a straight line segment consists of two disjoint subsegments of lengths a_1 and a_2, then the total length a is arithmetically computable as $a = a_1 + a_2$, but computing the total spread requires a call to the function $\sqrt{x}$: $s = s_1 + s_2 + 2\sqrt{s_1 \cdot s_2}$. Similarly, rotation by a given angle becomes more computationally difficult in the rational trigonometry. Is it possible to characterize such cases?

Acknowledgments. The author is very thankful to colleagues from the University of Technology, Sydney, for their hospitality.

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