Intro (Divine Proportions)

This book revolutionizes trigonometry, re-evaluates and expands Euclidean geometry, and gives a simpler and more elegant approach to many practical geometric problems. This new theory unites the three core areas of mathematics—geometry, number theory and algebra—and expels analysis and infinite processes from the foundations of the subject. Learning trigonometry and geometry should be easier than it currently is, and *Divine Proportions* attempts to develop a complete and precise alternative framework from which educators can draw.

The concise format is suitable for mathematicians, engineers, physical scientists, mathematics teachers and dedicated amateurs with a strong interest in geometry and good skills at algebraic manipulation. Gifted high school students and undergraduate mathematics majors will find most of the material accessible.

Advanced mathematical knowledge, such as linear algebra and group theory, is generally not needed. Indeed the geometry in this book often provides motivation for these subjects. Some brief familiarity with the most elementary aspects of fields is useful. The chapters on physics applications and rational polar coordinates require differential and integral calculus.

Rational trigonometry

The new form of trigonometry developed here is called *rational trigonometry*, to distinguish it from *classical trigonometry,* the latter involving $\cos\theta$, $\sin\theta$ and the many trigonometric relations currently taught to students. An essential point of rational trigonometry is that *quadrance* and *spread*, not distance and angle, are the right concepts for metrical geometry (i.e. a geometry in which measurement is involved).

Quadrance and spread are quadratic quantities, while distance and angle are almost, but not quite, linear ones. The quadratic view is the more general and powerful one. At some level, this is known by many mathematicians. When this insight is put into practice, as it is here, a new foundation for mathematics and mathematics education arises which simplifies Euclidean and non-Euclidean geometries, changes our understanding of algebraic geometry, and often simplifies difficult practical problems.

Quadrance measures the separation of points, and spread measures the separation of lines. It turns out that

$$\begin{aligned} \text{quadrance} &= (\text{distance})^2 \\ \text{spread} &= (\sin(\text{angle}))^2 \end{aligned}$$

although the actual definitions used in this text are independent of distance, angle and the trigonometric functions. They are ultimately very simple, based on finite arithmetic as taught in elementary school.

New laws now replace the Cosine law, the Sine law, and the dozens of other trigonometric formulas that often cause students difficulty. The most important new laws are the *Triple quad formula*, the *Spread law*, the *Cross law* and the *Triple spread formula*. *Pythagoras' theorem*, restated in terms of quadrances, also plays a key role. The derivation of these rules from first principles is straightforward, involving some moderate skill with basic algebra. The usual trigonometric functions, such as $\cos\theta$ and $\sin\theta$, play no role at all.

Rational trigonometry deals with many practical problems in an easier and more natural way than classical trigonometry, and often ends up with answers that are demonstrably more accurate. In fact rational trigonometry is so elementary that almost all calculations may be done by hand. Tables or calculators are not necessary, although the latter certainly speed up computations. It is a shame that this theory was not discovered earlier, since accurate tables were for many centuries not widely available.

Universal geometry

Because rational trigonometry uses only arithmetic and algebra, it allows the development of Euclidean geometry in a consistent and general way in an arbitrary field. This is *universal geometry*.

Historically metrical geometry has been difficult to develop outside the decimal numbers, due to the transcendental nature of angle. Once liberated from a dependence on analysis, the scope is much wider, and the opportunities to effectively harness the power of modern mathematical software, such as MAPLE, MAGMA, MUPAD, MATHEMATICA, MATLAB and others, increase dramatically.

Many classical geometrical subjects are here reformulated into this more general framework, such as Heron's, or Archimedes' formula, similar triangles, parallelograms, Pons Asinorum, centroids, orthocenters, theorems of Menelaus and Ceva, Stewart's theorem, circumcircles, Brahmagupta's formula for cyclic quadrilaterals, regular polygons, the Euler line and the nine point circle. Some topics, such as vertex bisectors and triangle incenters, become dependent on number theoretic considerations. Many more topics await the attention of researchers.

Universal geometry deals only with geometrical concepts and results which are common to all fields. The usual notions of *convexity* and *betweenness*, for example, hold only in the rational and decimal number fields and some closely related ones, so they are not as fundamental as say *perpendicularity*, which applies to all fields. Over finite fields a host of new geometrical configurations can now be studied, with many number theoretic and combinatorial implications.

Universal geometry does not require a prior understanding of classical geometry. It is a complete, independent system that stands logically separate from the existing treatment of Euclid. It neither contains classical geometry nor is contained by it, although there are many close connections. One of the main features of this new theory is its logical precision and clarity.

A subtle obstacle to beginning a serious study of geometry is the fact that many of the main concepts are already familiar from ordinary life. The temptation arises to dismiss the need for precise definitions, because everyone surely already knows what a point, line and triangle are! But without very clear and careful statements of the key concepts, mathematics necessarily ends up relying on doubtful arguments.

There are three rocks on which most attempts at developing Euclidean geometry founder. These are

- the ambiguity of defining the 'continuum', or the 'real number line'
- the problem of stating precisely what an 'angle' is
- the difficulty in making the jump from two to three dimensions.

The 'real number line' involves philosophising about infinite sets, and confusion with issues of computability. Angles require hand-waving about 'rotations', or 'lengths of circular arcs'. 'Three dimensional geometry' usually involves pictures and physically plausible arguments without proper mathematical basis.

All three are serious obstacles. The first two are overcome in this book, the third is not. To avoid the logical deficiencies with 'real numbers', universal geometry works over an arbitrary field. Furthermore the presentation given here avoids any mention of 'infinite sets'. To avoid angles, rational trigonometry deals with spreads.

As for the step from two to three dimensions, in principle this can be done, but it requires considerably more work. With universal geometry, one begins to appreciate the large gulf between these dimensions, both number theoretically and in terms of concepts and proofs required.

All theorems in this book will be presented only in the planar two dimensional case. The applications part of the book however does use three dimensions, so is in parts guilty of the usual logical obscurity.

Universal geometry naturally leads to a rebirth of the traditional idea of algebraic geometry as a study of the *metrical aspects of curves and varieties*, in the spirit of the classical investigations of the ancient Greeks, as well as the work of Fermat, Newton, Euler and their contemporaries. This is a more rigid subject than modern algebraic geometry, and possibly just as rich.

For example, many of the most important and beautiful properties of conics, going back to Apollonius and Archimedes, occur in a metrical setting. For more general curves and varieties, this way of thinking opens up a large potential area for investigation, including the possibility of using tangent conics and higher degree generalizations to classify points on curves, unifying aspects of algebraic and differential geometry, and for more accurate approximations in practical problems.

This view of algebraic geometry does not elevate algebraically closed fields to a position of dominance, and also reaffirms the ordinary intuition that curves and varieties are often most natural in an affine rather than a projective setting. In fact the metrical approach extends also to projective varieties.

The foundations of Euclidean geometry established here can be generalized to non-Euclidean geometries as well, including spherical (or elliptic) geometry, and hyperbolic geometry too. Universal geometry leads to a yet much broader vista, in which Euclidean and non-Euclidean geometries merge in a spectacular way to form *chromogeometry*. These topics are more advanced, and will be developed elsewhere.

As well as providing new directions for mathematics, universal geometry offers the possibility for new insights into physics. One reason is that physics is also largely a quadratic subject. Another is that Einstein's special theory of relativity fits naturally into this framework.

Organization of the book

Part I gives first of all an overview of rational trigonometry and how it differs from classical trigonometry. A brief statement of the main definitions and basic laws will allow the reader to get a preliminary feeling for the subject. (This material is developed carefully in Part II). A preliminary section clarifies terminology and reviews those aspects of basic mathematics that will be needed.

Cartesian planar geometry is introduced in a systematic manner. Fundamental concepts are clearly laid out, and the basic results of planar coordinate geometry are established over a general field. This material is mostly classical, but the definitions are often novel. Another chapter develops the theory of rotations in points and reflections in lines, along with the associated lineations, and includes important formulas that are often absent from geometry courses.

Part II is the heart of the book, and develops rational trigonometry over a general field, with characteristic two excluded for technical reasons. The notions of quadrance, spread and *quadrea* are introduced, as well as the concepts of *cross* and *twist* between lines. The Triple quad formula, Pythagoras' theorem, Spread law, Cross law and Triple spread formula are derived, as well as more complicated extensions involving four quadrances and spreads. The important *spread polynomials* naturally appear and have many interesting properties, valid over a general field.

Another chapter introduces *oriented triangles* and the concept of *turn* between two lines, closely related to the twist. The oriented version allows a notion of *signed areas* of triangles and n-gons. This completes the basic material on rational trigonometry.

Part III uses rational trigonometry to develop the fundamentals of universal geometry. Isosceles, equilateral and right triangles are studied, along with the procedures for solving general triangles. Then the laws of proportion for triangles and quadrilaterals are derived, along with Stewart's theorem, Menelaus' theorem and Ceva's theorem. Special lines and centers of triangles are discussed, such as medians, altitudes and vertex bisectors, along with circumcenters, orthocenters and incenters.

An introduction to general conics includes some surprises, as the usual definitions do not always generalize well to universal geometry. Circles and parabolas appear, as well as less familiar conics called *ribbons*, *quadrolas* and *grammolas*. The geometry of circles is studied. Some basic facts about quadrilaterals, especially cyclic ones, are included, as well as the Four point relation going back to Euler. The notions of a tangent line and a tangent conic are introduced, and illustrated with the folium of Descartes and the lemniscate of Bernoulli.

Every theorem in this book holds over an arbitrary field, with characteristic two excluded for technical reasons. Although sometimes parallel to existing theorems, and given the same names, the results are considerably more general. Note that no familiarity with classical trigonometry or geometry is assumed. The reader need not know what a circle or an ellipse is, what similar triangles are, or even what a line is. Indeed the reader should put aside fixed preconceptions in these directions.

Part IV shows how to apply the theory to a wide variety of practical problems using the decimal numbers, which in practice often means the rational numbers. There are applications to physics, surveying, including the problems of Snellius-Pothenot and Hansen, two and three dimensional situations, and Platonic solids.

The final chapter requires some knowledge of basic calculus, and derives the new *rational polar and spherical coordinates*, with applications to the Beta function, volumes and surface areas of spheres and some related situations. Some brief discussion shows how to extend this framework to four dimensional space and beyond. One Appendix contains a list of rational polar equations of some classical curves in the plane, and another briefly introduces the beautiful *ellipson*.

In this book, an equation of equality between previously defined objects uses the equal sign $=$, while an equation of assignment of a name or a value to a previously defined object uses the assignment sign $\equiv$. This follows E. Bishop's famous dictum that "Meaningful distinctions deserve to be maintained." Occasionally both $=$ and $\equiv$ are used in the same equation, hopefully without undue confusion.

All definitions are highlighted in bold. Italics are usually reserved for emphasis. Theorems are given descriptive names, and especially important material, such as a major definition, theorem or problem, is enclosed in a gray box. To further structure the content, the ends of Proofs and Solutions are marked by the symbol ■ while the ends of Examples and Exercises are marked by the symbol ⋄.

In universal geometry formulas are particularly significant, and proofs by calculation necessarily play an increasingly important role as you delve further into the subject. A computer with a modern algebra package, such as those mentioned earlier, becomes an indispensable tool into higher investigations, as in the approach of Zeilberger.

While much effort has been made to avoid ambiguities and to give careful proofs, there are undoubtedly places in this text where this goal has not been reached. I hope nevertheless that the framework is sufficiently solid for others to take the subject further with confidence.

To the reader

Mathematics is a conservative discipline, and it is not easy to acknowledge that traditional thinking might involve elements of misunderstanding.

In this context, the following analogy may be useful. In the Roman period, which saw the beginnings of classical trigonometry, arithmetic used Roman numerals (such as the page numbers in this introduction). Cities were built, students were taught, and an empire was administered, with an arithmetic that was cumbersome and hard to learn, at least when compared to the one we now use built from the Arabic-Hindu numerical system. Today we understand that the difficulty with arithmetic in Roman times was largely due to the awkward conceptual framework.

Much the same holds, in my opinion, for classical trigonometry—it has been such a hurdle to generations of students not because of the essential intractability of the subject, but rather because the basic notions used to study it for the last two thousand years are *not the right ones.*

By the time you have finished this book, you should be comfortable with the fact that *geometry is a quadratic subject, requiring quadratic mathematics.* Using more or less linear ideas, such as distance and angle, may be initially appealing but is ultimately inappropriate. With the natural approach of rational trigonometry, many more people should be able to appreciate the rich patterns of geometry and perhaps even experience the joy of mathematical discovery.

This book asks something from you, the reader—an openness to fresh ideas, attention to detail, and a willingness to explore. In return, it promises to transform your understanding of mathematics, to empower you with knowledge previously hidden, and to shed a new light on the beauty and unity of the world around us.