arXiv:0806.3617v1 [math.MG] 23 Jun 2008

# Chromogeometry

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June 23, 2008

**Abstract**

Chromogeometry brings together Euclidean geometry (called *blue*) and two relativistic geometries (called *red* and *green*), in a surprising three-fold symmetry. We show how the red and green 'Euler lines' and 'nine-point circles' of a triangle interact with the usual blue ones, and how the three orthocenters form an associated triangle with interesting collinearities. This is developed in the framework of rational trigonometry using *quadrance* and *spread* instead of *distance* and *angle*. The former are more suitable for relativistic geometries.

## Introduction

Three-fold symmetry is at the heart of a lot of interesting mathematics and physics. This paper shows that it also plays an unexpected role in planar geometry, in that the familiar Euclidean geometry is only one of a trio of interlocking metrical geometries. We refer to Euclidean geometry here as *blue* geometry; the other two geometries, called *red* and *green,* are relativistic in nature and are associated with the names of Lorentz, Einstein and Minkowski.

The three geometries support each other and interact in a rich way. This transcends Klein's Erlangen program, since there are now *three groups* acting on a space. Remarkable algebraic identities lie at the heart of the explanations.

The results described here are just the tip of an iceberg, leading to many rich generalizations of results of Euclidean geometry, with much waiting to be discovered and explored, see for example [4] for applications to conics and [3] for connections with one dimensional metrical geometry.

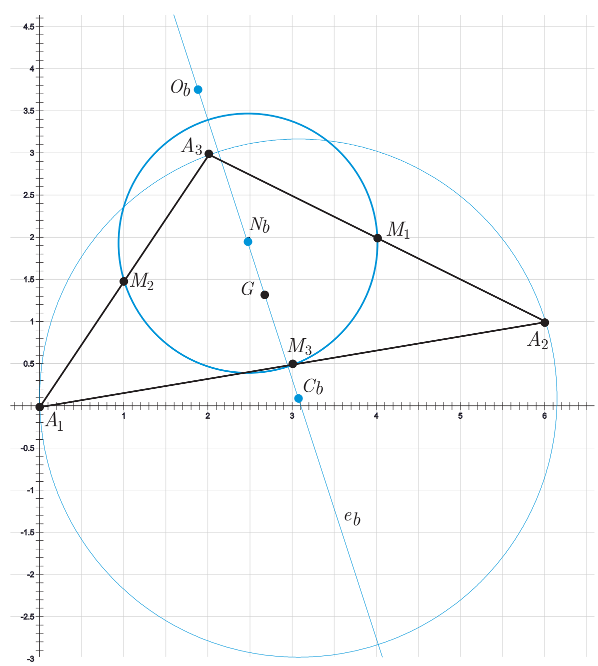
The basic structure of all three geometries are the same—they are ruled by the laws of *rational trigonometry* as developed recently in [1], which hold over a general field not of characteristic two. Although over the rational numbers (or the 'real numbers') there are significant differences between the Euclidean (blue) version and the other two (red and green), it is the *interaction* of all three which yields the biggest surprises.

To start the ball rolling, this paper first introduces the phenomenon in the context of the classical Euler line and nine-point circle of a triangle. Then we recall the main laws of rational trigonometry, introduce the basic facts about the three geometries and state some explicit formulas, and then show how chromogeometry allows us to enlarge our understanding of the geometry of a triangle. In particular we associate to each triangle $\overline{A_1A_2A_3}$ in the Cartesian plane a second interesting triangle which we call the $\Omega$-*triangle* of $\overline{A_1A_2A_3}$.

The results are verified by routine but sometimes lengthy computation, and they inevitably reduce to algebraic identities, some of which are quite lovely. The development takes place in the framework of *universal geometry*, so that we are interested primarily in what happens over *arbitrary fields*. The paper ([2]) shows that universal geometry also extends to arbitrary quadratic forms, and embraces both spherical and hyperbolic geometries in a projective version.

# Euler lines and nine-point circles in relativistic settings

Recall that for a triangle $\overline{A_1A_2A_3}$ the intersection of the medians is the **centroid** $G$, the intersection of the altitudes is the **orthocenter** $O$ and the intersection of the perpendicular bisectors of the sides is the **circumcenter** $C$, which is the center of the circumcircle of the triangle. Remarkably, it was left to Euler to discover that these three points are collinear, and that $G$ divides $\overline{OC}$ in the (affine) proportion $2:1$. Furthermore the center $N$ of the circumcircle of the triangle $\overline{M_1M_2M_3}$ of midpoints of the sides of $\overline{A_1A_2A_3}$ (called the **nine-point circle** of $\overline{A_1A_2A_3}$) also lies on the Euler line, and is the midpoint of $\overline{OC}$.



This is above shown for the triangle $\overline{A_1A_2A_3}$ with points

$$A_1 \equiv [0,0] \qquad A_2 \equiv [6,1] \qquad A_3 \equiv [2,3]\,.$$

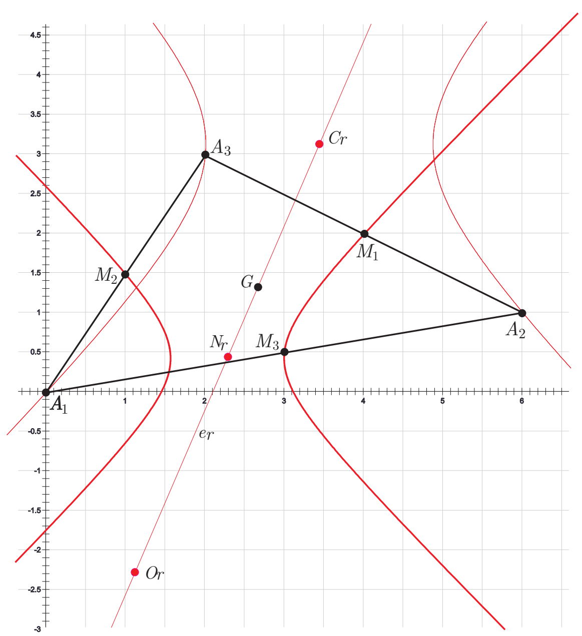
The triangle $\overline{A_1A_2A_3}$ is in black, while the circumcircle and nine-point circle are in blue (the latter more boldly), as are the Euler line and the points $O, C$ and $N$, which are given the subscript $b$ for blue, and henceforth referred to as the **blue Euler line**, the **blue orthocenter** etc.

Planar Euclidean geometry rests on the **blue quadratic form** $x^2 + y^2$ (or if you prefer the corresponding symmetric bilinear form, or dot product). It is also interesting to consider the **red quadratic form** $x^2 - y^2$ which figures prominently in two dimensional special relativity. In this case, two lines are red perpendicular precisely when one can be obtained from the other by ordinary Euclidean reflection in a **red null line,** which is red perpendicular to itself, and has usual slope $\pm 1$.

It turns out that for any triangle $\overline{A_1A_2A_3}$ the three red altitudes also intersect, now in a point called the **red orthocenter** and denoted $O_r$, and the three perpendicular bisectors also intersect in a point called the **red circumcenter** and denoted $C_r$. This latter point is the center of the unique red circle through the three points of the triangle, where a **red circle** is given by an equation of the form

$$(x - x_0)^2 - (y - y_0)^2 = K.$$

This is what we would usually call a rectangular hyperbola, with axes in the red null directions.



This diagram shows the same triangle $\overline{A_1A_2A_3}$, as well as the red circumcircle, the red nine-point circle and the red orthocenter, circumcenter, nine-point center and centroid $G$, the latter being independent of colour.

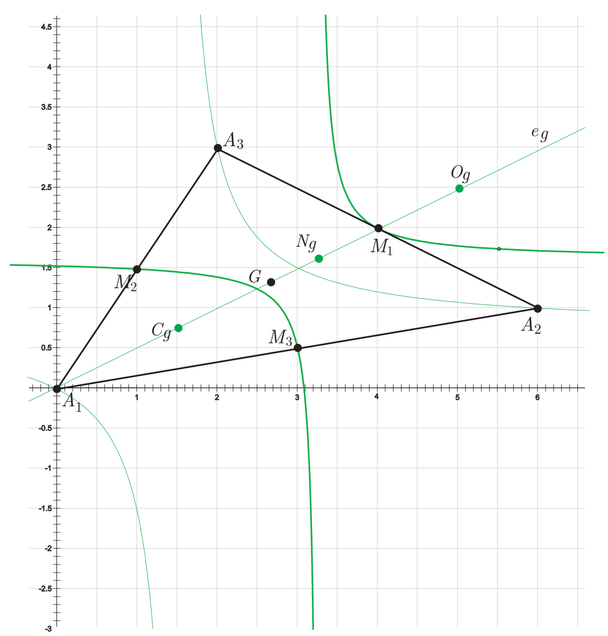
Note that these points all lie on a line—the **red Euler line**, and the affine relationships between these points is exactly the same as for the blue Euler line, so that for example $N_r$ is the midpoint of $\overline{O_rC_r}$.

In the classical framework, there are some difficulties in setting up this relativistic geometry, as 'distance' and 'angle' are problematic. In universal geometry one regards the *quadratic form* as primary, not its *square root*, and by expressing everything in terms of the algebraic concepts of *quadrance* and *spread*, Euclidean geometry can be built up so as to allow generalization to the relativistic framework, and indeed to geometries built from other quadratic forms.

This approach was introduced recently in [1], see also [5], and works over a general field with characteristic two excluded for technical reasons, as shown in [2]. The possibility of relativistic geometries over other fields seems particularly attractive.

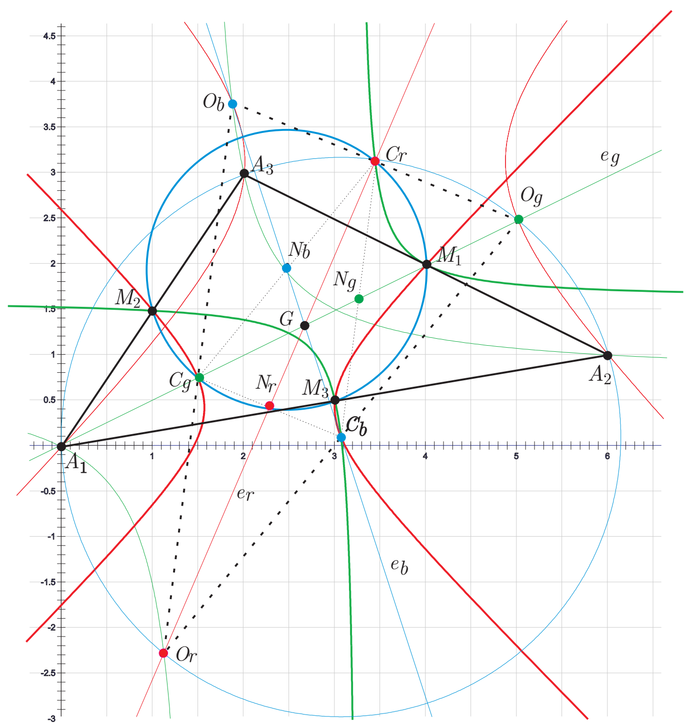
There is a third geometry, that associated to the **green quadratic form** $2xy$. Two lines are green perpendicular when one is the ordinary Euclidean reflection of the other in a line parallel to the axes, the latter being a **green null line**.

Since $x^2 - y^2$ and $2xy$ are conjugate by a simple change of variable, it should be no surprise that the corresponding relations between the green orthocenter $O_g$, green circumcenter $C_g$, green nine-point center $N_g$ and the centroid $G$ hold as well. Here is the relevant diagram for our triangle $\overline{A_1A_2A_3}$.



I conjecture that *most theorems of planar Euclidean geometry, when formulated algebraically in the context of universal geometry, extend to the red and green situations.* However there are exceptions. For example, over the 'real numbers' there are no equilateral triangles in the red or green geometries, so Napoleon's theorem and Morley's theorem will not have direct analogs.

Much could be said further to support this conjecture, but this is not what I wish to pursue here. Instead, let's consider a completely new phenomenon. Observe what happens when the three diagrams are put together!




We get remarkable collinearities, for example between $O_b$, $C_r$ and $O_g$, and between $C_b$, $N_r$ and $C_g$, with furthermore $C_r$ the midpoint of $\overline{O_bO_g}$, and $N_r$ the midpoint of $\overline{C_bC_g}$. Also we observe that, for example, $O_r$ and $O_g$ lie on the blue circumcircle of $\overline{A_1A_2A_3}$, while $C_r$ and $C_g$ lie on the blue nine-point circle of $\overline{A_1A_2A_3}$. The three colours generally interact symmetrically, so the same relations hold if we permute colours.

However there are some aspects of chromogeometry in which this symmetry is broken. The blue geometry as we shall see behaves somewhat differently from the red and the green in certain contexts, and when we come to explicit formulas we will see that the green geometry is often simpler. The red geometry seems less inclined to distinguish itself.

## Rational trigonometry

Let's now proceed more formally, beginning with the main definitions and laws of rational trigonometry. We work over a fixed field, not of characteristic two, whose elements will be called **numbers**. The **plane** will consist of the standard vector space of dimension two over this field. A **point**, or **vector**, is an ordered pair $A \equiv [x, y]$ of numbers. The origin is denoted $O \equiv [0, 0]$.

A **line** is a proportion $l \equiv \langle a : b : c \rangle$ where $a$ and $b$ are not both zero. The point $A \equiv [x, y]$ **lies on** the line $l \equiv \langle a : b : c \rangle$, or equivalently the line $l$ **passes through** the point $A$, precisely when

$$ax + by + c = 0.$$

This is not the only possible convention, and the reader should be aware that it is prejudiced towards the usual Euclidean (blue) geometry. For any two points $A_1 \equiv [x_1, y_1]$ and $A_2 \equiv [x_2, y_2]$ there is a unique line $l \equiv A_1A_2$ which passes through them both. Specifically we have

$$A_1A_2 = \langle y_1 - y_2 : x_2 - x_1 : x_1y_2 - x_2y_1 \rangle .$$

Three points $[x_1, y_1]$, $[x_2, y_2]$ and $[x_3, y_3]$ are **collinear** precisely when they lie on the same line, which amounts to the condition

$$x_1y_2 - x_1y_3 + x_2y_3 - x_3y_2 + x_3y_1 - x_2y_1 = 0. \tag{1}$$

Three lines $\langle a_1 : b_1 : c_1 \rangle$, $\langle a_2 : b_2 : c_2 \rangle$ and $\langle a_3 : b_3 : c_3 \rangle$ are **concurrent** precisely when they pass through the same point, which amounts to the condition

$$a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_3b_2c_1 + a_3b_1c_2 - a_2b_1c_3 = 0.$$

Fix a symmetric bilinear form, denoted by the dot product $A_1 \cdot A_2$. In practise we will take this bilinear form to be non-degenerate. The line $A_1A_2$ is **perpendicular** to the line $B_1B_2$ precisely when

$$(A_2 - A_1) \cdot (B_2 - B_1) = 0.$$

A point $A$ is a **null point** or **null vector** precisely when $A \cdot A = 0$. The **origin** $O$ is always a null point, but there may be others. A line $A_1A_2$ is a **null line** precisely when the vector $A_2 - A_1$ is a null vector.

A set $\{A_1, A_2, A_3\}$ of three distinct non-collinear points is a **triangle** and is denoted $\overline{A_1A_2A_3}$. The **lines** of the triangle are $l_3 \equiv A_1A_2$, $l_2 \equiv A_1A_3$ and $l_1 \equiv A_2A_3$. A triangle is **non-null** precisely when each of its lines is non-null. A **side** of the triangle is a subset of $\{A_1, A_2, A_3\}$ with two elements, and is denoted $\overline{A_1A_2}$ etc. A **vertex** of the triangle is a subset of $\{l_1, l_2, l_3\}$ with two elements, and is denoted $\overline{l_1l_2}$ etc.

The **quadrance** between the points $A_1$ and $A_2$ is the number

$$Q(A_1, A_2) \equiv (A_2 - A_1) \cdot (A_2 - A_1).$$

The line $A_1A_2$ is a null line precisely when $Q(A_1, A_2) = 0$.

The **spread** between the non-null lines $A_1A_2$ and $B_1B_2$ is the number

$$s(A_1A_2, B_1B_2) \equiv 1 - \frac{((A_2 - A_1) \cdot (B_2 - B_1))^2}{Q(A_1, A_2) Q(B_1, B_2)}.$$

This is independent of the choice of points lying on the two lines. Two non-null lines are perpendicular precisely when the spread between them is 1.

Here are the five main laws of planar rational trigonometry in this general setting, replacing the usual Sine law, Cosine law etc. Proofs can be found in [2]. Aside from giving new directions to geometry, these laws have the potential to change the teaching of high school mathematics, because they are simpler, and allow faster and more accurate calculations in practical problems. But the advantage for us here is that they *hold for general quadratic forms*, and in particular for each of the blue, red and green geometries.

**Theorem 1 (Triple quad formula)** *The points $A_1, A_2$ and $A_3$ are collinear precisely when the quadrances $Q_1 \equiv Q(A_2, A_3)$, $Q_2 \equiv Q(A_1, A_3)$ and $Q_3 \equiv Q(A_1, A_2)$ satisfy*

$$(Q_1 + Q_2 + Q_3)^2 = 2\left(Q_1^2 + Q_2^2 + Q_3^2\right).$$

**Theorem 2 (Pythagoras' theorem)** *For $A_1$, $A_2$ and $A_3$ three distinct points, $A_1A_3$ is perpendicular to $A_2A_3$ precisely when the quadrances $Q_1 \equiv Q(A_2, A_3)$, $Q_2 \equiv Q(A_1, A_3)$ and $Q_3 \equiv Q(A_1, A_2)$ satisfy*

$$Q_1 + Q_2 = Q_3.$$

**Theorem 3 (Spread law)** *Suppose the non-null triangle $\overline{A_1A_2A_3}$ has quadrances $Q_1 \equiv Q(A_2, A_3)$, $Q_2 \equiv Q(A_1, A_3)$ and $Q_3 \equiv Q(A_1, A_2)$, and spreads $s_1 \equiv s(A_1A_2, A_1A_3)$, $s_2 \equiv s(A_2A_1, A_2A_3)$ and $s_3 \equiv s(A_3A_1, A_3A_2)$. Then*

$$\frac{s_1}{Q_1} = \frac{s_2}{Q_2} = \frac{s_3}{Q_3}.$$

**Theorem 4 (Cross law)** *Suppose the non-null triangle $\overline{A_1A_2A_3}$ has quadrances $Q_1 \equiv Q(A_2, A_3)$, $Q_2 \equiv Q(A_1, A_3)$ and $Q_3 \equiv Q(A_1, A_2)$, and spreads $s_1 \equiv s(A_1A_2, A_1A_3)$, $s_2 \equiv s(A_2A_1, A_2A_3)$ and $s_3 \equiv s(A_3A_1, A_3A_2)$. Then*

$$(Q_1 + Q_2 - Q_3)^2 = 4Q_1Q_2(1 - s_3).$$

Note that the Cross law includes as special cases both the Triple quad formula and Pythagoras' theorem. The next result is the algebraic analog to the sum of the angles in a triangle formula.

**Theorem 5 (Triple spread formula)** *Suppose the non-null triangle $\overline{A_1A_2A_3}$ has spreads $s_1 \equiv s(A_1A_2, A_1A_3)$, $s_2 \equiv s(A_2A_1, A_2A_3)$ and $s_3 \equiv s(A_3A_1, A_3A_2)$. Then*

$$(s_1 + s_2 + s_3)^2 = 2\left(s_1^2 + s_2^2 + s_3^2\right) + 4s_1s_2s_3.$$

A useful observation is that the Triple spread formula shows that $s_3 = 1$ implies that

$$s_1 + s_2 = 1.$$

## Three fold symmetry

The vectors $A_1 \equiv [x_1, y_1]$ and $A_2 \equiv [x_2, y_2]$ are **parallel** precisely when

$$x_1 y_2 - x_2 y_1 = 0.$$

We will be interested in three main examples of symmetric bilinear forms. Define the **blue** dot product

$$[x_1, y_1] \cdot_b [x_2, y_2] \equiv x_1 x_2 + y_1 y_2,$$

the **red** dot product

$$[x_1, y_1] \cdot_r [x_2, y_2] \equiv x_1 x_2 - y_1 y_2$$

and the **green** dot product

$$[x_1, y_1] \cdot_g [x_2, y_2] = x_1 y_2 + x_2 y_1.$$

Note that between them these four expressions give all possible bilinear expressions in the two vectors that involve only coefficients $\pm 1$, up to sign. Two lines $l_1$ and $l_2$ are **blue**, **red** and **green perpendicular** respectively precisely when they are perpendicular with respect to the blue, red and green forms. For lines $l_1 \equiv \langle a_1 : b_1 : c_1 \rangle$ and $l_2 \equiv \langle a_2 : b_2 : c_2 \rangle$ these conditions amount to the respective conditions

$$a_1 a_2 + b_1 b_2 = 0 \quad \text{[blue]}$$

$$a_1 a_2 - b_1 b_2 = 0 \quad \text{[red]}$$

and

$$a_1 b_2 + b_1 a_2 = 0 \quad \text{[green]}.$$

In terms of coordinates, the formulas for the **blue**, **red** and **green quadrances** between points $A_1 \equiv [x_1, y_1]$ and $A_2 \equiv [x_2, y_2]$ are

$$\begin{aligned} Q_b(A_1, A_2) &= (x_2 - x_1)^2 + (y_2 - y_1)^2 \\ Q_r(A_1, A_2) &= (x_2 - x_1)^2 - (y_2 - y_1)^2 \\ Q_g(A_1, A_2) &= 2(x_2 - x_1)(y_2 - y_1). \end{aligned}$$

**Theorem 6 (Coloured quadrances)** *For any points $A_1$ and $A_2$ let $Q_b$, $Q_r$ and $Q_g$ be the blue, red and green quadrances between $A_1$ and $A_2$ respectively. Then*

$$Q_b^2 = Q_r^2 + Q_g^2.$$

**Proof.** This is a consequence of the identity

$$\left(r^2 + s^2\right)^2 = \left(r^2 - s^2\right)^2 + (2rs)^2.$$

■

The formulas for the **blue**, **red** and **green spreads** between lines $l_1 \equiv \langle a_1 : b_1 : c_1 \rangle$ and $l_2 \equiv \langle a_2 : b_2 : c_2 \rangle$ are

$$\begin{aligned} s_b(l_1, l_2) &= 1 - \frac{(a_1 a_2 + b_1 b_2)^2}{(a_1^2 + b_1^2)(a_2^2 + b_2^2)} = \frac{(a_1 b_2 - a_2 b_1)^2}{(a_2^2 + b_2^2)(a_1^2 + b_1^2)} \\ s_r(l_1, l_2) &= 1 - \frac{(b_1 b_2 - a_1 a_2)^2}{(b_1^2 - a_1^2)(b_2^2 - a_2^2)} = -\frac{(a_1 b_2 - a_2 b_1)^2}{(a_2^2 - b_2^2)(a_1^2 - b_1^2)} \\ s_g(l_1, l_2) &= 1 - \frac{(-a_1 b_2 - a_2 b_1)^2}{4 a_1 a_2 b_1 b_2} = -\frac{(a_1 b_2 - a_2 b_1)^2}{4 a_1 a_2 b_1 b_2}. \end{aligned}$$

Note carefully the *minus signs* that precede the final expressions in the red and green cases.

**Theorem 7 (Coloured spreads)** *For any lines $l_1$ and $l_2$ let $s_b$, $s_r$ and $s_g$ be the blue, red and green spreads between $l_1$ and $l_2$ respectively. Then*

$$\frac{1}{s_b} + \frac{1}{s_r} + \frac{1}{s_g} = 2.$$

**Proof.** This is a consequence of the identity

$$\left(a_1^2 + b_1^2\right)\left(a_2^2 + b_2^2\right) - \left(a_1^2 - b_1^2\right)\left(a_2^2 - b_2^2\right) - 4 a_1 a_2 b_1 b_2 = 2\left(a_1 b_2 - a_2 b_1\right)^2. \ \blacksquare$$

# Quadrances

The most important single quantity associated to a triangle $\overline{A_1 A_2 A_3}$ with quadrances $Q_1, Q_2$ and $Q_3$ is the **quadrea** $\mathcal{A}$ defined by

$$\mathcal{A} \equiv (Q_1 + Q_2 + Q_3)^2 - 2\left(Q_1^2 + Q_2^2 + Q_3^2\right).$$

By the Triple quad formula this is a measure of the non-collinearity of the points $A_1, A_2$ and $A_3$. We denote by $\mathcal{A}_b, \mathcal{A}_r$ and $\mathcal{A}_g$ the respective **blue**, **red** and **green quadreas** of a triangle $\overline{A_1 A_2 A_3}$.

**Theorem 8 (Quadrea )** *For three points $A_1 \equiv [x_1, y_1]$, $A_2 \equiv [x_2, y_2]$ and $A_3 \equiv [x_3, y_3]$, the three quadreas $\mathcal{A}_b, \mathcal{A}_r$ and $\mathcal{A}_g$ satisfy*

$$\mathcal{A}_b = -\mathcal{A}_r = -\mathcal{A}_g = 4\left(x_1 y_2 - x_1 y_3 + x_2 y_3 - x_3 y_2 + x_3 y_1 - x_2 y_1\right)^2.$$

**Proof.** A calculation. $\blacksquare$

So each quadrea of a triangle is $\pm 16$ times the square of its signed area, the latter being defined purely in an affine setting, without any need for metrical choices.

We now adopt the convention that if no proof is given, 'a calculation' is to be assumed.

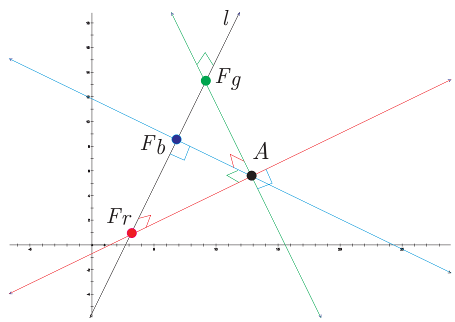
## Altitudes

**Theorem 9 (Altitudes to a line)** *For any point $A$ and any line $l$, there exist unique lines $n_b, n_r$ and $n_g$ through $A$ which are respectively blue, red and green perpendicular to $l$. If $A \equiv [x_0, y_0]$ and $l \equiv \langle a : b : c \rangle$ then*

$$n_b = \langle b : -a : -bx_0 + ay_0 \rangle$$
$$n_r = \langle b : a : -bx_0 - ay_0 \rangle$$
$$n_g = \langle a : -b : -ax_0 + by_0 \rangle . \quad \blacksquare$$

The lines $n_b, n_r, n_g$ are respectively the **blue**, **red** and **green altitudes** from $A$ to $l$, and they intersect $l$ at the **feet,** provided that $l$ is non-null.

**Theorem 10 (Perpendicularity of altitudes)** *For any point $A$ and any line $l$, let $n_b, n_r, n_g$ be the blue, red and green altitudes from $A$ to $l$ respectively. Then $n_b$ and $n_r$ are green perpendicular, $n_r$ and $n_g$ are blue perpendicular, and $n_g$ and $n_b$ are red perpendicular.* $\blacksquare$




The figure shows an example of the three colour altitudes from a point $A$ to a line $l$, and their feet $F_b$, $F_r$ and $F_g$.

**Theorem 11 (Pythagorean means)** *Let $l \equiv \langle a : b : c \rangle$ be a line which is non-null in each of the three geometries. If $A$ is a point and $F_b, F_r$ and $F_g$ are the respective feet of the altitudes $n_b, n_r$ and $n_g$ from $A$ to $l$, then we have the affine relation*

$$F_b = \frac{\left(a^2 - b^2\right)^2}{\left(a^2 + b^2\right)^2} F_r + \frac{4b^2a^2}{\left(a^2 + b^2\right)^2} F_g.$$

**Proof.** Suppose that $A \equiv [x_0, y_0]$ and $l \equiv \langle a : b : c \rangle$. Elimination yields

$$F_b = \left[ \frac{b^2x_0 - aby_0 - ac}{a^2 + b^2}, \frac{-abx_0 + a^2y_0 - bc}{a^2 + b^2} \right]$$
$$F_r = \left[ \frac{-b^2x_0 - aby_0 - ca}{a^2 - b^2}, \frac{abx_0 + a^2y_0 + bc}{a^2 - b^2} \right]$$
$$F_g = \left[ \frac{ax_0 - by_0 - c}{2a}, \frac{-ax_0 + by_0 - c}{2b} \right]$$

from which we deduce the result. $\blacksquare$

Note again the connection with Pythagorean triples.

## Anti-symmetric polynomials

We use notation for anti-symmetric polynomials introduced in [1, page 28]. If $m$ is a monomial in the variables $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3, \cdots$ with all indices in the range $1, 2$ and $3$, then we define $[m]^-$ to be the *anti-symmetric* polynomial consisting of the sum of all monomials obtained from $m$ by performing all six permutations of the indices and multiplying each term by the sign of the corresponding permutation.

We often write such polynomials in the order described by the successive transpositions

$$(23), (12), (23), (12), (23).$$

For example

$$\begin{aligned} [x_1y_2]^- &\equiv x_1y_2 - x_1y_3 + x_2y_3 - x_3y_2 + x_3y_1 - x_2y_1 \\ \left[x_1^2x_2y_2\right]^- &\equiv x_1^2x_2y_2 - x_1^2x_3y_3 + x_2^2x_3y_3 - x_3^2x_2y_2 + x_3^2x_1y_1 - x_2^2x_1y_1 \\ \left[x_1^3y_1\right]^- &\equiv x_1^3y_1 - x_1^3y_1 + x_2^3y_2 - x_3^3y_3 + x_3^3y_3 - x_2^3y_2 = 0. \end{aligned}$$

The polynomial $[x_1y_2]^-$ is of particular importance, since it occurs in (1), is twice the signed area of the triangle $\overline{A_1A_2A_3}$, appears in the Quadrea theorem, and is often a denominator in formulas in the subject.

## Orthocenters

Given a triangle $\overline{A_1A_2A_3}$, for each point $A_m$, $m = 1, 2$ and $3$ we may construct the blue, red and green altitudes $a_m^b, a_m^r$ and $a_m^g$ respectively to the opposite side.

**Theorem 12 (Orthocenter formulas)** *The three blue altitudes $a_1^b, a_2^b$ and $a_3^b$ meet in a point $O_b$ called the **blue orthocenter**. The three red altitudes $a_1^r, a_2^r$ and $a_3^r$ meet in a point $O_r$ called the **red orthocenter**. The three green altitudes $a_1^g, a_2^g$ and $a_3^g$ meet in a point $O_g$ called the **green orthocenter**. For $A_1 \equiv [x_1, y_1]$, $A_2 \equiv [x_2, y_2]$ and $A_3 \equiv [x_3, y_3]$ these points are given by*

$$\begin{aligned} O_b &= \left[\frac{[x_1x_2y_2]^- + \left[y_1y_2^2\right]^-}{[x_1y_2]^-}, \frac{[x_1y_1y_2]^- + \left[x_1^2x_2\right]^-}{[x_1y_2]^-}\right] \\ O_r &= \left[\frac{[x_1x_2y_2]^- - \left[y_1y_2^2\right]^-}{[x_1y_2]^-}, \frac{[x_1y_1y_2]^- - \left[x_1^2x_2\right]^-}{[x_1y_2]^-}\right] \\ O_g &= \left[\frac{\left[x_1^2y_2\right]^- + [x_1x_2y_1]^-}{[x_1y_2]^-}, \frac{\left[x_1y_2^2\right]^- - [x_1y_1y_2]^-}{[x_1y_2]^-}\right]. \ \blacksquare \end{aligned}$$

## Circumcenters

When $A_1$ and $A_2$ are distinct points with $l = A_1A_2$, and $A$ is the midpoint of $A_1$ and $A_2$, then the blue, red and green altitudes from $A$ to $l$ are respectively called the **blue**, **red** and **green perpendicular bisectors** of the side $\overline{A_1A_2}$.

**Theorem 13 (Perpendicular bisectors)** *If $A_1 \equiv [x_1, y_1]$ and $A_2 \equiv [x_2, y_2]$ are distinct points then the blue, red and green perpendicular bisectors of $\overline{A_1A_2}$ have respective equations*

$$\begin{aligned}
(x_1 - x_2)\,x + (y_1 - y_2)\,y &= \frac{x_1^2 - x_2^2 + y_1^2 - y_2^2}{2} \\
(x_1 - x_2)\,x - (y_1 - y_2)\,y &= \frac{x_1^2 - x_2^2 - y_1^2 + y_2^2}{2} \\
(y_2 - y_1)\,x + (x_2 - x_1)\,y &= y_2x_2 - x_1y_1. \quad \blacksquare
\end{aligned}$$

Given a triangle $\overline{A_1A_2A_3}$ we may construct the blue, red and green perpendicular bisectors of the three sides, denoted by $b_m^b, b_m^r$ and $b_m^g$ respectively for $m = 1, 2$ and 3, where $b_1^b$ for example is the blue perpendicular bisector of the side $\overline{A_2A_3}$ and so on.

**Theorem 14 (Circumcenter formulas)** *The three blue perpendicular bisectors $b_1^b, b_2^b$ and $b_3^b$ meet in a point $C_b$ called the **blue circumcenter**. The three red perpendicular bisectors $b_1^r, b_2^r$ and $b_3^r$ meet in a point $C_r$ called the **red circumcenter**. The three green perpendicular bisectors $b_1^g, b_2^g$ and $b_3^g$ meet in a point $C_g$ called the **green circumcenter**. For $A_1 \equiv [x_1, y_1]$, $A_2 \equiv [x_2, y_2]$ and $A_3 \equiv [x_3, y_3]$, these points are given by*

$$\begin{aligned}
C_b &= \left[\frac{\left[x_1^2y_2\right]^- + \left[y_1^2y_2\right]^-}{2\left[x_1y_2\right]^-}, \frac{\left[x_1y_2^2\right]^- + \left[x_1x_2^2\right]^-}{2\left[x_1y_2\right]^-}\right] \\
C_r &= \left[\frac{\left[x_1^2y_2\right]^- - \left[y_1^2y_2\right]^-}{2\left[x_1y_2\right]^-}, \frac{\left[x_1y_2^2\right]^- - \left[x_1x_2^2\right]^-}{2\left[x_1y_2\right]^-}\right] \\
C_g &= \left[\frac{\left[x_1x_2y_2\right]^-}{\left[x_1y_2\right]^-}, \frac{\left[x_1y_1y_2\right]^-}{\left[x_1y_2\right]^-}\right]. \quad \blacksquare
\end{aligned}$$

**Theorem 15 (Circumcenters as midpoints)** *For any triangle a coloured circumcenter is the midpoint of the two orthocenters of the other two colours.*

**Proof.** This follows from the Orthocenter formulas and Circumcenter formulas. ■

## Nine-point centres

Suppose that the respective midpoints of a triangle $\overline{A_1A_2A_3}$ are $M_m$ for $m = 1, 2$ and 3, where $M_1$ is the midpoint of the side $\overline{A_2A_3}$ and so on. We let $N_b, N_r$

and $N_g$ be the blue, red and green circumcenters respectively of the triangle $\overline{M_1M_2M_3}$, and call these the **blue**, **red** and **green nine-point centers** of the original triangle $\overline{A_1A_2A_3}$.

**Theorem 16 (Nine-point center formulas)** *For $A_1 \equiv [x_1, y_1]$, $A_2 \equiv [x_2, y_2]$ and $A_3 \equiv [x_3, y_3]$, the blue, red and green nine-point centers of $\overline{A_1A_2A_3}$ are*

$$N_b = \left[\frac{\left[x_1^2y_2\right]^- - \left[y_1^2y_2\right]^- + 2\left[x_1x_2y_2\right]^-}{4\left[x_1y_2\right]^-}, \frac{\left[x_1y_2^2\right]^- - \left[x_1x_2^2\right]^- + 2\left[x_1y_1y_2\right]^-}{4\left[x_1y_2\right]^-}\right]$$
$$N_r = \left[\frac{\left[x_1^2y_2\right]^- + \left[y_1^2y_2\right]^- + 2\left[x_1x_2y_2\right]^-}{4\left[x_1y_2\right]^-}, \frac{\left[x_1y_2^2\right]^- + \left[x_1x_2^2\right]^- + 2\left[x_1y_1y_2\right]^-}{4\left[x_1y_2\right]^-}\right]$$
$$N_g = \left[\frac{\left[x_1^2y_2\right]^-}{2\left[x_1y_2\right]^-}, \frac{\left[x_1y_2^2\right]^-}{2\left[x_1y_2\right]^-}\right]. \blacksquare$$

**Theorem 17 (Nine-point centers as midpoints)** *In any triangle a coloured nine-point center is the midpoint of the two circumcenters of the other two colours.*

**Proof.** This follows from the Circumcenter formulas and Nine-point center formulas. ■

## The Ω-triangle and the Euler lines

The **Ω-triangle** of a triangle $\overline{A_1A_2A_3}$ is the triangle $\Omega \equiv \Omega\left(\overline{A_1A_2A_3}\right) \equiv \overline{O_bO_rO_g}$ of orthocenters of $\overline{A_1A_2A_3}$. From the theorems of the last two sections, the corresponding midpoints of the sides of $\Omega$ are $C_b, C_r$ and $C_g$, with $C_b$ the midpoint of $O_r$ and $O_g$ etc., and the midpoints of the triangle $\overline{C_bC_rC_g}$ are $N_b, N_r$ and $N_g$, with $N_b$ the midpoint of $C_r$ and $C_g$ etc. We also know that the centroid of $\Omega$ is the same as the centroid $G$ of the original triangle $\overline{A_1A_2A_3}$.

**Theorem 18 (Blue Euler line)** *The points $O_b, N_b, G$ and $C_b$ lie on a line called the **blue Euler line**. Furthermore $N_b$ is the midpoint of $O_b$ and $C_b$, and we have the affine relations*

$$G = \frac{1}{3}O_b + \frac{2}{3}C_b = \frac{1}{3}C_b + \frac{2}{3}N_b.$$

**Proof.** This follows from the Orthocenter, Circumcenter and Nine-point center formulas. ■

**Theorem 19 (Red Euler line)** *The points $O_r, N_r, G$ and $C_r$ lie on a line called the **red Euler line**. Furthermore $N_r$ is the midpoint of $O_r$ and $C_r$, and*

$$G = \frac{1}{3}O_r + \frac{2}{3}C_r = \frac{1}{3}C_r + \frac{2}{3}N_r.$$

**Proof.** Likewise. ■

**Theorem 20 (Green Euler line)** *The points $O_g, N_g, G$ and $C_g$ lie on a line called the* ***green Euler line****. Furthermore $N_g$ is the midpoint of $O_g$ and $C_g$, and*

$$G = \frac{1}{3}O_g + \frac{2}{3}C_g = \frac{1}{3}C_g + \frac{2}{3}N_g.$$

**Proof.** Likewise. ■

The geometry of the $\Omega$-triangle clarifies the various ratios occurring along points on the Euler lines, since these are just the medians of $\Omega$. The lines joining the circumcenters are the lines of the medial triangle of $\Omega$, and so are parallel to the lines of $\Omega$.

## Circles

A **blue, red or green circle** is an equation $c$ in $x$ and $y$ of the form

$$(x - x_0)^2 + (y - y_0)^2 = K$$
$$(x - x_0)^2 - (y - y_0)^2 = K$$
$$2(x - x_0)(y - y_0) = K$$

respectively, where the point $[x_0, y_0]$ is then unique and called the **centre** of $c$, and $K$ is the **quadrance** of $c$. A blue circle is an ordinary Euclidean circle. Red and green circles are more usually described as rectangular hyperbolas. A red circle has asymptotes parallel to the lines with equations $y = \pm x$, and a green circle has asymptotes parallel to the coordinate axes.

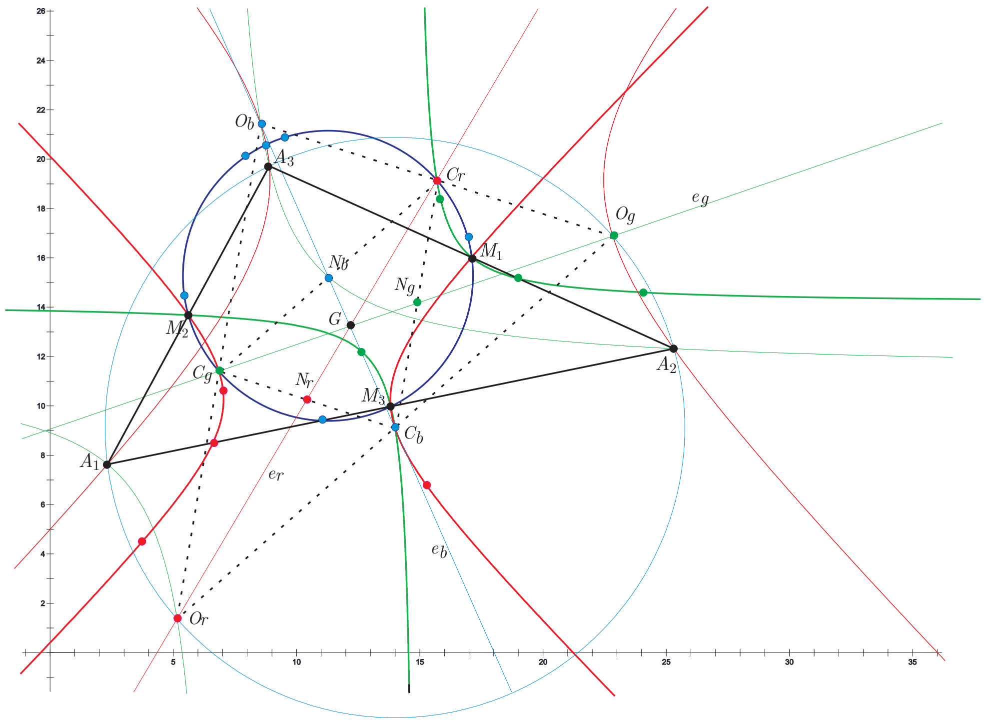
**Theorem 21 (Circumcircles)** *If $A_1, A_2$ and $A_3$ are three distinct non-collinear points, then there are unique blue, red and green circles passing through $A_1, A_2$ and $A_3$.* ■

The circles above will be called respectively the **blue**, **red** and **green circumcircles** of the triangle $\overline{A_1A_2A_3}$, while the circumcircles of the triangle of midpoints $M_1, M_2, M_3$ of the triangle $\overline{A_1A_2A_3}$ will be called respectively the **blue**, **red** and **green nine-point circles** of the triangle $\overline{A_1A_2A_3}$.

**Theorem 22 (Orthocenters on circumcircles)** *Any coloured orthocenter of a triangle $\overline{A_1A_2A_3}$ lies on the circumcircles of the other two colours.* ■

**Theorem 23 (Nine-point circles)** *Any coloured nine-point circle of a triangle $\overline{A_1A_2A_3}$ passes through the feet of the altitudes of that colour, as well as the midpoints of the segments from the same coloured orthocenter to the points $A_1, A_2$ and $A_3$. In addition it passes through the circumcenters of $\overline{A_1A_2A_3}$ of the other two colours.* ■

The following figure shows some of the other points on the nine-point circles of different colours. Others are off the page.




Hopefully this taste of chromogeometry will encourage others to explore this rich new realm. See [3] and [4] for more in this direction.

# References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