27

(from Divine Proportions) Rational spherical coordinates

One of the important traditional uses of angles and the transcendental trigonometric functions $\cos\theta$ and $\sin\theta$ is to establish polar coordinates in the plane, and spherical and cylindrical coordinates in three-dimensional space. This simplifies problems with rotational symmetry in advanced calculus, mechanics and engineering.

This chapter shows how to employ rational analogues to accomplish the same tasks, with examples chosen from some famous problems in the subject. The rational approach employs conventions that generalize well to higher dimensions.

27.1 Polar spread and quadrance

For a point $A \equiv [x, y]$ in Cartesian coordinates, introduce the **polar spread** s and the **quadrance** Q by

$$\begin{aligned} s &\equiv x^2/\left(x^2+y^2\right) \\ Q &\equiv x^2+y^2. \end{aligned}$$

Then $[s, Q]$ are the **rational polar coordinates** of the point $A \equiv [x, y]$. The spread s is defined between OA and the y *axis*. This convention

- corresponds to the usual practice in surveying and navigation
- integrates more smoothly with higher dimensional generalizations
- is natural for human beings, for whom *up* is more interesting than *right*.

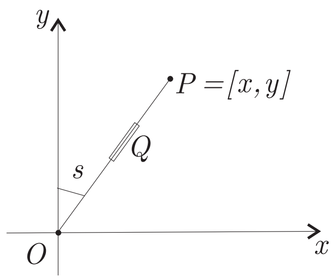


Figure 27.1: Rational polar coordinates

The rational polar coordinates s and Q determine x and y up to sign, so determine A uniquely in the first quadrant. This quadrant is better described by the respective signs of x and y, so call it also the **(++)-quadrant**.

To specify a general point A, the rational coordinates s and Q need to be augmented with two additional bits of information—the signs of x and y respectively. Now

$$\begin{aligned} x^2 &= sQ \\ y^2 &= (1-s)\,Q. \end{aligned} \tag{27.1}$$

Take differentials of these two relations to obtain

$$\begin{aligned} 2x\,dx &= Q\,ds + s\,dQ \\ 2y\,dy &= -Q\,ds + (1-s)\;dQ. \end{aligned}$$

Thus in the (++)-quadrant

$$\begin{aligned} 4xy\,dx\,dy &= \begin{vmatrix} Q & s \\ -Q & 1-s \end{vmatrix} ds\,dQ \\ &= \begin{vmatrix} Q & s \\ 0 & 1 \end{vmatrix} ds\,dQ = Q\,ds\,dQ. \end{aligned} \tag{27.2}$$

For future reference, note that the determinant is evaluated by adding the first row to the second to get a diagonal matrix. In the (++)-quadrant, use (27.1) to obtain

$$xy = \sqrt{s\,(1-s)}\,Q$$

so the element of area is

$$dx\,dy = \frac{1}{4\sqrt{s\,(1-s)}}\,ds\,dQ. \tag{27.3}$$

Example 27.1 The area a of the central circle of quadrance K is, by symmetry,

$$a = 4\int_0^K \int_0^1 \frac{1}{4\sqrt{s\,(1-s)}}\,ds\,dQ = K\int_0^1 \frac{1}{\sqrt{s\,(1-s)}}\,ds.$$

This is not an integral which can be evaluated explicitly using basic calculus, motivating the definition of the number

$$\pi = \int_0^1 \frac{1}{\sqrt{s\,(1-s)}}\,ds. \tag{27.4}$$

So the area of the central circle of quadrance K is πK. $\diamond$

Exercise 27.1 Use the substitutions $s \equiv r^2$ and $s \equiv 1/t$ to show that

$$\pi = 2\int_0^1 \frac{dr}{\sqrt{1-r^2}} = \int_1^\infty \frac{dt}{t\sqrt{t-1}}.$$

Then use the substitutions $r \equiv 2u/\left(1+u^2\right)$ and $v \equiv 1/u$ to show that

$$\pi = 4\int_0^1 \frac{du}{1+u^2} = 4\int_1^\infty \frac{dv}{1+v^2}. \quad \diamond$$

Example 27.2 A lemniscate of Bernoulli has Cartesian equation

$$\left(x^2+y^2\right)^2 = x^2 - y^2 \tag{27.5}$$

and polar equation

$$r^2 = \cos 2\theta.$$

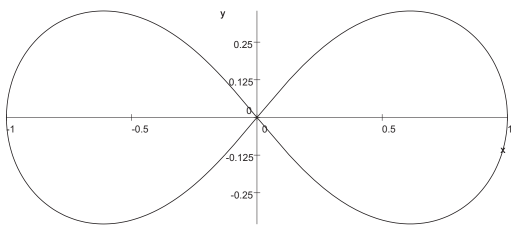


Figure 27.2: Lemniscate of Bernoulli

Replace x^2 and y^2 in (27.5) using (27.1) to get

$$\begin{aligned} Q^2 &= sQ - (1-s)\,Q \\ &= (2s-1)\,Q. \end{aligned}$$

So either of

$$Q = 2s - 1 \qquad \text{or} \qquad s = (Q+1)/2$$

is a **rational polar equation** of the lemniscate. Rational polar equations of some other classical curves are described in Appendix 1. For the lemniscate the polar spread varies in the range $1/2 \leq s \leq 1$, so the area is

$$\begin{aligned} a &= 4\int_{1/2}^{1}\int_{0}^{2s-1} \frac{1}{4\sqrt{s\,(1-s)}}\,dQ\,ds \\ &= \int_{1/2}^{1} \frac{2s-1}{\sqrt{s\,(1-s)}}\,ds = \int_{0}^{1/4} \frac{1}{\sqrt{u}}\,du = 1. \quad \diamond \end{aligned}$$

Example 27.3 The integral $I = \int_0^\infty e^{-x^2}dx$ is difficult to evaluate using only the calculus of one variable. Using rational polar coordinates, the idea is as follows, where the integral is over the (++)-quadrant.

$$\begin{aligned} I^2 &= \int_0^\infty e^{-x^2}\,dx \int_0^\infty e^{-y^2}\,dy \\ &= \int_0^\infty \int_0^\infty e^{-\left(x^2+y^2\right)}\,dx\,dy \\ &= \int_0^\infty \int_0^1 \frac{e^{-Q}}{4\sqrt{s\,(1-s)}}\,ds\,dQ \\ &= \int_0^\infty e^{-Q}\,dQ \int_0^1 \frac{1}{4\sqrt{s\,(1-s)}}\,ds \\ &= \left[-e^{-Q}\right]_{Q=0}^{\infty} \times \pi/4 \\ &= \pi/4 \end{aligned}$$

so that $I = \sqrt{\pi}/2$. $\diamond$

The rotationally invariant measure $d\mu$ on the circle of quadrance $Q = r^2$ is, since $dQ = 2r\,dr$, determined by the equation

$$dx\,dy = d\mu\,dr = \frac{d\mu\,dQ}{2r}.$$

Compare this with (27.3) to see that

$$d\mu = \frac{r}{2\sqrt{s\,(1-s)}}\,ds.$$

It follows that the quarter of the central circle of radius r in the (++)-quadrant has measure $\pi r/2$, and the full circle has measure $2\pi r$.

27.2 Evaluating $\pi^2/16$

The unit quarter circle has area $\pi/4$, so a squared area of

$$\pi^2/16 \approx 0.616\,850\,275\,068\ldots.$$

To evaluate this constant, we follow ideas of Archimedes. Approximate a quarter circle successively by first one, then two, then four isosceles triangles, and so on, each time subdividing each triangle into two by a vertex bisector, as shown in Figure 27.3.

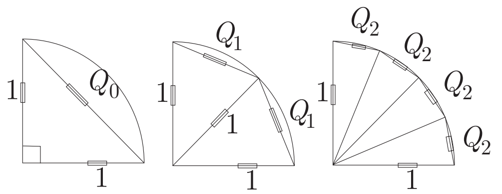


Figure 27.3: Approximations to a quarter circle

By the Quadrea spread theorem (page 82), the quadrea of an isosceles triangle with $Q_1 = Q_2 \equiv 1$ and spread $s_3 \equiv s$ is $\mathcal{A} = 4s$. After n divisions there are 2^n congruent isosceles triangles, each with spread s_n at the common point, and hence each with quadrea $4s_n$. This gives for the resulting (2^n+2)-gon a total quadrea of $\mathcal{A}_n = (2^n)^2 \times 4s_n$, and so a squared area of $a_n^2 = \mathcal{A}_n/16 = 2^{2n-2}s_n$. Now since

$$s_{n+1} = \frac{1-\sqrt{1-s_n}}{2}$$

it follows that

$$\begin{aligned} a_{n+1}^2 &= 2^{2n}s_{n+1} = 2^{2n-1}\left(1-\sqrt{1-s_n}\right) \\ &= 2^{2n-1}\left(1-2^{-n+1}\sqrt{2^{2n-2}-a_n^2}\right) \\ &= 2^{2n-1}-2^n\sqrt{2^{2n-2}-a_n^2}. \end{aligned}$$

Surprisingly, this recurrence relation yields a pleasant form for the general term a_n^2, as indicated by the following computations.

$$\begin{aligned} a_0^2 &= 2^{-2} = 0.25 \\ a_1^2 &= 2^{-1}-2^0\sqrt{2^{-2}-2^{-2}} = 2^{-1} = 0.5 \\ a_2^2 &= 2^1-2^1\sqrt{2^0-2^{-1}} = 2-\sqrt{2} \approx 0.585\,786 \\ a_3^2 &= 2^3-2^2\sqrt{2^2-2+\sqrt{2}} = 8-4\sqrt{2+\sqrt{2}} \approx 0.608\,964 \\ a_4^2 &= 2^5-2^3\sqrt{2^4-\left(8-4\sqrt{2+\sqrt{2}}\right)} = 32-16\sqrt{2+\sqrt{2+\sqrt{2}}} \approx 0.614\,871 \end{aligned}$$

Exercise 27.2 Show that this pattern continues, giving a closed expression for a_n^2. $\diamond$

27.3 Beta function

Following Euler, for decimal numbers $p > 0$ and $q > 0$ define the **Beta function,** or **Beta integral**,

$$B(p,q) \equiv \int_0^1 s^{p-1}(1-s)^{q-1}\,ds.$$

There is a standard expression for the Beta function in terms of the **Gamma function** defined for $t > 0$ by

$$\Gamma(t) \equiv \int_0^\infty e^{-u}u^{t-1}du = 2\int_0^\infty e^{-x^2}x^{2t-1}dx.$$

Integration by parts and direct calculation shows that

$$\begin{aligned} \Gamma(t+1) &= t\,\Gamma(t) \\ \Gamma(1) &= 1. \end{aligned}$$

This implies that

$$\Gamma(n) = (n-1)!$$

for any positive integer $n \geq 1$.

Use rational polar coordinates to rewrite the following integral over the (++)-quadrant

$$\begin{aligned} \Gamma(p)\,\Gamma(q) &= 4\int_0^\infty e^{-x^2}x^{2p-1}dx \int_0^\infty e^{-y^2}y^{2q-1}dy \\ &= \int_0^\infty \int_0^\infty e^{-\left(x^2+y^2\right)}x^{2(p-1)}y^{2(q-1)}\,4xy\,dx\,dy \\ &= \int_0^\infty \int_0^1 e^{-Q}(sQ)^{p-1}((1-s)\,Q)^{q-1}\,Q\,ds\,dQ \\ &= \int_0^\infty e^{-Q}Q^{p+q-1}dQ \int_0^1 s^{p-1}(1-s)^{q-1}\,ds \\ &= \Gamma(p+q)\,B(p,q) \end{aligned}$$

where (27.2) was used to go from the second to the third line. Thus

$$B(p,q) = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}. \qquad (27.6)$$

Values of the Beta function are particularly useful in calculations involving rational polar or spherical coordinates. Note that in particular

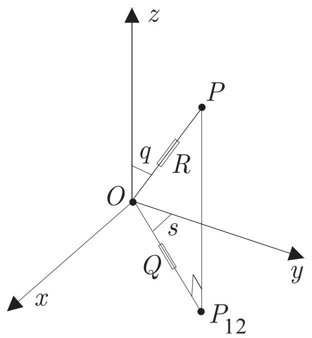
$$B(1/2, 1/2) = \pi = (\Gamma(1/2))^2$$

so that, recovering the computation of Example 27.3,

$$\Gamma(1/2) = 2\int_0^\infty e^{-x^2}dx = \sqrt{\pi}.$$

27.4 Rational spherical coordinates

Represent a point in three-dimensional space by $A \equiv [x, y, z]$, and define the **rational spherical coordinates** $[s, q, R]$ of A by



$$\begin{aligned} s &\equiv x^2 / \left(x^2 + y^2\right) \\ q &\equiv \left(x^2 + y^2\right) / \left(x^2 + y^2 + z^2\right) \\ R &\equiv x^2 + y^2 + z^2. \end{aligned}$$

Geometrically if $A_{12} = [x, y, 0]$ is the perpendicular projection of A onto the $x - y$ plane, then s is the polar spread between OA_{12} and the y axis, while the **second polar spread** q is the spread between OA and the z axis. Then R is the **three-dimensional quadrance**, and $Q \equiv x^2 + y^2 = qR$.

Then

$$x^2 = sqR \qquad y^2 = (1 - s)\, qR \qquad z^2 = (1 - q)\, R \tag{27.7}$$

so that x, y and z are determined, up to sign, by $[s, q, R]$. Take differentials to obtain

$$\begin{aligned} 2x\, dx &= qR\, ds + sR\, dq + sq\, dR \\ 2y\, dy &= -qR\, ds + (1 - s)\, R\, dq + (1 - s)\, q\, dR \\ 2z\, dz &= 0\, ds - R\, dq + (1 - q)\; dR. \end{aligned}$$

Thus in the $(+ + +)$-octant, where the signs of x, y and z are all positive,

$$\begin{aligned} 8xyz\, dx\, dy\, dz &= \begin{vmatrix} qR & sR & sq \\ -qR & (1 - s)\, R & (1 - s)\, q \\ 0 & -R & 1 - q \end{vmatrix} ds\, dq\, dR \\ &= \begin{vmatrix} qR & sR & sq \\ 0 & R & q \\ 0 & 0 & 1 \end{vmatrix} ds\, dq\, dR = qR^2\, ds\, dq\, dR \end{aligned}$$

where the determinant is evaluated by adding the first row to the second, and then the second row to the third, to obtain a diagonal matrix.

In the $(+ + +)$-octant, combine the equations of (27.7) to obtain

$$xyz = R^{3/2} q \sqrt{s\,(1 - s)\,(1 - q)}$$

so the element of volume is

$$dx\, dy\, dz = \frac{\sqrt{R}}{8\sqrt{s\,(1 - s)\,(1 - q)}}\, ds\, dq\, dR. \tag{27.8}$$

Example 27.4 The volume v of the central sphere of quadrance $K \equiv k^2$ $(k \geq 0)$ is eight times the volume in the $(+++)$-octant. It is thus

$$\begin{aligned} v &= 8\int_0^K \int_0^1 \int_0^1 \frac{\sqrt{R}}{8\sqrt{s(1-s)(1-q)}}\, ds\, dq\, dR \\ &= \int_0^1 \frac{ds}{\sqrt{s(1-s)}} \int_0^1 \frac{dq}{\sqrt{1-q}} \int_0^K \sqrt{R}\, dR \\ &= \pi\left[-2\sqrt{1-q}\right]_{q=0}^1 \left[\frac{2R^{\frac{3}{2}}}{3}\right]_{R=0}^K = \frac{4\pi K^{\frac{3}{2}}}{3} = \frac{4\pi k^3}{3}. \end{aligned}$$

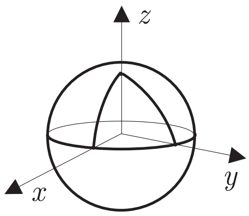


Figure 27.4: Volume of a sphere ⋄

Example 27.5 An 'ice cream cone' lies above the cone $z^2 = x^2 + y^2$, and inside the projective sphere $x^2 + y^2 + z^2 = z$ centered at $[0, 0, 1/2]$ with quadrance $1/4$. Write the cone as $q = 1/2$ and the sphere as $R = 1 - q$, so that the volume v is

$$\begin{aligned} v &= 4\int_0^1 \int_0^{1/2} \int_0^{1-q} \frac{\sqrt{R}}{8\sqrt{s(1-s)(1-q)}}\, dR\, dq\, ds \\ &= \frac{\pi}{2}\int_0^{1/2} \frac{1}{\sqrt{1-q}} \left[\frac{2R^{3/2}}{3}\right]_{R=0}^{1-q} dq \\ &= \frac{\pi}{3}\int_0^{1/2} (1-q)\, dq = \frac{\pi}{3}\left[q - \frac{q^2}{2}\right]_{q=0}^{1/2} = \frac{\pi}{8}. \end{aligned}$$

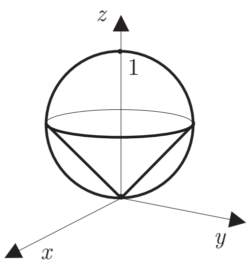


Figure 27.5: Volume of an ice cream cone ⋄

Example 27.6 To find the volume v of the spherical cap inside the sphere $x^2+y^2+z^2=K\equiv k^2$ $(k\geq 0)$ and lying above the plane $z=d\geq 0$, where $d\leq k$, use **rational cylindrical coordinates** $[s,Q,z]$

$$\begin{aligned} v &= 4\int_0^1\int_0^{K-d^2}\int_d^{\sqrt{K-Q}}\frac{1}{4\sqrt{s(1-s)}}\,dz\,dQ\,ds \\ &= \pi\int_0^{K-d^2}\left(\sqrt{K-Q}-d\right)dQ=\pi\left[-2(K-Q)^{3/2}/3-Qd\right]_{Q=0}^{K-d^2} \\ &= \frac{\pi}{3}\left(d^3-3dk^2+2k^3\right)=\frac{\pi}{3}(k-d)^2(2k+d)\,. \end{aligned}$$

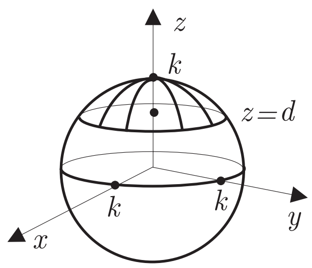


Figure 27.6: Volume of spherical cap ⋄

Example 27.7 The volume of the spherical ring remaining when a cylinder with axis the z-axis is removed from the central sphere of quadrance $K\equiv k^2$ $(k\geq 0)$, leaving a solid bounded by the planes $z=d$ and $z=-d$, where $d\leq k$, is

$$\begin{aligned} v &= 8\int_0^1\int_{K-d^2}^{K}\int_0^{\sqrt{K-Q}}\frac{1}{4\sqrt{s(1-s)}}\,dz\,dQ\,ds=2\pi\int_{K-d^2}^{K}\sqrt{K-Q}\,dQ \\ &= 2\pi\left[-2(K-Q)^{3/2}/3\right]_{Q=K-d^2}^{K}=\frac{4\pi}{3}d^3. \end{aligned}$$

Curiously, this is independent of the quadrance K of the sphere.

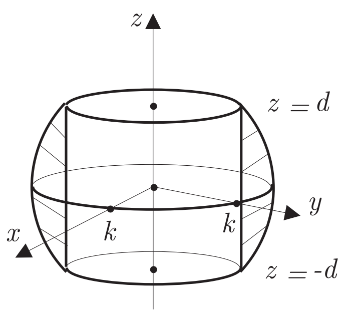


Figure 27.7: Volume of spherical ring ⋄

Example 27.8 To find the volume v above the paraboloid $z = x^2 + y^2$ and below the plane $z = r \geq 0$

$$v = 4 \int_0^1 \int_0^r \int_R^r \frac{1}{4\sqrt{s(1-s)}} \, dz \, dR \, ds = \pi \int_0^r (r - R) \, dR = \frac{\pi r^2}{2}.$$

As discovered by Archimedes, this is one half of the volume of the cylinder of height r and radius $\sqrt{r}$. $\diamond$

Example 27.9 The moment M_{xy} of the upper hemisphere of the unit sphere of density 1 and mass $M \equiv 2\pi/3$ with respect to the $xy-$plane is

$$M_{xy} = 4 \int_0^1 \int_0^1 \int_0^1 \frac{z\sqrt{R}}{8\sqrt{s(1-s)(1-q)}} \, ds \, dq \, dR$$

where $z = \sqrt{(1-q)R}$. Thus

$$M_{xy} = \frac{\pi}{2} \times 1 \times \int_0^1 R \, dR = \frac{\pi}{4}$$

and the centroid has z coordinate $\overline{z} \equiv M_{xy}/M = 3/8$, so is $[0, 0, 3/8]$.

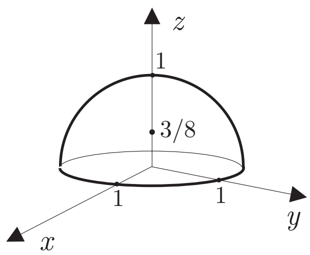


Figure 27.8: Center of mass of upper hemisphere $\diamond$

Example 27.10 The moment of inertia of the solid unit ball around the z axis is

$$\begin{aligned} I_z &= 8 \int_0^1 \int_0^1 \int_0^1 \frac{qR\sqrt{R}}{8\sqrt{s(1-s)(1-q)}} \, ds \, dq \, dR \\ &= \pi \int_0^1 R^{3/2} dR \int_0^1 \frac{q}{\sqrt{1-q}} \, dq \\ &= \pi \times \frac{2}{5} \times B\left(2, \frac{1}{2}\right) = \pi \times \frac{2}{5} \times \frac{4}{3} = \frac{8\pi}{15} \end{aligned}$$

since from (27.6)

$$B\left(2, \frac{1}{2}\right) = \frac{\Gamma(2)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{2}\right)} = \frac{\Gamma\left(\frac{1}{2}\right)}{\frac{3}{2} \times \Gamma\left(\frac{3}{2}\right)} = \frac{\Gamma\left(\frac{1}{2}\right)}{\frac{3}{2} \times \frac{1}{2} \times \Gamma\left(\frac{1}{2}\right)} = \frac{4}{3}. \quad \diamond$$

Example 27.11 The volume v of the hyperbolic cap shown in Figure 27.9, above the top sheet of the hyperboloid $z^2 - x^2 - y^2 = K \equiv k^2$ $(k \geq 0)$ and below the plane $z = d \geq 0$, where $d \geq k$, is

$$\begin{aligned} v &= 4\int_0^1 \int_0^{d^2-K} \int_{\sqrt{Q+K}}^{d} \frac{1}{4\sqrt{s(1-s)}}\, dz\, dQ\, ds = \pi \int_0^{d^2-K} \left(d - \sqrt{Q+K}\right) dQ \\ &= \pi \left[Qd - 2(Q+K)^{3/2}/3\right]_{Q=0}^{d^2-K} = \frac{\pi}{3}(k-d)^2(2k+d). \end{aligned}$$

This is the same formula as the volume of a spherical cap in Example 27.6!

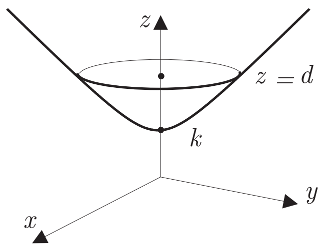


Figure 27.9: Volume of hyperbolic cap ◇

Example 27.12 The volume of the hyperbolic ring shown in Figure 27.10 inside a cylinder with axis the z-axis and outside the hyperboloid of one sheet $x^2 + y^2 - z^2 = K \equiv k^2$ $(k \geq 0)$ bounded by the planes $z = d$ and $z = -d$ is

$$\begin{aligned} v &= 8\int_0^1 \int_K^{d^2+K} \int_0^{\sqrt{Q-K}} \frac{1}{4\sqrt{s(1-s)}}\, dz\, dQ\, ds = 2\pi \int_K^{d^2+K} \sqrt{Q-K}\, dQ \\ &= 2\pi \left[2(Q-K)^{3/2}/3\right]_{Q=K}^{d^2+K} = \frac{4\pi}{3}d^3. \end{aligned}$$

Curiously, this is independent of K, and is the same as the volume of the spherical ring in Example 27.7!

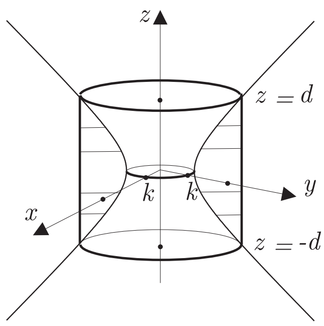


Figure 27.10: Volume of hyperbolic ring ◇

27.5 Surface measure on a sphere

For a fixed value K of R, the polar spreads s and q parametrize that part of the surface of the sphere of quadrance K contained in the $(+++)$-octant. To describe the full sphere these two spreads must be augmented by three additional bits of information, namely the signs of x, y and z.

The rotationally invariant surface measure $d\nu$ on the sphere $R \equiv r^2$ is, since $dR = 2r\,dr$, determined by

$$dx\,dy\,dz = d\nu\,dr = d\nu\,dR\,/2r.$$

Compare this with (27.8) to get

$$\frac{\sqrt{R}}{8\sqrt{s\,(1-s)\,(1-q)}}\,ds\,dq\,dR = \frac{1}{2\sqrt{R}}\,d\nu\,dR.$$

Thus

$$d\nu = \frac{R\,ds\,dq}{4\sqrt{s\,(1-s)\,(1-q)}}.$$

Example 27.13 The total surface area a of the sphere of quadrance $K \equiv k^2$ is

$$\begin{aligned} a &= 8\int_0^1\int_0^1 \frac{K}{4\sqrt{s\,(1-s)\,(1-q)}}\,ds\,dq \\ &= 2K\int_0^1 \frac{ds}{\sqrt{s\,(1-s)}}\int_0^1 \frac{dq}{\sqrt{1-q}} \\ &= 2K \times \pi \times 2 = 4\pi K = 4\pi k^2. \ \diamond \end{aligned}$$

Example 27.14 The surface area a of the spherical cap of the sphere $x^2+y^2+z^2 = K \equiv k^2$ $(k \geq 0)$ lying above the plane $z = d \geq 0$, where $d \leq k$, as shown in Figure 27.6, is

$$\begin{aligned} a &= 4\int_0^{\left(K-d^2\right)/K}\int_0^1 \frac{K}{4\sqrt{s\,(1-s)\,(1-q)}}\,ds\,dq \\ &= \pi K\left[-2\,(1-q)^{1/2}\right]_{q=0}^{\left(K-d^2\right)/K} \\ &= 2\pi k^2\left(1-\frac{d}{k}\right). \end{aligned}$$

The linear dependence of this expression on d is one of the most remarkable properties of a sphere, and is responsible for the fact that an egg slicer subdivides a sphere into strips of constant surface area. This fact is also important for harmonic analysis on a sphere, and for the representation theory of the rotation group. $\diamond$

27.6 Four dimensional rational spherical coordinates

For a point $A \equiv [x, y, z, w]$ in four dimensional space define

$$\begin{aligned} s &\equiv x^2 / \left(x^2 + y^2\right) \\ q &\equiv \left(x^2 + y^2\right) / \left(x^2 + y^2 + z^2\right) \\ r &\equiv \left(x^2 + y^2 + z^2\right) / \left(x^2 + y^2 + z^2 + w^2\right) \\ T &\equiv x^2 + y^2 + z^2 + w^2. \end{aligned}$$

Then T is the four-dimensional quadrance, and r is the **third polar spread** between OA and the new (fourth) $w-$axis. Then

$$\begin{aligned} x^2 &= sqrT \\ y^2 &= (1-s)\, qrT \\ z^2 &= (1-q)\, rT \\ w^2 &= (1-r)\, T. \end{aligned} \tag{27.9}$$

Take differentials and follow the established pattern to get

$$\begin{aligned} 16\, xyzw\, dx\, dy\, dz\, dw &= \begin{vmatrix} qrT & srT & sqT & sqr \\ -qrT & (1-s)\, rT & (1-s)\, qT & (1-s)\, qr \\ 0 & -rT & (1-q)\, T & (1-q)\, r \\ 0 & 0 & -T & 1-r \end{vmatrix} ds\, dq\, dr\, dT \\ &= \begin{vmatrix} qrT & srT & sqT & sqr \\ 0 & rT & qT & qr \\ 0 & 0 & T & r \\ 0 & 0 & 0 & 1 \end{vmatrix} ds\, dq\, dr\, dT \\ &= qr^2T^3\, ds\, dq\, dr\, dT. \end{aligned}$$

In the $(++++)$-octant, (27.9) yields

$$xyzw = s^{1/2} q\, r^{3/2} T^2 \sqrt{(1-s)(1-q)(1-r)}$$

so the element of content (four dimensional version of volume) is

$$dx\, dy\, dz\, dw = \frac{\sqrt{r}\, T}{16\sqrt{s\,(1-s)(1-q)(1-r)}}\, ds\, dq\, dr\, dT.$$

Example 27.15 The central sphere of quadrance $K \equiv k^2$ has content

$$\begin{aligned} c &= 16 \int_0^K \int_0^1 \int_0^1 \int_0^1 \frac{\sqrt{r}\, T}{16\sqrt{s\,(1-s)(1-q)(1-r)}}\, ds\, dq\, dr\, dT \\ &= \int_0^1 \frac{ds}{\sqrt{s\,(1-s)}} \int_0^1 \frac{dq}{\sqrt{1-q}} \int_0^1 \frac{\sqrt{r}\, dq}{\sqrt{1-r}} \int_0^K T\, dT \\ &= \pi \times 2 \times B\left(\frac{3}{2}, \frac{1}{2}\right) \times \frac{K^2}{2}. \end{aligned}$$

But
$$B\left(\frac{3}{2},\frac{1}{2}\right)=\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(2\right)}=\frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{1}=\frac{\pi}{2}$$
so that
$$c=\frac{\pi^2K^2}{2}=\frac{\pi^2k^4}{2}.\ \diamond$$

If $d\nu$ denotes spherical surface measure on the unit 3-sphere determined by
$$dx\,dy\,dz\,dw=d\nu\,dT\,/2$$
then (since $T=1$)
$$d\nu=\frac{\sqrt{r}}{8\sqrt{s\left(1-s\right)\left(1-q\right)\left(1-r\right)}}\,ds\,dq\,dr.$$

Exercise 27.3 Use this to show that the surface volume of the unit 3-sphere is $2\pi^2$. $\diamond$

It should now be clear how to extend rational spherical coordinates to higher dimensions. In n-dimensional space, rational spherical coordinates involve $(n-1)$ polar spreads, and one quadrance. The basic relations are algebraic, and so do not require an understanding and visualization of projections.