

Affine and Projective Universal Geometry

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Abstract

By recasting metrical geometry in a purely algebraic setting, both Euclidean and non-Euclidean geometries can be studied over a general field with an arbitrary quadratic form. Both an affine and a projective version of this new theory are introduced here, and the main formulas extend those of rational trigonometry in the plane. This gives a unified, computational model of both spherical and hyperbolic geometries, allows the extension of many results of Euclidean geometry to the relativistic setting, and provides a new metrical approach to algebraic geometry.

Introduction

Universal Geometry extends Euclidean and non-Euclidean geometries to general fields and quadratic forms. This new development is a natural outgrowth of *Rational Trigonometry* as described in the elementary text [5]. It was there developed in the planar context with an emphasis on the applications to Euclidean geometry. In this paper the subject is built up in two very general settings—the *affine* one in an n dimensional vector space over a general field with a metrical structure given by an arbitrary, but fixed, symmetric bilinear form, and the associated *projective* one involving the space of lines through the origin of a vector space with a symmetric bilinear form.

This allows us to dramatically simplify the usual trigonometric relations for both Euclidean and non-Euclidean geometries, to extend them to general bilinear forms, and to reveal the rich geometrical structure of projective space, with interesting implications for algebraic geometry.

It is pleasant that the main laws of planar rational trigonometry have affine and projective versions which turn out to hold simultaneously in elliptic geometry, in hyperbolic geometry, and indeed in any metrical geometry based on a symmetric bilinear form. The usual dichotomy between spherical and hyperbolic trigonometry deserves re-evaluation.

The first section introduces and motivates the new approach in a particularly simple but important special case—that of two dimensional hyperbolic geometry. The second section establishes the basic trigonometric laws in a general

affine setting, using n dimensional space with an arbitrary symmetric bilinear form over a general field. The *Triple quad formula*, *Pythagoras' theorem*, the *Spread law*, *Thales' theorem*, the *Cross law* and the *Triple spread formula* include algebraic analogs of the familiar Sine law and Cosine law, along with the fact that the sum of angles in a triangle is 180° . Then we derive the projective versions of these laws, which are seen to be deformations of the affine ones, along with formulas for projective right, isosceles and equilateral triangles, including general versions of Napier's rules for solving right triangles in spherical trigonometry. We briefly mention the important *Spread polynomials*, which are universal analogs of the Chebyshev polynomials of the first kind, but have an interpretation over any field. Two examples are shown, one affine and the other projective, one over the rationals and the other over a finite field. Finally, the Lambert quadrilateral from hyperbolic geometry is shown to be a general feature in the projective setting.

Because of the novelty of the approach, some remarks of a subjective nature may be useful to orient the reader. Metrical geometry is here presented as fundamentally an *algebraic subject* rather than an *analytic one*, and the main division in the subject is not between *Euclidean* and *non-Euclidean*, but rather between *affine* and *projective*. Elliptic and hyperbolic geometries should be considered as *projective theories*. Their natural home is the projective space of a vector space, with metrical structure—*not a metric in the usual sense*—determined by a bilinear or quadratic form. Over arbitrary fields the familiar close relation between spheres or hyperboloids and projective space largely disappears, and the projective space is almost always more basic. The fundamental formulas and theorems of metrical geometry are those which *hold over a general field* and are *independent of the choice of bilinear form*. Many results of Euclidean geometry *extend to the relativistic setting*, and beyond, once you have understood them in a universal framework.

This paper lays out the basic tools to begin a dramatic extension of Euclidean and non-Euclidean geometries, and to once again investigate those aspects of algebraic geometry concerned with the *metrical properties of curves and varieties*, in the spirit of Archimedes, Apollonius and the seventeenth and eighteenth century mathematicians.

A motivating example from the hyperbolic plane

Hyperbolic geometry is usually regarded as either a synthetic theory or an analytic theory. With the synthetic approach you replace Euclid's fifth axiom with an axiom that allows more than one line through a point parallel to a given line, and follow Bolyai, Gauss and Lobachevsky as described in [3]. With the more modern analytic approach, you introduce a Riemannian metric in the upper half plane or Poincaré disk (or sometimes the hyperboloid of two sheets in three dimensional space), calculate the geodesics, derive formulas for hyperbolic distances—often employing the group PSL_2 of isometries, and then develop hyperbolic trigonometry. This is described in many places, for example [1] and [4].

The initial interest is in the regular tessellations of the hyperbolic plane, complex analysis and Riemann surfaces, and the connections with number theory via quadratic forms and automorphic forms, although nowadays the applications extend much further.

The new approach to be described here is entirely algebraic and elementary, and allows us to formulate two dimensional hyperbolic geometry as a *projective theory* over a general field. There are numerous computational, pedagogical and conceptual advantages.

Begin with three dimensional space, with typical vector $[x, y, z]$ and bilinear form

$$[x_1, y_1, z_1] \cdot [x_2, y_2, z_2] = x_1x_2 + y_1y_2 - z_1z_2.$$

Define the **projective point** $a = [x : y : z]$ to be the line through the origin $O = [0, 0, 0]$ and the non-zero vector $[x, y, z]$. The **projective quadrance** between projective points $a_1 = [x_1 : y_1 : z_1]$ and $a_2 = [x_2 : y_2 : z_2]$ is the number

$$\begin{aligned} q(a_1, a_2) &= \frac{(x_2y_1 - x_1y_2)^2 - (y_1z_2 - z_1y_2)^2 - (z_1x_2 - x_1z_2)^2}{(x_1^2 + y_1^2 - z_1^2)(x_2^2 + y_2^2 - z_2^2)} \\ &= 1 - \frac{(x_1x_2 + y_1y_2 - z_1z_2)^2}{(x_1^2 + y_1^2 - z_1^2)(x_2^2 + y_2^2 - z_2^2)}. \end{aligned}$$

Define the **projective line** $L = (l : m : n)$ to be the plane through the origin (in three dimensional space) with equation

$$lx + my - nz = 0.$$

The **projective spread** between projective lines $L_1 = (l_1 : m_1 : n_1)$ and $L_2 = (l_2 : m_2 : n_2)$ is the number

$$S(L_1, L_2) = \frac{(l_1m_2 - l_2m_1)^2 - (m_1n_2 - m_2n_1)^2 - (n_1l_2 - n_2l_1)^2}{(l_1^2 + m_1^2 - n_1^2)(l_2^2 + m_2^2 - n_2^2)}.$$

To give a concrete example, consider the projective triangle $\overline{a_1a_2a_3}$ with projective points

$$a_1 = [1 : 0 : 2] \quad a_2 = [1 : -1 : 3] \quad a_3 = [2 : 1 : 5].$$

Over the real numbers these lines would correspond to (pairs of) points on the usual hyperboloid of two sheets inside the null cone $x^2 + y^2 - z^2 = 0$. The projective lines determined by these projective points are

$$L_1 = a_2a_3 = (8 : -1 : 3) \quad L_2 = a_1a_3 = (2 : 1 : 1) \quad L_3 = a_1a_2 = (2 : -1 : 1).$$

The projective quadrances of the projective triangle $\overline{a_1a_2a_3}$ are then

$$q_1 = -2/5 \quad q_2 = -1/15 \quad q_3 = -4/21$$

while the projective spreads are

$$S_1 = 3/4 \quad S_2 = 1/8 \quad S_3 = 5/14.$$

The analog of the hyperbolic Sine law is

$$\frac{3/4}{-2/5} = \frac{1/8}{-1/15} = \frac{5/14}{-4/21}$$

and there are also analogs of both types of hyperbolic Cosine law.

Over the real numbers, to convert this information into the familiar Poincaré model, we intersect the projective point $a = [x_0 : y_0 : 1]$ with the plane $z = 1/2$, yielding the point $A = [x_0/2, y_0/2, 1/2]$. If a lies inside the null cone $x^2 + y^2 - z^2 = 0$ then A lies in the open disk $x^2 + y^2 < 1/4$, $z = 1/2$, which is the equatorial disk of the sphere $x^2 + y^2 + (z - 1/2)^2 = 1/4$ with north pole $N = [0, 0, 1]$. Project A orthogonally onto the lower hemisphere of this sphere, giving the point

$$A' = \left[\frac{x_0}{2}, \frac{y_0}{2}, \frac{1}{2} - \frac{\sqrt{1 - x_0^2 - y_0^2}}{2} \right]$$

and then stereographically project A' from N to the Poincaré disk $x^2 + y^2 < 1$, $z = 0$ (viewed as the open unit disk in the complex plane) to get the point

$$z_A = \frac{x_0 \left(1 - \sqrt{1 - x_0^2 - y_0^2} \right)}{x_0^2 + y_0^2} + i \frac{y_0 \left(1 - \sqrt{1 - x_0^2 - y_0^2} \right)}{x_0^2 + y_0^2}.$$

Also project the point $[x_0, y_0, 1]$ in the plane $z = 1$ orthogonally onto the Poincaré disk to get $w_A = [x_0, y_0, 0]$. Then $z_A = \varphi(a)$ is the corresponding point in the Poincaré model to a , and 0 is the corresponding point to $o = [0 : 0 : 1]$. If ρ is the usual hyperbolic distance in the Poincaré disk then it is well-known (see for example [1, Chapter 7]) that

$$\rho(0, z_A) = \frac{1}{2} \rho(0, w_A) = \frac{1}{2} \log \frac{1 + \sqrt{x_0^2 + y_0^2}}{1 - \sqrt{x_0^2 + y_0^2}}$$

and

$$\sinh^2(\rho(0, z_A)) = \frac{x_0^2 + y_0^2}{1 - x_0^2 - y_0^2}.$$

Note that the latter is just the negative of $q(o, a)$.

Now returning to our example, you may use the above formulas to find the corresponding points in the Poincaré model to be

$$z_1 = 2 - \sqrt{3} \quad z_2 = \frac{3}{2} - \frac{1}{2}\sqrt{7} + i \left(-\frac{3}{2} + \frac{1}{2}\sqrt{7} \right) \quad z_3 = 2 - \frac{4}{5}\sqrt{5} + i \left(1 - \frac{2}{5}\sqrt{5} \right).$$

The standard formula

$$\rho(z, w) = \log \frac{|1 - z\bar{w}| + |z - w|}{|1 - z\bar{w}| - |z - w|}$$

gives (approximately) the hyperbolic distances

$$\rho_1 \approx 0.596455365 \quad \rho_2 \approx 0.255412812 \quad \rho_3 \approx 0.423648930.$$

The corresponding angles in the hyperbolic triangle $\overline{z_1 z_2 z_3}$ may then be calculated using the hyperbolic Cosine Rule

$$\cosh \rho_3 = \cosh \rho_1 \cosh \rho_2 - \sinh \rho_1 \sinh \rho_2 \cos \theta_3$$

to be, in radians, (approximately)

$$\theta_1 \approx 2.094395102 \approx 2\pi/3 \quad \theta_2 \approx 0.361367126 \quad \theta_3 \approx 0.640522314.$$

To check correctness, you can verify (approximately) the hyperbolic Sine Law

$$\frac{\sin 2.0943951}{\sinh 0.5964553} \approx \frac{\sin 0.361367}{\sinh 0.255412} \approx \frac{\sin 0.640522}{\sinh 0.423648} \approx 1.36931.$$

To relate the two approaches, the projective quadrance in the projective rational model is the *negative* of the square of the hyperbolic sine of the hyperbolic distance between the corresponding points in the Poincaré model, and the projective spread is the square of the sine of the angle between corresponding geodesics in the Poincaré model.

The advantages of the projective rational model of the hyperbolic plane include a cleaner derivation of the theory, simpler and more precise calculations, with no approximations to transcendental functions required, a more complete symmetry between rational formulations of the two hyperbolic Cosine laws, a view of the usual hyperbolic plane as part of a larger picture involving all projective points, thus accessing the ‘line at infinity’ (the null cone) as well as the ‘exterior hyperbolic plane’ corresponding to the hyperboloid of one sheet, and the existence of a beautiful duality between projective points and projective lines that greatly simplifies hyperbolic geometry in two dimensions. And as stated previously the theory generalizes to higher dimensions, to arbitrary fields, and to general symmetric bilinear forms, and so unifies elliptic and hyperbolic trigonometry.

A more complete account of the two dimensional case, with emphasis on duality, isometries and applications to tessellations, will be given elsewhere. Now we turn to develop the general affine theory, and after that the projective theory which generalizes the above situation.

Affine rational trigonometry

Trigonometry studies the *measurement of triangles*. We work in an n dimensional vector space over a field, not of characteristic two. Elements of the field are referred to as **numbers**. Elements of the vector space are called **points** or **vectors** (the two terms will be used interchangeably) and are denoted by U, V, W and so on. The zero vector or point is denoted O . The unique line l through distinct points U and V is denoted UV . For a non-zero point U the line OU is also denoted $[U]$. The unique plane Π through non-collinear points U, V and W is denoted UVW . The plane OUV is also denoted $[U, V]$. The unique

affine 3-flat (translate of a three dimensional subspace) δ through non-planar points U, V, W and Z is denoted $UVWZ$, and so on.

Fix a symmetric bilinear form and represent it by

$$U \cdot V.$$

In terms of this form, the line UV is **perpendicular** to the line WZ precisely when

$$(V - U) \cdot (Z - W) = 0.$$

A point U is a **null point** or **null vector** precisely when $U \cdot U = 0$. The **origin** O is always a null point, but in general there are others as well. A line UV is a **null line** precisely when the vector $U - V$ is a null vector. Null points and lines are themselves well worth studying, but in this paper we concentrate on non-null points and lines. Some definitions will be empty when applied to null points or null lines.

For any vector V the number $V \cdot V$ will be denoted a_V while more generally for any vectors U and V the number $U \cdot V$ will be denoted b_{UV} . Thus $b_{VV} = a_V$.

Given vectors V and U the **projection of U onto the line $[V]$** is

$$\frac{U \cdot V}{V \cdot V} V.$$

Then a line perpendicular to $[V]$ in the plane $[U, V]$ is

$$\left[U - \frac{U \cdot V}{V \cdot V} V \right].$$

Motivated by this, we define the intersecting planes $[U, W]$ and $[V, W]$ to be **perpendicular** precisely when

$$\left(U - \frac{U \cdot W}{W \cdot W} W \right) \cdot \left(V - \frac{V \cdot W}{W \cdot W} W \right) = 0$$

which is the same as

$$(W \cdot W)(U \cdot V) - (U \cdot W)(V \cdot W) = 0 \tag{1}$$

or

$$a_W b_{UV} - b_{UW} b_{VW} = 0.$$

A set $\{U, V, W\}$ of three distinct non-collinear points is a **triangle** and is denoted \overline{UVW} . The **lines** of the triangle \overline{UVW} are UV , VW and UW . Such a triangle is **non-null** precisely when each of its lines is non-null.

The **quadrance** between the points U and V is the number

$$Q(U, V) = (V - U) \cdot (V - U).$$

Then clearly

$$\begin{aligned} Q(U, V) &= Q(V, U) = V \cdot V - 2U \cdot V + U \cdot U \\ &= a_V - 2b_{UV} + a_U. \end{aligned}$$

The line UV is a null line precisely when $Q(U, V) = 0$, or equivalently when it is perpendicular to itself.

In ordinary Euclidean geometry distance along a line is additive, assuming you know what the correct order of points is. With universal geometry, the important relation between the quadrances formed by three collinear points is described by a quadratic symmetric function which goes back essentially to Archimedes' discovery of what is usually called Heron's formula.

Theorem 1 (Triple quad formula) *If U, V and W are collinear then the quadrances $Q_W = Q(U, V)$, $Q_U = Q(V, W)$ and $Q_V = Q(U, W)$ satisfy*

$$(Q_U + Q_V + Q_W)^2 = 2(Q_U^2 + Q_V^2 + Q_W^2).$$

Proof. First note that the equation can be rewritten as

$$(Q_U + Q_V - Q_W)^2 = 4Q_U Q_V.$$

Assume U, V and W are collinear and say U and V are distinct so that $W - U = \lambda(V - U)$ for some number λ . In this case

$$\begin{aligned} Q_V &= Q(U, W) = (W - U) \cdot (W - U) = \lambda^2 Q_W \\ Q_U &= Q(V, W) = (W - V) \cdot (W - V) = (\lambda - 1)^2 Q_W. \end{aligned}$$

If UV is a null line then the result is automatic, as both sides are zero, and otherwise the equation amounts to the identity

$$\left((\lambda - 1)^2 + \lambda^2 - 1 \right)^2 = 4\lambda^2 (\lambda - 1)^2.$$

■

The next theorem is a restatement and generalization of the most important theorem in mathematics.

Theorem 2 (Pythagoras' theorem) *If U, V and W are three distinct points then UW is perpendicular to VW precisely when the quadrances $Q_W = Q(U, V)$, $Q_U = Q(V, W)$ and $Q_V = Q(U, W)$ satisfy*

$$Q_W = Q_U + Q_V.$$

Proof. The condition UW perpendicular to VW means that

$$(W - U) \cdot (W - V) = 0$$

or

$$a_W - b_{UW} - b_{VW} + b_{UV} = 0.$$

The condition $Q_W = Q_U + Q_V$ is

$$a_V - 2b_{UV} + a_U = (a_V - 2b_{VW} + a_W) + (a_U - 2b_{UW} + a_W). \quad (2)$$

After a division by two, these conditions are seen to be the same. ■

In Euclidean geometry, the separation of lines is traditionally measured by the transcendental notion of *angle*. The difficulties in defining an angle precisely, and in extending the concept, are eliminated in rational trigonometry by using instead the notion of *spread*—in Euclidean geometry the square of the sine of the angle between two rays lying on those lines. Fortunately we can formulate the concept completely algebraically.

The **spread** between the non-null lines UW and VZ is the number

$$s(UW, VZ) = 1 - \frac{((W - U) \cdot (Z - V))^2}{Q(U, W)Q(V, Z)}.$$

This depends only on the two lines, not the choice of points lying on them. The spread between two non-null lines is 1 precisely when they are perpendicular.

If W is a non-null point then the **spread** between the two planes OUW and OVW may be defined to be the spread between the lines

$$\left[U - \frac{U \cdot W}{W \cdot W}W\right] \quad \text{and} \quad \left[V - \frac{V \cdot W}{W \cdot W}W\right].$$

The next result is an algebraic generalization of the Sine law in planar trigonometry.

Theorem 3 (Spread law) *Suppose the non-null triangle \overline{UVW} has quadrances $Q_W = Q(U, V)$, $Q_U = Q(V, W)$ and $Q_V = Q(U, W)$, and spreads $s_U = s(UV, UW)$, $s_V = s(VW, VU)$ and $s_W = s(WU, WV)$. Then*

$$\frac{s_U}{Q_U} = \frac{s_V}{Q_V} = \frac{s_W}{Q_W}.$$

Proof. Some straightforward simplification shows that the spread $s_W = s(WU, WV)$ is

$$1 - \frac{(a_W - b_{VW} - b_{UW} + b_{UV})^2}{(a_W - 2b_{UW} + a_U)(a_W - 2b_{VW} + a_V)} \quad (3)$$

$$= \frac{\left(\begin{array}{c} a_U a_V + a_U a_W + a_V a_W + (b_{UV} + b_{UW} + b_{VW})^2 \\ -2(b_{UV}^2 + b_{UW}^2 + b_{VW}^2) - 2a_U b_{VW} - 2a_V b_{UW} - 2a_W b_{UV} \end{array} \right)}{(a_W - 2b_{UW} + a_U)(a_W - 2b_{VW} + a_V)}. \quad (4)$$

The numerator in the expression (4) is symmetric in the points U, V and W , while the denominator is $Q_U Q_V$. It follows that by dividing by Q_W you get an expression which is symmetric in the three points. ■

Thales' theorem is the basis of similar triangles in universal geometry.

Theorem 4 (Thales' theorem) *If the non-null triangle \overline{UVW} has spread $s_W = 1$ then*

$$s_U = \frac{Q_U}{Q_W}.$$

Proof. Immediate from the Spread law. ■

The next result is an algebraic generalization of the Cosine law.

Theorem 5 (Cross law) *Suppose the non-null triangle \overline{UVW} has quadrances $Q_W = Q(U, V)$, $Q_U = Q(V, W)$ and $Q_V = Q(U, W)$, and spreads $s_U = s(UV, UW)$, $s_V = s(VW, VU)$ and $s_W = s(WU, WV)$. Then*

$$(Q_U + Q_V - Q_W)^2 = 4Q_U Q_V (1 - s_W).$$

Proof. From (2) we have the expression

$$\begin{aligned} Q_U + Q_V - Q_W &= (a_V - 2b_{VW} + a_W) + (a_U - 2b_{UW} + a_W) - (a_V - 2b_{UV} + a_U) \\ &= 2(a_W + b_{UV} - b_{UW} - b_{VW}) \end{aligned}$$

while from (3) we have

$$1 - s_W = \frac{(a_W - b_{VW} - b_{UW} + b_{UV})^2}{Q_U Q_V}.$$

The result follows. ■

Note that the Cross law includes as a special case both the Triple quad formula and Pythagoras' theorem. The next result is an algebraic analog of the fact that the sum of the angles in a triangle is 180° .

Theorem 6 (Triple spread formula) *Suppose the non-null triangle \overline{UVW} has spreads $s_U = s(UV, UW)$, $s_V = s(VW, VU)$ and $s_W = s(WU, WV)$. Then*

$$(s_U + s_V + s_W)^2 = 2(s_U^2 + s_V^2 + s_W^2) + 4s_U s_V s_W.$$

Proof. Assume the quadrances of \overline{UVW} are $Q_W = Q(U, V)$, $Q_U = Q(V, W)$ and $Q_V = Q(U, W)$. Write the Spread law as

$$\frac{s_U}{Q_U} = \frac{s_V}{Q_V} = \frac{s_W}{Q_W} = \frac{1}{D}$$

for some non-zero number D . Now substitute $Q_U = Ds_U$, $Q_V = Ds_V$ and $Q_W = Ds_W$ into the Cross law, and cancel a factor of D^2 , yielding

$$(s_U + s_V - s_W)^2 = 4s_U s_V (1 - s_W).$$

Rearrange this to get

$$(s_U + s_V + s_W)^2 = 2(s_U^2 + s_V^2 + s_W^2) + 4s_U s_V s_W.$$

■

The Triple spread formula can be reinterpreted as a statement about three non-parallel coplanar lines. If three lines lie in a (two dimensional) plane then the spread between any two of them is unaffected if one or more of the lines

is translated. In particular we can arrange that the three lines are concurrent, and so the Triple spread formula still applies.

Another useful observation is that if say $S_W = 1$, then the Triple spread formula becomes

$$(s_U + s_V - 1)^2 = 0$$

so that

$$s_U + s_V = 1.$$

Secondary results of planar rational trigonometry, some developed in [5], are consequences of the main laws of this section, and so still hold in this general setting.

An affine example over the rational numbers

Here is an example of trigonometry in four dimensional space over the rational numbers (the most important field) with bilinear form

$$U \cdot V = U M V^T$$

where

$$M = \begin{pmatrix} 0 & 1 & 0 & 3 \\ 1 & 1 & 2 & -1 \\ 0 & 2 & 1 & 0 \\ 3 & -1 & 0 & -1 \end{pmatrix}.$$

Consider the triangle \overline{UVW} where

$$U = [1, 2, 4, 3/2] \quad V = [-1, 0, 1/2, 3] \quad W = [2, 2, 1, 5].$$

Then the quadrances are

$$Q_U = \frac{177}{4} \quad Q_V = \frac{71}{4} \quad Q_W = 38$$

and the spreads are

$$s_U = \frac{10263}{10792} \quad s_V = \frac{3421}{8968} \quad s_W = \frac{3421}{4189}.$$

Then you may verify the Spread law

$$\frac{10263/10792}{177/4} = \frac{3421/8968}{71/4} = \frac{3421/4189}{38} = \frac{3421}{159182}$$

(one of the forms of) the Cross law

$$\left(\frac{177}{4} + \frac{71}{4} - 38 \right)^2 = 4 \times \frac{177}{4} \times \frac{71}{4} \times \left(1 - \frac{3421}{4189} \right).$$

The Triple spread law becomes

$$\begin{aligned} \left(\frac{10\,263}{10\,792} + \frac{3421}{8968} + \frac{3421}{4189} \right)^2 &= \frac{29\,258\,102\,500}{6334\,727\,281} \\ &= 2 \left(\left(\frac{10\,263}{10\,792} \right)^2 + \left(\frac{3421}{8968} \right)^2 + \left(\frac{3421}{4189} \right)^2 \right) + 4 \times \frac{10\,263}{10\,792} \times \frac{3421}{8968} \times \frac{3421}{4189}. \end{aligned}$$

Geometry in such a setting has many familiar features. Here are the circumcenter C and circumquadrance K of the triangle

$$\begin{aligned} C &= \left[\frac{144}{311}, \frac{3789}{3421}, \frac{18\,773}{13\,684}, \frac{46\,709}{13\,684} \right] \\ K &= \frac{79\,591}{6842}. \end{aligned}$$

The orthocenter of the triangle is

$$O = \left[\frac{334}{311}, \frac{6106}{3421}, \frac{9429}{3421}, \frac{9145}{3421} \right]$$

the centroid is

$$G = \left[\frac{2}{3}, \frac{4}{3}, \frac{11}{6}, \frac{19}{6} \right]$$

and the nine-point center is

$$N = \left[\frac{239}{311}, \frac{9895}{6842}, \frac{56\,489}{27\,368}, \frac{83\,289}{27\,368} \right].$$

All four points C, G, N and O are collinear, lying on the **Euler line** of the triangle, and as expected

$$\begin{aligned} G &= \frac{2}{3}C + \frac{1}{3}O \\ N &= \frac{1}{2}C + \frac{1}{2}O. \end{aligned}$$

Many other aspects of Euclidean geometry may be explored.

Projective rational trigonometry

Fix an $(n + 1)$ dimensional vector space over a field with a symmetric bilinear form $U \cdot V$ as in the previous section. A line through the origin O will now be called a **projective point** and denoted by a small letter such as u . The space of such projective points is called n **dimensional projective space**. This is a natural domain for algebraic geometry, and the metrical structure we will introduce gives new directions for this subject.

If V is a non-zero vector (or point) in the vector space, then $v = [V]$ will denote the projective point OV . A projective point is a **null projective point** precisely when some non-zero null point lies on it. Two projective points $u = [U]$ and $v = [V]$ are **perpendicular** precisely when they are perpendicular as lines. This is equivalent to the condition $U \cdot V = 0$.

A plane through the origin (two dimensional subspace) will be called a **projective line** and denoted by a capital letter such as L . A three dimensional subspace will be called a **projective plane**. If V and W are independent vectors then $L = [V, W]$ will denote the projective line OVW . If V, W and Z are independent vectors then $\pi = [V, W, Z]$ will denote the projective plane $OVWZ$.

A projective point u **lies on** a projective line L (or equivalently L **passes through** u) precisely when the line u lies on the plane L . Similar terminology applies to projective points lying on projective planes, or projective lines lying on projective planes etc. There is a unique projective line $L = uv$ which passes through any two distinct projective points u and v . Three or more projective points which lie on a single projective line are **collinear**. Three or more projective lines which all pass through a single projective point are **concurrent**.

Two intersecting projective lines uw and vw are **perpendicular** precisely when they are perpendicular as intersecting planes. If $u = [U]$, $v = [V]$ and $w = [W]$ then this is equivalent to the condition (1)

$$(W \cdot W)(U \cdot V) - (U \cdot W)(V \cdot W) = 0.$$

A set $\{u, v, w\}$ of three distinct non-collinear projective points is a **projective triangle**, and is denoted \overline{uvw} . The projective triangle \overline{uvw} is **null** if one or more of its projective points is null. The **projective lines** of the projective triangle \overline{uvw} are uv , vw and uw .

The rich metrical structure of projective space arises with the correct notion of the separation of two projective points u and v . Since each is a line through the origin in the ambient vector space, we may apply the notion of spread between these two lines, as developed in the previous section.

The **projective quadrance** between the non-null projective points $u = [U]$ and $v = [V]$ is the number

$$q(u, v) = 1 - \frac{(U \cdot V)^2}{(U \cdot U)(V \cdot V)}$$

This is the same as the spread $s(OU, OV)$, and has the value 1 precisely when the projective points are perpendicular. In terms of the a 's and b 's of the previous section

$$q(u, v) = \frac{a_U a_V - b_{UV}^2}{a_U a_V}.$$

Note the use of the small letter q for a projective quadrance to suggest that it is really a spread, and to distinguish it from a quadrance Q .

Theorem 7 (Projective triple quad formula) *If $u = [U]$, $v = [V]$ and $w = [W]$ are collinear projective points then the projective quadrances $q_w =$*

$q(u, v)$, $q_u = q(v, w)$ and $q_v = q(u, w)$ satisfy

$$(q_u + q_v + q_w)^2 = 2(q_u^2 + q_v^2 + q_w^2) + 4q_u q_v q_w.$$

Proof. If $u = [U]$, $v = [V]$ and $w = [W]$ are collinear then the three vectors U, V and W are dependent and the lines OU, OV and OW are coplanar. Since the projective quadrance between projective points is just the spread between these lines, the theorem is an immediate consequence of the (affine) Triple spread formula of the previous section. ■

Here is the projective version of the most important theorem in mathematics. Not surprisingly, it has consequences in many directions.

Theorem 8 (Projective Pythagoras' theorem) *Suppose that u, v and w are three distinct non-null projective points and that uw is perpendicular to vw . Then the projective quadrances $q_w = q(u, v)$, $q_u = q(v, w)$ and $q_v = q(u, w)$ satisfy*

$$q_w = q_u + q_v - q_u q_v.$$

Proof. If $u = [U]$, $v = [V]$ and $w = [W]$ for some vectors U, V and W , then

$$\begin{aligned} & q_w - q_u - q_v + q_u q_v \\ &= \frac{a_V a_U - b_{UV}^2}{a_U a_V} - \frac{a_V a_W - b_{VW}^2}{a_V a_W} - \frac{a_W a_U - b_{UW}^2}{a_U a_W} \\ &+ \frac{(a_V a_W - b_{VW}^2)(a_W a_U - b_{UW}^2)}{a_V a_W a_U a_W} \\ &= \frac{(a_W b_{UV} + b_{UW} b_{VW})(b_{UW} b_{VW} - a_W b_{UV})}{a_W^2 a_V a_U}. \end{aligned}$$

But we have seen above that the condition that uw is perpendicular to vw is equivalent to

$$a_W b_{UV} - b_{UW} b_{VW} = 0.$$

Thus this implies that

$$q_w - q_u - q_v + q_u q_v = 0.$$

■

Note that the converse does not in general follow.

The **projective spread** between the intersecting projective lines $wu = [W, U]$ and $wv = [W, V]$ is defined to be the spread between these intersecting planes, namely the number

$$\begin{aligned} & S(wu, wv) \\ &= 1 - \frac{\left(\left(U - \frac{U \cdot W}{W \cdot W} W \right) \cdot \left(V - \frac{V \cdot W}{W \cdot W} W \right) \right)^2}{\left(\left(U - \frac{U \cdot W}{W \cdot W} W \right) \cdot \left(U - \frac{U \cdot W}{W \cdot W} W \right) \right) \left(\left(V - \frac{V \cdot W}{W \cdot W} W \right) \cdot \left(V - \frac{V \cdot W}{W \cdot W} W \right) \right)}. \end{aligned}$$

It is undefined if one of the denominators is zero, and is 1 precisely when the two projective lines are perpendicular. In terms of the a 's and b 's

$$S(wu, wv) = 1 - \frac{(a_W b_{UV} - b_{UW} b_{VW})^2}{(a_U a_W - b_{UW}^2)(a_V a_W - b_{VW}^2)}. \quad (5)$$

The projective form of the spread law has the same form as the affine one.

Theorem 9 (Projective spread law) *Suppose the non-null projective triangle \overline{uvw} has projective quadrances $q_u = q(v, w)$, $q_v = q(u, w)$ and $q_w = q(u, v)$, and projective spreads $S_u = S(uv, uw)$, $S_v = S(vw, vu)$ and $S_w = S(wu, wv)$. Then*

$$\frac{S_u}{q_u} = \frac{S_v}{q_v} = \frac{S_w}{q_w}.$$

Proof. Assume that $u = [U]$, $v = [V]$ and $w = [W]$. After expansion and simplification,

$$S(wu, wv) = \frac{a_W (a_U a_W a_V - a_V b_{UW}^2 - a_U b_{VW}^2 - a_W b_{UV}^2 + 2b_{UW} b_{UV} b_{VW})}{(a_U a_W - b_{UW}^2)(a_V a_W - b_{VW}^2)} \quad (6)$$

Together with

$$q_w = q(u, v) = \frac{a_V a_U - b_{UV}^2}{a_U a_V}$$

(6) shows that the quotient S_w/q_w is actually symmetric in the three variables U, V and W . ■

The next result is a simple but surprising consequence of the Projective spread law. Even in the simple context of two dimensional elliptic trigonometry, it reveals that there is an aspect of *similar triangles in spherical geometry*. This interesting point helps explain why the spread ratio (opposite quadrance over hypotenuse quadrance) is so important in rational trigonometry.

Theorem 10 (Projective Thales' theorem) *If the projective triangle \overline{uvw} has projective spread $S_w = 1$ then*

$$S_u = \frac{q_u}{q_w}.$$

Proof. This follows from the previous theorem. ■

Here is the projective version of the Cross law. Unlike the affine result, it is a *quadratic* equation in the projective spread S_w .

Theorem 11 (Projective cross law) *Suppose the non-null projective triangle \overline{uvw} has projective quadrances $q_u = q(v, w)$, $q_v = q(u, w)$ and $q_w = q(u, v)$, and projective spreads $S_u = S(uv, uw)$, $S_v = S(vw, vu)$ and $S_w = S(wu, wv)$. Then*

$$(S_w q_u q_v - q_u - q_v - q_w + 2)^2 = 4(1 - q_u)(1 - q_v)(1 - q_w).$$

Proof. If $u = [U]$, $v = [V]$ and $w = [W]$ for some vectors U, V and W , then

$$\begin{aligned} 4(1 - q_u)(1 - q_v)(1 - q_w) &= 4 \frac{b_{UV}^2}{a_U a_V} \frac{b_{VW}^2}{a_V a_W} \frac{b_{UW}^2}{a_U a_W} \\ &= \frac{4b_{VW}^2 b_{UW}^2 b_{UV}^2}{a_W^2 a_U^2 a_V^2}. \end{aligned}$$

Also

$$\begin{aligned} S_w q_u q_v &= \frac{a_W (a_U a_W a_V - a_V b_{UW}^2 - a_U b_{VW}^2 - a_W b_{UV}^2 + 2b_{UW} b_{UV} b_{VW})}{(a_U a_W - b_{UW}^2)(a_V a_W - b_{VW}^2)} \\ &\times \frac{(a_V a_W - b_{VW}^2)(a_U a_W - b_{UW}^2)}{a_V a_W a_U a_W} \\ &= \frac{a_U a_W a_V - a_V b_{UW}^2 - a_U b_{VW}^2 - a_W b_{UV}^2 + 2b_{UW} b_{UV} b_{VW}}{a_U a_V a_W}. \end{aligned}$$

Thus

$$\begin{aligned} &S_w q_u q_v - q_u - q_v - q_w + 2 \\ &= \frac{a_U a_W a_V - a_V b_{UW}^2 - a_U b_{VW}^2 - a_W b_{UV}^2 + 2b_{UW} b_{UV} b_{VW}}{a_U a_V a_W} \\ &- \frac{a_V a_W - b_{VW}^2}{a_V a_W} - \frac{a_U a_W - b_{UW}^2}{a_U a_W} - \frac{a_V a_U - b_{UV}^2}{a_V a_U} + 2 \\ &= \frac{2b_{VW} b_{UW} b_{UV}}{a_W a_U a_V}. \end{aligned}$$

The result follows. ■

Note that the **projective quadrea** $\mathcal{A} = S_w q_u q_v = S_u q_v q_w = S_v q_u q_w$ is symmetric in U, V and W , also because of the Projective spread law.

Theorem 12 (Dual projective cross law) *Suppose the non-null projective triangle \overline{uvw} has projective quadrances $q_u = q(v, w)$, $q_v = q(u, w)$ and $q_w = q(u, v)$, and projective spreads $S_u = S(uv, uw)$, $S_v = S(vw, vu)$ and $S_w = S(wu, vw)$. Then*

$$(q_w S_u S_v - S_u - S_v - S_w + 2)^2 = 4(1 - S_u)(1 - S_v)(1 - S_w).$$

Proof. If $C_u = 1 - S_u$, $C_v = 1 - S_v$ and $C_w = 1 - S_w$ then the required identity can be rewritten as

$$(q_w (1 - C_u)(1 - C_v) + C_u + C_v + C_w - 1)^2 = 4C_u C_v C_w. \quad (7)$$

If $u = [U]$, $v = [V]$ and $w = [W]$ for some vectors U, V and W , then using (5) the left hand side of (7) is the square of an expression which simplifies in a pleasant fashion to

$$\frac{2(a_W b_{UV} - b_{UW} b_{VW})(a_V b_{UW} - b_{UV} b_{VW})(a_U b_{VW} - b_{UV} b_{UW})}{(a_V a_W - b_{VW}^2)(a_U a_W - b_{UW}^2)(a_U a_V - b_{UV}^2)}.$$

The square of this is exactly the right hand side of (7). ■

Theorem 13 (Dual projective Pythagoras' theorem) *If $q_w = 1$ in the notation of the previous proof, then*

$$S_w = S_u + S_v - S_u S_v.$$

Proof. This follows from the previous result and the polynomial identity

$$\begin{aligned} & (S_u S_v - S_u - S_v - S_w + 2)^2 - 4(1 - S_u)(1 - S_v)(1 - S_w) \\ &= (S_w - S_u - S_v + S_u S_v)^2. \end{aligned}$$

■

The proof of the next result utilized a computer, although it could be checked by hand.

Theorem 14 (Projective triple spread formula) *Suppose that u, v, w and z are coplanar projective points with projective spreads $R_u = S(zv, zw)$, $R_v = S(zu, zw)$ and $R_w = S(zu, zv)$. Then*

$$(R_u + R_v + R_w)^2 = 2(R_u^2 + R_v^2 + R_w^2) + 4R_u R_v R_w.$$

Proof. (Using a computer) Suppose that $u = [U]$, $v = [V]$, $w = [W]$ and $z = [Z]$ for vectors U, V, W and Z . Then

$$\begin{aligned} R_u &= \frac{a_Z (a_V a_W a_Z - a_Z b_{VW}^2 - a_V b_{WZ}^2 - a_W b_{VZ}^2 + 2b_{VW} b_{VZ} b_{WZ})}{(a_V a_Z - b_{VZ}^2) (a_Z a_W - b_{WZ}^2)} \\ R_v &= \frac{a_Z (a_U a_W a_Z - a_Z b_{UW}^2 - a_U b_{WZ}^2 - a_W b_{UZ}^2 + 2b_{UW} b_{UZ} b_{WZ})}{(a_U a_Z - b_{UZ}^2) (a_Z a_W - b_{WZ}^2)} \\ R_w &= \frac{a_Z (a_U a_V a_Z - a_Z b_{UV}^2 - a_U b_{VZ}^2 - a_V b_{UZ}^2 + 2b_{UV} b_{UZ} b_{VZ})}{(a_U a_Z - b_{UZ}^2) (a_Z a_V - b_{VZ}^2)} \end{aligned}$$

A computer calculation shows that

$$(R_u + R_v + R_w)^2 - 2(R_u^2 + R_v^2 + R_w^2) - 4R_u R_v R_w$$

has a factor which is the determinant

$$\begin{vmatrix} a_U & b_{UV} & b_{UW} & b_{UZ} \\ b_{UV} & a_V & b_{VW} & b_{VZ} \\ b_{UW} & b_{VW} & a_W & b_{WZ} \\ b_{UZ} & b_{VZ} & b_{WZ} & a_Z \end{vmatrix}.$$

But if u, v, w and z are coplanar then U, V, W and Z are linearly dependent, so this determinant is zero. ■

It is worth pointing out that for the case of a non-degenerate bilinear form and $n = 3$ there is a duality between projective points and projective lines, so the previous three theorems can be deduced from the corresponding earlier results. However in general this duality is not available.

Special projective triangles

The following theorem gives universal analogs of Napier's rules in spherical trigonometry. The various formulas in this proof are fundamental for projective trigonometry, and they can all be derived easily enough from the basic equations of the theorem.

Theorem 15 (Napier's rules) *Suppose the projective triangle \overline{uvw} has projective quadrances $q_u = q(v, w)$, $q_v = q(u, w)$ and $q_w = q(u, v)$ and projective spreads $S_u = S(uv, uw)$, $S_v = S(vu, vw)$ and $S_w = S(wu, wv)$. If $S_w = 1$ then any two of the five quantities $[q_u, q_v, q_w, S_u, S_v]$ determine the other three, solely through the three **basic equations***

$$q_w = q_u + q_v - q_u q_v \quad S_u = q_u / q_w \quad S_v = q_v / q_w.$$

Proof. Two of the projective quadrances allow you to determine the third via the Projective Pythagoras' theorem $q_w = q_u + q_v - q_u q_v$, and then the other two Projective Thales' equations $S_u = q_u / q_w$ and $S_v = q_v / q_w$ give the projective spreads.

Given the two projective spreads S_u and S_v , use the Projective Pythagoras' theorem and the Thales' equations $S_u = q_u / q_w$ and $S_v = q_v / q_w$ to obtain

$$1 = S_u + S_v - S_u S_v q_w.$$

Thus

$$q_u = S_u q_w = \frac{S_u + S_v - 1}{S_v}$$

$$q_v = S_v q_w = \frac{S_u + S_v - 1}{S_u}$$

and

$$q_w = \frac{S_u + S_v - 1}{S_u S_v}.$$

If you know a projective spread, say S_u , and a projective quadrance, then there are three possibilities. If the projective quadrance is q_w , then $q_u = S_u q_w$,

$$q_v = \frac{q_w - q_u}{1 - q_u} = \frac{q_w - S_u q_w}{1 - S_u q_w}$$

and

$$S_v = \frac{q_v}{q_w} = \frac{1 - S_u}{1 - S_u q_w}.$$

If the projective quadrance is q_u , then $q_w = q_u / S_u$,

$$q_v = \frac{q_w - q_u}{1 - q_u} = \frac{q_u (1 - S_u)}{S_u (1 - q_u)}$$

and

$$S_v = \frac{q_v}{q_w} = \frac{1 - S_u}{1 - q_u}.$$

If the projective quadrance is q_v , then substitute $q_u = S_u q_w$ into the Projective Pythagoras equation to get

$$q_w = S_u q_w + q_v - S_u q_v q_w.$$

So

$$q_w = \frac{q_v}{1 - S_u(1 - q_v)}$$

$$q_u = \frac{S_u q_v}{1 - S_u(1 - q_v)}$$

and

$$S_v = \frac{q_v}{q_w} = 1 - S_u(1 - q_v).$$

■

A projective triangle is **isosceles** precisely when at least two of its projective quadrances are equal.

Theorem 16 (Pons Asinorum) *Suppose a non-null projective triangle \overline{uvw} has projective quadrances q_u , q_v and q_w , and projective spreads S_u , S_v and S_w . Then $q_u = q_v$ precisely when $S_u = S_v$.*

Proof. Follows immediately from the Projective spread law. ■

It follows from the Projective Pythagoras' theorem that if $S_u = S_v = 1$ then $q_w = 0$ or $q_u = q_v = 1$.

Theorem 17 (Isosceles projective triangle) *Suppose an isosceles projective triangle has non-zero projective quadrances $q_u = q_v = q$ and q_w , and projective spreads $S_u = S_v = S$ and S_w . Then*

$$q_w = \frac{4q(1-S)(1-q)}{(1-Sq)^2}$$

$$S_w = \frac{4S(1-S)(1-q)}{(1-Sq)^2}.$$

Proof. Use the Projective spread law in the form

$$S_w = \frac{S q_w}{q}$$

to replace S_w in the Projective cross law

$$(q^2 S_w - (2q + q_w - 2))^2 = 4(1-q)^2(1-q_w).$$

This yields a quadratic equation in q_w with solutions $q_w = 0$, which is impossible by assumption, and

$$q_w = \frac{4q(1-S)(1-q)}{(1-Sq)^2}.$$

Thus

$$S_w = \frac{Sq_w}{q} = \frac{4S(1-S)(1-q)}{(1-Sq)^2}.$$

■

A projective triangle is **equilateral** precisely when all its quadrances are equal. The following formula appeared in the Euclidean spherical case as Exercise 24.1 in [5].

Theorem 18 (Equilateral projective triangles) *Suppose that a projective triangle is equilateral with common non-zero projective quadrance $q_1 = q_2 = q_3 = q$, and with common projective spread $S_1 = S_2 = S_3 = S$. Then*

$$(1 - Sq)^2 = 4(1 - S)(1 - q).$$

Proof. From the Isosceles projective triangle theorem

$$q = \frac{4q(1-S)(1-q)}{(1-Sq)^2}.$$

Since $q \neq 0$ this yields

$$(1 - Sq)^2 = 4(1 - S)(1 - q).$$

■

The above result is symmetric in S and q . Note that if $S = 3/4$ then $q = 8/9$. This value is important in chemistry—it is the tetrahedral spread found for example in the methane molecule, and corresponds to an angle which is approximately 109.47° . As I will show elsewhere, rational trigonometry provides a much more refined analysis of the geometry of the Platonic solids, but some basic results in this direction can be found in [5].

Spread polynomials

We have seen that both affine and projective trigonometry involve the Triple spread formula

$$(a + b + c)^2 = 2(a^2 + b^2 + c^2) + 4abc.$$

If $a = b = s$ then c turns out to be either 0 or $4s(1-s)$. If $a = 4s(1-s)$ and $b = s$ then c turns out to be either s or $s(3-4s)^2$. There is then a sequence of polynomials $S_n(s)$ for $n = 0, 1, 2, \dots$ with the property that $S_{n-1}(s), s$ and $S_n(s)$ always satisfy the Triple spread formula. They play a role in all metrical geometries, independent of the nature of the symmetric bilinear form,

and are defined over the integers. These are rational analogs of the Chebyshev polynomials of the first kind, and they have many remarkable properties.

The **spread polynomial** $S_n(s)$ is defined recursively by $S_0(s) = 0$, $S_1(s) = s$ and the rule

$$S_n(s) = 2(1 - 2s)S_{n-1}(s) - S_{n-2}(s) + 2s.$$

The coefficient of s^n in $S_n(s)$ is a power of four, so the degree of the polynomial $S_n(s)$ is n in any field of characteristic not two. It turns out that in the decimal number field

$$S_n(s) = \frac{1 - T_n(1 - 2s)}{2}$$

where T_n is the n -th Chebyshev polynomial of the first kind. The first few spread polynomials are $S_0(s) = 0$, $S_1(s) = s$, $S_2(s) = 4s - 4s^2$, $S_3(s) = 9s - 24s^2 + 16s^3$, $S_4(s) = 16s - 80s^2 + 128s^3 - 64s^4$ and $S_5(s) = 25s - 200s^2 + 560s^3 - 640s^4 + 256s^5$. Note that $S_2(s)$ is the logistic map.

As shown in [5], $S_n \circ S_m = S_{nm}$ for $n, m \geq 1$, and the spread polynomials have interesting orthogonality properties over finite fields. S. Goh [2] observed that there is a sequence of ‘spread-cyclotomic’ polynomials $\phi_k(s)$ of degree $\phi(k)$ with integer coefficients such that for any $n = 1, 2, 3, \dots$

$$S_n(s) = \prod_{k|n} \phi_k(s).$$

A projective example over \mathbb{F}_{11}

Consider a five dimensional vector space over the field \mathbb{F}_{11} with bilinear form

$$U \cdot V = U M V^T$$

where

$$M = \begin{pmatrix} 1 & 10 & 1 & 0 & 0 \\ 10 & 2 & 5 & 2 & 0 \\ 1 & 5 & 1 & 4 & 3 \\ 0 & 2 & 4 & 7 & 2 \\ 0 & 0 & 3 & 2 & 8 \end{pmatrix}.$$

The geometry of the associated four dimensional projective space is pleasantly accessible. The use of a finite field simplifies calculations and provides an ideal laboratory for geometrical explorations, even if you are interested in other fields. Consider the triangle \overline{uvw} where

$$u = [1 : 4 : 2 : 6 : 1] \quad v = [1 : 2 : 3 : 4 : 1] \quad w = [0 : 8 : 8 : 3 : 1].$$

The projective quadrances are then $q_u = 9$, $q_v = 8$ and $q_w = 1$, and the projective spreads are $S_u = 2$, $S_v = 3$ and $S_w = 10$. Since $q_w = 1$, \overline{uvw} is a **dual right triangle**, so that $S_w = S_u + S_v - S_u S_v$.

The Projective spread law is verified to be

$$\frac{2}{9} = \frac{3}{8} = \frac{10}{1}$$

while the Projective cross law takes the form

$$(10 \times 9 \times 8 - 9 - 8 - 1 + 2)^2 = 0 = 4(1 - 9)(1 - 8)(1 - 1)$$

and the Dual projective cross law takes the form

$$(1 \times 2 \times 3 - 2 - 3 - 10 + 2)^2 = 5 = 4(1 - 2)(1 - 3)(1 - 10).$$

Note that the squares in F_{11} are $0, 1, 3, 4, 5, 9$, so S_v is the only projective spread of the triangle \overline{uvw} which is a square. As in the discussion in [5], this implies that only the vertex at v has a bisector, and there are two such. The points $b_1 = [3 : 0 : 5 : 8 : 7]$ and $b_2 = [3 : 2 : 7 : 6 : 10]$ lie on uw and

$$S(vu, vb_1) = S(vw, vb_1) = 10 \quad \text{and} \quad S(vu, vb_2) = S(vw, vb_2) = 2.$$

While bisectors and medians are dependent on number theoretic considerations, the orthocenter of the projective triangle turns out not to be. You can check that here it is

$$O = [9 : 1 : 0 : 4 : 1].$$

Lambert quadrilaterals

Here are two (of many) results from hyperbolic geometry (see [1, Chapter 7]) that hold more generally.

Theorem 19 (Lambert quadrilateral) *Suppose the projective points u, v, w and z are coplanar and form projective spreads*

$$S(uv, uz) = S(vu, vw) = S(wv, wz) = 1$$

and projective quadrances $q(u, v) = q$ and $q(v, w) = p$. Then

$$\begin{aligned} q(w, z) = y = q(1 - p) / (1 - qp) & & q(u, z) = x = p(1 - q) / (1 - qp) \\ q(u, w) = s = q + p - qp & & q(v, z) = r = (q + p - 2qp) / (1 - qp) \end{aligned}$$

and

$$\begin{aligned} S(vu, vz) = x/r & & S(vw, vz) = y/r & & S(wv, wu) = q/s \\ S(uw, uv) = p/s & & S(uw, uz) = q(1 - p)/s & & S(wu, wz) = p(1 - q)/s \end{aligned}$$

and

$$S(zu, zw) = S = 1 - pq.$$

Proof. The fact that the four points are coplanar implies that the Projective triple spread theorem applies to any three projective lines of the projective quadrilateral \overline{uvwz} meeting at a projective point. Furthermore it implies that where three projective lines meet and one of the spreads is 1, the other two spreads must sum to 1.

The expressions for $S(vu, vz)$, $S(vw, vz)$, $S(wv, wu)$ and $S(uw, uv)$ follow from the Projective Thales' theorem. The expression for s follows from the Projective Pythagoras theorem applied to \overline{uvw} . The same theorem applied to \overline{uvz} and \overline{vwz} gives the equations

$$\begin{aligned} r &= q + x - qx \\ r &= p + y - py \end{aligned}$$

and since $S(vu, vw) = 1$

$$\frac{x}{r} + \frac{y}{r} = 1.$$

These three equations can then be solved to yield the stated values for r , x and y . Also the equations

$$S(uw, uv) + S(uw, uz) = 1 = S(wv, wu) + S(wu, wz)$$

can be used to solve for $S(uw, uz)$ and $S(wu, wz)$. Finally use the Projective spread law in \overline{uvwz} to get

$$S(zu, zw) = S = 1 - pq.$$

■

Theorem 20 (Right hexagon) *Suppose a projective hexagon $\overline{a_1a_2a_3a_4a_5a_6}$ is planar, meaning that all the projective points lie in some fixed projective plane, and that all successive projective spreads are equal to 1, that is*

$$S(a_1a_2, a_1a_6) = S(a_2a_3, a_2a_1) = \cdots = S(a_6a_1, a_6a_5) = 1.$$

Then

$$\frac{q(a_1, a_2)}{q(a_4, a_5)} = \frac{q(a_2, a_3)}{q(a_5, a_6)} = \frac{q(a_3, a_4)}{q(a_1, a_6)}.$$

Proof. Follows by repeated use of the last formula from the previous theorem.

■

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