67. An identity of Ramanujan, and applications

Dedicated to Dick Askey on the occasion of his 65th Birthday

1. Introduction

The identity of the title is, in modern notation,

$$(q)_\infty = (q^{25})_\infty \left(R(q^5) - q - q^2 R(q^5)^{-1} \right) \tag{1}$$

where

$$R(q) = \left(\begin{matrix} q^2, q^3 \\ q, \ q^4 \end{matrix} ; q^5 \right)_\infty .$$

Ramanujan [7, p.212] stated this identity without proof ("It can be shewn that...") on the way to proving the identity Hardy [7, p.xxxv] regarded as Ramanujan's most beautiful,

$$\sum_{n \geq 0} p(5n+4) q^n = 5 \frac{(q^5)_\infty^5}{(q)_\infty^6}, \tag{2}$$

where $p(n)$ is the number of partitions of n, given by $\sum_{n \geq 0} p(n) q^n = 1/(q)_\infty$.

It is my intention to give as direct a proof as is possible of (1). I will then apply (1) to give a more direct proof than Ramanujan of (2), to give a more direct derivation than Watson of the modular equation of 5^{th} order, to give an elementary derivation of a result involving Ramanujan's tau function, $\tau(n)$, defined by $\sum_{n \geq 1} \tau(n) q^n = q(q)_\infty^{24}$ and to prove a pair of identities from the lost notebook.

2. Proof of principal identity

It should be noted that Watson [8, p.102] gives a proof of (1) using the quintuple product identity. For a history of the quintuple product identity, see Hirschhorn [3] and the review of [3] by Bressoud [1], and for a succinct proof see Hirschhorn [4].

We prove a result slightly more general than (1). We note that this more general result is a special case of Hirschhorn [2, (2.1)], but the proof I now give is slicker. We prove

(3)

$$(a, a^2, a^{-2}q, a^{-1}q, q; q)_\infty = (q^5)_\infty \left\{ \left(\begin{matrix} a^5 q, a^{-5} q^4 \\ q, \ q^4 \end{matrix} ; q^5 \right)_\infty - a \left(\begin{matrix} a^5 q^2, a^{-5} q^3 \\ q^2, q^3 \end{matrix} ; q^5 \right)_\infty \right.$$

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$$-a^2\begin{pmatrix} a^{-5}q^2, a^5q^3 \\ q^2, q^3 \end{pmatrix}; q^5\Big)_\infty + a^3\begin{pmatrix} a^{-5}q,\ a^5q^4 \\ q,\ q^4 \end{pmatrix}; q^5\Big)_\infty\Bigg\}.$$

(1) follows on replacing q by q^5 and setting $a = q$.

We have

$$\begin{aligned}(a, a^2, a^{-2}q, a^{-1}q, q; q)_\infty &= \frac{(a, a^{-1}q, q; q)_\infty (a^2, a^{-2}q, q; q)_\infty}{(q)_\infty} \\ &= \frac{1}{(q)_\infty}\sum_{-\infty}^{\infty}(-1)^r a^r q^{(r^2-r)/2}\sum_{-\infty}^{\infty}(-1)^s a^{2s} q^{(s^2-s)/2} \\ &= \sum_{-\infty}^{\infty} a^n c_n(q)\end{aligned}$$

where

$$c_n(q) = \frac{1}{(q)_\infty}\sum_{r+2s=n}(-1)^{r+s}q^{(r^2-r+s^2-s)/2}.$$

If we put $r = n - 2t,\ s = 2n + t$ we find

$$\begin{aligned}c_{5n}(q) &= \frac{1}{(q)_\infty}\sum_{t=-\infty}^{\infty}(-1)^{n+t}q^{((n-2t)^2-(n-2t)+(2n+t)^2-(2n+t))/2} \\ &= (-1)^n q^{(5n^2-3n)/2}\frac{1}{(q)_\infty}\sum_{-\infty}^{\infty}(-1)^t q^{(5t^2+t)/2} \\ &= \frac{(-1)^n q^{(5n^2-3n)/2}}{(q, q^4; q^5)_\infty}.\end{aligned}$$

Similarly,

$$\begin{aligned}c_{5n+1}(q) &= \frac{1}{(q)_\infty}\sum_{t=-\infty}^{\infty}(-1)^{n+t+1}q^{((n+1-2t)^2-(n+1-2t)+(2n+t)^2-(2n+t))/2} \\ &= (-1)^{n+1} q^{(5n^2-n)/2}\frac{1}{(q)_\infty}\sum_{-\infty}^{\infty}(-1)^t q^{(5t^2-3t)/2} \\ &= -\frac{(-1)^n q^{(5n^2-n)/2}}{(q^2, q^3; q^5)_\infty},\end{aligned}$$

$$\begin{aligned}c_{5n+2}(q) &= \frac{1}{(q)_\infty}\sum_{t=-\infty}^{\infty}(-1)^{n+t+1}q^{((n-2t)^2-(n-2t)+(2n+1+t)^2-(2n+1+t))/2} \\ &= (-1)^{n+1} q^{(5n^2+n)/2}\frac{1}{(q)_\infty}\sum_{-\infty}^{\infty}(-1)^t q^{(5t^2+3t)/2}\end{aligned}$$

$$= -\frac{(-1)^n q^{(5n^2+n)/2}}{(q^2, q^3; q^5)_\infty},$$

$$\begin{aligned} c_{5n+3}(q) &= \frac{1}{(q)_\infty} \sum_{t=-\infty}^{\infty} (-1)^{n+t} q^{((n+1-2t)^2-(n+1-2t)+(2n+1+t)^2-(2n+1+t))/2} \\ &= (-1)^n q^{(5n^2+3n)/2} \frac{1}{(q)_\infty} \sum_{-\infty}^{\infty} (-1)^t q^{(5t^2-t)/2} \\ &= \frac{(-1)^n q^{(5n^2+3n)/2}}{(q,\ q^4; q^5)_\infty}, \end{aligned}$$

$$\begin{aligned} c_{5n-1}(q) &= \frac{1}{(q)_\infty} \sum_{t=-\infty}^{\infty} (-1)^{n+t+1} q^{((n-1-2t)^2-(n-1-2t)+(2n+t)^2-(2n+t))/2} \\ &= (-1)^{n+1} q^{(5n^2-5n+2)/2} \frac{1}{(q)_\infty} \sum_{-\infty}^{\infty} (-1)^t q^{(5t^2+5t)/2} \\ &= 0. \end{aligned}$$

Thus,

$$\begin{aligned} &(a, a^2, a^{-2}q, a^{-1}q, q; q)_\infty \\ &= \frac{1}{(q, q^4; q^5)_\infty} \sum_{-\infty}^{\infty} (-1)^n a^{5n} q^{(5n^2-3n)/2} - \frac{1}{(q^2, q^3; q^5)_\infty} \sum_{-\infty}^{\infty} (-1)^n a^{5n+1} q^{(5n^2-n)/2} \\ &- \frac{1}{(q^2, q^3; q^5)_\infty} \sum_{-\infty}^{\infty} (-1)^n a^{5n+2} q^{(5n^2+n)/2} + \frac{1}{(q, q^4; q^5)_\infty} \sum_{-\infty}^{\infty} (-1)^n a^{5n+3} q^{(5n^2+3n)/2}. \end{aligned}$$

If we now sum using the triple product identity, we obtain (3).

3. Ramanujan's most beautiful identity

We can write (1)

$$(q^5)_\infty (q, q^2, q^3, q^4; q^5)_\infty = (q^{25})_\infty \left(R(q^5) - q - q^2 R(q^5)^{-1} \right). \tag{4}$$

If ω is a fifth root of unity and we substitute ωq for q in (4), we find

$$(q^5)_\infty (\omega q, \omega^2 q^2, \omega^3 q^3, \omega^4 q^4; q^5)_\infty = (q^{25})_\infty \left(R(q^5) - \omega q - \omega^2 q^2 R(q^5)^{-1} \right). \tag{5}$$

If we write (5) for each of the five fifth roots of unity and multiply the five results, we obtain

$$\frac{(q^5)_\infty^6}{(q^{25})_\infty} = (q^{25})_\infty^5 \left(R(q^5)^5 - 11q^5 - q^{10} R(q^5)^{-5} \right).$$

or,

$$1 = \frac{(q^{25})_\infty^6}{(q^5)_\infty^6}\left(R(q^5)^5 - 11q^5 - q^{10}R(q^5)^{-5}\right). \tag{6}$$

If we divide (6) by (1), we find

$$\frac{1}{(q)_\infty} = \frac{(q^{25})_\infty^5}{(q^5)_\infty^6}\frac{R(q^5)^5 - 11q^5 - q^{10}R(q^5)^{-5}}{R(q^5) - q - q^2R(q^5)^{-1}}.$$

That is,

$$\sum_{n\geq 0} p(n)q^n = \frac{(q^{25})_\infty^5}{(q^5)_\infty^6}\left(R(q^5)^4 + qR(q^5)^3 + 2q^2R(q^5)^2 + 3q^3R(q^5)\right.$$
$$\left. + 5q^4 - 3q^5R(q^5)^{-1} + 2q^6R(q^5)^{-2} - q^7R(q^5)^{-3} + q^8R(q^5)^{-4}\right).$$

If we extract those terms in which the power of q is congruent to 4 modulo 5, divide by q^4 and replace q^5 by q we obtain

$$\sum_{n\geq 0} p(5n+4)q^n = 5\frac{(q^5)_\infty^5}{(q)_\infty^6},$$

as desired.

4. The modular equation of 5^{th} order

The modular equation of 5^{th} order appears below as (7). It was first proved by Watson [8],p.103. Our proof is along similar lines, but is more straightforward – we use a different elementary identity.

Let us write

$$\zeta = \frac{(q)_\infty}{q(q^{25})_\infty} = q^{-1}R(q^5) - 1 - qR(q^5)^{-1},$$
$$\tau = \frac{(q^5)_\infty^6}{q^5(q^{25})_\infty^6} = q^{-5}R(q^5)^5 - 11 - q^5R(q^5)^{-5}.$$

From the elementary identity

$$\alpha^5 - 11 - \alpha^{-5} = (\alpha - 1 - \alpha^{-1})^5 + 5(\alpha - 1 - \alpha^{-1})^4 + 15(\alpha - 1 - \alpha^{-1})^3$$
$$+ 25(\alpha - 1 - \alpha^{-1})^2 + 25(\alpha - 1 - \alpha^{-1})$$

we obtain the modular equation

$$\tau = \zeta^5 + 5\zeta^4 + 15\zeta^3 + 25\zeta^2 + 25\zeta.$$

We can write this

$$\left(\frac{\tau}{\zeta}\right)^5 = 25\left(\frac{\tau}{\zeta}\right)^4 + 25\tau\left(\frac{\tau}{\zeta}\right)^3 + 15\tau^2\left(\frac{\tau}{\zeta}\right)^2 + 5\tau^3\left(\frac{\tau}{\zeta}\right) + \tau^4. \tag{7}$$

This enables one to find appropriate generating functions and prove Ramanujan's partition congruences for powers of 5,

$$5^\alpha | p\left(5^\alpha n - \frac{5^{2\alpha}-1}{24}\right)$$

but that is another story [5],[6].

5. Ramanujan's tau function

Ramanujan's tau function, $\tau(n)$, is defined by

$$\sum_{n\geq 1} \tau(n) q^n = q(q)_\infty^{24}.$$

We will use (1) to give an elementary demonstration of the fact that

$$\tau(5n) = 4830\tau(n) - 5^{11}\tau\left(\frac{n}{5}\right) \tag{8}$$

and hence that

$$5^\alpha | \tau(5^\alpha n).$$

It should be noted that (8) is a special case of the relation, where p is any prime,

$$\tau(pn) = \tau(p)\tau(n) - p^{11}\tau\left(\frac{n}{p}\right)$$

which can be proved from the theory of modular forms and Hecke operators.

We have by (1) that

$$\begin{aligned}
&\sum_{n\geq 1} \tau(n)q^n = q(q)_\infty^{24} = q(q^{25})_\infty^{24}\left(R(q^5) - q - q^2R(q^5)^{-1}\right)^{24} \\
&= (q^{25})_\infty^{24} \\
&\times \left(qR(q^5)^{24} - 24q^2R(q^5)^{23} + 252q^3R(q^5)^{22} - 1472q^4R(q^5)^{21} + 4830q^5R(q^5)^{20} - \cdots\right. \\
&\quad -212520q^{10}R(q^5)^{15} + \cdots + 3487260q^{15}R(q^5)^{10} - \cdots - 25077360q^{20}R(q^5)^5 + \cdots \\
&\quad +1490375q^{25} + \cdots + 25077360q^{30}R(q^5)^{-5} + \cdots + 3487260q^{35}R(q^5)^{-10} - \cdots \\
&\quad \left. +212520q^{40}R(q^5)^{-15} - \cdots + 4830q^{45}R(q^5)^{-20} + \cdots + q^{49}R(q^5)^{-24}\right).
\end{aligned}$$

It follows that

$$
\begin{aligned}
&\sum_{n\geq 1} \tau(5n)q^n \\
&= (q^5)_\infty^{24}\big(4830qR(q)^{20} - 212520q^2R(q)^{15} + 3487260q^3R(q)^{10} - 25077360q^4R(q)^5 \\
&\qquad\qquad +14903725q^5 + 25077360q^6R(q)^{-5} + 3487260q^7R(q)^{-10} + 212520q^8R(q)^{-15} \\
&\qquad\qquad +4830q^9R(q)^{-20}\big) \\
&= (q^5)_\infty^{24}\big(4830q(R(q)^5 - 11q - q^2R(q)^{-5})^4 - 48828125q^5\big) \\
&= (q^5)_\infty^{24}\left(4830q\left(\frac{(q)_\infty^6}{(q^5)_\infty^6}\right)^4 - 5^{11}q^5\right) \\
&= 4830q(q)_\infty^{24} - 5^{11}q^5(q^5)_\infty^{24} \\
&= 4830\sum_{n\geq 1}\tau(n)q^n - 5^{11}\sum_{n\geq 1}\tau\left(\frac{n}{5}\right)q^n.
\end{aligned}
$$

Hence

$$\tau(5n) = 4830\tau(n) - 5^{11}\tau\left(\frac{n}{5}\right),$$

as claimed.

It follows that $5|\tau(5n)$, $25|\tau(25n)$ and for $\alpha \geq 1$

$$\tau(5^{\alpha+2}n) = 4830\tau(5^{\alpha+1}n) - 5^{11}\tau(5^\alpha n).$$

Hence by induction on α

$$5^\alpha|\tau(5^\alpha n).$$

6. Two identities from the lost notebook (unpublished)

Cubing (1) gives

$$
\begin{aligned}
&(q^{25})_\infty^3(R(q^5) - q - q^2R(q^5)^{-1})^3 \\
&= (q)_\infty^3 \\
&= 1 - 3q + 5q^3 - 7q^6 + 9q^{10} - 11q^{15} + 13q^{21} - 15q^{28} + - \ \cdots\ . \\
&= (1 + 9q^{10} - 11q^{15} - 19q^{45} + \ \cdots\ ) \\
&\qquad - q(3 + 7q^5 - 13q^{20} + 23q^{65} + \ \cdots\ )
\end{aligned}
$$

$$+5q^3(1-3q^{25}+5q^{75}-+\ \cdots)$$

$$=\sum_{-\infty}^{\infty}(-1)^n(10n+1)q^{5(5n^2+n)/2}-q\sum_{-\infty}^{\infty}(-1)^n(10n+3)q^{5(5n^2+3n)/2}+5q^3(q^{25})_\infty^3.$$

It follows that

$$(q^5)_\infty^3(R(q)^3-3qR(q)^{-2})=\sum_{-\infty}^{\infty}(-1)^n(10n+1)q^{(5n^2+n)/2}$$

and

$$(q^5)_\infty^3(3R(q)^2+qR(q)^{-3})=\sum_{-\infty}^{\infty}(-1)^n(10n+3)q^{(5n^2+3n)/2}.$$

It is not hard to show that these can be written

$$(q^5)_\infty^3(R(q)^3-3qR(q)^{-2})=(q^2,q^3,q^5;q^5)_\infty\left\{1+10\sum_{n\geq 0}\left(\frac{q^{5n+2}}{1-q^{5n+2}}-\frac{q^{5n+3}}{1-q^{5n+3}}\right)\right\}$$

and

$$(q^5)_\infty^3(3R(q)^2+qR(q)^{-3})=(q,q^4,q^5;q^5)_\infty\left\{3+10\sum_{n\geq 0}\left(\frac{q^{5n+1}}{1-q^{5n+1}}-\frac{q^{5n+4}}{1-q^{5n+4}}\right)\right\}.$$

If we divide one of these by the other, we find

$$\frac{R(q)^5-3q}{3R(q)^5+q}=\frac{1+10\displaystyle\sum_{-\infty}^{\infty}\frac{q^{5n+2}}{1-q^{5n+2}}}{3+10\displaystyle\sum_{-\infty}^{\infty}\frac{q^{5n+1}}{1-q^{5n+1}}}.$$

If we now make $R(q)^5$ the subject, we find

$$R(q)^5=\frac{1+3\displaystyle\sum_{-\infty}^{\infty}\frac{q^{5n+1}}{1-q^{5n+1}}+\sum_{-\infty}^{\infty}\frac{q^{5n+2}}{1-q^{5n+2}}}{\displaystyle\sum_{-\infty}^{\infty}\frac{q^{5n}}{1-q^{5n+1}}-3\sum_{-\infty}^{\infty}\frac{q^{5n+1}}{1-q^{5n+2}}}.$$

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