A PROBLEM IN DYNAMICS

1. The problem

Three masses, $m_1,\ m_2,\ m_3$ are placed in line on a smooth horizontal surface. Initially, m_1 and m_3 are at rest, and m_2 is moving towards m_3 with speed v. Assuming elastic collisions between masses, determine the final velocities of the three masses.

This problem was posed in a first-year Physics course at UNSW, and the students were asked to consider the case in which $m_1,\ m_3 >> m_2$. The given solution,

$$v_1 = -v\sqrt{\frac{m_2 m_3}{m_1(m_1+m_3)}}, \quad v_2 = v_3 = v\sqrt{\frac{m_1 m_2}{m_3(m_1+m_3)}}$$

was unsatisfactory to me, particularly because it was counter-intuitive that we should have $v_2 = v_3$. So it seemed desirable to solve the general problem exactly, and that is what I have done.

2. The solution

Let $(v_1)_n,\ (v_2)_n,\ (v_3)_n$ be the velocities of the masses $m_1,\ m_2,\ m_3$ after n collisions, and let

$$\mathbf{v}_n = \begin{pmatrix} (v_1)_n \\ (v_2)_n \\ (v_3)_n \end{pmatrix}.$$

Then

$$\mathbf{v}_0 = \begin{pmatrix} 0 \\ v \\ 0 \end{pmatrix}.$$

Upon collision, the velocities of the masses change, and the changes are governed by conservation of momentum and conservation of energy.

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From the odd-numbered collisions (those between m_2 and m_3) $(v_1)_{2n+1},\ (v_2)_{2n+1},\ (v_3)_{2n+1}$ are determined by

$$\begin{aligned} (v_1)_{2n+1} &= (v_1)_{2n}, \\ m_2(v_2)_{2n+1} + m_3(v_3)_{2n+1} &= m_2(v_2)_{2n} + m_3(v_3)_{2n}, \\ m_2(v_2)^2_{2n+1} + m_3(v_3)^2_{2n+1} &= m_2(v_2)^2_{2n} + m_3(v_3)^2_{2n}. \end{aligned}$$

It follows that

$$(v_2)_{2n+1} + (v_2)_{2n} = (v_3)_{2n+1} + (v_3)_{2n},$$

$$\begin{pmatrix} m_2 & m_3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} (v_2)_{2n+1} \\ (v_3)_{2n+1} \end{pmatrix} = \begin{pmatrix} m_2 & m_3 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} (v_2)_{2n} \\ (v_3)_{2n} \end{pmatrix}$$

and

$$\begin{pmatrix} (v_2)_{2n+1} \\ (v_3)_{2n+1} \end{pmatrix} = \begin{pmatrix} \dfrac{m_2 - m_3}{m_2 + m_3} & \dfrac{2m_3}{m_2 + m_3} \\ \dfrac{2m_2}{m_2 + m_3} & \dfrac{m_3 - m_2}{m_2 + m_3} \end{pmatrix} \begin{pmatrix} (v_2)_{2n} \\ (v_3)_{2n} \end{pmatrix}.$$

Thus

$$\mathbf{v}_{2n+1} = R\mathbf{v}_{2n}$$

where

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \dfrac{m_2 - m_3}{m_2 + m_3} & \dfrac{2m_3}{m_2 + m_3} \\ 0 & \dfrac{2m_2}{m_2 + m_3} & -\dfrac{m_2 - m_3}{m_2 + m_3} \end{pmatrix}$$

Similarly, from the even-numbered collisions (those between m_2 and m_1),

$$\mathbf{v}_{2n+2} = L\mathbf{v}_{2n+1}$$

where

$$L = \begin{pmatrix} \dfrac{m_1 - m_2}{m_1 + m_2} & \dfrac{2m_2}{m_1 + m_2} & 0 \\ \dfrac{2m_1}{m_1 + m_2} & -\dfrac{m_1 - m_2}{m_1 + m_2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Observe that if we write

$$m_3/m_2 = r, \quad m_1/m_2 = l$$

then

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\dfrac{r-1}{r+1} & \dfrac{2r}{r+1} \\ 0 & \dfrac{2}{r+1} & \dfrac{r-1}{r+1} \end{pmatrix}, \quad L = \begin{pmatrix} \dfrac{l-1}{l+1} & \dfrac{2}{l+1} & 0 \\ \dfrac{2l}{l+1} & -\dfrac{l-1}{l+1} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It thus becomes clear from the mathematics (if it was not already so from the physics) that the behaviour of the velocities depends not on the <u>masses</u>, but on their <u>ratios</u>.

It follows that

$$\mathbf{v}_{2n} = (LR)^n \mathbf{v}_0,$$
$$\mathbf{v}_{2n+1} = R(LR)^n \mathbf{v}_0.$$

Now,

$$LR = \begin{pmatrix} \dfrac{l-1}{l+1} & -\dfrac{2(r-1)}{(l+1)(r+1)} & \dfrac{4r}{(l+1)(r+1)} \\ \dfrac{2l}{l+1} & \dfrac{(l-1)(r-1)}{(l+1)(r+1)} & -\dfrac{2(l-1)r}{(l+1)(r+1)} \\ 0 & \dfrac{2}{r+1} & \dfrac{r-1}{r+1} \end{pmatrix}.$$

The characteristic polynomial of LR is

$$(\lambda - 1)\left(\lambda^2 - 2\left(\frac{lr - l - r - 1}{(l+1)(r+1)}\right)\lambda + 1\right).$$

If ϕ is defined by

$$\cos\phi = \sqrt{\frac{lr}{(l+1)(r+1)}}, \quad \sin\phi = \sqrt{\frac{(l+r+1)}{(l+1)(r+1)}},$$

then the characteristic polynomial of LR becomes

$$(\lambda - 1)(\lambda^2 - 2\cos 2\phi\lambda + 1).$$

Thus the eigenvalues of LR are

$$1, \quad e^{\pm 2\phi i}.$$

If $\mathbf{p}_1, \ \mathbf{p}_2, \ \mathbf{p}_3$ are defined by

$$\mathbf{p}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{p}_2 = \begin{pmatrix} 1 \\ -(l+r) \\ 1 \end{pmatrix}, \quad \mathbf{p}_3 = \begin{pmatrix} \sqrt{r(l+r+1)/l} \\ 0 \\ -\sqrt{l(l+r+1)/r} \end{pmatrix}$$

then

$$\begin{aligned} LR\mathbf{p}_1 &= \mathbf{p}_1, \\ LR\mathbf{p}_2 &= \cos 2\phi\ \mathbf{p}_2 + \sin 2\phi\ \mathbf{p}_3, \\ LR\mathbf{p}_3 &= -\sin 2\phi\ \mathbf{p}_2 + \cos 2\phi\ \mathbf{p}_3. \end{aligned}$$

Thus, if P is the matrix whose columns are $\{\mathbf{p}_1,\ \mathbf{p}_2,\ \mathbf{p}_3\}$ and M is the matrix

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\phi & -\sin 2\phi \\ 0 & \sin 2\phi & \cos 2\phi \end{pmatrix}$$

then

$$LR = PMP^{-1}.$$

It follows that

$$\mathbf{v}_{2n} = (LR)^n \mathbf{v}_0 = PM^nP^{-1}\mathbf{v}_0 = PM^nP^{-1} \begin{pmatrix} 0 \\ v \\ 0 \end{pmatrix}$$

and

$$\mathbf{v}_{2n+1} = R\mathbf{v}_{2n}.$$

It is clear that we need only the central column of P^{-1}, and that is easily verified to be

$$\begin{pmatrix} 1/(l+r+1) \\ -1/(l+r+1) \\ 0 \end{pmatrix}.$$

It follows that

$$P^{-1}\mathbf{v}_0 = P^{-1} \begin{pmatrix} 0 \\ v \\ 0 \end{pmatrix} = \begin{pmatrix} v/(l+r+1) \\ -v/(l+r+1) \\ 0 \end{pmatrix}$$

and

$$M^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2n\phi & -\sin 2n\phi \\ 0 & \sin 2n\phi & \cos 2n\phi \end{pmatrix}$$

so

$$M^nP^{-1}\mathbf{v}_0 = \begin{pmatrix} \dfrac{v}{l+r+1} \\ -\dfrac{v}{l+r+1} \cos 2n\phi \\ -\dfrac{v}{l+r+1} \sin 2n\phi \end{pmatrix}$$

and

$$\mathbf{v}_{2n} = PM^nP^{-1}\mathbf{v}_0 = \begin{pmatrix} \frac{v}{l+r+1} - \frac{v}{l+r+1}\Big(\cos 2n\phi + \sqrt{\frac{r(l+r+1)}{l}}\sin 2n\phi\Big) \\ \frac{v}{l+r+1} + \frac{(l+r)v}{l+r+1}\cos 2n\phi \\ \frac{v}{l+r+1} + \frac{v}{l+r+1}\Big(\sqrt{\frac{l(l+r+1)}{r}}\sin 2n\phi - \cos 2n\phi\Big) \end{pmatrix}.$$

If we write

$$\overline{v} = \frac{v}{l+r+1} = \frac{m_2 v}{m_1+m_2+m_3}$$

then $\overline{v}$ is the velocity of the centre of mass of the system (this velocity is constant), and

(1)

$$\begin{aligned} (v_1)_{2n} &= \overline{v}\left\{1 - \left(\cos 2n\phi + \sqrt{\frac{r(l+r+1)}{l}}\sin 2n\phi\right)\right\} \\ (v_2)_{2n} &= \overline{v}\Big\{1 + (l+r)\cos 2n\phi\Big\} \\ (v_3)_{2n} &= \overline{v}\left\{1 + \left(\sqrt{\frac{l(l+r+1)}{r}}\sin 2n\phi - \cos 2n\phi\right)\right\} \end{aligned}$$

and from $\mathbf{v}_{2n+1} = R\mathbf{v}_{2n}$,

$$\begin{aligned} (v_1)_{2n+1} &= \overline{v}\left\{1 - \left(\cos 2n\phi + \sqrt{\frac{r(l+r+1)}{l}}\sin 2n\phi\right)\right\} \\ (v_2)_{2n+1} &= \overline{v}\left\{1 - \left(\frac{lr+r^2+r-l}{r+1}\cos 2n\phi - \frac{2\sqrt{lr(l+r+1)}}{r+1}\sin 2n\phi\right)\right\} \\ (v_3)_{2n+1} &= \overline{v}\left\{1 + \left(\frac{2l+r+1}{r+1}\cos 2n\phi + \frac{r-1}{r+1}\sqrt{\frac{l(l+r+1)}{r}}\sin 2n\phi\right)\right\}. \end{aligned}$$

These results can be recast as

$$\begin{aligned} (v_1)_{2n} &= \overline{v}\left\{1 - \sqrt{\frac{(l+r)(r+1)}{l}}\sin(2n\phi + \epsilon)\right\} \\ (v_2)_{2n} &= \overline{v}\Big\{1 + (l+r)\cos 2n\phi\Big\} \end{aligned}$$

$$(v_3)_{2n} = \overline{v}\left\{1 + \sqrt{\frac{(l+1)(l+r)}{r}}\sin\left(2n\phi - \epsilon'\right)\right\}$$

$$(v_1)_{2n+1} = \overline{v}\left\{1 - \sqrt{\frac{(l+r)(r+1)}{l}}\sin\left(2n\phi + \epsilon\right)\right\}$$

$$(v_2)_{2n+1} = \overline{v}\Big\{1 - (l+r)\cos\left(2n\phi + \epsilon''\right)\Big\}$$

$$(v_3)_{2n+1} = \overline{v}\left\{1 + \sqrt{\frac{(l+1)(l+r)}{r}}\sin\left(2n\phi + \epsilon'''\right)\right\}$$

where

$$\cos\epsilon = \sqrt{\frac{r(l+r+1)}{(l+r)(r+1)}}, \quad \sin\epsilon = \sqrt{\frac{l}{(l+r)(r+1)}},$$

$$\cos\epsilon' = \sqrt{\frac{l(l+r+1)}{(l+1)(l+r)}}, \quad \sin\epsilon' = \sqrt{\frac{r}{(l+1)(l+r)}},$$

$$\cos\epsilon'' = \frac{lr + r^2 + r - l}{(l+r)(r+1)}, \quad \sin\epsilon'' = \frac{2\sqrt{lr(l+r+1)}}{(l+r)(r+1)},$$

$$\cos\epsilon''' = \frac{r-1}{r+1}\sqrt{\frac{l(l+r+1)}{(l+1)(l+r)}}, \quad \sin\epsilon''' = \frac{2l+r+1}{r+1}\sqrt{\frac{r}{(l+1)(l+r)}}.$$

and

$$\cos\phi = \sqrt{\frac{lr}{(l+1)(r+1)}}, \quad \sin\phi = \sqrt{\frac{l+r+1}{(l+1)(r+1)}}.$$

Expansion of $\cos 2\epsilon,\ \sin 2\epsilon,\ \cos(\phi+\epsilon),\ \sin(\phi+\epsilon),\ \cos(\epsilon+\epsilon'),\ \sin(\epsilon+\epsilon')$ shows that

$$\epsilon'' = 2\epsilon, \quad \epsilon''' = \phi + \epsilon, \quad \phi = \epsilon + \epsilon'.$$

Thus, the results may be recast solely in terms of $\epsilon,\ \epsilon'$ as follows.

(2)

If n is even,

$$(v_1)_n = \overline{v}\left\{1 - \frac{\sin\left((n+1)\epsilon + n\epsilon'\right)}{\sin\epsilon}\right\}$$

$$(v_2)_n = \overline{v}\left\{1 + \frac{\cos(\epsilon+\epsilon')\cos(n\epsilon+n\epsilon')}{\sin\epsilon\sin\epsilon'}\right\}$$
$$(v_3)_n = \overline{v}\left\{1 + \frac{\sin(n\epsilon+(n-1)\epsilon')}{\sin\epsilon'}\right\}$$

while if n is odd,

$$(v_1)_n = \overline{v}\left\{1 - \frac{\sin(n\epsilon+(n-1)\epsilon')}{\sin\epsilon}\right\}$$
$$(v_2)_n = \overline{v}\left\{1 - \frac{\cos(\epsilon+\epsilon')\cos((n+1)\epsilon+(n-1)\epsilon')}{\sin\epsilon\sin\epsilon'}\right\}$$
$$(v_3)_n = \overline{v}\left\{1 + \frac{\sin((n+1)\epsilon+n\epsilon')}{\sin\epsilon'}\right\}$$

where

$$\overline{v} = v\tan\epsilon\tan\epsilon'.$$

Now define "catch-up" velocities δ_n by

$$\delta_{2n+1} = (v_2)_{2n} - (v_3)_{2n}, \quad \delta_{2n+2} = (v_1)_{2n+1} - (v_2)_{2n+1}.$$

From the formulae (1)

$$\begin{aligned}
\delta_{2n+1} &= v\left\{\cos 2n\phi - \sqrt{\frac{l}{r(l+r+1)}}\sin 2n\phi\right\} \\
&= v\sqrt{\frac{(l+r)(r+1)}{r(l+r+1)}}\cos(2n\phi+\epsilon) \\
&= v\frac{\cos((2n+1)\epsilon+2n\epsilon')}{\cos\epsilon},
\end{aligned}$$

$$\begin{aligned}
\delta_{2n+2} &= v\left\{\frac{r-1}{r+1}\cos 2n\phi - (2l+r+1)\sqrt{\frac{r(l+r+1)}{l}}\sin 2n\phi\right\} \\
&= v\sqrt{\frac{(l+1)(l+r)}{l(l+r+1)}}\cos(2n\phi+\epsilon''') \\
&= v\frac{\cos((2n+2)\epsilon+(2n+1)\epsilon')}{\cos\epsilon'}.
\end{aligned}$$

Thus

$$\delta_n = C_n v \cos(n\epsilon + (n-1)\epsilon')$$

where

$$C_n = \begin{cases} \sec\epsilon \text{ if } n \text{ is odd} \\ \sec\epsilon' \text{ if } n \text{ is even.} \end{cases}$$

The number N of collisions is given by

$$\delta_1, \cdots, \delta_N > 0, \quad \delta_{N+1} \le 0,$$

that is,

$$\cos\epsilon, \ \cdots, \ \cos(N\epsilon + (N-1)\epsilon') > 0, \quad \cos((N+1)\epsilon + N\epsilon') \le 0,$$

$$N\epsilon + (N-1)\epsilon' < \frac{\pi}{2}, \quad (N+1)\epsilon + N\epsilon' \ge \frac{\pi}{2},$$

$$(N-1)(\epsilon + \epsilon') < \frac{\pi}{2} - \epsilon, \quad N(\epsilon + \epsilon') \ge \frac{\pi}{2} - \epsilon,$$

$$N - 1 < (\frac{\pi}{2} - \epsilon)/(\epsilon + \epsilon') \le N,$$

$$\begin{aligned} N &= \lceil (\frac{\pi}{2} - \epsilon)/(\epsilon + \epsilon') \rceil \\ &= \left\lceil \tan^{-1}\sqrt{\frac{r(l+r+1)}{l}} \Big/ \tan^{-1}\sqrt{\frac{(l+r+1)}{lr}} \right\rceil \end{aligned}$$

and the final velocities are given by (2) with $n = N$.

3. Two numerical examples

If $l = 64, \ r = 16$, then

$$\epsilon = \tan^{-1}(\frac{2}{9}) \approx 0.218669, \quad \epsilon' = \tan^{-1}(\frac{1}{18}) \approx 0.055499$$

$$N = \left\lceil (\frac{\pi}{2} - \epsilon)/(\epsilon + \epsilon') \right\rceil = 5$$

$$\overline{v} = \frac{1}{81} v$$

$$(v_1)_5 = \frac{1}{81}v\left\{1 - \frac{\sin(5\epsilon + 4\epsilon')}{\sin\epsilon}\right\} \approx -0.042718v,$$
$$(v_2)_5 = \frac{1}{81}v\left\{1 - \frac{\cos(\epsilon + \epsilon')\cos(6\epsilon + 4\epsilon')}{\sin\epsilon\sin\epsilon'}\right\} \approx -0.023981v,$$
$$(v_3)_5 = \frac{1}{81}v\left\{1 + \frac{\sin(6\epsilon + 5\epsilon')}{\sin\epsilon'}\right\} \approx 0.234872v.$$

The exact values for the final velocities, calculated from

$$\mathbf{v}_5 = RLRLR\mathbf{v}_0$$

are

$$(v_1)_5 = -\frac{177344}{4151485}v, \quad (v_2)_5 = -\frac{99555}{4151485}v, \quad (v_3)_5 = \frac{975066}{4151485}v.$$

If $l = 16,\ r = 64$, then

$$\epsilon = \tan^{-1}(\frac{1}{18}) \approx 0.055499, \quad \epsilon' = \tan^{-1}(\frac{2}{9}) \approx 0.218669$$

$$N = \lceil(\frac{\pi}{2} - \epsilon)/(\epsilon + \epsilon')\rceil = 6$$

$$\overline{v} = \frac{1}{81}v$$

$$(v_1)_6 = \frac{1}{81}v\left\{1 - \frac{\sin(7\epsilon + 6\epsilon')}{\sin\epsilon}\right\} \approx -0.208350v,$$
$$(v_2)_6 = \frac{1}{81}v\left\{1 + \frac{\cos(\epsilon + \epsilon')\cos(6\epsilon + 6\epsilon')}{\sin\epsilon\sin\epsilon'}\right\} \approx -0.060879v,$$
$$(v_3)_6 = \frac{1}{81}v\left\{1 + \frac{\sin(6\epsilon + 5\epsilon')}{\sin\epsilon'}\right\} \approx 0.068664v.$$

The exact values are

$$(v_1)_6 = -\frac{281112126}{1349232625}v, \ (v_2)_6 = -\frac{82140335}{1349232625}v, \ (v_3)_6 = \frac{92643234}{1349232625}v.$$

4. Asymptotic analysis

We can show that if $l,\ r \geq 1$, that is, $m_1,\ m_3 \geq m_2$, then

$$(v_1)_N = -v\sqrt{\frac{m_2 m_3}{m_1(m_1+m_3)}}\left\{1 - \frac{13}{8}\theta_1\left(\sqrt{\frac{m_2}{m_1}} + \sqrt{\frac{m_2}{m_3}}\right)\right\},$$
$$(v_3)_N = v\sqrt{\frac{m_1 m_2}{m_3(m_1+m_3)}}\left\{1 + \frac{3}{2}\theta_2\left(\sqrt{\frac{m_2}{m_1}} + \sqrt{\frac{m_2}{m_3}}\right)\right\}$$

where $-\frac{4}{13} < \theta_1 < 1, \quad -\frac{5}{12} < \theta_2 < 1.$

We have

$$(v_1)_N = -T + \overline{v}$$

where

$$T = \overline{v}\frac{\sin((N+1)\epsilon + N\epsilon')}{\sin\epsilon} \quad \text{or} \quad \overline{v}\frac{\sin(N\epsilon + (N-1)\epsilon'}{\sin\epsilon}.$$

Now,

$$N\epsilon + (N-1)\epsilon' < \frac{\pi}{2} \leq (N+1)\epsilon + N\epsilon'.$$

So

$$\frac{\pi}{2} \leq (N+1)\epsilon + N\epsilon' < \frac{\pi}{2} + (\epsilon+\epsilon') \quad \text{and} \quad \frac{\pi}{2} - (\epsilon+\epsilon') \leq N\epsilon + (N-1)\epsilon' < \frac{\pi}{2}.$$

It follows that

$$1 \geq \sin((N+1)\epsilon + N\epsilon') > \cos(\epsilon+\epsilon') \quad \text{and} \quad \cos(\epsilon+\epsilon') \leq \sin(N\epsilon + (N-1)\epsilon') < 1.$$

Thus

$$\overline{v}\frac{\cos(\epsilon+\epsilon')}{\sin\epsilon} \leq T \leq \frac{\overline{v}}{\sin\epsilon}.$$

Now,

$$\begin{aligned}
\frac{\overline{v}}{\sin\epsilon} &= \overline{v}\sqrt{\frac{lr + r^2 + r + l}{l}} \\
&= \overline{v}\sqrt{\frac{r(l+r+1)}{l}}\sqrt{1 + \frac{l}{r(l+r+1)}} \\
&= v\sqrt{\frac{r}{l(l+r+1)}}\sqrt{1 + \frac{l}{r(l+r+1)}}
\end{aligned}$$

$$= v\sqrt{\frac{r}{l(l+r)}}\sqrt{1+\frac{l}{r(l+r+1)}}\Big/\sqrt{1+\frac{1}{l+r}}$$

$$= v\sqrt{\frac{r}{l(l+r)}}\left(1+\frac{1}{2}\theta_3\Big(\frac{1}{r}\Big)\right)\left(1-\frac{1}{2}\theta_4\frac{1}{l+r}\right), \qquad 0<\theta_3<1,\ 0<\theta_4<1$$

$$= v\sqrt{\frac{r}{l(l+r)}}\left(1+\frac{1}{2}\theta_5\Big(\frac{1}{l}+\frac{1}{r}\Big)\right)\left(1-\frac{1}{8}\theta_6\Big(\frac{1}{l}+\frac{1}{r}\Big)\right), \qquad 0<\theta_5<1,\ 0<\theta_6<1$$

$$= v\sqrt{\frac{r}{l(l+r)}}\left(1+\frac{1}{2}\theta_7\Big(\frac{1}{l}+\frac{1}{r}\Big)\right), \qquad -\frac{1}{4}<\theta_7<1.$$

Also,

$$\cos\phi = \frac{1}{\sqrt{1+\frac{1}{l}}}\frac{1}{\sqrt{1+\frac{1}{r}}}$$

$$= \left(1-\frac{1}{2}\theta_8\Big(\frac{1}{l}\Big)\right)\left(1-\frac{1}{2}\theta_9\Big(\frac{1}{r}\Big)\right), \qquad 0<\theta_8<1,\ 0<\theta_9<1$$

$$= \left(1-\frac{1}{2}\theta_{10}\Big(\frac{1}{l}+\frac{1}{r}\Big)\right), \qquad 0<\theta_{10}<1.$$

So

$$\overline{v}\frac{\cos\phi}{\sin\epsilon} = v\sqrt{\frac{r}{l(l+r)}}\left(1+\frac{1}{2}\theta_7\Big(\frac{1}{l}+\frac{1}{r}\Big)\right)\left(1-\frac{1}{2}\theta_{10}\Big(\frac{1}{l}+\frac{1}{r}\Big)\right)$$

$$= v\sqrt{\frac{r}{l(l+r)}}\left(1-\frac{5}{8}\theta_{11}\Big(\frac{1}{l}+\frac{1}{r}\Big)\right), \qquad -\frac{4}{5}<\theta_{11}<1.$$

It follows that

$$T = v\sqrt{\frac{r}{l(l+r)}}\left(1-\frac{5}{8}\theta_{12}\Big(\frac{1}{l}+\frac{1}{r}\Big)\right), \qquad -\frac{4}{5}<\theta_{12}<1.$$

Also,

$$\overline{v} = \frac{v}{l+r+1} = v\sqrt{\frac{r}{l(l+r+1)}}\sqrt{\frac{l}{r(l+r+1)}}$$

$$= v\sqrt{\frac{r}{l(l+r)}}\sqrt{\frac{l+r}{l+r+1}}\theta_{13}\Big(\frac{1}{\sqrt{r}}\Big), \qquad 0<\theta_{13}<1$$

$$= v\sqrt{\frac{r}{l(l+r)}}\theta_{14}\Big(\frac{1}{\sqrt{l}}+\frac{1}{\sqrt{r}}\Big), \qquad 0<\theta_{14}<1.$$

Therefore

$$
\begin{aligned}
(v_1)_N &= -T + \overline{v} = -v\sqrt{\frac{r}{l(l+r)}}\left(1 - \theta_{14}\Big(\frac{1}{\sqrt{l}} + \frac{1}{\sqrt{r}}\Big) - \frac{5}{8}\theta_{12}\Big(\frac{1}{l} + \frac{1}{r}\Big)\right) \\
&= -v\sqrt{\frac{r}{l(l+r)}}\left(1 - \theta_{14}\Big(\frac{1}{\sqrt{l}} + \frac{1}{\sqrt{r}}\Big) - \frac{5}{8}\theta_{15}\Big(\frac{1}{\sqrt{l}} + \frac{1}{\sqrt{r}}\Big)\right), \qquad -\frac{4}{5} < \theta_{15} < 1 \\
&= -v\sqrt{\frac{r}{l(l+r)}}\left(1 - \frac{13}{8}\theta_{16}\Big(\frac{1}{\sqrt{l}} + \frac{1}{\sqrt{r}}\Big)\right), \qquad -\frac{4}{13} < \theta_{16} < 1,
\end{aligned}
$$

as claimed. The proof of the result for $(v_3)_N$ is similar. ■