57. THE PIZZA THEOREM

JEREMY, MICHAEL, JEREMY K., ANDREW AND PHILIP HIRSCHHORN

We think the following important fact deserves wider appreciation.

Theorem. If a circular pizza is cut into $4n$ slices by $2n$ concurrent cuts (which run right across the pizza) at equal angles to each other, and n people share the pizza by taking every n'th slice (thus receiving four slices each) then they receive equal shares.

Proof. Let $\alpha = \dfrac{\pi}{2n}$ and let $r(\theta)$ be the distance from P, the point of concurrency, to the edge of the pizza.
Then the k'th person's share is

$$\int_{(k-1)\alpha}^{k\alpha} \frac{1}{2}\left(r(\theta)^2 + r(\theta + \frac{\pi}{2})^2 + r(\theta+\pi)^2 + r(\theta + \frac{3\pi}{2})^2 \right) d\theta.$$

But, as we shall show, the integrand is simply

$$2R^2$$

where R is the radius of the pizza.
So the k'th person's share is

$$\int_{(k-1)\alpha}^{k\alpha} 2R^2 \; d\theta = 2R^2\alpha = \pi R^2/n,$$

or one n'th of the pizza. ■

To show that the integrand is $2R^2$, let $r(\theta) = a,\ r(\theta+\dfrac{\pi}{2}) = b,\ r(\theta+\pi) = c,\ r(\theta+\dfrac{3\pi}{2}) = d.$ (See figure 1.)

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Then, as can be seen from figure 2 with PA=a, PB=b, PC=c, PD=d, we have both

$$R^2 = \mathrm{OD}^2 = \mathrm{OM}^2 + \mathrm{MD}^2 = \mathrm{NP}^2 + \mathrm{MD}^2 = \left(\frac{|a-c|}{2}\right)^2 + \left(\frac{b+d}{2}\right)^2$$

and

$$R^2 = \mathrm{OC}^2 = \mathrm{ON}^2 + \mathrm{NC}^2 = \mathrm{MP}^2 + \mathrm{NC}^2 = \left(\frac{|b-d|}{2}\right)^2 + \left(\frac{a+c}{2}\right)^2.$$

Adding these gives

$$2R^2 = \frac{1}{2}\left(a^2 + b^2 + c^2 + d^2\right),$$

as claimed.

Corollary. If the pizza has one or more toppings, each covering a circular region, the regions not necessarily concentric, and the point of concurrency of the cuts lies inside all the regions, then every person receives equal shares of each topping, and of the crust.

(See figure 3.)