An amazing identity of Ramanujan

In the so-called "lost notebook" of Ramanujan ([1], p. 341), one finds the following amazing statement:

If

$$\frac{1+53x+9x^2}{1-82x-82x^2+x^3} = \sum_{n\geq 0} a_n x^n,$$

$$\frac{2-26x-12x^2}{1-82x-82x^2+x^3} = \sum_{n\geq 0} b_n x^n$$

and

$$\frac{2+8x-10x^2}{1-82x-82x^2+x^3} = \sum_{n\geq 0} c_n x^n,$$

then

$$a_n^3 + b_n^3 = c_n^3 + (-1)^n.$$

What is amazing about this result is not only that it is true, but that anyone could think of it at all. The purpose of this note is twofold: to show that the result is true, and to show how Ramanujan may have found it.

It is easy enough to show via partial fractions that

$$a_n = \frac{1}{85}\left\{(64+8\sqrt{85})\alpha^n + (64-8\sqrt{85})\beta^n - 43(-1)^n\right\},$$

$$b_n = \frac{1}{85}\left\{(77+7\sqrt{85})\alpha^n + (77-7\sqrt{85})\beta^n + 16(-1)^n\right\},$$

$$c_n = \frac{1}{85}\left\{(93+9\sqrt{85})\alpha^n + (93-9\sqrt{85})\beta^n - 16(-1)^n\right\},$$

where $\alpha = \dfrac{83+9\sqrt{85}}{2}$, $\beta = \dfrac{83-9\sqrt{85}}{2}$.

It follows that

$$a_n^3 = \frac{1}{85^3}\Big\{(1306624+141824\sqrt{85})\alpha^{3n} + (1306624-141824\sqrt{85})\beta^{3n}$$

$$-(1230144+132096\sqrt{85})(-\alpha^2)^n-(1230144-132096\sqrt{85})(-\beta^2)^n$$
$$+(96960+12120\sqrt{85})\alpha^n+(96960-12120\sqrt{85})\beta^n+267245(-1)^n\Big\},$$

$$b_n^3=\frac{1}{85^3}\Big\{(1418648+153664\sqrt{85})\alpha^{3n}+(1418648-153664\sqrt{85})\beta^{3n}$$
$$+(484512+51744\sqrt{85}(-\alpha^2)^n+(484512-51744\sqrt{85})(-\beta^2)^n$$
$$+(466620+42420\sqrt{85})\alpha^n+(466620-42420\sqrt{85})\beta^n+173440(-1)^n\Big\},$$

$$c_n^3=\frac{1}{85^3}\Big\{(2725272+295488\sqrt{85})\alpha^{3n}+(2725272-295488\sqrt{85})\beta^{3n}$$
$$-(745632+80352\sqrt{85})(-\alpha^2)^n-(745632-80352\sqrt{85})(-\beta^2)^n$$
$$+(563580+54540\sqrt{85})\alpha^n+(563580-54540\sqrt{85})\beta^n-173440(-1)^n\Big\},$$

and $a_n^3+b_n^3=c_n^3+(-1)^n$.

How did Ramanujan discover the result? I believe he did the following.

He started with an identity such as

$$(A^2+7AB-9B^2)^3+(2A^2-4AB+12B^2)^3$$
$$=(2A^2+10B^2)^3+(A^2-9AB-B^2)^3.$$

Several similar identities appear in Ramanujan's work.) Now define the sequence $\{h_n\}$ by

$$h_0=0,\ h_1=1,\ h_{n+2}=9h_{n+1}+h_n.$$

Then

$$h_{n+1}^2-h_{n+2}h_n=(-1)^n.$$

Set

$$A=h_{n+1},\ B=h_n.$$

Then

$$A^2-9AB-B^2=h_{n+1}^2-h_n(9h_{n+1}+h_n)$$
$$=h_{n+1}^2-h_nh_{n+2}=(-1)^n.$$

Let

$$\begin{aligned}
a_n &= A^2 + 7AB - 9B^2 = h_{n+1}^2 + 7h_{n+1}h_n - 9h_n^2, \\
b_n &= 2A^2 - 4AB + 12B^2 = 2h_{n+1}^2 - 4h_{n+1}h_n + 12h_n^2, \text{ and} \\
c_n &= 2A^2 + 10B^2 = 2h_{n+1}^2 + 10h_n^2.
\end{aligned}$$

Then

$$a_n^3 + b_n^3 = c_n^3 + (-1)^n.$$

Now it can easily be shown that

$$h_n = \frac{1}{\sqrt{85}} \left\{ \left(\frac{9+\sqrt{85}}{2} \right)^n - \left(\frac{9-\sqrt{85}}{2} \right)^n \right\},$$

and hence that

$$h_n^2 = \frac{1}{85} \left\{ \left(\frac{83+9\sqrt{85}}{2} \right)^n + \left(\frac{83-9\sqrt{85}}{2} \right)^n - 2(-1)^n \right\},$$

$$h_{n+1}^2 = \frac{1}{85} \left\{ \left(\frac{83+9\sqrt{85}}{2} \right)^{n+1} + \left(\frac{83-9\sqrt{85}}{2} \right)^{n+1} + 2(-1)^n \right\},$$

and

$$h_n h_{n+1} = \frac{1}{85} \left\{ \left(\frac{9+\sqrt{85}}{2} \right) \left(\frac{83+9\sqrt{85}}{2} \right)^n + \left(\frac{9-\sqrt{85}}{2} \right) \left(\frac{83-9\sqrt{85}}{2} \right)^n - 9(-1)^n \right\}.$$

It follows that

$$\sum_{n \geq 0} h_n^2 x^n = \frac{x - x^2}{1 - 82x - 82x^2 + x^3},$$

$$\sum_{n \geq 0} h_{n+1}^2 x^n = \frac{1 - x}{1 - 82x - 82x^2 + x^3},$$

and

$$\sum_{n\geq 0} h_n h_{n+1} x^n = \frac{9x}{1-82x-82x^2+x^3},$$

and hence that

$$\sum_{n\geq 0} a_n x^n = \frac{(1-x)+7(9x)-9(x-x^2)}{1-82x-82x^2+x^3}$$
$$= \frac{1+53x+9x^2}{1-82x-82x^2+x^3},$$

$$\sum_{n\geq 0} b_n x^n = \frac{2(1-x)-4(9x)+12(x-x^2)}{1-82x-82x^2+x^3}$$
$$= \frac{2-26x-12x^2}{1-82x-82x^2+x^3},$$

and

$$\sum_{n\geq 0} c_n x^n = \frac{2(1-x)+10(x-x^2)}{1-82x-82x^2+x^3}$$
$$= \frac{2+8x-10x^2}{1-82x-82x^2+x^3},$$

as stated by Ramanujan.

REFERENCE

S. Ramanujan Aiyangar, *The Lost Notebook and Other Unpublished Papers*, New Delhi, Narosa, 1988.