The Ramanujan Journal manuscript No.
(will be inserted by the editor)

Factorizations of certain q-series identities of Ramanujan and others

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Received: date / Accepted: date

Dedicated to my good friends Mourad Ismail and Dennis Stanton

Abstract B. Yuttanan has shown that certain identities of Ramanujan and of Horie and Kanou can be factorized, that is, obtained by multiplying together certain pairs of identities. We give alternative, simpler factorizations of the same identities, and more.

Keywords Factorizations of q-series identities

1 Introduction

I am motivated to write this paper by Boonrod Yuttanan's paper [3]. Most, but not all, of our results are versions of results in [3], but written very differently.

Ramanujan stated [2], p. 229, that for

$$v = q^{\frac{1}{2}} \begin{pmatrix} q, \ q^7 \\ q^3, \ q^5 \end{pmatrix}; q^8 \Big)_\infty, \tag{1.1}$$

$$\frac{1}{v} - v = \frac{\phi(q^2)}{q^{\frac{1}{2}}\psi(q^4)}, \tag{1.2}$$

$$\frac{1}{v} + v = \frac{\phi(q)}{q^{\frac{1}{2}}\psi(q^4)}. \tag{1.3}$$

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It is an immediate consequence of (1.2) and (1.3) that

$$\frac{1}{v^2} - 6 + v^2 = 2\left(\frac{1}{v} - v\right)^2 - \left(\frac{1}{v} + v\right)^2 = \frac{2\phi(q^2)^2 - \phi(q)^2}{q\psi(q^4)^2} = \frac{\phi(-q)^2}{q\psi(q^4)^2}, \tag{1.4}$$

a result given in Horie and Kanou [1], as part of Theorem 2.2.

Further, from (1.2) we deduce

$$\frac{1}{v} + 2 - v = \frac{\phi(q^2) + 2q^{\frac{1}{2}}\psi(q^4)}{q^{\frac{1}{2}}\psi(q^4)} = \frac{\phi(q^{\frac{1}{2}})}{q^{\frac{1}{2}}\psi(q^4)}, \tag{1.5}$$

$$\frac{1}{v} - 2 - v = \frac{\phi(q^2) - 2q^{\frac{1}{2}}\psi(q^4)}{q^{\frac{1}{2}}\psi(q^4)} = \frac{\phi(-q^{\frac{1}{2}})}{q^{\frac{1}{4}}\psi(q^4)}, \tag{1.6}$$

of which the product is (1.4). The pair (1.5), (1.6) is said to be a “factorization” of (1.4).

Another factorization of (1.4), which does not have its counterpart in [3], is

$$\frac{1}{v} - 2\sqrt{2} + v = \frac{\phi(-q)}{q^{\frac{1}{2}}\psi(q^4)} \prod_{n \text{ odd}} \frac{1 - \sqrt{2}\, q^{\frac{n}{2}} + q^n}{1 + \sqrt{2}\, q^{\frac{n}{2}} + q^n}, \tag{1.7}$$

$$\frac{1}{v} + 2\sqrt{2} + v = \frac{\phi(-q)}{q^{\frac{1}{2}}\psi(q^4)} \prod_{n \text{ odd}} \frac{1 + \sqrt{2}\, q^{\frac{n}{2}} + q^n}{1 - \sqrt{2}\, q^{\frac{n}{2}} + q^n}. \tag{1.8}$$

In this note, we obtain factorizations of all of (1.2), (1.3), (1.5), (1.6), (1.7) and (1.8). We prove

$$\frac{1}{\sqrt{v}} + \sqrt{v} = \sqrt{\frac{\phi(q^2)}{q^{\frac{1}{2}}\psi(q^4)} \left(\begin{matrix} -q^{\frac{1}{2}}, q^{\frac{3}{2}}, q^{\frac{5}{2}}, -q^{\frac{7}{2}} \\ q^{\frac{1}{2}}, -q^{\frac{3}{2}}, -q^{\frac{5}{2}}, q^{\frac{7}{2}} \end{matrix}; q^4\right)_\infty}, \tag{1.9}$$

$$\frac{1}{\sqrt{v}} - \sqrt{v} = \sqrt{\frac{\phi(q^2)}{q^{\frac{1}{2}}\psi(q^4)} \left(\begin{matrix} q^{\frac{1}{2}}, -q^{\frac{3}{2}}, -q^{\frac{5}{2}}, q^{\frac{7}{2}} \\ -q^{\frac{1}{2}}, q^{\frac{3}{2}}, q^{\frac{5}{2}}, -q^{\frac{7}{2}} \end{matrix}; q^4\right)_\infty}, \tag{1.10}$$

$$\frac{1}{\sqrt{v}} + i\sqrt{v} = \sqrt{\frac{\phi(q)}{q^{\frac{1}{2}}\psi(q^4)} \left(\begin{matrix} -iq^{\frac{1}{2}}, iq^{\frac{3}{2}} \\ iq^{\frac{1}{2}}, -iq^{\frac{3}{2}} \end{matrix}; q^2\right)_\infty}, \tag{1.11}$$

$$\frac{1}{\sqrt{v}} - i\sqrt{v} = \sqrt{\frac{\phi(q)}{q^{\frac{1}{2}}\psi(q^4)} \left(\begin{matrix} iq^{\frac{1}{2}}, -iq^{\frac{3}{2}} \\ -iq^{\frac{1}{2}}, iq^{\frac{3}{2}} \end{matrix}; q^2\right)_\infty}, \tag{1.12}$$

$$\frac{1}{\sqrt{v}} - (\sqrt{2}+1)\sqrt{v} = \sqrt{\frac{\phi(-q^{\frac{1}{2}})}{q^{\frac{1}{2}}\psi(q^4)} \prod_{n \geq 1} \frac{1 - \sqrt{2}\, q^{\frac{n}{2}} + q^n}{1 + \sqrt{2}\, q^{\frac{n}{2}} + q^n}}, \tag{1.13}$$

$$\frac{1}{\sqrt{v}} + (\sqrt{2}-1)\sqrt{v} = \sqrt{\frac{\phi(-q^{\frac{1}{2}})}{q^{\frac{1}{2}}\psi(q^4)} \prod_{n \geq 1} \frac{1 + \sqrt{2}\, q^{\frac{n}{2}} + q^n}{1 - \sqrt{2}\, q^{\frac{n}{2}} + q^n}}, \tag{1.14}$$

$$\frac{1}{\sqrt{v}} + (\sqrt{2}+1)\sqrt{v} = \sqrt{\frac{\phi(q^{\frac{1}{2}})}{q^{\frac{1}{2}}\psi(q^4)} \prod_{n \text{ odd}} \frac{1+\sqrt{2}\,q^{\frac{n}{2}}+q^n}{1-\sqrt{2}\,q^{\frac{n}{2}}+q^n} \prod_{n \text{ even}} \frac{1-\sqrt{2}\,q^{\frac{n}{2}}+q^n}{1+\sqrt{2}\,q^{\frac{n}{2}}+q^n}}, \tag{1.15}$$

$$\frac{1}{\sqrt{v}} - (\sqrt{2}-1)\sqrt{v} = \sqrt{\frac{\phi(q^{\frac{1}{2}})}{q^{\frac{1}{2}}\psi(q^4)} \prod_{n \text{ odd}} \frac{1-\sqrt{2}\,q^{\frac{n}{2}}+q^n}{1+\sqrt{2}\,q^{\frac{n}{2}}+q^n} \prod_{n \text{ even}} \frac{1+\sqrt{2}\,q^{\frac{n}{2}}+q^n}{1-\sqrt{2}\,q^{\frac{n}{2}}+q^n}}. \tag{1.16}$$

All the factorizations are clear, except that (1.7) is the product of (1.13) and (1.16), (1.8) the product of (1.14) and (1.15).

2 Proofs

We shall make use of the following elementary lemma.
Lemma 1.

$$If\ xy = P\ and\ x/y = Q\ then\ x = \sqrt{PQ}\ and\ y = \sqrt{P/Q}.$$

We shall also make frequent use of Jacobi's triple product identity

$$(a^{-1}q, aq, q^2; q^2)_\infty = \sum_{-\infty}^{\infty} (-1)^n a^n q^{n^2}. \tag{2.1}$$

Proofs of (1.9), (1.10).

We start by observing that

$$\begin{aligned}
(-q, q^3, q^5, -q^7, q^8, q^8; q^8)_\infty &= \sum_{r,s=-\infty}^{\infty} (-1)^s q^{4r^2-3r+4s^2-s} \\
&= \sum_{r\equiv s \pmod 2} (-1)^s q^{4r^2-3r+4s^2-s} + \sum_{r\not\equiv s \pmod 2} (-1)^s q^{4r^2-3r+4s^2-s} \\
&= \sum_{t,u=-\infty}^{\infty} (-1)^{t-u} q^{4(t+u)^2-3(t+u)+4(t-u)^2-(t-u)} \\
&\quad + \sum_{t.u=-\infty}^{\infty} (-1)^{t-u} q^{4(t+u+1)^2-3(t+u+1)+4(t-u)^2-(t-u)}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{t,u=-\infty}^{\infty} (-1)^{t+u} q^{8t^2-4t+8u^2-2u} + \sum_{t,u=-\infty}^{\infty} (-1)^{t+u} q^{8t^2+4t+8u^2+6u+1} \\
&= (q^4, q^6, q^{10}, q^{12}, q^{16}, q^{16}; q^{16})_\infty + q(q^2, q^4, q^{12}, q^{14}, q^{16}, q^{16}; q^{16})_\infty \\
&= (q^4, q^{12}, q^{16}; q^{16})_\infty \Big((q^6, q^{10}, q^{16}; q^{16})_\infty + q(q^2, q^{14}, q^{16}; q^{16})_\infty \Big) \\
&= \frac{(q^4;q^4)_\infty (q^{16};q^{16})_\infty^2}{(q^8;q^8)_\infty} \Big((q^6, q^{10}; q^{16})_\infty + q(q^2, q^{14}; q^{16})_\infty \Big).
\end{aligned}
$$

If we put $-q$ for q, we obtain

$$
\begin{aligned}
&(q, -q^3, -q^5, q^7, q^8, q^8; q^8)_\infty \\
&\qquad = \frac{(q^4;q^4)_\infty (q^{16};q^{16})_\infty^2}{(q^8;q^8)_\infty} \Big((q^6, q^{10}; q^{16})_\infty - q(q^2, q^{14}; q^{16})_\infty \Big).
\end{aligned}
$$

It follows that

$$
(q^3, q^5; q^8)_\infty + q^{\frac{1}{2}} (q, q^7; q^8)_\infty = \frac{(q^4;q^4)_\infty^3}{(q^2;q^2)_\infty (q^8;q^8)_\infty^2} (-q^{\frac{1}{2}}, q^{\frac{3}{2}}, q^{\frac{5}{2}}, -q^{\frac{7}{2}}; q^4)_\infty,
$$

$$
(q^3, q^5; q^8)_\infty - q^{\frac{1}{2}} (q, q^7; q^8)_\infty = \frac{(q^4;q^4)_\infty^3}{(q^2;q^2)_\infty (q^8;q^8)_\infty^2} (q^{\frac{1}{2}}, -q^{\frac{3}{2}}, -q^{\frac{5}{2}}, q^{\frac{7}{2}}; q^4)_\infty,
$$

$$
(q^3, q^5; q^8)_\infty^2 - q(q, q^7; q^8)_\infty^2 = \frac{(q^4;q^4)_\infty^6}{(q^2;q^2)_\infty^2 (q^8;q^8)_\infty^4} (q, q^3, q^5, q^7; q^8)_\infty
$$

and

$$
\frac{1}{q^{\frac{1}{2}}} \left(\begin{matrix} q^3, q^5 \\ q, q^7 \end{matrix} ; q^8 \right)_\infty - q^{\frac{1}{2}} \left(\begin{matrix} q, q^7 \\ q^3, q^5 \end{matrix} ; q^8 \right)_\infty = \frac{(q^4;q^4)_\infty^6}{q^{\frac{1}{2}} (q^2;q^2)_\infty^2 (q^8;q^8)_\infty^4} = \frac{\phi(q^2)}{q^{\frac{1}{2}} \psi(q^4)}.
$$

That is,

$$
\left(\frac{1}{\sqrt{v}} + \sqrt{v} \right) \left(\frac{1}{\sqrt{v}} - \sqrt{v} \right) = \frac{1}{v} - v = \frac{\phi(q^2)}{q^{\frac{1}{2}} \psi(q^4)}, \tag{2.2}
$$

while

$$
\begin{aligned}
\left(\frac{1}{\sqrt{v}} + \sqrt{v} \right) \Big/ \left(\frac{1}{\sqrt{v}} - \sqrt{v} \right) &= \frac{1+v}{1-v} = \frac{(q^3, q^5; q^8)_\infty + q^{\frac{1}{2}} (q, q^7; q^8)_\infty}{(q^3, q^5; q^8)_\infty - q^{\frac{1}{2}} (q, q^7; q^8)_\infty} \\
&= \left(\begin{matrix} -q^{\frac{1}{2}}, q^{\frac{3}{2}}, q^{\frac{5}{2}}, -q^{\frac{7}{2}} \\ q^{\frac{1}{2}}, -q^{\frac{3}{2}}, -q^{\frac{5}{2}}, q^{\frac{7}{2}} \end{matrix} ; q^4 \right)_\infty . \qquad (2.3)
\end{aligned}
$$

The results (1.9) and (1.10) now follow from (2.2) and (2.3) by Lemma 1.

Proofs of (1.11), (1.12).

If in Jacobi's triple product identity (2.1) we put q^2 for q, aq for a, we find

$$(a^{-1}q, aq^3, q^4; q^4)_\infty = \sum_{-\infty}^{\infty} (-1)^n a^n q^{2n^2+n}.$$

If we now set $a = i$, we obtain

$$\begin{aligned}(-iq, iq^3, q^4; q^4)_\infty &= \sum_{-\infty}^{\infty} (-1)^n i^n q^{2n^2+n} \\ &= \sum_{-\infty}^{\infty} (-1)^n q^{8n^2+2n} + i \sum_{-\infty}^{\infty} (-1)^n q^{8n^2+6n+1} \\ &= (q^6, q^{10}, q^{16}; q^{16})_\infty + iq(q^2, q^{14}, q^{16}; q^{16})_\infty. \quad (2.4)\end{aligned}$$

If we put $-q$ for q (alternatively $-i$ for i) in (2.4), we obtain

$$(iq, -iq^3, q^4; q^4)_\infty = (q^6, q^{10}, q^{16}; q^{16})_\infty - iq(q^2, q^{14}, q^{16}; q^{16})_\infty.$$

Thus we have

$$\begin{aligned}(q^3, q^5; q^8)_\infty + iq^{\frac12}(q, q^7; q^8)_\infty &= \frac{(q^2; q^2)_\infty}{(q^8; q^8)_\infty}(-iq^{\frac12}, iq^{\frac32}; q^2)_\infty, \\ (q^3, q^5; q^8)_\infty - iq^{\frac12}(q, q^7; q^8)_\infty &= \frac{(q^2; q^2)_\infty}{(q^8; q^8)_\infty}(iq^{\frac12}, -iq^{\frac32}; q^2)_\infty, \\ (q^3, q^5; q^8)^2_\infty + q(q, q^7; q^8)^2_\infty &= \frac{(q^2; q^2)^2_\infty}{(q^8; q^8)^2_\infty}(-q, -q^3; q^4)_\infty \\ &= \frac{(q^2; q^2)^4_\infty}{(q; q)_\infty (q^4; q^4)_\infty (q^8; q^8)^2_\infty}, \\ \frac{1}{q^{\frac12}}\left(\begin{matrix} q^3, q^5 \\ q, q^7 \end{matrix}; q^8\right)_\infty + q^{\frac12}\left(\begin{matrix} q, q^7 \\ q^3, q^5 \end{matrix}; q^8\right)_\infty &= \frac{(q^2; q^2)^5_\infty}{(q; q)^2_\infty (q^4; q^4)_\infty (q^8; q^8)^2_\infty} \\ &= \frac{\phi(q)}{q^{\frac12}\psi(q^4)}.\end{aligned}$$

Thus,

$$\left(\frac{1}{\sqrt{v}} + i\sqrt{v}\right)\left(\frac{1}{\sqrt{v}} - i\sqrt{v}\right) = \frac{1}{v} + v = \frac{\phi(q)}{q^{\frac12}\psi(q^4)} \quad (2.5)$$

and

$$\begin{aligned}\left(\frac{1}{\sqrt{v}} + i\sqrt{v}\right)\Big/\left(\frac{1}{\sqrt{v}} - i\sqrt{v}\right) = \frac{1+iv}{1-iv} &= \frac{(q^3, q^5; q^8)_\infty + iq^{\frac12}(q, q^7; q^8)_\infty}{(q^3, q^5; q^8)_\infty - iq^{\frac12}(q, q^7; q^8)_\infty} \\ &= \left(\begin{matrix} -iq^{\frac12}, iq^{\frac32} \\ iq^{\frac12}, -iq^{\frac32} \end{matrix}; q^2\right)_\infty. \quad (2.6)\end{aligned}$$

The results (1.11) and (1.12) now follow from (2.5) and (2.6) by Lemma 1.

Proofs of (1.13), (1.14), (1.15), (1.16).

If in Jacobi's triple product identity (2.1) we put aq for a, then replace q by $q^{\frac{1}{2}}$, we obtain

$$\begin{aligned}(1-a^{-1})(a^{-1}q,aq,q;q)_\infty &= \sum_{-\infty}^{\infty}(-1)^n a^n q^{(n^2+n)/2} \\ &= 1-a^{-1}+\sum_{n\geq 1}(-1)^n(a^n-a^{-n-1})q^{(n^2+n)/2}.\end{aligned}$$

If $a\neq 1$ and we divide by $1-a^{-1}$, we obtain

$$\begin{aligned}(a^{-1}q,aq,q;q)_\infty &= 1+\sum_{n\geq 1}\left(\frac{a^n-a^{-n-1}}{a-a^{-1}}\right)q^{(n^2+n)/2} \\ &= 1+\sum_{n\geq 1}\left(\frac{a^{n+\frac{1}{2}}-a^{-n-\frac{1}{2}}}{a^{\frac{1}{2}}-a^{-\frac{1}{2}}}\right)q^{(n^2+n)/2}.\end{aligned}$$

For $a=\exp\{2i\theta\}$, $\theta\neq k\pi$, this becomes

$$\prod_{n\geq 1}(1-2\cos 2\theta\, q^n+q^{2n})(1-q^n)=\sum_{n\geq 0}(-1)^n\frac{\sin{(2n+1)\theta}}{\sin\theta}\,q^{(n^2+n)/2}. \quad (2.7)$$

In particular, if we put $\theta=\dfrac{\pi}{8}$ in (2.7), we obtain

$$\begin{aligned}\prod_{n\geq 1}(1-\sqrt{2}\,q^n+q^{2n})(1-q^n) &= 1-(\sqrt{2}+1)q+(\sqrt{2}+1)q^3-q^6-q^{10}+\ \cdots \\ &= (1-q^6-q^{10}+\ \cdots\ )-(\sqrt{2}+1)q(1-q^2-q^{14}+\ \cdots\ ) \\ &= (q^6,q^{10},q^{16};q^{16})_\infty-(\sqrt{2}+1)q(q^2,q^{14},q^{16};q^{16})_\infty,\end{aligned} \quad (2.8)$$

and if we put $\theta=\dfrac{3\pi}{8}$ in (2.7), or simply change $\sqrt{2}$ to $-\sqrt{2}$ in (2.8), we obtain

$$\begin{aligned}&\prod_{n\geq 1}(1+\sqrt{2}\,q^n+q^{2n})(1-q^n) \\ &\qquad = (q^6,q^{10},q^{16};q^{16})_\infty+(\sqrt{2}-1)q(q^2,q^{14},q^{16};q^{16})_\infty.\end{aligned} \quad (2.9)$$

It follows that

$$(q^3,q^5;q^8)_\infty - (\sqrt{2}+1)q^{\frac12}(q,q^7;q^8)_\infty = \frac{(q^{\frac12};q^{\frac12})_\infty}{(q^8;q^8)_\infty}\prod_{n\ge1}(1-\sqrt{2}\,q^{\frac{n}{2}}+q^n),$$

$$(q^3,q^5;q^8)_\infty + (\sqrt{2}-1)q^{\frac12}(q,q^7;q^8)_\infty = \frac{(q^{\frac12};q^{\frac12})_\infty}{(q^8;q^8)_\infty}\prod_{n\ge1}(1+\sqrt{2}\,q^{\frac{n}{2}}+q^n),$$

$$\begin{aligned}(q^3,q^5;q^8)_\infty^2 - 2q^{\frac12}(q,q^3,q^5,q^7;q^8)_\infty - q(q,q^7;q^8)_\infty^2 &\\ = \frac{(q^{\frac12};q^{\frac12})_\infty^2}{(q^8;q^8)_\infty^2}\prod_{n\ge1}(1+q^{2n}) &\\ = \frac{(q^{\frac12};q^{\frac12})_\infty^2(q^4;q^4)_\infty}{(q^2;q^2)_\infty(q^8;q^8)_\infty^2}, &\end{aligned}$$

$$\frac{1}{q^{\frac12}}\left(\begin{matrix}q^3,q^5\\ q,\,q^7\end{matrix};q^8\right)_\infty - 2 - q^{\frac12}\left(\begin{matrix}q,\,q^7\\ q^3,q^5\end{matrix};q^8\right)_\infty = \frac{(q^{\frac12};q^{\frac12})_\infty^2(q^4;q^4)_\infty}{q^{\frac12}(q;q)_\infty(q^8;q^8)_\infty^2} = \frac{\phi(-q^{\frac12})}{q^{\frac12}\psi(q^4)}.$$

Thus we have

$$\left(\frac{1}{\sqrt{v}}-(\sqrt{2}+1)\sqrt{v}\right)\left(\frac{1}{\sqrt{v}}+(\sqrt{2}-1)\sqrt{v}\right) = \frac1v - 2 - v = \frac{\phi(-q^{\frac12})}{q^{\frac12}\psi(q^4)} \tag{2.10}$$

and

$$\begin{aligned}\left(\frac{1}{\sqrt{v}}-(\sqrt{2}+1)\sqrt{v}\right)\Big/\left(\frac{1}{\sqrt{v}}+(\sqrt{2}-1)\sqrt{v}\right) &= \frac{1-(\sqrt{2}+1)v}{1+(\sqrt{2}+1)v}\\ &= \frac{(q^3,q^5;q^8)_\infty - (\sqrt{2}+1)q^{\frac12}(q,q^7;q^8)_\infty}{(q^3,q^5;q^8)_\infty + (\sqrt{2}-1)q^{\frac12}(q,q^7;q^8)_\infty}\\ &= \prod_{n\ge1}\frac{1-\sqrt{2}\,q^{\frac{n}{2}}+q^n}{1+\sqrt{2}\,q^{\frac{n}{2}}+q^n}.\end{aligned} \tag{2.11}$$

The results (1.13) and (1.14) now follow from (2.10) and (2.11) by Lemma 1.

If we put $-q$ for q in (2.8) and (2.9), we obtain

$$\begin{aligned}\prod_{n\ \text{odd}}(1+\sqrt{2}\,q^n+q^{2n})(1+q^n)\prod_{n\ \text{even}}(1-\sqrt{2}\,q^n+q^{2n})(1-q^n)&\\ = (q^6,q^{10},q^{16};q^{16})_\infty + (\sqrt{2}+1)q(q^2,q^{14},q^{16};q^{16})_\infty&\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}\prod_{n\ \text{odd}}(1-\sqrt{2}\,q^n+q^{2n})(1+q^n)\prod_{n\ \text{even}}(1+\sqrt{2}\,q^n+q^{2n})(1-q^n)&\\ = (q^6,q^{10},q^{16};q^{16})_\infty - (\sqrt{2}-1)q(q^2,q^{14},q^{16};q^{16})_\infty.&\end{aligned} \tag{2.13}$$

The results (1.15) and (1.16) follow from (2.12) and (2.13) as did (1.13) and (1.14) from (2.10) and (2.11).

3 Remark

The identities (1.9) and (1.10) can be written

$$\frac{1}{\sqrt{v}} + \sqrt{v} = \sqrt{\frac{\phi(q^2)}{q^{\frac{1}{2}}\psi(q^4)}\frac{v(-q^{\frac{1}{2}})}{v(q^{\frac{1}{2}})}}, \qquad \frac{1}{\sqrt{v}} - \sqrt{v} = \sqrt{\frac{\phi(q^2)}{q^{\frac{1}{2}}\psi(q^4)}\frac{v(q^{\frac{1}{2}})}{v(-q^{\frac{1}{2}})}}.$$

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AMS Classification Numbers: 11P83