

ESTIMATING THE APÉRY NUMBERS II

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ABSTRACT. We describe a method to find the complete asymptotic expansion of the n th Apéry number, and find the first few terms.

1. INTRODUCTION

In 1983, Apéry stunned the mathematical world by proving that $\zeta(3)$ is irrational. Alf van der Poorten gave an entertaining account of Apéry's presentation [3].

Apéry's proof involves the eponymous numbers,

$$A_n = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2,$$

that satisfy the recurrence

$$(n+1)^3 A_{n+1} - (34n^3 + 51n^2 + 27n + 5)A_n + n^3 A_{n-1} = 0$$

together with $A_0 = 1$, $A_1 = 5$.

In a previous note [1] I demonstrated how one can find the dominant term in the asymptotic expansion of A_n . Indeed, I found that

$$A_n \sim C n^{-\frac{3}{2}} \alpha^n \text{ as } n \rightarrow \infty,$$

where

$$C = \frac{3 + 2\sqrt{2}}{2^{\frac{9}{4}} \pi^{\frac{3}{2}}} \text{ and } \alpha = 17 + 12\sqrt{2}.$$

In this note I show how one can use the recurrence given above to find as many terms of the asymptotic expansion of A_n as one might want.

Indeed, I show that

$$A_n \sim C n^{-\frac{3}{2}} \alpha^n \left(1 + \frac{c_1}{n} + \frac{c_2}{n^2} \dots \right) \text{ as } n \rightarrow \infty$$

where

$$c_1 = \frac{15\sqrt{2} - 48}{64}, \quad c_2 = \frac{2057 - 1200\sqrt{2}}{4096}, \quad c_3 = \frac{62917\sqrt{2} - 87024}{262144},$$

and so on.

Note that the same method can be used to find the asymptotic expansion of a large class of recurrent sequences, even if one doesn't have the solution in closed form!

2. THE CALCULATION

We have the recurrence

$$(n+1)^3 A_{n+1} - (34n^3 + 51n^2 + 27n + 5)A_n + n^3 A_{n-1} = 0.$$

If we divide by n^3 , we find

$$\left(1 + \frac{3}{n} + \frac{3}{n^2} + \frac{1}{n^3}\right) A_{n+1} - \left(34 + \frac{51}{n} + \frac{27}{n^2} + \frac{5}{n^3}\right) A_n + A_{n-1} = 0.$$

Now suppose

$$A_n = Cn^{-\frac{3}{2}}\alpha^n \left(1 + \frac{c_1}{n} + \frac{c_2}{n^2} + \dots\right).$$

Then

$$\begin{aligned} A_{n+1} &= C(n+1)^{-\frac{3}{2}}\alpha^{n+1} \left(1 + \frac{c_1}{n+1} + \frac{c_2}{(n+1)^2} + \dots\right) \\ &= Cn^{-\frac{3}{2}} \left(1 + \frac{1}{n}\right)^{-\frac{3}{2}} \alpha^{n+1} \left(1 + \frac{c_1}{n} \left(1 + \frac{1}{n}\right)^{-1} + \frac{c_2}{n^2} \left(1 + \frac{1}{n}\right)^{-2} + \dots\right) \\ &= Cn^{-\frac{3}{2}}\alpha^{n+1} \left(1 - \frac{3}{2n} + \frac{15}{8n^2} - \frac{35}{16n^3} + \dots\right) \\ &\quad \times \left(1 + \frac{c_1}{n} + \frac{c_2 - c_1}{n^2} + \frac{c_3 - 2c_2 + c_1}{n^3} + \dots\right), \end{aligned}$$

while

$$\begin{aligned} A_{n-1} &= C(n-1)^{-\frac{3}{2}}\alpha^{n-1} \left(1 + \frac{c_1}{n-1} + \frac{c_2}{(n-1)^2} + \dots\right) \\ &= Cn^{-\frac{3}{2}} \left(1 - \frac{1}{n}\right)^{-\frac{3}{2}} \alpha^{n-1} \left(1 + \frac{c_1}{n} \left(1 - \frac{1}{n}\right)^{-1} + \frac{c_2}{n^2} \left(1 - \frac{1}{n}\right)^{-2} + \dots\right) \\ &= Cn^{-\frac{3}{2}}\alpha^{n-1} \left(1 + \frac{3}{2n} + \frac{15}{8n^2} + \frac{35}{16n^3} + \dots\right) \\ &\quad \times \left(1 + \frac{c_1}{n} + \frac{c_2 + c_1}{n^2} + \frac{c_3 + 2c_2 + c_1}{n^3} + \dots\right). \end{aligned}$$

If we substitute these results into the recurrence relation and divide by $Cn^{-\frac{3}{2}}\alpha^n$, we obtain

$$\begin{aligned} &\alpha \left(1 + \frac{3}{n} + \frac{3}{n^2} + \frac{1}{n^3}\right) \left(1 - \frac{3}{2n} + \frac{15}{8n^2} - \frac{35}{16n^3} + \dots\right) \\ &\quad \times \left(1 + \frac{c_1}{n} + \frac{c_2 - c_1}{n^2} + \frac{c_3 - 2c_2 + c_1}{n^3} + \dots\right) \\ &\quad - \left(34 + \frac{51}{n} + \frac{27}{n^2} + \frac{5}{n^3}\right) \left(1 + \frac{c_1}{n} + \frac{c_2}{n^2} + \frac{c_3}{n^3} + \dots\right) \\ &\quad + \alpha^{-1} \left(1 + \frac{3}{2n} + \frac{15}{8n^2} + \frac{35}{16n^3} + \dots\right) \left(1 + \frac{c_1}{n} + \frac{c_2 + c_1}{n^2} + \frac{c_3 + 2c_2 + c_1}{n^3} + \dots\right) \\ &= 0, \end{aligned}$$

or,

$$\begin{aligned} & \alpha \left(1 + \frac{3}{2n} + \frac{3}{8n^2} - \frac{1}{16n^3} + \dots \right) \left(1 + \frac{c_1}{n} + \frac{c_2 - c_1}{n^2} + \frac{c_3 - 2c_2 + c_1}{n^3} + \dots \right) \\ & - \left(34 + \frac{51}{n} + \frac{27}{n^2} + \frac{5}{n^3} \right) \left(1 + \frac{c_1}{n} + \frac{c_2}{n^2} + \frac{c_3}{n^3} + \dots \right) \\ & + \alpha^{-1} \left(1 + \frac{3}{2n} + \frac{15}{8n^2} + \frac{35}{16n^3} + \dots \right) \left(1 + \frac{c_1}{n} + \frac{c_2 + c_1}{n^2} + \frac{c_3 + 2c_2 + c_1}{n^3} + \dots \right) \\ & = 0. \end{aligned}$$

If we now compare coefficients, we obtain the equations

$$\begin{aligned} & \alpha - 34 + \alpha^{-1} = 0, \\ & \alpha \left(\frac{3}{2} + c_1 \right) - (51 + 34c_1) + \alpha^{-1} \left(\frac{3}{2} + c_1 \right) = 0, \\ & \alpha \left(\frac{3}{8} + \frac{3}{2}c_1 + (c_2 - c_1) \right) - (27 + 51c_1 + 34c_2) + \alpha^{-1} \left(\frac{15}{8} + \frac{3}{2}c_1 + (c_2 + c_1) \right) = 0, \\ & \alpha \left(-\frac{1}{16} + \frac{3}{8}c_1 + \frac{3}{2}(c_2 - c_1) + (c_3 - 2c_2 + c_1) \right) - (5 + 27c_1 + 51c_2 + 34c_3) \\ & \quad + \alpha^{-1} \left(\frac{35}{16} + \frac{15}{8}c_1 + \frac{3}{2}(c_2 + c_1) + (c_3 + 2c_2 + c_1) \right) = 0, \\ & \alpha \left(\frac{3}{128} - \frac{1}{16}c_1 + \frac{3}{8}(c_2 - c_1) + \frac{3}{2}(c_3 - 2c_2 + c_1) + (c_4 - 3c_3 + 3c_2 - c_1) \right) \\ & \quad - (5c_1 + 27c_2 + 51c_3 + 34c_4) \\ & \quad + \alpha^{-1} \left(\frac{315}{128} + \frac{35}{16}c_1 + \frac{15}{8}(c_2 + c_1) + \frac{3}{2}(c_3 + 2c_2 + c_1) + (c_4 + 3c_3 + 3c_2 + c_1) \right) \\ & = 0, \end{aligned}$$

and so on.

The solutions of these are

$$c_1 = \frac{15\sqrt{2} - 48}{64}, \quad c_2 = \frac{2057 - 1200\sqrt{2}}{4096}, \quad c_3 = \frac{62917\sqrt{2} - 87024}{262144},$$

and so on. Thus we have the result stated earlier.

Note added: Shalosh B. Ekhad and Doron Zeilberger [2] have automated this whole process (apart from the exact determination of C).

REFERENCES

- [1] M. D. Hirschhorn, Estimating the Apéry numbers, *The Fibonacci Quarterly*, **50(2)** (2012) 129–131.
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