A naive proof that $F_{5n} \equiv 0 \pmod 5$

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Introduction

We give a new and simple proof of the fact that

$$F_{5n} \equiv 0 \pmod 5,$$

as well as the facts that

$$F_{5n+1} \equiv F_{5n+2} \equiv F_{n+1} + 2F_n = F_{n+2} + F_n,$$
$$F_{5n+3} \equiv -F_{5n+4} \equiv 2F_{n+1} - F_n = F_{n+1} + F_{n-1}.$$

Proofs

We have

$$\begin{aligned}
\sum_{n\geq 0} F_n x^n &= \frac{x}{1-x-x^2} \\
&= \frac{x(1-x-x^2)^4}{(1-x-x^2)^5} \\
&= \frac{x(1-4x+2x^2+8x^3-5x^4-8x^5+2x^6+4x^7+x^8)}{1-5x+5x^2+10x^3-15x^4-11x^5+15x^6+10x^7-5x^8-5x^9-x^{10})} \\
&\equiv \frac{x+x^2+2x^3-2x^4+2x^6+2x^7-x^8+x^9}{1-x^5-x^{10}} \pmod 5.
\end{aligned}$$

It follows that, modulo 5,

$$\begin{aligned}
\sum_{n\geq 0} F_{5n+1} x^n &\equiv \frac{1+2x}{1-x-x^2}, \\
\sum_{n\geq 0} F_{5n+2} x^n &\equiv \frac{1+2x}{1-x-x^2}, \\
\sum_{n\geq 0} F_{5n+3} x^n &\equiv \frac{2-x}{1-x-x^2}, \\
\sum_{n\geq 0} F_{5n+4} x^n &\equiv \frac{-2+x}{1-x-x^2} \\
\text{and } \sum_{n\geq 0} F_{5n} x^n &\equiv 0.
\end{aligned}$$

It follows that, modulo 5,

$$F_{5n} \equiv 0 \tag{1}$$

and

$$\begin{aligned} F_{5n+1} \equiv F_{5n+2} &\equiv F_{n+1} + 2F_n = F_{n+2} + F_n, \\ F_{5n+3} \equiv -F_{5n+4} &\equiv 2F_{n+1} - F_n = F_{n+1} + F_{n-1}. \end{aligned} \tag{2}$$

Comments

The ususal proof of (1) goes as follows.

We have

$$\begin{aligned} \sum_{n\geq 0} F_n x^n &= \frac{x}{1-x-x^2} \\ &= \frac{x(1+x+2x^2+3x^3+5x^4-3x^5+2x^6-x^7+x^8)}{1-11x^5-x^{10}} \end{aligned} \tag{3}$$

so

$$\begin{aligned} \sum_{n\geq 0} F_{5n+1} x^n &= \frac{1-3x}{1-11x-x^2}, \\ \sum_{n\geq 0} F_{5n+2} x^n &= \frac{1+2x}{1-11x-x^2} \\ \sum_{n\geq 0} F_{5n+3} x^n &= \frac{2-x}{1-11x-x^2}, \\ \sum_{n\geq 0} F_{5n+4} x^n &= \frac{3+x}{1-11x-x^2} \\ \text{and} \quad \sum_{n\geq 0} F_{5n} x^n &= \frac{5x}{1-11x-x^2}. \end{aligned}$$

The result (1) follows directly, but (2) does not.

The result (3) can be checked by "cross–multiplication", but is obtained using fifth roots of unity. Thus we have, with $\eta^5 = 1, \quad \eta \neq 1$,

$$\begin{aligned} &\frac{x}{1-x-x^2} \\ &= \frac{x(1-\eta x-\eta^2x^2)(1-\eta^2x-\eta^4x^2)(1-\eta^3x-\eta^6x^2)(1-\eta^4x-\eta^8x^2)}{(1-x-x^2)(1-\eta x-\eta^2x^2)(1-\eta^2x-\eta^4x^2)(1-\eta^3x-\eta^6x^2)(1-\eta^4x-\eta^8x^2)} \\ &= \frac{x(1+x+2x^2+3x^3+5x^4-3x^5+2x^6-x^7+x^8)}{1-11x^5-x^{10}}. \end{aligned}$$

The proofs presented in this paper are more naive, in that they do not require reference to complex numbers.