141. More on π

Abstract

We give simple proofs that $\pi < \frac{355}{113}$ and $3\frac{10}{71} < \pi < 3\frac{1}{7}$.

It has been known since Archimedes (c. 250BC) that π is roughly $\dfrac{22}{7}$, and that $\pi < \dfrac{22}{7}$. A really neat proof of these facts was found, perhaps by Kurt Mahler in the 1960's,* and that is

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx = \frac{22}{7} - \pi. \tag{$*$}$$

It is clear that the integral is positive, and since the denominator of the integrand is at least 1, the integral is less than $\dfrac{1}{630}$.

It has been known since Zhu Chongzhi (5th C.) that an approximation to π, better than $\dfrac{22}{7}$, is $\dfrac{355}{113}$, which also is in excess. (π and $\dfrac{355}{113}$ agree to 6 decimals.)

In a recent article, Stephen Lucas sought a simple integral which is obviously positive, and whose value is $\dfrac{355}{113} - \pi$. Perhaps the nicest he came up with is

$$\int_0^1 \frac{x^8(1-x)^8(25+816x^2)}{3164(1+x^2)}\,dx = \frac{355}{113} - \pi.$$

I noticed that the idea involved in $(*)$ can be extended to prove not only the desired result, but more.

Consider

$$\int_0^1 \frac{x^{4n}(1-x)^4}{1+x^2}\,dx.$$

To evaluate this integral, we use partial fractions:

$$\frac{x^{4n}(1-x)^4}{1+x^2} = x^{4n+2} - 4x^{4n+1} + 5x^{4n} - 4x^{4n-2} + 4x^{4n-4} - + \;\cdots\; + 4 - \frac{4}{1+x^2}.$$

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It follows that

$$\int_0^1 \frac{x^{4n}(1-x)^4}{1+x^2}\,dx = 4\left(1-\frac{1}{3}+\frac{1}{5}-+\ \cdots\ -\frac{1}{4n-1}\right)+\frac{5}{4n+1}-\frac{4}{4n+2}+\frac{1}{4n+3}-\pi$$
$$= \text{rational} - \pi,$$

where the denominator of the rational is (before any cancellation) divisible by all primes up to $4n+3$.

Thus, for example,

$$\int_0^1 \frac{x^{112}(1-x)^4}{1+x^2}\,dx = \frac{P}{113Q} - \pi,$$

where, using my trusty computer I find

$$P = 46922045053712930642150903262788670879977081355826$$
$$Q = 1321747859964364573442354864845520400714361911 75,$$
$$P < 355Q = 46922049028734942357203597702015974225359847867125$$

and

$$0 < \int_0^1 \frac{x^{112}(1-x)^4}{1+x^2}\,dx < \frac{355}{113} - \pi,$$

which gives

$$\pi < \frac{355}{113}.$$

In similar vein,

$$\int_0^1 \frac{x^{4n+2}(1-x)^4}{1+x^2}\,dx = \pi - \text{rational},$$

where the denominator is (before any cancellation) divisible by all primes up to $4n+5$. In particular,

$$\int_0^1 \frac{x^{70}(1-x)^4}{1+x^2}\,dx = \pi - \frac{R}{71S},$$

where

$$R = 3100473885152861164004910599081,$$
$$S = 13900161851314672380764590350,$$
$$R > 223S = 3099736092843171940910503648050$$

and

$$0 < \int_0^1 \frac{x^{70}(1-x)^4}{1+x^2} < \pi - \frac{223}{71},$$

from which

$$\pi > \frac{223}{71} = 3\frac{10}{71},$$

a fact known to Archimedes.

*This is wrong. It was D. P. Dalzell (1944).

Reference

[1] Stephen K. Lucas, Integral proofs that $355/113 > \pi$, This Gazette, 32 (2005), 263–266.