CHAPTER 5

VECTOR GEOMETRY

1. The dot product, lengths and angles

Given $n$-vectors $a$ and $b$, we define

$$a \cdot b = a^T b = \sum_{i=1}^{n} a_i b_i.$$  

This is known as the dot (or inner) product of $a$ and $b$. It is a number.

**Lemma 30** (The Cauchy–Schwarz inequality). For all vectors $a, b \in \mathbb{R}^n$,

$$|a \cdot b| \leq \left( \sum_{i=1}^{n} a_i^2 \right)^{1/2} \left( \sum_{i=1}^{n} b_i^2 \right)^{1/2}.$$  

**Proof.** Define the quadratic polynomial $Q$ by

$$Q(\lambda) = \sum_{i=1}^{n} (a_i + \lambda b_i)^2$$

$$= \sum_{i=1}^{n} (a_i^2 + 2\lambda a_i b_i + b_i^2)$$

$$= \left( \sum_{i=1}^{n} a_i^2 \right) + 2\lambda \left( \sum_{i=1}^{n} a_i b_i \right) + \lambda^2 \left( \sum_{i=1}^{n} b_i^2 \right),$$

for all $\lambda \in \mathbb{R}$. Clearly $Q \geq 0$, so the discriminant of $Q$ is nonnegative. Thus

$$\left( \sum_{i=1}^{n} a_i b_i \right)^2 \leq \left( \sum_{i=1}^{n} a_i^2 \right) \left( \sum_{i=1}^{n} b_i^2 \right).$$

Taking square roots proves the result. \qed

We define the length $\|v\|$ of a vector $v$ in $\mathbb{R}^n$ by

$$\|v\| = \left( \sum_{i=1}^{n} v_i^2 \right)^{1/2}.$$  

This extends the natural length in $\mathbb{R}^2$ and $\mathbb{R}^3$ to $\mathbb{R}^n$.

**Proposition 31.** Given nonzero vectors $v$ and $w$ in $\mathbb{R}^2$ or $\mathbb{R}^3$, the angle $\theta$ between them is given by

$$\cos \theta = \frac{v \cdot w}{\|v\| \|w\|}.$$
Proof. The vectors \(v, w\) and \(v - w\) form a triangle, with the angle \(\theta\) opposite the third side. Thus, from the cosine rule,

\[
2 \|v\| \|w\| \cos \theta = \|v\|^2 + \|w\|^2 - \|v - w\|^2
\]

\[
= \sum_j v_j^2 + \sum_j w_j^2 - \sum_j (v_j - w_j)^2
\]

\[
= 2 \sum_j v_j w_j.
\]

The right hand side is just \(2v \cdot w\), and the result follows. \(\square\)

We define the angle \(\theta\) between nonzero vectors \(v\) and \(w\) in \(\mathbb{R}^n\) to be

\[
\cos^{-1} \left( \frac{v \cdot w}{\|v\| \|w\|} \right).
\]

Challenge Problem. Suppose that \(M\) is an \(n \times n\) matrix such that \(M^T M = I\). Show that, for nonzero vectors \(v\) and \(w\) in \(\mathbb{R}^n\), the angle between them is equal to the angle between the vectors \(Mv\) and \(Mw\).

Proposition 32. The length has the following properties:

1. \(\|v\| \geq 0\)
2. \(\|v\| = 0 \iff v = 0\).
3. \(\|\lambda v\| = |\lambda| \|v\|\)
4. \(\|v + w\| \leq \|v\| + \|w\|\)

for all vectors \(v\) and \(w\) and scalars \(\lambda\).

Proof. The first three properties are easy to prove. To prove (4), observe that, from the Cauchy–Schwarz inequality,

\[
\|v + w\|^2 = \sum_i (v_i + w_i)^2
\]

\[
= \sum_i v_i^2 + 2 \sum_i v_i w_i + \sum_i w_i^2
\]

\[
= \|v\|^2 + 2v \cdot w + \|w\|^2
\]

\[
\leq \|v\|^2 + 2 \|v\| \|w\| + \|w\|^2
\]

\[
= (\|v\| + \|w\|)^2,
\]

whence \(\|v + w\| \leq \|v\| + \|w\|\). \(\square\)

Proposition 33. The dot product has the following properties:

1. \(\|a\|^2 = a \cdot a\)
2. \((a + b) \cdot (c + d) = a \cdot c + a \cdot d + b \cdot c + b \cdot d\)
3. \((\lambda a) \cdot b = \lambda (a \cdot b) = a \cdot (\lambda b)\)
4. \(|a \cdot b| \leq \|a\| \|b\|\)
5. \(a \cdot b = b \cdot a\)
6. \(a \cdot 0 = 0\)
for all vectors \( \mathbf{a} \) and \( \mathbf{b} \) and all scalars \( \lambda \).

**Proof.** Property 1 holds by definition, and 4 is the Cauchy–Schwarz inequality. Observe that

\[
(a + b) \cdot (c + d) = \sum_i [(a + b)_i (c + d)_i]
= \sum_i (a_i + b_i)(c_i + d_i)
= \sum_i a_i c_i + \sum_i a_i d_i + \sum_i b_i c_i + \sum_i b_i d_i
= a \cdot c + a \cdot d + b \cdot c + b \cdot d,
\]
proving 2. The proofs of 3 and 5 are similar, and the proof of 6 is obvious.

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**2. A geometric application: orthogonality**

We say that vectors \( \mathbf{v} \) and \( \mathbf{w} \) are orthogonal or perpendicular, and write \( \mathbf{v} \perp \mathbf{w} \), if \( \mathbf{v} \cdot \mathbf{w} = 0 \). This is when the angle between them is \( \pi/2 \). An altitude of a triangle is the perpendicular from one vertex to the opposite side.

**Theorem 34.** The altitudes of a triangle are concurrent.

**Proof.** Choose \( D \) on \( BC \) and \( E \) on \( AC \) so that \( AD \perp BC \) and \( BE \perp AC \). Let \( P \) be where these two altitudes meet.

Represent \( \overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC} \) and \( \overrightarrow{OP} \) by \( \mathbf{a}, \mathbf{b}, \mathbf{c} \) and \( \mathbf{p} \).

Since \( AD \perp BC \), we have \( AP \perp BC \), and similarly, \( BP \perp AC = 0 \). Then

\[
(p - a) \cdot (c - b) = 0
(p - b) \cdot (c - a) = 0.
\]

Expand out these expressions:

\[
p \cdot c - p \cdot b - a \cdot c + a \cdot b = 0
p \cdot b - p \cdot a + b \cdot c + b \cdot a = 0.
\]
Subtracting,
\[ p \cdot a - p \cdot b + b \cdot c - a \cdot c = 0 \]
\[ p \cdot (a - b) + c \cdot (b - a) = 0 \]
\[(p - c) \cdot (a - b) = 0 \]
i.e.,
\[ CP \perp BA. \]
It follows that \( CP \) (produced if necessary) is perpendicular to \( BA \), i.e., \( CP \) is the altitude through \( C \). Hence all altitudes meet at \( P \).

Another important idea is projection. We define the projection \( P_b a \) of \( a \) onto \( b \) by
\[ P_b a = \frac{a \cdot b}{b \cdot b} b. \]

**Theorem 35.** Given vectors \( a \) and \( b \), such that \( b \neq 0 \),
\[ \|P_b a - a\| \leq \|\lambda b - a\| \]
for all \( \lambda \) in \( \mathbb{R} \), and
\[ (P_b a - a) \cdot b = 0. \]

**Proof.** We take a vector \( a \) and a nonzero vector \( b \), and find when \( \|\lambda b - a\| \) is minimum.

Actually, \( \|\lambda b - a\| \) is minimum when its square is minimum, so we study
\[ \|\lambda b - a\|^2 = (\lambda b - a) \cdot (\lambda b - a) \]
\[ = \lambda^2 (b \cdot b) - 2\lambda (a \cdot b) + (a \cdot a). \]
This is minimum when
\[ 2\lambda (b \cdot b) - 2(a \cdot b) = 0, \]
i.e., \( \lambda b \cdot b = a \cdot b \). In this case, \( \lambda b = P_b a \), and
\[ (\lambda b - a) \cdot b = \frac{a \cdot b}{b \cdot b} (b \cdot b) - a \cdot b = 0, \]
i.e., \( \lambda b - a \perp b. \)

The theorem tells us that \( P_b a \) is the component of \( a \) in the direction of \( b \), and that \( a - P_b a \) is perpendicular to \( b \), as indicated in the following figure.
Problem 33. Suppose that \( \mathbf{a}, \mathbf{b} \neq 0 \) and that \( P_b \mathbf{a} = P_a \mathbf{b} \). What does this tell us about \( \mathbf{a} \) and \( \mathbf{b} \)?

**Answer.** We are given that

\[
\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}} = \frac{\mathbf{b} \cdot \mathbf{a}}{\mathbf{a} \cdot \mathbf{a}}
\]

so, if \( \mathbf{a} \cdot \mathbf{b} \neq 0 \), then

\[
(\mathbf{a} \cdot \mathbf{a}) \mathbf{b} = (\mathbf{b} \cdot \mathbf{b}) \mathbf{a}.
\]

Take lengths:

\[
\| (\mathbf{a} \cdot \mathbf{a}) \mathbf{b} \| = \mathbf{a} \cdot \mathbf{a} \| \mathbf{b} \| = \| \mathbf{a} \|^2 \| \mathbf{b} \|
\]

and

\[
\| (\mathbf{b} \cdot \mathbf{b}) \mathbf{a} \| = \mathbf{b} \cdot \mathbf{b} \| \mathbf{a} \| = \| \mathbf{b} \|^2 \| \mathbf{a} \|.
\]

Then

\[
\| \mathbf{a} \|^2 \| \mathbf{b} \| = \| \mathbf{b} \|^2 \| \mathbf{a} \|,
\]

so

\[
\| \mathbf{a} \| = \| \mathbf{b} \|
\]

and \( \mathbf{a} \cdot \mathbf{a} = \mathbf{b} \cdot \mathbf{b} \). Thus \( \mathbf{a} \cdot \mathbf{b} = 0 \), or \( \mathbf{a} = \mathbf{b} \). \( \triangle \)

3. The Cross Product

In \( \mathbb{R}^3 \), and only in \( \mathbb{R}^3 \), it is possible to define a product which, given two vectors, produces another vector, and which is mathematically and physically important. The cross (or vector) product of 3-vectors \( \mathbf{v} \) and \( \mathbf{w} \) is defined in several different but equivalent ways, as follows:

1. In coordinates, we write

\[
(v_1, v_2, v_3) \times (w_1, w_2, w_3) = (v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1)
\]

2. Using determinants, we write

\[
\mathbf{v} \times \mathbf{w} = \det \begin{pmatrix} e_1 & e_2 & e_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}
\]
or, alternatively,

\[ \mathbf{v} \times \mathbf{w} = \det \begin{pmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} \]

3. We may write \( \mathbf{v} \) as \( v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k} \), and express \( \mathbf{w} \) similarly, and then

\[ \mathbf{v} \times \mathbf{w} = v_1 w_1 \mathbf{i} \times \mathbf{i} + v_2 w_1 \mathbf{j} \times \mathbf{i} + v_3 w_1 \mathbf{k} \times \mathbf{i} + v_1 w_2 \mathbf{i} \times \mathbf{j} + v_2 w_2 \mathbf{j} \times \mathbf{j} + v_3 w_2 \mathbf{k} \times \mathbf{j} + v_1 w_3 \mathbf{i} \times \mathbf{k} + v_2 w_3 \mathbf{j} \times \mathbf{k} + v_3 w_3 \mathbf{k} \times \mathbf{k}, \]

which we simplify using the following table:

\[
\begin{array}{ccc}
i \times i &=& 0 \\
j \times i &=& -k \\
k \times i &=& j \\
i \times j &=& k \\
j \times j &=& 0 \\
k \times j &=& -i \\
i \times k &=& -j \\
j \times k &=& i \\
k \times k &=& 0
\end{array}
\]

We consider some of the algebraic properties of the cross product.

**Proposition 36.** For all \( \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{R}^3 \) and \( \lambda \in \mathbb{R} \),

1. \( \mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} \)
2. \((\lambda \mathbf{a}) \times \mathbf{b} = \lambda (\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (\lambda \mathbf{b}) \)
3. \((\mathbf{a} + \mathbf{b}) \times (\mathbf{c} + \mathbf{d}) = \mathbf{a} \times \mathbf{c} + \mathbf{a} \times \mathbf{d} + \mathbf{b} \times \mathbf{c} + \mathbf{b} \times \mathbf{d} \)
4. \((\lambda \mathbf{a}) \times \mathbf{a} = 0 \)
5. \( \mathbf{a} \times \mathbf{0} = 0 \times \mathbf{a} = 0 \).

**Proof.** Properties 1, 2 and 3 follow from the definitions: using the determinant form is probably the easiest way. From 1 and 2,

\[(\lambda \mathbf{a}) \times \mathbf{a} = \lambda (\mathbf{a} \times \mathbf{a}) = -\lambda (\mathbf{a} \times \mathbf{a}),\]

whence \((\lambda \mathbf{a}) \times \mathbf{a} = 0\), proving 4. Property 5 follows from 2 by putting \( \lambda = 0 \). \( \square \)

**Proposition 37.** For \( \mathbf{a}, \mathbf{b} \in \mathbb{R}^3 \),

1. \((\mathbf{a} \times \mathbf{b}) \cdot (\lambda \mathbf{a} + \mu \mathbf{b}) = 0 \) for all \( \lambda, \mu \in \mathbb{R} \)
2. \( \|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \|\mathbf{b}\| \sin \theta \), where \( \theta \) is the angle between \( \mathbf{a} \) and \( \mathbf{b} \)
3. the direction of \( \mathbf{a} \times \mathbf{b} \) is given by the right hand rule: if the thumb and first two fingers on a right hand are held perpendicular to each other, and the thumb points in the direction of \( \mathbf{a} \) while the first finger points in the direction of \( \mathbf{b} \), then the second finger points in the direction of \( \mathbf{a} \times \mathbf{b} \).
Proof. First, take \( c \) in \( \mathbb{R}^3 \). Then

\[
(a \times b) \cdot c = \det \begin{pmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} \cdot c
\]

\[
= \left( e_1 \det \begin{pmatrix} a_2 & a_3 \\ b_2 & b_3 \end{pmatrix} - e_2 \det \begin{pmatrix} a_1 & a_3 \\ b_1 & b_3 \end{pmatrix} + e_3 \det \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \right) \cdot c
\]

\[
= c_1 \det \begin{pmatrix} a_2 & a_3 \\ b_2 & b_3 \end{pmatrix} - c_2 \det \begin{pmatrix} a_1 & a_3 \\ b_1 & b_3 \end{pmatrix} + c_3 \det \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}
\]

\[
= \det \begin{pmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}.
\]

If \( c = \lambda a + \mu b \), the first row of this determinant is a linear combination of the other two rows, so the determinant is 0.

To prove the second part, we expand out:

\[
\|a \times b\|^2 + (a \cdot b)^2 = (a_2 b_3 - b_2 a_3)^2 + (a_3 b_1 - a_1 b_3)^2
\]

\[
+ (a_1 b_2 - a_2 b_1)^2 + (a_1 b_1 + a_2 b_2 + a_3 b_3)^2
\]

\[
= \ldots
\]

\[
= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2).
\]

Therefore,

\[
\|a \times b\|^2 = \|a\|^2 \|b\|^2 - (a \cdot b)^2
\]

\[
= \|a\|^2 \|b\|^2 (1 - \cos^2 \theta)
\]

\[
= \|a\|^2 \|b\|^2 \sin^2 \theta.
\]

Now take the square root of both sides.

The last part may be checked with a few examples. A proof can be given, using rotations, but will not be done here.

\[
\square
\]

4. Geometric applications: area and volume

The area of the parallelogram generated by vectors \( a \) and \( b \) may be computed using cross products.
The area is equal to
\[
\|a\| \|b\| \sin \theta = \| a \times b \|.
\]

Three vectors define a parallelepiped.

Its volume is the product of the area of the base and the height, i.e.,
\[
\|a \times b\| \|c\| \cos \phi = (a \times b) \cdot c,
\]
since \( a \times b \) points upwards. If the vectors \( a, b \) and \( c \) are oriented differently, \((a \times b) \cdot c\) may be negative. The volume is then \( |(a \times b) \cdot c| \). It is not necessary to write parentheses in \( a \times b \cdot c \).

We define the *scalar triple product* of vectors \( a, b \) and \( c \in \mathbb{R}^3 \) to be
\[
\det \begin{pmatrix}
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3 \\
c_1 & c_2 & c_3
\end{pmatrix}.
\]
It is equal to \( a \cdot b \times c = a \times b \cdot c = -a \cdot c \times b \), etc. It is the signed volume of a parallelepiped. The scalar triple product is equal to \( \|a\| \|b\| \|c\| \) if and only if \( a, b \) and \( c \) are mutually orthogonal and oriented the right way.

5. Describing planes in \( \mathbb{R}^3 \)

We may use the dot and cross product to describe planes in \( \mathbb{R}^3 \). The *point-normal form* of a plane in \( \mathbb{R}^3 \) is an equation of the form
\[
(x - p) \cdot n = 0,
\]
i.e.,
\[
x \cdot n = p \cdot n,
\]
i.e.,
\[
n_1 x_1 + n_2 x_2 + n_3 x_3 = c,
\]
where \( c = p \cdot n \). The point \( p \) is in the plane, and \( n \) is normal (perpendicular to the plane).
Problem 34. Find a point-normal form for the plane
\[3x_1 - 2x_2 + x_3 = 4.\]

Answer. Take \(n = (3, -2, 1)^T\). The left hand side is then \(x \cdot n\). Find \(p\) such that \(p \cdot n = 4\), for example,
\[p = (1, 0, 1)^T.\]
Then
\[x \cdot n = p \cdot n\]
so
\[(x - p) \cdot n = 0.\]
\(\triangle\)

Problem 35. Suppose that \(p = (2, 0, 1)^T\) and \(n = (0, 3, 0)^T\). Find the Cartesian equation of the plane \((x - p) \cdot n = 0\).

Answer. The point-normal form is
\[x \cdot n = p \cdot n,\]
so
\[3x_2 = 0\]
i.e.,
\[x_2 = 0.\]
\(\triangle\)
We have seen the point-normal form of a plane:

$$(x - p) \cdot n = 0,$$

and how to pass between this and the Cartesian form. It is easy to go between point-normal and parametric form too.

Given the parametric form

$$x = p + \lambda u + \mu v,$$

take $n$ to be $u \times v$, so $n \perp u$ and $n \perp v$. Then

$$n \cdot (x - p) = n \cdot (\lambda u + \mu v) = 0.$$

Conversely, given a point-normal form

$$n \cdot (x - p) = 0,$$

find any nonzero $u$ so that $u \times n = 0$ (i.e., $u \perp n$). Take $v = n \times u$. Then $v$, $n$ and $u$ are mutually orthogonal. The equation of the plane in parametric form is $x = p + \lambda u + \mu v$.

Observe that, if a plane is given by

$$n \cdot (x - p) = 0,$$

i.e.,

$$n \cdot x = n \cdot p,$$

and if $n \cdot p = 0$, then the plane goes through the origin. If $n \cdot p \neq 0$, then

$$\left(\frac{1}{n \cdot p} n\right) \cdot x = 1,$$

i.e., we have written the plane in the form

$$m \cdot x = 1,$$

where $m = (n \cdot p)^{-1} n$. If we have the equation

$$m_1 x_1 + m_2 x_2 + m_3 x_3 = 1,$$

then the intercepts with the axes are $(1/m_1, 0, 0)$, $(0, 1/m_2, 0)$, and $(0, 0, 1/m_3)$. If $m_i = 0$, the plane does not meet the $x_i$ axis, i.e., this axis is parallel to the plane.

### 6. Finding distances

We can use dot and cross products to find distances between geometric objects, such as points, lines and planes. We already know how to find the distance between two points. The following lemma is a key fact.

**Lemma 38.** Given nonzero vectors $v$ and $w$, then

$$\|P_w v\| = \frac{|v \cdot w|}{\|w\|}$$

and

$$\|v - P_w v\| = \frac{\|v \times w\|}{\|w\|}.$$
Proof. Let $\theta$ be the angle between $v$ and $w$. Then

$$\|P_w v\| = \|v\| |\cos \theta|$$

$$= \frac{\|v\| |v \cdot w|}{\|v\| \|w\|}$$

$$= \frac{|v \cdot w|}{\|w\|}.$$ 

Similarly,

$$\|v - P_w v\| = \|v\| \sin \theta$$

$$= \frac{\|v\| |v \times w|}{\|v\| \|w\|}$$

$$= \frac{|v \times w|}{\|w\|},$$

as required. \qed

In the following, $a$ will represent $\overrightarrow{OA}$, $b$ will represent $\overrightarrow{OB}$, etc.

6.1. The distance from a point to a line. Consider the point $P$ and the line $l$, given by $x = a + \lambda d$. 

We can see that
\[
\text{dist}(P, l) = \| \mathbf{p} - \mathbf{a} - P_d(p - a) \| = \left\| \frac{(\mathbf{p} - \mathbf{a}) \times \mathbf{d}}{\| \mathbf{d} \|} \right\|.
\]

6.2. The distance from a point to a plane. Consider the point \( P \) and the plane \( \Pi \), given by \( (\mathbf{x} - \mathbf{a}) \cdot \mathbf{n} = 0 \).

We can see that
\[
\text{dist}(P, \Pi) = \| P_n(p - a) \| = \frac{|(p - a) \cdot \mathbf{n}|}{\| \mathbf{n} \|}.
\]
6.3. The distance between two planes. Consider two planes in $\mathbb{R}^3$: $\Pi$, given by $(x - a) \cdot m = 0$, and $\Sigma$, given by $(x - b) \cdot n = 0$. There are two possibilities: they may be parallel or not. They are parallel if and only if their normal vectors are parallel, i.e., $m$ is a multiple of $n$.

Case 1: The planes are not parallel. In this case, they meet, i.e.,

$$\text{dist}(\Pi, \Sigma) = 0.$$ 

Case 2: The planes are parallel. In this case, the distance between them is the same as the distance between any point, $B$ say, in $\Sigma$ and the plane $\Pi$, i.e.,

$$\text{dist}(\Pi, \Sigma) = \text{dist}(\Pi, B) = \|P_n (b - a)\| = \frac{|(b - a) \cdot n|}{\|n\|}.$$ 

6.4. The distance between a line and a plane. Consider the line $l$, given by $x = a + \lambda d$, and the plane $\Pi$, given by $(x - p) \cdot n = 0$, in $\mathbb{R}^3$. There are two possibilities: they may be parallel or not. The line $l$ is parallel to the plane $\Pi$ if and only if $d \cdot n = 0$.

Case 1: The line and the plane are not parallel. In this case, they meet, i.e.,

$$\text{dist}(l, \Pi) = 0.$$ 

Case 2: The line and the plane are parallel. In this case, the distance between them is the same as the distance from the point $A$ on the line to the plane, i.e.,

$$\text{dist}(l, \Pi) = \text{dist}(A, \Pi) = \|P_n (a - p)\|.$$ 

6.5. The distance between two lines. Consider lines $l$ and $m$ in $\mathbb{R}^3$, given by $x = a + \lambda d$ and $x = b + \mu e$. Again, there are two possibilities: the lines may be parallel or not. Clearly, they are parallel if and only if $d$ is a multiple of $e$.

Case 1: Parallel lines. In this case, the distance between the two lines is equal to the distance between any point, say $B$, on one line and the other line $l$. Then

$$\text{dist}(l, m) = \text{dist}(l, B)$$
$$= \|b - a - P_d (b - a)\|$$
$$= \frac{|(b - a) \times d|}{\|d\|}.$$ 

Case 2: Skew lines. The shortest line segment between the two lines is perpendicular to both, so parallel to $d \times e$. The lines lie in two parallel planes, namely $(x - a) \cdot (d \times e) = 0$ and $(x - b) \cdot (d \times e) = 0$. The distance between the two lines is the same as the distance between these two planes, i.e.,

$$\text{dist}(l, m) = \|P_{d \times e} (a - b)\| = \frac{|(a - b) \cdot d \times e|}{\|d \times e\|}.$$