A Simple Finite Element Code written in Julia

Bill McLean, UNSW

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Purpose

Required for the computational component of an honours course (3 hours/week for 6 weeks) on finite elements.

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Students also use the code in an assignment.

Why Julia?

- Modern alternative to Matlab.
- MIT licence and good documentation.
- Cross-platform: Linux, Windows and OS X.
- Easy, student-friendly syntax.
- Fast execution thanks to LLVM.
- Extensive bindings to standard numerical libraries.

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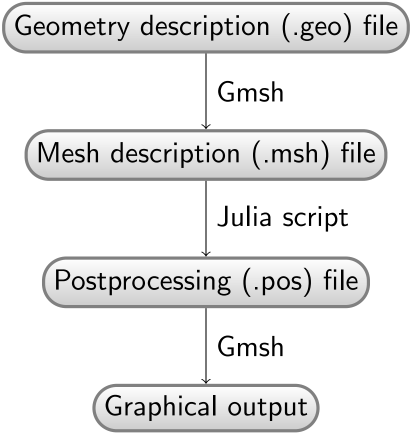
Ported code from previous version written in Python.

Overview of FEM solution process



Gmsh

C. Geuzaine and J.-F. Remacle. Gmsh: a three-dimensional finite element mesh generator with built-in pre- and post-processing facilities. *International Journal for Numerical Methods in Engineering* 79(11), pp. 1309–1331, 2009.

- GPL software with comprehensive documentation.
- Cross-platform: Linux, Windows and OS X.
- Fast and robust meshes in 2D and 3D.
- Convenient data format for FEM.
- Extensive visualisation features.
- Reasonably easy to handle simple geometries.

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FEM code has no other software dependency.

Package modules

Gmsh.jl Handles reading and writing of Gmsh data files.

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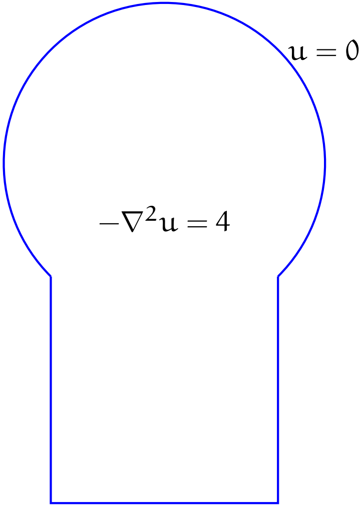
How many lines of code?

```
$ wc *.jl
 229   657  6469 FEM.jl
 249   823  7850 Gmsh.jl
 152   518  4423 PlanarPoisson.jl
 630  1998 18742 total
```

Simple example



Weak formulation and finite element approximation

Sobolev space $H_0^1(\Omega)$ consists of those $u \in L_2(\Omega)$ such that $\partial_x u$ and $\partial_y u \in L_2(\Omega)$, with $u = 0$ on Ω.
Weak solution $u \in H_0^1(\Omega)$ satisfies

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Approximate Ω by triangulated domain Ω_h.

Finite element space $\mathcal{S}_h$ consists of all continuous, piecewise-linear functions that vanish on $\partial\Omega_h$; thus, $\mathcal{S}_h \subseteq H_0^1(\Omega_h)$.

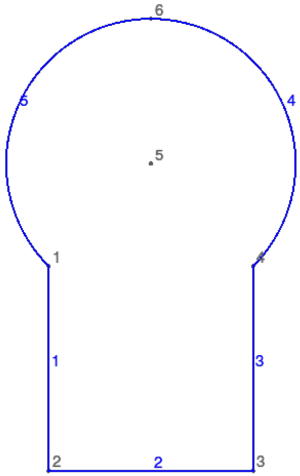
Finite element solution $u_h \in \mathcal{S}_h$ satisfies

$$\int_{\Omega_h} \nabla u_h \cdot \nabla v = \int_{\Omega_h} 4v \quad \text{for all } v \in \mathcal{S}_h.$$

Describe the geometry

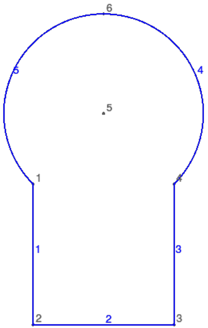
Create file `keyhole.geo` containing.

```
Point(1) = {-1, 0, 0};
Point(2) = {-1,-2, 0};
Point(3) = { 1,-2, 0};
Point(4) = { 1, 0, 0};
Point(5) = { 0, 1, 0};
Point(6) = { 0, 1+Sqrt(2), 0};

Line(1) = {1, 2};
Line(2) = {2, 3};
Line(3) = {3, 4};
Circle(4) = {4, 5, 6};
Circle(5) = {6, 5, 1};
```



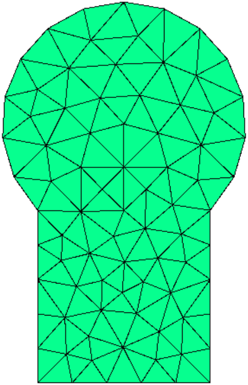
Label the domain and boundary

```
Line Loop(7) = {1, 2, 3, 4, 5};
Plane Surface(1) = { 7 };

Physical Surface("Omega") = { 1 };
Physical Line("Gamma") = { 1, 2, 3, 4, 5 };
```



Triangulate the domain

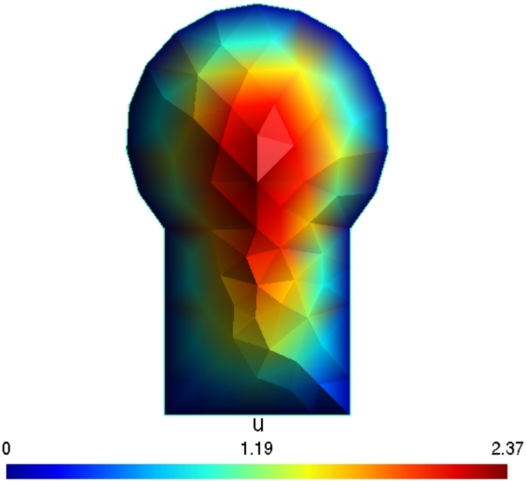
Use Gmsh GUI or CLI to create file `keyhole.msh`.



Solver script

```
using Gmsh
using FEM
using PlanarPoisson
mesh = read_msh_file("keyhole.msh")
essential_bc = [ "Gamma" ]
f(x) = 4.0
vp = VariationalProblem(mesh, essential_bc)
add_bilin_form!(vp,  "Omega", grad_dot_grad!)
add_lin_functnl!(vp, "Omega", source_times_func!, f)
A, b = assembled_linear_system(vp)
ufree = A \ b
u = complete_soln(ufree, vp)
open("keyhole.pos", "w") do fid
    write_format_version(fid)
    save_warp_nodal_scalar_field(u, "u", mesh, fid)
end
```

Visualisation



Geometric element type

In the Gmsh module.

```
immutable GeomType
    gmsh_code :: Integer
    dimen     :: Integer
    nonodes   :: Integer
end

const LINE        = GeomType(1, 1, 2)
const TRIANGLE    = GeomType(2, 2, 3)
const TETRAHEDRON = GeomType(4, 3, 4)

const GETGEOMTYPE = { 1 => LINE,  2 => TRIANGLE,
                      4 => TETRAHEDRON }
```

Mesh data structure

```
immutable Mesh
    coord    :: Array{Float64, 2}
    physdim  :: Dict{String, Integer}
    physnum  :: Dict{String, Integer}
    physname :: Dict{Integer, String}
    elmtype  :: Dict{String, GeomType}
    elms_of  :: Dict{String, Matrix{Integer}}
    nodes_of :: Dict{String, Set{Integer}}
end
```

For example,

$$\begin{aligned}\texttt{mesh.coord[:,n]} &= x,\ y,\ z \text{ coordinates of } n\text{th node},\\ \texttt{mesh.elms\_of["Omega"]} &= \text{connectivity matrix for elements in } \Omega.\end{aligned}$$

FEM data structures

```
immutable DoF
    isfree     :: Vector{Bool}
    freenode   :: Vector{Integer}
    fixednode  :: Vector{Integer}
    node2free  :: Vector{Integer}
    node2fixed :: Vector{Integer}
end

immutable VariationalProblem
    mesh         :: Mesh
    dof          :: DoF
    essential_bc :: Vector{ASCIIString}
    bilin_form   :: Vector{Any}
    lin_functnl  :: Vector{Any}
    ufixed       :: Vector{Float64}
end
```

Inhomogeneous Dirichlet data

```
function assign_bdry_vals!(vp::VariationalProblem,
                           name::String, g::Function)
    if !(name in vp.essential_bc)
        error("$name: not listed in essential_bc")
    end
    for nd in vp.mesh.nodes_of[name]
        i = vp.dof.node2fixed[nd]
        x = vp.mesh.coord[:,nd]
        vp.ufixed[i] = g(x)
    end
end
```

Matrix assembly

```
A = sparse(Int64[], Int64[], Float64[], nofree, nofree)
b = zeros(nofree)
for (name, elm_mat!, coef) in vp.bilin_form
    next = assembled_matrix(name, elm_mat!,
                            coef, mesh, dof)
    A += next[:,1:nofree]
    if nofixed > 0
        b -= next[:,nofree+1:end] * vp.ufixed
    end
end
for (name, elm_vec!, f) in vp.lin_functnl
    next = assembled_vector(name, elm_vec!, f,
                            mesh, dof)
    b += next
end
```

A more complicated example

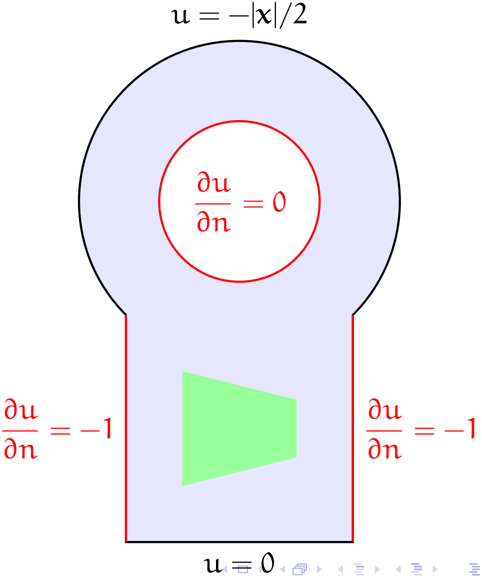
$$-\nabla \cdot (a\nabla u) = f$$

$a = 1$

$f = 1$

$a = 10$

$f = 4$



Weak formulation and finite element approximation

$$\mathcal{S} = \{ v \in H^1(\Omega) : v = -|x|/2 \text{ for } x \in \text{Black} \},$$
$$\mathcal{T} = \{ u \in H^1(\Omega) : v = 0 \text{ for } x \in \text{Black} \}.$$

Weak solution $u \in \mathcal{H}$ satisfies

$$\int_{\text{Blue}} \nabla u \cdot \nabla v + 10 \int_{\text{Green}} \nabla u \cdot \nabla v = \int_{\text{Blue}} v + \int_{\text{Green}} 4v - \int_{\text{Red}} v$$

for all $v \in \mathcal{T}$.

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for all $v \in \mathcal{T}$.

Finite element solution $u_h \in \mathcal{S}_h$ satisfies

$$\int_{\text{Blue}_h} \nabla u_h \cdot \nabla v + 10 \int_{\text{Green}_h} \nabla u_h \cdot \nabla v = \int_{\text{Blue}_h} v + \int_{\text{Green}_h} 4v - \int_{\text{Red}_h} v$$

for all $v \in \mathcal{T}_h$.

Setting up the variational problem

```
g(x) = -hypot(x[1],x[2])/2
assign_bdry_vals!(vp, "North", g)
add_bilin_form!(vp,  "Major",
                grad_dot_grad!, 1.0)
add_lin_functnl!(vp, "Major",
                 source_times_func!, x->1.0)
add_bilin_form!(vp,  "Minor",
                grad_dot_grad!, 10.0)
add_lin_functnl!(vp, "Minor",
                 source_times_func!, x->4.0)
add_lin_functnl!(vp, "East",
                 bdry_source_times_func!, x->-1.0)
add_lin_functnl!(vp, "West",
                 bdry_source_times_func!, x->-1.0)
```

Visualisation

4604 nodes, 9291 triangles.