A Second-Order Accurate Scheme for a Fractional Wave Equation

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References:

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Riemann–Liouville fractional integral:

$$\mathcal{I}_\alpha v(t) := \int_0^t \beta(t-s)v(s)\,ds, \quad \beta(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad 0 < \alpha < 1.$$

Initial-boundary value problem for fractional wave equation: find $u = u(x,t)$ such that

$$\frac{\partial u}{\partial t} + \mathcal{I}_\alpha A u = f(t) \quad \text{for } t > 0, \text{ with } u(0) = u_0, \tag{1}$$

where $A = -\nabla^2$ on a domain $\Omega \subseteq \mathbb{R}^d$ subject to homogeneous Dirichlet or Neumann boundary conditions.

Energy argument shows that (1) admits a unique *mild solution*, stable in the norm of $L_2(\Omega)$,

$$\|u(t)\| \le \|u_0\| + \int_0^t \|f(s)\|\,ds, \quad t > 0.$$

Introduce time levels $0 = t_0 < t_1 < t_2 < \cdots$ with non-uniform step size $k_n = t_n - t_{n-1}$, and generate $U^n \approx u(t_n)$ using a generalized Crank–Nicolson scheme

$$\frac{U^n - U^{n-1}}{k_n} + \mathcal{I}_\alpha^{n-1/2} AU = f^{n-1/2}, \quad \text{for } n \geq 1.$$

We use an averaged, product-integration method to define

$$\mathcal{I}_\alpha v(t_{n-1/2}) \approx \mathcal{I}_\alpha^{n-1/2} V := \frac{1}{k_n} \int_{t_{n-1}}^{t_n} \mathcal{I}_\alpha \bar{V}(t)\, dt$$

where

$$\bar{V}(t) := \begin{cases} V^1 & \text{for } 0 < t < t_1, \\ V^{n-1/2} := \frac{1}{2}(V^n + V^{n-1}) & \text{for } t_{n-1} < t < t_n \text{ and } n \geq 2. \end{cases}$$

Note: $A\bar{U}$ well-defined even if $U^0 \notin D(A)$.

The definition of $\mathcal{I}_\alpha^{n-1/2}$ allows us to apply an energy argument and prove stability for an arbitrary choice of time levels:

$$\|U^n\| \le \|U^0\| + 2\sum_{j=1}^{n} \|f^{j-1/2}\| k_j \quad \text{for } n \ge 1.$$

The most singular modes of $u(t)$ behave like $t^{1+\alpha}$ as $t \to 0$, so u_{tt} is not bounded. To compensate for the singular behaviour at $t = 0$, we take

$$t_n = (nk)^\gamma \quad \text{for } 0 \le n \le N \text{ with } k = \frac{T^{1/\gamma}}{N},$$

for suitable $\gamma > 1$. However, the obvious error analysis still yields a suboptimal error bound,

$$\|U^n - u(t_n)\| \le Ck^{1+\alpha} \quad \text{for } 0 \le t_n \le T.$$

A much more elaborate analysis shows that

$$
\begin{aligned}
\|U^n - u(t_n)\| \le \|U^0 - u_0\| &+ 2\sum_{j=1}^{n} \left\| f^{j-1/2} k_j - \int_{t_{j-1}}^{t_j} f(t)\,dt \right\| \\
&+ C\left(\int_0^{t_1} t\|Au'(t)\|\,dt + k_2 \int_{t_1}^{t_2} \|Au'(t)\|\,dt + \sum_{j=2}^{n} k_j^2 \int_{t_{j-1}}^{t_j} \|Au''(t)\|\,dt \right) \\
&+ C \sum_{j=2}^{n-1} \left(k_j \int_{t_j}^{t_j^*} \|Au'(t)\|\,dt + (k_{j+1} - k_j) \int_{t_{j-1}}^{t_j^*} \|Au'(t)\|\,dt \right),
\end{aligned}
$$

where $t_j^* = \frac{1}{2}(t_{j-1} + t_{j+1})$. Assuming

$$
t\|Au'(t)\| + t^2\|Au''(t)\| + t\|f'(t)\| + t^2\|f''(t)\| \le Mt^{\sigma-1},
$$

with $\sigma > 0$, it follows that

$$
\|U^n - u(t_n)\| \le \|U^0 - u_0\| + CMk^2 \quad \text{if } \gamma > 2/\sigma.
$$

If we discretise in space using continuous piecewise-linear finite elements, and if

$$\|u(t)\|_\kappa + t\|u'(t)\|_\kappa \le Mt^\nu \quad \text{for } t > 0\text{, with } 0 \le \kappa \le 2 \text{ and } \nu \ge 0,$$

then the additional error is bounded by

$$\|U_h^0 - R_h u_0\| + CMh^\kappa \begin{cases} 1 + \log(t_n/t_1) & \nu = 0, \\ 1 & \nu > 0, \end{cases}$$

where R_h denotes the Ritz projector.

Practical conclusion: for smooth f and u_0, the error is $O(h^2 + k^2)$ provided $\gamma > 2/\sigma$ and u_0 satisfies the boundary conditions.

For a simple numerical example, take

$$\begin{aligned} u_t - \mathcal{I}_{1/2} u_{xx} &= f && \text{for } 0 < x < 1 \text{ and } 0 < t < 1, \\ u(x,0) &= \cos(\pi x) && \text{for } t = 0, \\ u_x(0,t) = 0 &= u_x(1,t) && \text{for } 0 < t < 1, \end{aligned}$$

with $f(x,t)$ chosen so that the exact solution is

$$u = \cos(\pi x) - \frac{4t^{3/2}}{3\sqrt{\pi}}\Big(1 + \cos(2\pi x)\Big).$$

With $k = h = 1/N$ and $f^{n-1/2} = f(t_{n-1/2})$, we find that the maximum error in $L_2(0,1)$ is as follows:

N	$\gamma = 1$		$\gamma = 4/3$		$\gamma = 2$	
20	5.42e-03		5.53e-03		6.37e-03	
40	1.39e-03	1.95	1.39e-03	1.99	1.63e-03	1.99
80	5.17e-04	1.43	3.47e-04	1.99	4.00e-04	1.99
160	1.95e-04	1.40	8.14e-05	2.09	9.54e-05	2.07
320	7.26e-05	1.42	2.04e-05	1.99	2.39e-05	1.99