The Conditioning of Boundary Element Equations

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Outline:

1. Bounds for extremal eigenvalues
2. General assumptions on the mesh
3. Quasi-uniform meshes
4. Shape-regular meshes
5. Anisotropic refinement

1. Bounds for extremal eigenvalues

Consider a bounded bilinear form

$$B : \widetilde{H}^m(\Gamma) \times \widetilde{H}^m(\Gamma) \to \mathbb{R},$$

where

$$\Gamma = \begin{cases} \Omega, & \text{for FEM,} \\ \partial\Omega, & \text{for BEM.} \end{cases}$$

Assume

$$B(v,u) = B(u,v) \quad \text{and} \quad B(v,v) \simeq \|v\|^2_{\widetilde{H}^m(\Gamma)}.$$

We want to find $u \in \widetilde{H}^m(\Gamma)$ satisfying

$$B(u,v) = \langle f, v \rangle \quad \text{for all } v \in \widetilde{H}^m(\Gamma).$$

1.1 Example. Poisson equation on $\Omega \subseteq \mathbb{R}^d$:

$$
\begin{aligned}
-\nabla^2 u &= f \ \text{ in } \Omega, \\
u &= 0 \ \text{ on } \partial\Omega.
\end{aligned}
$$

Solution space: $\widetilde{H}^1(\Omega) = H_0^1(\Omega)$, i.e., $m = 1$.

First Green identity:

$$
\begin{aligned}
\int_\Omega f v \, dx = \int_\Omega (-\nabla^2 u) v \, dx &= \int_\Omega \big[\nabla u \cdot \nabla v - \nabla \cdot (v \nabla u) \big] \, dx \\
&= \int_\Omega \nabla u \cdot \nabla v \, dx - \int_{\partial\Omega} v \, \frac{\partial u}{\partial n} \, d\sigma,
\end{aligned}
$$

so if $v = 0$ on $\partial\Omega$ then

$$
\underbrace{\int_\Omega \nabla u \cdot \nabla v \, dx}_{B(u,v)} = \underbrace{\int_\Omega f v \, dx}_{\langle f, v \rangle} .
$$

1.2 Example. The 3D screen problem for the Laplacian with Dirichlet boundary condition on Γ leads to the weakly-singular integral equation

$$\frac{1}{4\pi}\int_{\Gamma} \frac{u(y)}{|x-y|}\, d\sigma_y = f(x), \qquad x \in \Gamma.$$

The associated bilinear form

$$B(u,v) = \iint_{\Gamma\times\Gamma} \frac{u(y)v(x)}{|x-y|}\, dx\, dy$$

satisfies

$$B(v,v) \simeq \|v\|^2_{\widetilde{H}^{-1/2}(\Gamma)}.$$

Our assumptions are satisfied with $m = -1/2$ and $d = 2$.

Use standard h-version of FEM or BEM: find $u_X \in X \subseteq \widetilde{H}^m(\Gamma)$ satisfying

$$B(u_X, v) = \langle f, v \rangle \quad \text{for all } v \in X.$$

Set of nodes $\{\, x_k : k \in \mathcal{N} \,\}$ and corresponding nodal basis functions ϕ_k such that

$$\phi_j(x_k) = \delta_{jk}.$$

Galerkin solution

$$u_X(x) = \sum_{k \in \mathcal{N}} \alpha_k \phi_k(x), \qquad \alpha_k = u_X(x_k),$$

found by solving $N \times N$ linear system,

$$\boldsymbol{B}\boldsymbol{\alpha} = \boldsymbol{f},$$

where

$$N = \#\mathcal{N}, \quad \boldsymbol{B} = [B(\phi_k, \phi_j)], \quad \boldsymbol{\alpha} = [\alpha_k], \quad \boldsymbol{f} = [\langle f, \phi_j \rangle].$$

For the symmetric, positive-definite matrix $\boldsymbol{B}$ we want to estimate

$$\mathrm{cond}(\boldsymbol{B}) = \|\boldsymbol{B}\|_2 \|\boldsymbol{B}^{-1}\|_2 = \frac{\lambda_{\max}(\boldsymbol{B})}{\lambda_{\min}(\boldsymbol{B})}.$$

Observation: since

$$\lambda_{\max}(\boldsymbol{B}) = \max_{\boldsymbol{\alpha}\neq\boldsymbol{0}} \frac{\boldsymbol{\alpha}^T\boldsymbol{B}\boldsymbol{\alpha}}{\boldsymbol{\alpha}^T\boldsymbol{\alpha}} \quad \text{and} \quad \lambda_{\min}(\boldsymbol{B}) = \min_{\boldsymbol{\alpha}\neq\boldsymbol{0}} \frac{\boldsymbol{\alpha}^T\boldsymbol{B}\boldsymbol{\alpha}}{\boldsymbol{\alpha}^T\boldsymbol{\alpha}},$$

if we show

$$\lambda_X \boldsymbol{\alpha}^T\boldsymbol{\alpha} \le \boldsymbol{\alpha}^T\boldsymbol{B}\boldsymbol{\alpha} \le \Lambda_X \boldsymbol{\alpha}^T\boldsymbol{\alpha} \quad \text{for all } \boldsymbol{\alpha} \in \mathbb{R}^N, \tag{1}$$

then we obtain the one-sided bounds

$$\lambda_{\max}(\boldsymbol{B}) \le \Lambda_X, \qquad \lambda_{\min}(\boldsymbol{B}) \ge \lambda_X, \qquad \mathrm{cond}(\boldsymbol{B}) \le \frac{\Lambda_X}{\lambda_X}.$$

Writing $v(x) = \sum_{k\in\mathcal{N}} \alpha_j \phi_j(x)$ we see that (1) is equivalent to

$$\lambda_X \sum_{j\in\mathcal{N}} v(x_j)^2 \lesssim \|v\|^2_{\widetilde{H}^m(\Gamma)} \lesssim \Lambda_X \sum_{j\in\mathcal{N}} v(x_j)^2 \quad \text{for all } v \in X.$$

Let $\boldsymbol{D} = \text{diag}(\boldsymbol{B})$ and consider the diagonally-scaled linear system

$$\boldsymbol{D}^{-1}\boldsymbol{B}\boldsymbol{\alpha} = \boldsymbol{D}^{-1}\boldsymbol{f}.$$

Put

$$\boldsymbol{B}' = \boldsymbol{D}^{-1/2}\boldsymbol{B}\boldsymbol{D}^{-1/2} = \boldsymbol{D}^{1/2}(\boldsymbol{D}^{-1}\boldsymbol{B})\boldsymbol{D}^{-1/2}$$

so that $\text{cond}(\boldsymbol{D}^{-1}\boldsymbol{B}) = \text{cond}(\boldsymbol{B}')$ with $\boldsymbol{B}'$ symmetric and positive-definite.

Diagonal scaling is equivalent to rescaling the nodal basis functions to have unit energy.

If we show

$$\lambda'_X \boldsymbol{\alpha}^T \boldsymbol{\alpha} \le \boldsymbol{\alpha}^T \boldsymbol{B}' \boldsymbol{\alpha} \le \Lambda'_X \boldsymbol{\alpha}^T \boldsymbol{\alpha} \quad \text{for all } \boldsymbol{\alpha} \in \mathbb{R}^N,$$

or equivalently

$$\lambda'_X \boldsymbol{\alpha}^T \boldsymbol{D} \boldsymbol{\alpha} \le \boldsymbol{\alpha}^T \boldsymbol{B} \boldsymbol{\alpha} \le \Lambda'_X \boldsymbol{\alpha}^T \boldsymbol{D} \boldsymbol{\alpha} \quad \text{for all } \boldsymbol{\alpha} \in \mathbb{R}^N, \tag{2}$$

then

$$\lambda_{\max}(\boldsymbol{B}') \le \Lambda'_X, \qquad \lambda_{\min}(\boldsymbol{B}') \ge \lambda'_X, \qquad \operatorname{cond}(\boldsymbol{B}') \le \frac{\Lambda'_X}{\lambda'_X}.$$

Putting

$$v_j(x) = \alpha_j \phi_j(x) = v(x_j)\phi_j(x)$$

we find that

$$\boldsymbol{\alpha}^T \boldsymbol{D} \boldsymbol{\alpha} = \sum_{j \in \mathcal{N}} B(v_j, v_j) \simeq \sum_{j \in \mathcal{N}} \|v_j\|^2_{\widetilde{H}^m(\Gamma)},$$

so (2) is equivalent to

$$\lambda'_X \sum_{j \in \mathcal{N}} \|v_j\|^2_{\widetilde{H}^m(\Gamma)} \lesssim \|v\|^2_{\widetilde{H}^m(\Gamma)} \lesssim \Lambda'_X \sum_{j \in \mathcal{N}} \|v_j\|^2_{\widetilde{H}^m(\Gamma)} \quad \text{for all } v \in X.$$

2. General Assumptions on the Mesh

For each element K in a partition $\mathcal{P}$, let

$$\begin{aligned} h_K &= \text{diameter of } K, \\ \rho_K &= \text{diameter of largest ball contained in } K, \\ \mathcal{N}(K) &= \{\, j \in \mathcal{N} : \operatorname{supp} \phi_j \text{ intersects } K \,\}, \end{aligned}$$

and for each node label $j \in \mathcal{N}$, let

$$\begin{aligned} h_j &= \text{average of those } h_K \text{ with } j \in \mathcal{N}(K), \\ \rho_j &= \text{average of those } \rho_K \text{ with } j \in \mathcal{N}(K). \end{aligned}$$

We assume that the element shapes vary smoothly across the mesh, so that

$$h_j \simeq h_K \quad \text{and} \quad \rho_j \simeq \rho_K \quad \text{whenever } j \in \mathcal{N}(K),$$

uniformly over our family of partitions $\{\mathcal{P}\}$. Further, we assume that $\#\mathcal{N}(K)$ is uniformly bounded, and so is L, the number of disjoint subsets $\mathcal{N}_1$, ..., $\mathcal{N}_L$ of $\mathcal{N}$ required to ensure that $\mathcal{N} = \mathcal{N}_1 \cup \cdots \cup \mathcal{N}_L$ and

$$\operatorname{supp} \phi_j \cap \operatorname{supp} \phi_{j'} = \emptyset \text{ if } j,\ j' \in \mathcal{N}_\ell,\ j \neq j'.$$

3. Quasi-uniform Meshes

Consider the simplest case where

$$h_K \simeq \rho_K \simeq h$$

and, with d denoting the dimension of Γ,

$$\|v\|_{L_2(\Gamma)}^2 \simeq h^d \sum_{j \in \mathcal{N}} [v(x_j)]^2.$$

Folklore:

$$\text{cond}(\boldsymbol{B}) \lesssim h^{-2|m|} \simeq N^{2|m|/d}.$$

Diagonal scaling does not affect the condition number in this case because all diagonal entries

$$B(\phi_j, \phi_j) \simeq \|\phi_j\|_{\widetilde{H}^m(\Gamma)}^2 \simeq h^{d-2m}$$

are of comparable size.

3.1 Theorem. For a quasi-uniform mesh, if $m \le 0$ then

$$h^{d-2m} \sum_{j \in \mathcal{N}} v(x_j)^2 \lesssim \|v\|^2_{\widetilde{H}^m(\Gamma)} \lesssim h^d \sum_{j \in \mathcal{N}} v(x_j)^2 \quad \text{for all } v \in X,$$

whereas if $m \ge 0$ then

$$h^d \sum_{j \in \mathcal{N}} v(x_j)^2 \lesssim \|v\|^2_{\widetilde{H}^m(\Gamma)} \lesssim h^{d-2m} \sum_{j \in \mathcal{N}} v(x_j)^2 \quad \text{for all } v \in X.$$

Proof: use inverse estimates:

$$\|v\|_{\widetilde{H}^m(\Gamma)} \lesssim \|v\|_{L_2(\Gamma)} \lesssim h^m \|v\|_{\widetilde{H}^m(\Gamma)} \quad \text{when } m \le 0$$

or

$$\|v\|_{L_2(\Gamma)} \lesssim \|v\|_{\widetilde{H}^m(\Gamma)} \lesssim h^{-m} \|v\|_{L_2(\Gamma)} \quad \text{when } m \ge 0.$$

3.2 Theorem. Recall that $v_j(x) = v(x_j)\phi_j(x)$. For a quasi-uniform mesh, if $m \le 0$ then

$$\sum_{j\in\mathcal{N}} \|v_j\|^2_{\widetilde{H}^m(\Gamma)} \lesssim \|v\|^2_{\widetilde{H}^m(\Gamma)} \lesssim h^{2m} \sum_{j\in\mathcal{N}} \|v_j\|^2_{\widetilde{H}^m(\Gamma)} \quad \text{for all } v \in X,$$

whereas if $m \ge 0$ then

$$h^{2m} \sum_{j\in\mathcal{N}} \|v_j\|^2_{\widetilde{H}^m(\Gamma)} \lesssim \|v\|^2_{\widetilde{H}^m(\Gamma)} \lesssim \sum_{j\in\mathcal{N}} \|v_j\|^2_{\widetilde{H}^m(\Gamma)} \quad \text{for all } v \in X.$$

Proof: Can show $\|\phi_j\|^2_{\widetilde{H}^m(\Gamma)} \simeq h^{d-2m}$ so

$$v(x_j)^2 \simeq h^{2m-d}\|v(x_j)\phi_j\|^2_{\widetilde{H}^m(\Gamma)} = h^{2m-d}\|v_j\|^2_{\widetilde{H}^m(\Gamma)}.$$

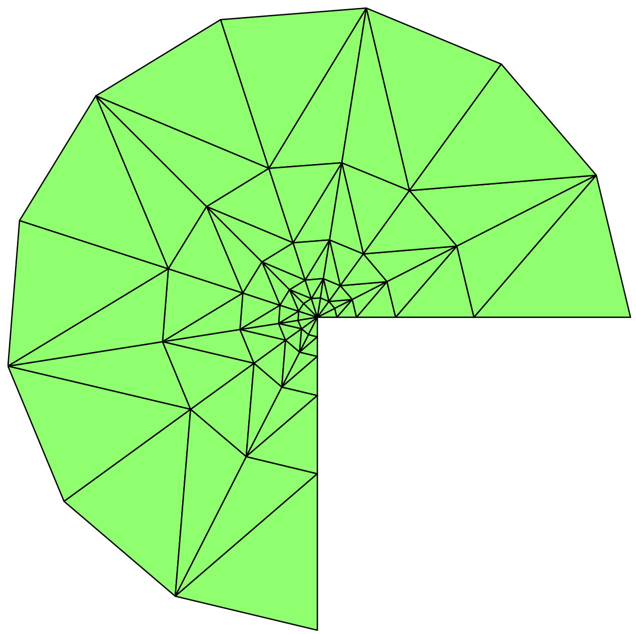
4. Shape-Regular Meshes

Now replace quasi-uniformity with the weaker assumption that

$$h_K \lesssim \rho_K \quad \text{for all } K \in \mathcal{P}.$$

This allows local refinement but rules out long, thin elements.

90 faces and 56 vertices



References:

Isaac Fried, Condition of finite element matrices generated from nonuniform meshes, *AIAA J.* **10** (1972), 219–221.

R. E. Bank and L. R. Scott, On the conditioning of finite element equations with highly refined meshes, *SIAM J. Numer. Anal.* **26** (1989), 1383–1394.

Mark Ainworth, William McLean and Thanh Tran, The conditioning of boundary element equations on locally refined meshes and preconditioning by diagonal scaling, *SIAM. J. Numer. Anal.* **36** (1999), 1901–1932.

Recall that

$$N = \#\mathcal{N} = \text{number of degrees of freedom},$$

and let

$$h_{\max} = \max_{K \in \mathcal{P}} h_K, \qquad h_{\min} = \min_{K \in \mathcal{P}} h_K.$$

We find the following one-sided bounds for the extremal eigenvalues of $\boldsymbol{B}$ and $\boldsymbol{B}'$.

	λ_X	Λ_X	λ'_X	Λ'_X
$-d < 2m \le 0$	$h_{\min}^{d-2m}$	$N^{-2m/d} h_{\max}^{d-2m}$	1	$N^{-2m/d}$
$0 \le 2m < d$	$N^{-2m/d} h_{\min}^{d-2m}$	$h_{\max}^{d-2m}$	$N^{-2m/d}$	1

Conclusion: for a shape-regular mesh,

$$\operatorname{cond}(\boldsymbol{B}) \lesssim N^{|2m|/d} \left(\frac{h_{\max}}{h_{\min}} \right)^{d-2m} \quad \text{but} \quad \operatorname{cond}(\boldsymbol{B}') \lesssim N^{|2m|/d}.$$

4.1 Example. Using FEM for the Poisson problem, $m = 1$ so

$$\mathrm{cond}(\boldsymbol{B}) \simeq \mathrm{cond}(\boldsymbol{B}') \lesssim N\Big[1 + |\log(N h_{\mathrm{min}}^2)|\Big] \quad \text{if } d = 2,$$

whereas

$$\mathrm{cond}(\boldsymbol{B}) \lesssim N^{2/3} \frac{h_{\mathrm{max}}}{h_{\mathrm{min}}} \quad \text{and} \quad \mathrm{cond}(\boldsymbol{B}') \lesssim N^{2/3} \quad \text{if } d = 3.$$

4.2 Example. In the BEM screen problem (Laplacian with Dirichlet boundary conditions),

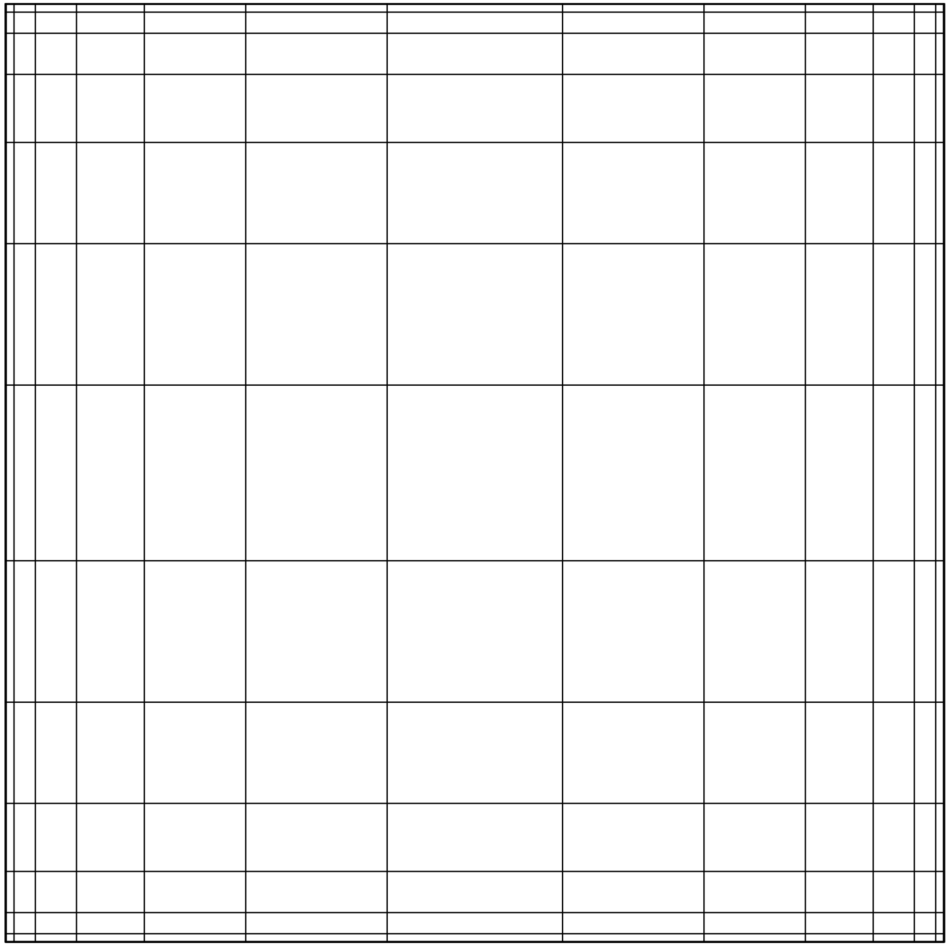
$$\frac{1}{4\pi} \int_\Gamma \frac{u(y)}{|x-y|} \, d\sigma_y = f(x), \quad x \in \Gamma,$$

we have $d = 2$ and $m = -1/2$, so

$$\mathrm{cond}(\boldsymbol{B}) \lesssim N^{1/2} \left(\frac{h_{\mathrm{max}}}{h_{\mathrm{min}}}\right)^3 \quad \text{but} \quad \mathrm{cond}(\boldsymbol{B}') \lesssim N^{1/2}.$$

5. Anisotropic Refinement

For 3D problems it is natural to use anisotropic mesh refinement to resolve edge singularities.



We are able to show one-sided bounds on the extremal eigenvalues of $\boldsymbol{B}$ and $\boldsymbol{B}'$. For instance, if $-d < 2m \leq 0$ then

$$\lambda_{\mathsf{min}}(\boldsymbol{B}) \gtrsim \min_{j \in \mathcal{N}} |\Gamma_j| \rho_j^{-2m}$$

and

$$\lambda_{\mathsf{max}}(\boldsymbol{B}) \lesssim N^{-2m/d} \max_{j \in \mathcal{N}} |\Gamma_j|^{1-2m/d},$$

where $\Gamma_j = \operatorname{supp} \phi_j$. In particular, taking $d = 2$ and $m = -1/2$, we see that for the weakly-singular boundary integral equation,

$$\lambda_{\mathsf{min}}(\boldsymbol{B}) \gtrsim \min_{j \in \mathcal{N}} |\Gamma_j| \rho_j \quad \text{and} \quad \lambda_{\mathsf{max}}(\boldsymbol{B}) \lesssim N^{1/2} \max_{j \in \mathcal{N}} |\Gamma_j|^{3/2}.$$

Consider these bounds for a special but realistic mesh.

Define a *power-graded mesh* on the unit interval with *grading exponent* $\beta \geq 1$,

$$t_j = \begin{cases} \dfrac{1}{2}\left(\dfrac{2j}{n}\right)^{\beta}, & 0 \leq j \leq n/2, \\ 1 - t_{n-j}, & n/2 < j \leq n. \end{cases}$$

The length of the jth interval is

$$\Delta t_j \simeq \frac{1}{n}\left(\frac{j}{n}\right)^{\beta-1} \simeq \Delta t_{n-j}, \quad 1 \leq j \leq n/2.$$

The corresponding *tensor-product mesh* on the unit square $\Gamma = (0,1)^2$ has vertices

$$t_{(i,j)} = (t_i, t_j), \quad 0 \leq i \leq n, 0 \leq j \leq n.$$

The maximum aspect ratio of the rectangles is

$$\frac{\Delta t_{n/2}}{\Delta t_1} \simeq \frac{1/n}{1/n^{\beta}} = n^{\beta-1}.$$

For piecewise-constants on the tensor-product mesh, $N = n^2$ and we find that, in the case of the weakly-singular boundary integral equation,

$$\lambda_{\text{max}}(\boldsymbol{B}) \lesssim N^{-1}, \quad \lambda_{\text{min}}(\boldsymbol{B}) \gtrsim N^{-3\beta/2}, \quad \text{cond}(\boldsymbol{B}) \lesssim N^{3\beta/2-1},$$

whereas, for the diagonally-scaled matrix, we are able to show

$$\lambda_{\text{max}}(\boldsymbol{B}') \lesssim N^{1/2} \begin{cases} 1, & 1 \le \beta \le 2, \\ (1+\log N)^{1/2}, & \beta = 2, \\ (1+\log N)^2, & \beta > 2, \end{cases}$$

and

$$\lambda_{\text{min}}(\boldsymbol{B}') \gtrsim \begin{cases} 1, & \beta = 1, \\ (1+\log N)^{-1}, & \beta > 1, \end{cases}$$

so

$$\text{cond}(\boldsymbol{B}') \lesssim N^{1/2}(1+\log N)^{\kappa}.$$

Numerical experiments indicate that these bounds are sharp. The following results are for a mesh grading exponent $\beta = 3$.

N	$\lambda_{\mathtt{max}}(\boldsymbol{B})$		$\lambda_{\mathtt{min}}(\boldsymbol{B})$		$\mathrm{cond}(\boldsymbol{B})$	
4	7.43E-01		1.87E-01		3.97E+00	
16	5.07E-01	-0.276	6.52E-04	-4.083	7.78E+02	3.807
64	1.78E-01	-0.755	1.28E-06	-4.496	1.39E+05	3.741
256	4.90E-02	-0.932	2.50E-09	-4.500	1.96E+07	3.568
1024	1.26E-02	-0.982	4.89E-12	-4.500	2.57E+09	3.518
Theory	$\lesssim N^{-1}$		$\gtrsim N^{-9/2}$		$\lesssim N^{7/2}$	

N	$\lambda_{\mathtt{max}}(\boldsymbol{B}')$		$\lambda_{\mathtt{min}}(\boldsymbol{B}')$		$\mathrm{cond}(\boldsymbol{B}')$	
4	2.00E+00		5.04E-01		3.97E+00	
16	3.38E+00	0.379	3.71E-01	-0.220	9.11E+00	0.599
64	6.43E+00	0.463	3.25E-01	-0.096	1.98E+01	0.559
256	1.26E+01	0.488	2.45E-01	-0.204	5.16E+01	0.692
1024	2.51E+01	0.496	1.86E-01	-0.200	1.35E+02	0.696
Theory	$\lesssim N^{1/2}(\log N)^2$		$\gtrsim (\log N)^{-1}$		$\lesssim N^{1/2}(\log N)^3$	

6. A Curiosity

Consider piecewise-linear FEM for the 1D problem

$$-u'' = f \quad \text{on } (0,1), \qquad u(0) = 0 = u(1).$$

(Here, $m = 1 = d$ so $2m > d$.)

Fix $0 < \alpha < 1$ and define a geometrically-graded mesh

$$x_0 = 0, \quad x_j = \alpha^{N-j}, \quad 1 \le j \le N.$$

Thus,

$$h_1 = \alpha^{N-1}, \quad h_j = x_j - x_{j-1} = (1-\alpha)\alpha^{N-j}, \quad 2 \le j \le N,$$

and

$$\boldsymbol{B} = \begin{bmatrix} h_1^{-1} + h_2^{-1} & -h_2^{-1} & & \\ -h_2^{-1} & h_2^{-1} + h_3^{-1} & -h_3^{-1} & \\ & \ddots & \ddots & \ddots \end{bmatrix}.$$

Diagonally-scaled matrix

$$\boldsymbol{B}' = \boldsymbol{D}^{-1/2}\boldsymbol{B}\boldsymbol{D}^{-1/2} = \begin{bmatrix} 1 & -a & & & \\ -a & 1 & -b & & \\ & -b & 1 & -b & \\ & & \ddots & \ddots & \ddots \end{bmatrix}$$

where

$$a = \frac{-B_{12}}{\sqrt{B_{11}B_{22}}} = \frac{1}{\sqrt{(h_2/h_1+1)(1+h_2/h_3)}} = \frac{1}{\sqrt{\alpha^{-1}+1}}$$

and, for $j \geq 2$,

$$b = \frac{-B_{j,j+1}}{\sqrt{B_{jj}B_{j+1,j+1}}} = \frac{1}{\sqrt{(\alpha+1)(1+\alpha^{-1})}} = \frac{\sqrt{\alpha}}{1+\alpha}.$$

We have $a < 1$, $2b < 1$ and $a + b < 1$ for $\alpha < 0{\cdot}4196433776\cdots$, implying

$$\text{cond}(\boldsymbol{B}') \leq \text{constant}.$$