Lectures on Boundary Integral Equations

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Plan

1. PDEs and Sobolev spaces.

2. Fredholm property for elliptic BVPs.

3. Surface potentials.

4. Boundary integral equations.

1. PDEs and Sobolev Spaces

Basic Notations

Second-order partial differential operator

$$\mathcal{P}u = -\sum_{j=1}^{n} \partial_j\left(\sum_{k=1}^{n} A_{jk}\partial_k u - A_j u\right) + \sum_{j=1}^{n} A_j \partial_j u + Au$$

with $\mathbb{C}^{m\times m}$-valued coefficients (possibly functions of x)

$$A_{jk} = [a_{pq}^{jk}], \quad A_j = [a_{pq}^{j}], \quad A = [a_{pq}], \quad 1 \le p \le m, \quad 1 \le q \le m.$$

Thus, with $u = [u_q]$, and using the summation convention,

$$\begin{aligned}(\mathcal{P}u)_p &= -\partial_j\left(a_{pq}^{jk}\partial_k u_q - a_{pq}^{j} u_q\right) + a_{pq}^{j}\partial_j u_q + a_{pq}u_q \\ &= -\partial_j\left(a_{pq}^{jk}\partial_k u_q\right) + \partial_j\left(a_{pq}^{j} u_q\right) + a_{pq}^{j}\partial_j u_q + a_{pq}u_q \\ &= -\partial_j\left(a_{pq}^{jk}\partial_k u_q\right) + 2a_{pq}^{j}\partial_j u_q + \left(\partial_j a_{pq}^{j} + a_{pq}\right)u_q.\end{aligned}$$

Formal adjoint:

$$\mathcal{P}^*u = -\sum_{j=1}^{n} \partial_j\left(\sum_{k=1}^{n} A_{kj}^*\partial_k u + A_j^* u\right) - \sum_{j=1}^{n} A_j^*\partial_j u + A^* u$$

satisfies

$$\int_\Omega (\mathcal{P}u)^* v\,dx = \int_\Omega u^*(\mathcal{P}^* v)\,dx \quad \text{for } u,\ v \in C^2_{\text{comp}}(\Omega)^m,$$

because if u, v vanish on $\Gamma = \partial\Omega$ then

$$\int_\Omega \partial_j (Au)^* v\,dx = -\int_\Omega (Au)^*\partial_j v\,dx = -\int_\Omega u^*(A^*\partial_j v)\,dx.$$

We say $\mathcal{P}$ is *formally self-adjoint* if $\mathcal{P}^* = \mathcal{P}$, i.e., if

$$A_{kj}^* = A_{jk}, \quad A_j^* = -A_j, \quad A^* = A.$$

(Wrongly stated in [SESBIE, p. 116].)

Conormal derivative w.r.t. a Lipschitz domain $\Omega \subseteq \mathbb{R}^n$ with outward unit normal ν on boundary $\Gamma = \partial\Omega$:

$$\mathcal{B}_\nu u = \mathcal{B}_{\nu,\Omega} u = \sum_{j=1}^{n} \nu_j \biggl(\sum_{k=1}^{n} A_{jk} \partial_k u - A_j u \biggr) \quad \text{on } \Gamma.$$

Conormal derivative for $\mathcal{P}^*$:

$$\tilde{\mathcal{B}}_\nu u = \tilde{\mathcal{B}}_{\nu,\Omega} u = \sum_{j=1}^{n} \nu_j \biggl(\sum_{k=1}^{n} A_{kj}^* \partial_k u + A_j^* u \biggr) \quad \text{on } \Gamma.$$

Sesquilinear form determined by $\mathcal{P}$ and Ω:

$$\Phi(u,v) = \Phi_\Omega(u,v) = \int_\Omega \biggl(\sum_{j=1}^{n} \sum_{k=1}^{n} (A_{jk}\partial_k u)^* \partial_j v + \sum_{j=1}^{n} \Bigl((A_j \partial_j u)^* v - (A_j u)^* \partial_j v \Bigr) + (Au)^* v \biggr) \, dx.$$

First Green identity: if u, $v \in C^2_{\mathsf{comp}}(\overline{\Omega})^m$ then

$$\Phi_\Omega(u,v) = \int_\Omega (\mathcal{P}u)^* v \, dx + \int_\Gamma (\mathcal{B}_\nu u)^* v \, d\sigma$$
$$= \int_\Omega u^* (\mathcal{P}^* v) \, dx + \int_\Gamma u^* (\tilde{\mathcal{B}}_\nu v) \, d\sigma.$$

Proof uses the divergence theorem:

$$\int_\Omega \operatorname{div} F \, dx = \int_\Gamma \nu \cdot F \, d\sigma,$$

i.e.,

$$\int_\Omega \sum_{j=1}^n \partial_j F_j \, dx = \int_\Gamma \sum_{j=1}^n \nu_j F_j \, d\sigma.$$

Key step:

$$\int_\Omega (A_{jk}\partial_k u)^* \partial_j v \, dx = \int_\Omega \partial_j\Big((A_{jk}\partial_k u)^* v\Big) \, dx - \int_\Omega \partial_j (A_{jk}\partial_k u)^* v \, dx$$
$$= \int_\Gamma (\nu_j A_{jk}\partial_k u)^* v \, d\sigma - \int_\Omega \partial_j (A_{jk}\partial_k u)^* v \, dx.$$

Laplace Operator

The *Laplacian* $(m = 1)$

$$\mathcal{P}u = -\nabla^2 u = -\sum_{j=1}^{n} \partial_j^2 u = -\left(\frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2}\right),$$

is formally self-adjoint ($\mathcal{P}^* = \mathcal{P}$) with

$$\mathcal{B}_\nu u = \partial_\nu u = \frac{\partial u}{\partial \nu} = \sum_{j=1}^{n} \nu_j \partial_j u = \nu \cdot \nabla u,$$

$$\Phi(u, v) = \int_\Omega \sum_{j=1}^{n} (\partial_j \bar{u}) \partial_j v \, dx = \int_\Omega \nabla \bar{u} \cdot \nabla v \, dx,$$

and

$$\int_\Omega \nabla \bar{u} \cdot \nabla v \, dx = \int_\Omega (-\nabla^2 \bar{u}) v \, dx + \int_\Gamma (\partial_\nu \bar{u}) v \, d\sigma.$$

Linear Elasticity

Strain tensor:

$$E_{jk}(u) = \frac{1}{2}(\partial_j u_k + \partial_k u_j).$$

Stress tensor, for a homogeneous and isotropic elastic medium:

$$\Sigma_{jk}(u) = 2\mu E_{jk}(u) + \lambda(\operatorname{div} u)\delta_{jk},$$

where the *Lamé coefficients* λ and μ are real constants. In equilibrium,

$$-\partial_j \Sigma_{jk}(u) = f_k \quad \text{where} \quad f = \text{body force density.}$$

Since

$$\partial_j \Sigma_{jk}(u) = \mu \partial_j(\partial_j u_k + \partial_k u_j) + \lambda \partial_k(\operatorname{div} u) = \mu \partial_j \partial_j u_k + (\mu + \lambda)\partial_k(\operatorname{div} u)$$

we have $\mathcal{P}u = f$ where

$$\mathcal{P}u = -\mu \nabla^2 u - (\nu + \lambda)\nabla(\operatorname{div} u).$$

Conormal derivative

$$\mathcal{B}_\nu u = \nu_j \Sigma_{jk}(u) e_k = \text{surface traction.}$$

Bilinear form

$$\Phi(u,v) = \int_\Omega \Sigma_{jk}(u)\partial_j v_k \, dx.$$

Since

$$2E_{jk}(u)\partial_j v_k = E_{jk}(u)(\partial_j v_k + \partial_k v_j) = E_{jk}(u)E_{jk}(v),$$

we have

$$\Phi(u,v) = \int_\Omega \Big(\mu E_{jk}(u)E_{jk}(v) + \lambda(\operatorname{div} u)(\operatorname{div} v)\Big)\, dx.$$

Quadratic form

$$\Phi(u,u) = 2 \times \text{(free energy).}$$

Sobolev Spaces — First Definition

Consider scalar-valued functions $u : \mathbb{R}^n \to \mathbb{C}$.

Test functions $\mathcal{D}(\Omega) = C^\infty_{\text{comp}}(\Omega)$.

Imbed $L_{1,\text{loc}}(\Omega)$ in space of *Schwartz distributions* $\mathcal{D}^*(\Omega)$ by writing

$$\langle u, \phi \rangle_\Omega = \int_\Omega u(x)\phi(x)\,dx \quad \text{for } \phi \in \mathcal{D}(\Omega).$$

Also write

$$(u, \phi)_\Omega = \langle \bar{u}, \phi \rangle_\Omega = \int_\Omega \overline{u(x)}\phi(x)\,dx.$$

Multi-index notation

$$\partial^\alpha u(x) = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} u(x) = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$$

for

$$\alpha = (\alpha_1, \ldots, \alpha_n), \quad |\alpha| = \alpha_1 + \cdots + \alpha_n.$$

Sobolev space of integer order $r \geq 0$ based on $L_p(\Omega)$ ($1 \leq p \leq \infty$):

$$W_p^r(\Omega) = \{\, u \in L_p(\Omega) : \partial^\alpha u \in L_p(\Omega) \text{ for } |\alpha| \leq r \,\}.$$

This is a Banach space with norm

$$\|u\|_{W_p^r(\Omega)} = \left(\sum_{|\alpha| \leq r} \int_\Omega |\partial^\alpha u(x)|^p \, dx \right)^{1/p}, \quad 1 \leq p < \infty,$$

with usual modification for $p = \infty$.

Slobodeckiĭ seminorm

$$|u|_{\mu,p,\Omega} = \left(\iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+p\mu}}\,dx\,dy\right)^{1/p}, \quad 0 < \mu < 1.$$

Here, integrand is $\left(|u(x)-u(y)|/|x-y|^{\mu+n/p}\right)^p$ so for $p=\infty$ we get a Hölder inequality:

$$|u(x)-u(y)| \le |u|_{\mu,\infty,\Omega}|x-y|^{\mu}.$$

Sobolev space of *fractional order* $s = r + \mu$,

$$W_p^s(\Omega) = \{\, u \in W_p^r(\Omega) : |\partial^\alpha u|_{\mu,p,\Omega} < \infty \text{ for } |\alpha| = r \,\}.$$

Obvious norm

$$\|u\|_{W_p^s(\Omega)} = \left(\|u\|^p_{W_p^r(\Omega)} + \sum_{|\alpha|=r} |\partial^\alpha u|^p_{\mu,p,\Omega}\right)^{1/p}.$$

Negative-order space $W_p^{-r}(\Omega)$ consists of all distributions $u \in \mathcal{D}^*(\Omega)$ of the form

$$u = \sum_{|\alpha| \le r} \partial^\alpha f_\alpha \quad \text{for some } f_\alpha \in L_p(\Omega).$$

Norm

$$\|u\|_{W_p^{-r}(\Omega)} = \inf \Bigl(\sum_{|\alpha| \le r} \|f_\alpha\|^p_{L_p(\Omega)} \Bigr)^{1/p}.$$

In this way,

$$|\langle u, v \rangle_\Omega| \le \|u\|_{W_{p^*}^{-r}(\Omega)} \|v\|_{W_p^r(\Omega)}$$

for

$$u \in W_{p^*}^{-r}(\Omega), \qquad v \in \mathcal{D}(\Omega), \qquad \frac{1}{p} + \frac{1}{p^*} = 1, \qquad 1 \le p < \infty.$$

Sobolev Spaces — Second Definition

Denote the *Fourier transform* of $u : \mathbb{R}^n \to \mathbb{C}$ by

$$\widehat{u}(\xi) = \int_{\mathbb{R}^n} e^{-i2\pi\xi\cdot x} u(x)\, dx$$

and recall the inversion formula

$$u(x) \sim \int_{\mathbb{R}^n} \widehat{u}(\xi) e^{i2\pi\xi\cdot x}\, d\xi.$$

Note that

$$\widehat{\partial^\alpha u}(\xi) = (i2\pi\xi_1)^{\alpha_1} \cdots (i2\pi\xi_n)^{\alpha_n} = (i2\pi\xi)^\alpha \widehat{u}(\xi).$$

Bessel potential of order $s \in \mathbb{R}$:

$$\mathcal{J}^s u(x) = \int_{\mathbb{R}^n} (1 + |\xi|^2)^{s/2} \widehat{u}(\xi) e^{i2\pi\xi\cdot x}\, dx, \qquad x \in \mathbb{R}^n.$$

Notice

$$\mathcal{J}^{s+t} = \mathcal{J}^s \mathcal{J}^t = \mathcal{J}^t \mathcal{J}^s, \quad (\mathcal{J}^s)^{-1} = \mathcal{J}^{-s}, \quad \mathcal{J}^0 = \text{identity}.$$

In fact,

$$\mathcal{J}^s = \left(1 - (2\pi)^{-2}\nabla^2\right)^{s/2}.$$

Space of *rapidly decreasing test functions* $\mathcal{S}(\mathbb{R}^n)$.

Space of *temperate distributions* $\mathcal{S}^*(\mathbb{R}^n)$.

Sobolev space of order s (second definition):

$$H^s(\mathbb{R}^n) = \{\, u \in \mathcal{S}^*(\mathbb{R}^n) : \mathcal{J}^s u \in L_2(\mathbb{R}^n) \,\}.$$

This is a Hilbert space with inner product

$$(u, v)_{H^s(\mathbb{R}^n)} = (\mathcal{J}^s u, \mathcal{J}^s v).$$

Parseval–Plancherel identity,

$$(\widehat{u}, \widehat{v})_{\mathbb{R}^n} = (u, v)_{\mathbb{R}^n},$$

easily implies that, for $r = 0, 1, 2, \ldots,$

$$H^r(\mathbb{R}^n) = W_2^r(\mathbb{R}^n) \quad \text{with} \quad \|u\|_{H^r(\mathbb{R}^n)} \sim \|u\|_{W_2^r(\mathbb{R}^n)}$$

Fractional-order case is a bit harder, but find $H^s(\mathbb{R}^n) = W_2^s(\mathbb{R}^n)$ for all real $s \geq 0$.

For any closed set $F \subseteq \mathbb{R}^n$ and any open set $\Omega \subseteq \mathbb{R}^n$ define

$$\begin{aligned} H_F^s &= \{\, u \in H^s(\mathbb{R}^n) : \operatorname{supp} u \subseteq F \,\}, \\ H^s(\Omega) &= \{\, u \in \mathcal{D}^*(\Omega) : u = U|_\Omega \text{ for some } U \in H^s(\mathbb{R}^n) \,\}, \\ \widetilde{H}^s(\Omega) &= \text{completion of } \mathcal{D}(\Omega) \text{ in } H^s(\mathbb{R}^n), \\ H_0^s(\Omega) &= \text{completion of } \mathcal{D}(\Omega) \text{ in } H^s(\Omega). \end{aligned}$$

Always have

$$\widetilde{H}^s(\Omega) \subseteq H^s_{\overline{\Omega}} \quad \text{and} \quad H^s_0(\Omega) \subseteq H^s(\Omega).$$

1.1 Theorem. If Ω is a Lipschitz domain then

1. $H^s(\Omega) = W^s_2(\Omega)$ for all real $s \geq 0$ and all integer $s < 0$;

2. $H^s(\Omega)^* = \widetilde{H}^{-s}(\Omega)$ and $\widetilde{H}^s(\Omega)^* = H^{-s}(\Omega)$ for all $s \in \mathbb{R}$;

3. $\widetilde{H}^s(\Omega) = H^s_{\overline{\Omega}}$ for all $s \in \mathbb{R}$;

4. $\widetilde{H}^s(\Omega) \subseteq H^s_0(\Omega)$ for all $s \geq 0$;

5. $\widetilde{H}^s(\Omega) = H^s_0(\Omega)$ for all real $s \geq 0$ except $s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$.

Vector-Valued Functions

If $u : \Omega \to \mathbb{C}^m$ then write

$$\langle u, \phi \rangle = \int_\Omega u(x)^T \phi(x)\, dx = \int_\Omega u(x) \cdot \phi(x)\, dx = \int_\Omega u_i(x) \phi_i(x)\, dx$$

and

$$(u, \phi) = \int_\Omega u(x)^* \phi(x)\, dx = \int_\Omega \bar{u}(x) \cdot \phi(x)\, dx = \int_\Omega \bar{u}_i(x) \phi_i(x)\, dx.$$

Denote vector function spaces by

$$\mathcal{D}(\Omega)^m = C^\infty_{\text{comp}}(\Omega)^m, \quad W^s_p(\Omega)^m, \quad H^s(\Omega)^m, \quad \text{etc.}$$

2. Fredholm Property and Elliptic BVPs

Boundary Values

2.1 Lemma. Consider the linear operator $\gamma : \mathcal{D}(\mathbb{R}^n) \to \mathcal{D}(\mathbb{R}^{n-1})$ defined by

$$\gamma u(x_1, \dots, x_{n-1}) = u(x_1, \dots, x_{n-1}, 0).$$

If $s > 1/2$ then γ has a unique extension to a bounded linear operator

$$\gamma : H^s(\mathbb{R}^n) \to H^{s-1/2}(\mathbb{R}^{n-1}).$$

Moreover, this extension has a continuous right inverse, i.e., there exists a bounded linear operator

$$\eta : H^{s-1/2}(\mathbb{R}^{n-1}) \to H^s(\mathbb{R}^n)$$

such that $\gamma\eta v = v$ for $v \in H^{s-1/2}(\mathbb{R}^{n-1})$.

For any subset $S \subseteq \mathbb{R}^n$ we write

$$\mathcal{D}(S) = \{\, u : u = U|_S \text{ for some } U \in \mathcal{D}(\mathbb{R}^n) \,\}.$$

We define the *trace operator* for Ω,

$$\gamma : \mathcal{D}(\overline{\Omega}) \to \mathcal{D}(\Gamma)$$

by putting

$$\gamma u = u|_\Gamma.$$

2.2 Theorem. If Ω is a $C^{k-1,1}$ domain ($k \geq 0$), then its trace operator has a unique extension to a bounded linear operator

$$\gamma : H^s(\Omega) \to H^{s-1/2}(\Gamma) \quad \text{for } \frac{1}{2} < s \leq k.$$

Moreover, this extension has a continuous right inverse.

2.3 Theorem. (Martin Costabel) If Ω is a Lipschitz (i.e., $C^{0,1}$) domain, then its trace operator is bounded for $\frac{1}{2} < s < \frac{3}{2}$.

2.4 Theorem. If Ω is a $C^{k-1,1}$ domain then

$$H_0^s(\Omega) = \left\{ u \in H^s(\Omega) : \gamma(\partial^\alpha u) = 0 \text{ for } |\alpha| < s - \tfrac{1}{2} \right\}$$

for $0 \le s \le k$. In particular,

$$H_0^s(\Omega) = H^s(\Omega) \quad \text{for } 0 \le s \le \tfrac{1}{2}$$

Recall the sesquilinear form

$$\Phi(u,v) = \Phi_\Omega(u,v) = \int_\Omega \Biggl(\sum_{j=1}^n \sum_{k=1}^n (A_{jk}\partial_k u)^* \partial_j v + \sum_{j=1}^n \Bigl((A_j \partial_j u)^* v - (A_j u)^* \partial_j v \Bigr) + (Au)^* v \Biggr) \, dx.$$

Easy to see that

$$|\Phi(u,v)| \le C\|u\|_{H^1(\Omega)^m}\|v\|_{H^1(\Omega)^m} \quad \text{for } u,\ v \in H^s(\Omega)^m.$$

If $u \in H^1(\Omega)^m$ then its restriction $u|_\Gamma$, or rather the trace γu, makes sense as an element of $H^{1/2}(\Gamma)^m$.

The next result allows us to define the *generalized conormal derivative* $\mathcal{B}_\nu u = g$ in such a way that the first Green identity is valid for H^1 functions.

2.5 Lemma. Assume that Ω is a Lipschitz domain. If $u \in H^1(\Omega)^m$ and $f \in \widetilde{H}^{-1}(\Omega)^m$ satisfy

$$\mathcal{P}u = f \quad \text{on } \Omega,$$

then there exists a unique $g \in H^{-1/2}(\Gamma)^m$ such that

$$\Phi(u,v) = (f,v)_\Omega + (g,\gamma v)_\Gamma \quad \text{for all } v \in H^1(\Omega)^m.$$

Abstract Variational Problems

Let V be a complex Hilbert space with inner product $(u,v)_V$ and consider a sesquilinear form

$$\Phi : V \times V \to \mathbb{C}.$$

Assume that Φ is *bounded*, i.e.,

$$|\Phi(u,v)| \le C\|u\|_V\|v\|_V \quad \text{for all } u,\ v \in V.$$

We can define a bounded linear operator $A : V \to V^*$ by

$$(Au,v) = \Phi(u,v) \quad \text{for all } u,\ v \in V,$$

where (f,v) is the dual pairing on $V^* \times V$.

Note: $(\cdot,\cdot)$ may differ from $(\cdot,\cdot)_V$, so we cannot identify V^* with V, although we will freely take $V^{**} = V$ so that $A^* : V \to V^*$ is given by

$$(A^*u, v) = (u, Av) = \overline{(Av, u)} = \overline{\Phi(v, u)}.$$

Given $f \in V^*$, we seek $u \in V$ satisfying the linear equation

$$Au = f,$$

or equivalently, the *variational problem*

$$\Phi(u, v) = (f, v) \quad \text{for all } v \in V.$$

The *dual problem*

$$A^*v = g$$

is equivalent to

$$\overline{\Phi(u, v)} = (g, u) \quad \text{for all } u \in V.$$

We say that Φ and A are *positive and bounded below* (PBB) if

$$\operatorname{Re}\Phi(v,v) \geq c\|v\|_V^2 \quad \text{for all } v \in V.$$

Obviously, A is PBB iff A^* is PBB.

2.6 Lemma. (Lax–Milgram) If Φ is positive and bounded below then $A : V \to V^*$ has a bounded inverse $A^{-1} : V^* \to V$.

Constructive proof due to Hildebrandt and Wienholtz (1964) uses *Galerkin's method*: take any increasing sequence of finite dimensional subspaces

$$V_1 \subseteq V_2 \subseteq V_3 \subseteq \cdots \subseteq V$$

such that $\bigcup_{n=1}^{\infty} V_n$ is dense in V and find $u_n \in V_n$ satisfying

$$\Phi(u_n, v) = (f, v) \quad \text{for all } v \in V_n,$$

then $u_n \to u$.

2.7 Theorem. (Fredholm Alternative) Assume that $A = A_0 + K$, where $A_0 : V \to V^*$ is PBB and $K : V \to V^*$ is compact. Either

> the homogeneous equation $Au = 0$ has only the trivial solution $u = 0$, in which case $A : V \to V^*$ has a bounded inverse $A^{-1} : V^* \to V$ and $A^* : V \to V^*$ has a bounded inverse $(A^*)^{-1} = (A^{-1})^* : V^* \to V$.

or else

> the homogeneous equation $Au = 0$ has p linearly independent solutions $u_1, \ldots, u_p$ for some (finite) $p \geq 1$, in which case
>
> > the homogeneous adjoint equation $A^* v = 0$ has the same number of linearly independent solutions $v_1, \ldots, v_p$;
> >
> > for each $f \in V^*$, the inhomogeneous equation $Au = f$ is solvable iff $(f, v_j) = 0$ for $j = 1, \ldots, p$;
> >
> > for each $g \in V^*$, the inhomogeneous adjoint equation $A^* v = g$ is solvable iff $(g, u_j) = 0$ for $j = 1, \ldots, p$.

Compatibility conditions: if $Au = f$ then

$$(f, v_j) = (Au, v_j) = (u, A^* v_j) = (u, 0) = 0$$

and likewise if $A^* v = g$ then

$$(g, u_j) = (A^* v, u_j) = (v, Au_j) = (v, 0) = 0.$$

Strong Ellipticity

Principal part of $\mathcal{P}$:

$$\mathcal{P}_0 u = -\sum_{j=1}^{n}\sum_{k=1}^{n} \partial_j(A_{jk}\partial_k u),$$

so

$$\mathcal{P}u = \mathcal{P}_0 u + \text{lower-order terms.}$$

If $u \in C^\infty_{\text{comp}}(\Omega)^m$ then

$$\mathcal{P}_0 u(x) \sim \int_{\mathbb{R}^n} P_0(x,\xi)\widehat{u}(\xi)e^{i2\pi\xi\cdot x}\, d\xi$$

where the *principal symbol* of $\mathcal{P}$ is given by

$$P_0(x,\xi) = (2\pi)^2 \sum_{j=1}^{n}\sum_{k=1}^{n} A_{jk}(x)\xi_j\xi_k, \qquad x \in \Omega, \xi \in \mathbb{R}^n.$$

We say that $\mathcal{P}$ is *strongly elliptic* if

$$\mathrm{Re}\Big(P_0(x,\xi)\eta\Big)^*\eta \geq c|\xi|^2|\eta|^2$$

for all $x \in \Omega$, $\xi \in \mathbb{R}^n$ and $\eta \in \mathbb{C}^m$.

2.8 Example. If $\mathcal{P}u = -\mu\nabla^2 u - (\nu + \lambda)\nabla(\operatorname{div} u)$ then

$$P_0(\xi) = (2\pi)^2\Big(\mu|\xi|^2 I + (\mu + \lambda)\xi\xi^T\Big)$$

so

$$\Big(P_0(\xi)\eta\Big)^*\eta = (2\pi)^2\Big(\mu|\xi|^2|\eta|^2 + (\mu + \lambda)|\eta \cdot \xi|^2\Big)$$

for $\xi \in \mathbb{R}^3$ and $\eta \in \mathbb{C}^3$. Hence, $\mathcal{P}$ is strongly elliptic iff the Lamé coefficients satisfy

$$\mu > 0 \quad \text{and} \quad 2\mu + \lambda > 0.$$

When $m = 1$, strong ellipticity reduces to

$$\mathrm{Re}\, P_0(x, \xi) \geq c|\xi|^2.$$

2.9 Example. If $\mathcal{P}u = -\nabla^2 u$ then $P_0(\xi) = (2\pi)^2|\xi|^2$ so $\mathcal{P}$ is strongly elliptic.

We say that $\mathcal{P}$ (or Φ) is *coercive* on a subspace $V \subseteq H^1(\Omega)^m$ if

$$\mathrm{Re}\, \Phi(v, v) \geq c\|v\|^2_{H^1(\Omega)^m} - C\|v\|^2_{L_2(\Omega)^m} \quad \text{for all } v \in V.$$

2.10 Theorem. $\mathcal{P}$ is strongly elliptic iff it is coercive on $H^1_0(\Omega)^m$. In the scalar case $m = 1$, if $\mathcal{P}$ is strongly elliptic then it is coercive on all of $H^1(\Omega)^m$.

For linear elasticity, *Korn's inequality* implies that $\mathcal{P}$ is coercive on all of $H^1(\Omega)^m$.

Dirichlet Problem

Assume that Ω is a bounded, Lipschitz domain in $\mathbb{R}^n$ with boundary $\Gamma = \partial\Omega$, and let $f \in H^{-1}(\Omega)^m$ and $g \in H^{1/2}(\Gamma)^m$ be given.

We seek $u \in H^1(\Omega)^m$ satisfying

$$\mathcal{P}u = f \quad \text{in } \Omega \qquad \text{and} \qquad \gamma u = g \quad \text{on } \Gamma. \tag{1}$$

To formulate the Fredholm alternative, we consider also the *homogeneous problem*,

$$\mathcal{P}u = 0 \quad \text{in } \Omega \qquad \text{and} \qquad \gamma u = 0 \quad \text{on } \Gamma. \tag{2}$$

and the *homogeneous adjoint problem*,

$$\mathcal{P}^* v = 0 \quad \text{in } \Omega \qquad \text{and} \qquad \gamma v = 0 \quad \text{on } \Gamma. \tag{3}$$

2.11 Theorem. If $\mathcal{P}$ is strongly elliptic, then either

> the Dirichlet problem (1) has a unique solution $u \in H^1(\Omega)^m$ for each $f \in H^{-1}(\Omega)^m$ and $g \in H^{1/2}(\Gamma)^m$ and the homogeneous adjoint problem (3) has only the trivial solution $v = 0$,

or else

> the homogeneous Dirichlet problem (2) has p linearly independent solutions in $H_0^1(\Omega)^m$;
>
> the homogeneous adjoint problem (3) has the same number of linearly independent solutions $v_1, \ldots, v_p \in H_0^1(\Omega)^m$;
>
> the inhomogeneous Dirichlet problem (1) is solvable iff f and g satisfy the compatibility conditions
>
> $$(f, v_j)_\Omega = (g, \tilde{\mathcal{B}}_\nu v_j)_\Gamma \quad \text{for } 1 \le j \le p.$$

Sketch of proof: by the Rellich theorem we have a compact imbedding

$$H_0^1(\Omega) \subseteq L_2(\Omega)^m$$

and by coercivity, $\Phi + C$ is PBB on $H_0^1(\Omega)^m$, so we can apply the abstract Fredholm alternative in $H_0^1(\Omega)^m$. This proves the result in the special case $g = 0$ because then, by the first Green identity, $u \in H_0^1(\Omega)^m$ satisfies $\Phi(u, v) = (f, v)_\Omega$ for all $v \in H_0^1(\Omega)^m$.

If $g \neq 0$, then $u = u_0 + u_1$ where $u_0 \in H_0^1(\Omega)^m$ is a solution of

$$\mathcal{P}u_0 = f - \mathcal{P}\eta g \quad \text{in } \Omega \qquad \text{and} \qquad \gamma u_0 = 0 \quad \text{on } \Gamma,$$

and $u_1 = \eta g$ with $\eta : H^{1/2}(\Gamma)^m \to H^1(\Omega)^m$ a continuous right inverse of γ. By the first Green identity,

$$(g, \tilde{\mathcal{B}}_\nu v_j)_\Gamma = \Phi(\eta g, v_j) = (\mathcal{P}\eta g, v_j)_\Omega,$$

so $(f - \mathcal{P}\eta g, v_j)_\Omega = 0$ iff $(f, v_j)_\Omega = (g, \tilde{\mathcal{B}}_\nu v_j)_\Gamma$.

Neumann Problem

Assume that Ω is a bounded, Lipschitz domain in $\mathbb{R}^n$ with boundary $\Gamma = \partial\Omega$, and let $f \in \widetilde{H}^{-1}(\Omega)^m$ and $g \in H^{-1/2}(\Gamma)^m$ be given.

We seek $u \in H^1(\Omega)^m$ satisfying

$$\mathcal{P}u = f \quad \text{in } \Omega \qquad \text{and} \qquad \mathcal{B}_\nu u = g \quad \text{on } \Gamma. \tag{4}$$

To formulate the Fredholm alternative, we consider also the homogeneous problem,

$$\mathcal{P}u = 0 \quad \text{in } \Omega \qquad \text{and} \qquad \mathcal{B}_\nu u = 0 \quad \text{on } \Gamma. \tag{5}$$

and the homogeneous adjoint problem,

$$\mathcal{P}^* v = 0 \quad \text{in } \Omega \qquad \text{and} \qquad \tilde{\mathcal{B}}_\nu v = 0 \quad \text{on } \Gamma. \tag{6}$$

2.12 Theorem. If Φ is coercive on $H^1(\Omega)^m$, then either

the Neumann problem (4) has a unique solution $u \in H^1(\Omega)^m$ for each $f \in \widetilde{H}^{-1}(\Omega)^m$ and $g \in H^{-1/2}(\Gamma)^m$ and the homogeneous adjoint problem (3) has only the trivial solution $v = 0$,

or else

the homogeneous Neumann problem (5) has p linearly independent solutions in $H^1_0(\Omega)^m$;

the homogeneous adjoint problem (6) has the same number of linearly independent solutions $v_1, \ldots, v_p \in H^1_0(\Omega)^m$;

the inhomogeneous Neumann problem (4) is solvable iff f and g satisfy the compatibility conditions

$$(f, v_j)_\Omega + (g, \gamma v_j)_\Gamma = 0 \quad \text{for } 1 \le j \le p.$$

2.13 Example. In linear elasticity,

$$\mathcal{P}u = -\lambda\nabla^2 u - (\nu + \lambda)\nabla(\operatorname{div} u) = \mathcal{P}^* u$$

and, for a connected Lipschitz domain $\Omega \subseteq \mathbb{R}^3$,

$$\mathcal{P}v = 0 \quad \text{in } \Omega$$

iff v is an infinitesimal rigid motion, i.e.,

$$v(x) = a + b \times x$$

for some constant vectors a, $b \in \mathbb{R}^3$. For v of this form,

$$\gamma v = 0 \quad \text{on } \Gamma \qquad \text{only if } a = b = 0,$$

but

$$\mathcal{B}_\nu v = 0 \quad \text{on } \Gamma \qquad \text{for all } a,\ b \in \mathbb{R}^3.$$

3. Surface Potentials

Fundamental Solutions

Assume that $\mathcal{P}$ is a strongly elliptic, second-order partial differential operator on $\mathbb{R}^n$.

A *volume potential* for $\mathcal{P}$ is an integral operator

$$\mathcal{G}u(x) = \int_{\mathbb{R}^n} G(x,y)u(y)\,dy, \quad x \in \mathbb{R}^n,$$

satisfying

$$\mathcal{P}\mathcal{G}u = u = \mathcal{G}\mathcal{P}u \quad \text{on } \mathbb{R}^n,$$

for all distributions u with compact support. The kernel $G(x,y)$ is said to be a *fundamental solution* for $\mathcal{P}$.

If $\mathcal{P}$ has constant coefficients then $\mathcal{P}$ is *translation invariant* so it is natural to seek G in the form

$$G(x,y) = \mathsf{G}(x-y).$$

3.1 Theorem. Assume that $\mathcal{P}$ has constant coefficients and no lower-order terms, so that its symbol is a homogeneous polynomial

$$P(\xi) = \sum_{j=1}^{n} \sum_{k=1}^{n} A_{jk}\xi_j\xi_k.$$

1. If $n = 2$ then a fundamental solution for $\mathcal{P}$ is given by

$$\mathsf{G}(z) = \int_{|\omega|=1} \Big[\Gamma'(1) - \log(2\pi|\omega \cdot z|)\Big] P(\omega)^{-1}\, d\omega.$$

2. If $n = 3$ then a fundamental solution for $\mathcal{P}$ is given by

$$\mathsf{G}(z) = \frac{1}{2|z|} \int_{\mathbb{S}_z^{\perp}} P(\omega_\perp)^{-1}\, d\omega_\perp,$$

where $\mathbb{S}_z^{\perp} = \{\, \omega_\perp \in \mathbb{S}^2 : \omega_\perp \cdot z = 0 \,\}$ is the unit circle in the plane normal to z and $d\omega_\perp$ is the element of arc length along $\mathcal{S}_z^{\perp}$.

3.2 Example. In the case of the Laplacian $\mathcal{P} = -\nabla^2$ we have

$$\mathsf{G}(z) = \frac{1}{2\pi} \log \frac{1}{|z|} \quad \text{if } n = 2,$$

and

$$\mathsf{G}(z) = \frac{1}{4\pi|z|} \quad \text{if } n = 3.$$

In fact,

$$\mathsf{G}(z) = \frac{1}{(n-2)|\mathbb{S}^{n-1}||z|^{n-2}} \quad \text{if } n \geq 3.$$

3.3 Example. For linear elasticity in $\mathbb{R}^3$,

$$\mathcal{P}u = -\lambda\nabla^2 u - (\nu + \lambda)\nabla(\operatorname{div} u),$$

we have

$$\mathsf{G}_{jk}(z) = \frac{1}{8\pi\mu(2\mu + \lambda)}\left((3\mu + \lambda)\frac{\delta_{jk}}{|z|} + (\mu + \lambda)\frac{z_j z_k}{|z|^3}\right).$$

Third Green Identity

Notation:

$$\begin{aligned} \Omega^- &= \text{bounded, Lipschitz domain,} \\ \Omega^+ &= \mathbb{R}^n \setminus \overline{\Omega^-} = \text{complementary, \textit{un}bounded domain,} \\ \Gamma &= \partial\Omega^- = \partial\Omega^+ = \text{common boundary.} \end{aligned}$$

One-sided trace operators

$$\gamma^\pm : H^s(\Omega^\pm)^m \to H^{s-1/2}(\Gamma)^m, \qquad \tfrac{1}{2} < s < \tfrac{3}{2}.$$

One-sided (generalized) conormal derivatives $\mathcal{B}_\nu^\pm$ and $\tilde{\mathcal{B}}_\nu^\pm$.

For u defined on $\mathbb{R}^n$, write $u^\pm = u|_{\Omega^\pm}$ but use $\gamma^+ u$ in preference to $\gamma^+ u^+$.

Jumps accross Γ denoted by

$$[u]_\Gamma = \gamma^+ u - \gamma^- u, \quad [\mathcal{B}_\nu u]_\Gamma = \mathcal{B}_\nu^+ u - \mathcal{B}_\nu^- u, \quad [\tilde{\mathcal{B}}_\nu u]_\Gamma = \tilde{\mathcal{B}}_\nu^+ u - \tilde{\mathcal{B}}_\nu^- u.$$

May drop $+$ or $-$ superscript when jump vanishes, e.g.,

$$\gamma u = \gamma^+ u = \gamma^- u \quad \text{if } [u]_\Gamma = 0.$$

Adjoint of two-sided trace:

$$(\gamma^* \psi, \phi)_{\mathbb{R}^n} = (\psi, \gamma\phi)_\Gamma, \quad \psi \in \mathcal{D}(\Gamma), \phi \in C^\infty(\mathbb{R}^n)^m.$$

Clear that

$$\operatorname{supp} \gamma^* \psi \subseteq \operatorname{supp} \psi \subseteq \Gamma.$$

Adjoint of two-sided conormal derivative:

$$(\mathcal{B}_\nu^* \psi, \phi)_{\mathbb{R}^n} = (\psi, \mathcal{B}_\nu \phi)_\Gamma \quad \text{and} \quad (\tilde{\mathcal{B}}_\nu^* \psi, \phi)_{\mathbb{R}^n} = (\psi, \tilde{\mathcal{B}}_\nu \phi)_\Gamma.$$

3.4 Lemma. Let $f^{\pm} \in \widetilde{H}^{-1}(\Omega^{\pm})^m$ and put $f = f^{+} + f^{-} \in H^{-1}(\mathbb{R}^n)$. Suppose that $u \in L_2(\mathbb{R}^n)^m$ with $u^{\pm} \in H^1(\Omega^{\pm})^m$. If

$$\mathcal{P}u^{\pm} = f^{\pm} \quad \text{on } \Omega^{\pm},$$

then

$$\mathcal{P}u = f + \tilde{\mathcal{B}}_\nu^*[u]_\Gamma - \gamma^*[\mathcal{B}_\nu u]_\Gamma \quad \text{on } \mathbb{R}^n.$$

Proof: the definition of $\mathcal{P}u$ as a distribution on $\mathbb{R}^n$ means that

$$(\mathcal{P}u, \phi)_{\mathbb{R}^n} = (u, \mathcal{P}^*\phi)_{\mathbb{R}^n} = (u^+, \mathcal{P}^*\phi)_{\Omega^+} + (u^-, \mathcal{P}^*\phi)_{\Omega^-},$$

and by the first Green identity,

$$(f^{\pm}, \phi)_{\Omega^{\pm}} \mp (\mathcal{B}_\nu^{\pm} u, \gamma\phi)_\Gamma = \Phi^{\pm}(u^{\pm}, \phi) = (u^{\pm}, \mathcal{P}^*\phi)_{\Omega^{\pm}} \mp (\gamma^{\pm} u, \tilde{\mathcal{B}}_\nu\phi)_\Gamma,$$

so

$$\begin{aligned}(\mathcal{P}u, \phi)_{\mathbb{R}^n} &= \Phi^+(u, \phi) + \Phi^-(u, \phi) + ([u]_\Gamma, \tilde{\mathcal{B}}_\nu\phi)_\Gamma \\ &= (f, \phi)_{\mathbb{R}^n} - ([\mathcal{B}_\nu u]_\Gamma, \gamma\phi)_\Gamma + ([u]_\Gamma, \tilde{\mathcal{B}}_\nu\phi)_\Gamma.\end{aligned}$$

We define the *single-layer potential* SL and the *double-layer potential* DL by

$$\mathsf{SL}\,\psi = \mathcal{G}\gamma^*\psi \quad \text{and} \quad \mathsf{DL}\,\psi = \mathcal{G}\tilde{\mathcal{B}}_\nu^*\psi,$$

so that

$$\begin{aligned}(\mathsf{SL}\,\psi, \phi)_{\mathbb{R}^n} &= (\psi, \gamma\mathcal{G}^*\phi)_\Gamma = \int_\Gamma \psi(y)^*\mathcal{G}^*\phi(y)\,d\sigma_y \\ &= \int_\Gamma \psi(y)^* \int_{\mathbb{R}^n} G(x,y)^*\phi(x)\,dx\,d\sigma_y \\ &= \int_{\mathbb{R}^n}\left(\int_\Gamma G(x,y)\psi(y)\,d\sigma_y\right)^*\phi(x)\,dx,\end{aligned}$$

and similarly,

$$\begin{aligned}(\mathsf{DL}\,\psi, \phi)_{\mathbb{R}^n} &= (\psi, \tilde{\mathcal{B}}_\nu\mathcal{G}^*\phi)_\Gamma = \int_\Gamma \psi(y)^*\left(\tilde{\mathcal{B}}_\nu\mathcal{G}^*\phi\right)(y)\,d\sigma_y \\ &= \int_\Gamma \psi(y)^*\tilde{\mathcal{B}}_{\nu,y}\int_{\mathbb{R}^n} G(x,y)^*\phi(x)\,dx\,d\sigma_y \\ &= \int_{\mathbb{R}^n}\left(\int_\Gamma\left(\tilde{\mathcal{B}}_{\nu,y}G(x,y)^*\right)^*\psi(y)\,d\sigma_y\right)^*\phi(x)\,dx.\end{aligned}$$

Thus,

$$\mathsf{SL}\,\psi(x) = \int_\Gamma G(x,y)\psi(y)\,d\sigma_y.$$

and

$$\mathsf{DL}\,\psi(x) = \int_\Gamma \Big(\tilde{\mathcal{B}}_{\nu,y} G(x,y)^*\Big)^* \psi(y)\,d\sigma_y, \qquad x \notin \Gamma.$$

3.5 Theorem. (Third Green Identity) Let $f^\pm \in \widetilde{H}^{-1}(\Omega^\pm)^m$ and put $f = f^+ + f^- \in H^{-1}(\mathbb{R}^n)$. Suppose that $u \in L_2(\mathbb{R}^n)^m$ with $u^\pm \in H^1(\Omega^\pm)^m$. If

$$\mathcal{P}u^\pm = f^\pm \quad \text{on } \Omega^\pm,$$

and if u has compact support, then

$$u = \mathcal{G}f + \mathsf{DL}[u]_\Gamma - \mathsf{SL}[\mathcal{B}_\nu u]_\Gamma \quad \text{on } \mathbb{R}^n.$$

Proof: $u = \mathcal{G}\mathcal{P}u = \mathcal{G}\Big(f + \tilde{\mathcal{B}}_\nu^*[u]_\Gamma - \gamma^*[\mathcal{B}_\nu u]_\Gamma\Big)$.

Jump Relations

Since $\mathcal{P}\,\mathsf{SL}\,\psi = \mathcal{P}\mathcal{G}\gamma^*\psi = \gamma^*\psi$ and $\mathcal{P}\,\mathsf{DL}\,\psi = \tilde{\mathcal{B}}_\nu^*\psi$ we have

$$\mathcal{P}\,\mathsf{SL}\,\psi = 0 = \mathcal{P}\,\mathsf{DL}\,\psi \quad \text{on } \mathbb{R}^n \setminus \Gamma.$$

We describe some results of Nedelec and Planchard (1973), Le Roux (1974), Hsiao and Wendland (1977) and Costabel (1988).

3.6 Theorem. Fix a cutoff function $\chi \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$. The single-layer potential gives rise to bounded linear operators

$$\begin{aligned} \chi\,\mathsf{SL} &: H^{-1/2}(\Gamma)^m \to H^1(\mathbb{R}^n)^m, \\ \gamma\,\mathsf{SL} &: H^{-1/2}(\Gamma)^m \to H^{1/2}(\Gamma)^m, \\ \mathcal{B}_\nu^\pm\,\mathsf{SL} &: H^{-1/2}(\Gamma)^m \to H^{-1/2}(\Gamma)^m, \end{aligned}$$

satisfying the jump relations

$$[\mathsf{SL}\,\psi]_\Gamma = 0 \quad \text{and} \quad [\mathcal{B}_\nu\,\mathsf{SL}\,\psi]_\Gamma = -\psi \quad \text{for } \psi \in H^{-1/2}(\Gamma)^m.$$

Sketch of Proof: Fix a second cutoff function $\chi_1 \in C^\infty_{\mathsf{comp}}(\mathbb{R}^n)$ satisfying $\chi_1 = 1$ on a neighbourhood of $\overline{\Omega^-}$. We have

$$(\chi \,\mathsf{SL}\, \psi, \phi)_{\mathbb{R}^n} = (\chi \mathcal{G} \chi_1 \gamma^* \psi, \phi)_{\mathbb{R}^n} = (\psi, \gamma \chi_1 \mathcal{G}^* \chi \phi)_{\mathbb{R}^n}$$

and

$$H^{-1}(\mathbb{R}^n)^m \xrightarrow{\chi_1 \mathcal{G}^* \chi} H^1(\mathbb{R}^n)^m \xrightarrow{\gamma} H^{1/2}(\Gamma)^m.$$

Thus,

$$\begin{aligned} \left|(\chi \,\mathsf{SL}\, \psi, \phi)_{\mathbb{R}^n}\right| &\le C\|\psi\|_{H^{-1/2}(\Gamma)^m} \|\gamma \chi_1 \mathcal{G}^* \chi \phi\|_{H^{1/2}(\Gamma)^m} \\ &\le C\|\psi\|_{H^{-1/2}(\Gamma)^m} \|\phi\|_{H^{-1}(\mathbb{R}^n)^m} \end{aligned}$$

which gives $\|\chi \,\mathsf{SL}\, \psi\|_{H^1(\mathbb{R}^n)^m} \le C\|\psi\|_{H^{-1/2}(\Gamma)^m}$, proving the mapping property for $\chi \,\mathsf{SL}$. The mapping property for $\gamma \,\mathsf{SL}$ follows at once, and since $\chi_1 \,\mathsf{SL}\, \psi \in H^1(\mathbb{R}^n)^m$,

$$[\gamma \,\mathsf{SL}\, \psi]_\Gamma = \Big[\gamma(\chi_1 \,\mathsf{SL}\, \psi)\Big]_\Gamma = 0.$$

Let $u = \chi_1 \,\mathsf{SL}\, \psi$. We have

$$\mathcal{P}u^{\pm} = f^{\pm} \quad \text{on } \Omega^{\pm},$$

with $f^{-} = 0$ and $f^{+} \in C^{\infty}_{\text{comp}}(\Omega^{-})^m$. By the definition of $\mathcal{B}_\nu u$,

$$\Phi^{\pm}(u, v) = (f^{\pm}, v)_{\Omega^{\pm}} \mp \Big(\mathcal{B}_\nu^{\pm} u, \gamma^{\pm} v\Big)_{\Gamma} \quad \text{for all } v \in H^1(\Omega^{\pm})^m,$$

and given $\phi \in H^{1/2}(\Gamma)^m$ we put $v = \eta\phi$ to obtain

$$\Big|(\mathcal{B}_\nu^{\pm} u, \phi)_{\Gamma}\Big| \le C\|u\|_{H^1(\Omega^{\pm})^m}\|\phi\|_{H^{1/2}(\Gamma)^m} \le C\|\psi\|_{H^{-1/2}(\Gamma)^m}\|\phi\|_{H^{1/2}(\Gamma)^m}.$$

Hence, $\|\mathcal{B}_\nu^{\pm} \,\mathsf{SL}\, u\|_{H^{-1/2}(\Gamma)^m} \le C\|\psi\|_{H^{-1/2}(\Gamma)^m}$.

The definition of SL gives $\mathcal{P}u = \mathcal{P}\chi_1 \,\mathsf{SL}\, \psi = \gamma^*\psi + f^{+}$ on $\mathbb{R}^n$, so

$$\gamma^*\psi + f^{+} = \mathcal{P}u = f + \tilde{\mathcal{B}}_\nu^*[u]_{\Gamma} - \gamma^*[\mathcal{B}_\nu u]_{\Gamma} = f^{+} - \gamma^*[\mathcal{B}_\nu u]_{\Gamma},$$

and hence $\gamma^*\Big(\psi + [\mathcal{B}_\nu u]_{\Gamma}\Big) = 0$ on $\mathbb{R}^n$. i.e.,

$$\Big(\psi + [\mathcal{B}_\nu u]_{\Gamma}, \phi\Big)_{\Omega'} = 0 \quad \text{for all } \phi \in C^{\infty}_{\text{comp}}(\mathbb{R}^n)^m.$$

3.7 Theorem. Fix a cutoff function $\chi \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$. The double-layer potential gives rise to bounded linear operators

$$\begin{aligned} \chi\,\mathsf{DL} &: H^{1/2}(\Gamma)^m \to H^1(\Omega^\pm)^m, \\ \gamma\,\mathsf{DL} &: H^{1/2}(\Gamma)^m \to H^{1/2}(\Gamma)^m, \\ \mathcal{B}_\nu^\pm\,\mathsf{DL} &: H^{1/2}(\Gamma)^m \to H^{-1/2}(\Gamma)^m, \end{aligned}$$

satisfying the jump relations

$$[\mathsf{DL}\,\psi]_\Gamma = \psi \quad \text{and} \quad [\mathcal{B}_\nu\,\mathsf{DL}\,\psi]_\Gamma = 0 \quad \text{for } \psi \in H^{1/2}(\Gamma)^m.$$

Proof: Choose $\lambda > 0$ such that the interior Dirichlet problem

$$(\mathcal{P} + \lambda)u = 0 \quad \text{in } \Omega^-, \qquad \gamma^- u = g \quad \text{on } \Gamma,$$

is uniquely solvable for all $g \in H^{1/2}(\Gamma)^m$, and denote the solution operator by $\mathcal{U} : g \mapsto u$.

Let $\psi \in H^{1/2}(\Gamma)^m$ and define $u \in L_2(\mathbb{R}^n)^m$ by

$$u = \begin{cases} \mathcal{U}\psi & \text{in } \Omega^-, \\ 0 & \text{in } \Omega^+. \end{cases}$$

Since $\mathcal{P}u^\pm = -\lambda u^\pm$ on $\Omega^\pm$, the third Green identity gives

$$u = -\lambda \mathcal{G}u + \mathrm{DL}[u]_\Gamma - \mathrm{SL}[\mathcal{B}_\nu u]_\Gamma = -\lambda \mathcal{G}u - \mathrm{DL}\,\psi + \mathrm{SL}\,\mathcal{B}_\nu^- \mathcal{U}\psi$$

so

$$\mathrm{DL}\,\psi = \mathrm{SL}\,\mathcal{B}_\nu^- \mathcal{U}\psi - u - \lambda \mathcal{G}u \quad \text{on } \mathbb{R}^n.$$

Mapping properties for DL follow from those for SL and $\mathcal{U}$.

Since $\chi \mathcal{G}u \in H^2(\mathbb{R}^n)^m$ we have $[\mathcal{G}u]_\Gamma = [\mathcal{B}_\nu \mathcal{G}u]_\Gamma = 0$, and the jump relations for SL give $[\mathrm{SL}\,\mathcal{B}_\nu^- \mathcal{U}\psi]_\Gamma = 0$ and $[\mathcal{B}_\nu^-\,\mathrm{SL}\,\mathcal{B}_\nu^- \mathcal{U}\psi]_\Gamma = -\mathcal{B}_\nu^- \mathcal{U}\psi$, so

$$[\mathrm{DL}\,\psi]_\Gamma = -[u]_\Gamma = \psi \quad \text{and} \quad [\mathcal{B}_\nu\,\mathrm{DL}\,\psi]_\Gamma = -\mathcal{B}_\nu^- \mathcal{U}\psi + \mathcal{B}_\nu^- u = 0.$$

Fundamental solution for $\mathcal{P}^*$ is just $\mathcal{G}^*$.

Single- and double-layer potentials for $\mathcal{P}^*$ denoted by

$$\widetilde{\mathsf{SL}}\psi = \mathcal{G}^*\gamma^*\psi \quad \text{and} \quad \widetilde{\mathsf{DL}}\psi = \mathcal{G}^*\mathcal{B}_\nu^*\psi.$$

3.8 Theorem. Let $\psi = \gamma u$ for some $u \in H^1(\mathbb{R}^n)^m$.

1. For all $\phi \in H^{-1/2}(\Gamma)^m$,

$$\Big(\gamma^\pm \, \mathsf{DL}\, \psi, \phi\Big)_\Gamma = \pm\Phi^\mp\Big(u, \widetilde{\mathsf{SL}}\phi\Big) = \Big(\psi, \tilde{\mathcal{B}}_\nu^\mp \widetilde{\mathsf{SL}}\phi\Big)_\Gamma.$$

2. For all $\phi \in H^{1/2}(\Gamma)^m$,

$$\Big(\mathcal{B}_\nu \, \mathsf{DL}\, \psi, \phi\Big)_\Gamma = \pm\Phi^\mp\Big(u, \widetilde{\mathsf{DL}}\phi\Big) = \Big(\psi, \tilde{\mathcal{B}}_\nu \widetilde{\mathsf{DL}}\phi\Big)_\Gamma.$$

4. Boundary Integral Equations

Operators on the Boundary

For simplicity, we assume from now on that $\mathcal{P} = \mathcal{P}^*$, i.e.,

$$A_{kj} = A_{jk}^*, \quad A_j^* = -A_j, \quad A^* = A.$$

Note that consequently $\tilde{\mathcal{B}}_\nu = \mathcal{B}_\nu$, $\mathcal{G}^* = \mathcal{G}$ and

$$G(y,x)^* = G(x,y).$$

Recall that

$$\mathsf{SL}\,\psi(x) = \int_\Gamma G(x,y)\psi(y)\,d\sigma_y = \widetilde{\mathsf{SL}}\psi(x),$$
$$\mathsf{DL}\,\psi(x) = \int_\Gamma \Big(\mathcal{B}_{\nu,y} G(y,x)\Big)^* \psi(y)\,d\sigma_y = \widetilde{\mathsf{DL}}\psi(x).$$

Define boundary integral operators

$$
\begin{aligned}
R &= -\mathcal{B}_\nu \, \mathsf{DL} : H^{1/2}(\Gamma)^m \to H^{-1/2}(\Gamma)^m, \\
S &= \gamma \, \mathsf{SL} : H^{-1/2}(\Gamma)^m \to H^{1/2}(\Gamma)^m, \\
T &= \gamma^+ \, \mathsf{DL} + \gamma^- \, \mathsf{DL} : H^{1/2}(\Gamma)^m \to H^{1/2}(\Gamma)^m,
\end{aligned}
$$

and note that

$$R^* = R, \qquad S^* = S,$$

but

$$T^* = \mathcal{B}_\nu^- \, \mathsf{SL} + \mathcal{B}_\nu^+ \, \mathsf{SL} : H^{-1/2}(\Gamma)^m \to H^{-1/2}(\Gamma)^m.$$

The jump relations imply that

$$
\begin{aligned}
\gamma \, \mathsf{SL} \, \psi &= S\psi, & \gamma^\pm \, \mathsf{DL} \, \psi &= \tfrac{1}{2}(\pm\psi + T\psi), \\
\mathcal{B}_\nu^\pm \, \mathsf{SL} \, \psi &= \tfrac{1}{2}(\mp\psi + T^*\psi), & \mathcal{B}_\nu \, \mathsf{DL} \, \psi &= -R\psi,
\end{aligned}
$$

since, for example,

$$
\begin{aligned}
\gamma^+ \, \mathsf{DL} \, \psi + \gamma^- \, \mathsf{DL} \, \psi &= T\psi, \\
\gamma^+ \, \mathsf{DL} \, \psi - \gamma^- \, \mathsf{DL} \, \psi &= \psi.
\end{aligned}
$$

4.1 Example. If $\mathcal{P} = -\nabla^2$ and $n = 3$, then

$$\mathsf{SL}\,\psi(x) = \frac{1}{4\pi}\int_\Gamma \frac{\psi(y)}{|x-y|}\,d\sigma_y,$$

$$\mathsf{DL}\,\psi(x) = \frac{1}{4\pi}\int_\Gamma \frac{\nu_y\cdot(x-y)}{|x-y|^3}\,\psi(y)\,d\sigma_y,$$

and

$$R\psi(x) = \frac{-1}{4\pi}\,\mathsf{fp}\int_\Gamma \frac{\nu_x\cdot\nu_y}{|x-y|^3}\,\psi(y)\,d\sigma_y - \frac{3}{4\pi}\int_\Gamma \frac{\nu_x\cdot(y-x)\,\nu_y\cdot(x-y)}{|x-y|^5}\,\psi(y)\,d\sigma_y,$$

$$S\psi(x) = \frac{1}{4\pi}\int_\Gamma \frac{\psi(y)}{|x-y|}\,d\sigma_y,$$

$$T\psi(x) = \frac{1}{2\pi}\int_\Gamma \frac{\nu_y\cdot(x-y)}{|x-y|^3}\,\psi(y)\,d\sigma_y.$$

Note: anywhere Γ is C^2 we have $\nu_y\cdot(x-y) = O(|x-y|^2)$.

The Dirichlet Problem

4.2 Theorem. Let $f \in \widetilde{H}^{-1}(\Omega^-)^m$ and $g \in H^{1/2}(\Gamma)^m$.

1. If $u \in H^1(\Omega^-)^m$ is a solution of the interior Dirichlet problem

$$\mathcal{P}u = f \quad \text{in } \Omega^-, \qquad \gamma^- u = g \quad \text{on } \Gamma, \tag{7}$$

   then the conormal derivative $\psi = \mathcal{B}_\nu^- u \in H^{-1/2}(\Gamma)^m$ is a solution of the boundary integral equation

$$S\psi = \tfrac{1}{2}(g + Tg) - \gamma\mathcal{G}f \quad \text{on } \Gamma, \tag{8}$$

   and u has the integral representation

$$u = \mathcal{G}f - \mathrm{DL}\, g + \mathrm{SL}\, \psi \quad \text{in } \Omega^-. \tag{9}$$

2. Conversely, if $\psi \in H^{-1/2}(\Gamma)^m$ is a solution of the boundary integral equation (8), then the formula (9) defines a solution $u \in H^1(\Omega^-)^m$ of (7).

Proof: Suppose that u satisfies (7) and define $u = 0$ and $f = 0$ on Ω^+. By the third Green identity,

$$u = \mathcal{G}f + \mathsf{DL}[\gamma u]_\Gamma - \mathsf{SL}[\mathcal{B}_\nu u]_\Gamma = \mathcal{G}f - \mathsf{DL}\, g + \mathsf{SL}\, \mathcal{B}_\nu^- u \quad \text{on } \Omega^-,$$

so, taking traces of both sides,

$$g = \gamma^- u = \gamma\mathcal{G}f - \tfrac{1}{2}(-g + Tg) + S\mathcal{B}_\nu^- u \quad \text{on } \Gamma,$$

and hence $\psi = \mathcal{B}_\nu^- u$ satisfies

$$S\psi = \tfrac{1}{2}(g + Tg) - \gamma\mathcal{G}f \quad \text{on } \Gamma.$$

Conversely, given ψ satisfying (8) we see that $u = \mathcal{G}f - \mathsf{DL}\, g + \mathsf{SL}\, \psi$ satisfies

$$\mathcal{P}u = \mathcal{P}\mathcal{G}f = f \quad \text{on } \Omega^-$$

and

$$\gamma^- u = \gamma\mathcal{G}f - \tfrac{1}{2}(-g + Tg) + S\psi = g.$$

4.3 Theorem. There exists a decomposition $S = S_0 + L$ such that

$$\mathrm{Re}(S_0\phi, \phi)_\Gamma \ge c\|\phi\|^2_{H^{-1/2}(\Gamma)^m} \quad \text{for all } \phi \in H^{-1/2}(\Gamma)^m$$

and $L : H^{-1/2}(\Gamma)^m \to H^{1/2}(\Gamma)^m$ is a compact linear operator.

Idea of proof for the case $\mathcal{P} = -\nabla^2$ and $n = 3$.

The jump relations and the first Green identity give

$$\begin{aligned}(S\psi, \phi)_\Gamma &= \Big(\gamma\,\mathrm{SL}\,\psi, \partial_\nu^-\,\mathrm{SL}\,\phi - \partial_\nu^+\,\mathrm{SL}\,\phi\Big)_\Gamma \\ &= \Phi^-(\mathrm{SL}\,\psi, \mathrm{SL}\,\phi) + (\mathrm{SL}\,\psi, \underbrace{\nabla^2\,\mathrm{SL}\,\phi}_{=0})_{\Omega^-} \\ &\qquad + \Phi^+(\mathrm{SL}\,\psi, \mathrm{SL}\,\phi) + (\mathrm{SL}\,\psi, \underbrace{\nabla^2\,\mathrm{SL}\,\phi}_{=0})_{\Omega^+},\end{aligned}$$

so

$$(S\phi, \phi)_\Gamma = \int_{\mathbb{R}^n} |\nabla\,\mathrm{SL}\,\phi|^2\,dx.$$

Hence, again using one of the jump relations,

$$\|\phi\|^2_{H^{-1/2}(\Gamma)^m} = \left\|[\mathcal{B}_\nu \,\mathsf{SL}\, \phi]_\Gamma\right\|^2_{H^{-1/2}(\Gamma)^m} \le C\|\chi \,\mathsf{SL}\, \phi\|^2_{H^1(\mathbb{R}^n)^m}.$$

Also, for $\mathcal{P} = -\nabla^2$ and $n = 3$,

$$\begin{aligned}(S\psi, \psi)_\Gamma = 0 \quad &\Rightarrow \quad \nabla \,\mathsf{SL}\, \psi = 0 \text{ on } \Omega^\pm \\ &\Rightarrow \quad \psi = -[\partial_\nu \,\mathsf{SL}\, \psi]_\Gamma = 0,\end{aligned}$$

so $S\psi = 0$ has only the trivial solution, and hence S has a bounded inverse

$$S^{-1} : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma).$$

However, for $n = 2$,

$$S \text{ is invertible} \iff \text{logarithmic capacity of } \Gamma \ne 1.$$

The Neumann Problem

4.4 Theorem. Let $f \in \widetilde{H}^{-1}(\Omega^-)^m$ and $g \in H^{-1/2}(\Gamma)^m$.

1. If $u \in H^1(\Omega^-)^m$ is a solution of the interior Neumman problem

$$\mathcal{P}u = f \quad \text{in } \Omega^-, \qquad \mathcal{B}_\nu^- u = g \quad \text{on } \Gamma, \tag{10}$$

   then the trace $\psi = \gamma^- u \in H^{1/2}(\Gamma)^m$ is a solution of the boundary integral equation

$$R\psi = \tfrac{1}{2}(g - T^* g) - \mathcal{B}_\nu \mathcal{G} f \quad \text{on } \Gamma, \tag{11}$$

   and u has the integral representation

$$u = \mathcal{G} f - \mathrm{DL}\,\psi + \mathrm{SL}\,g \quad \text{in } \Omega^-. \tag{12}$$

2. Conversely, if $\psi \in H^{1/2}(\Gamma)^m$ is a solution of the boundary integral equation (11), then the formula (12) defines a solution $u \in H^1(\Omega^-)^m$ of (10).

Notation: $\Omega_\rho^+ = \{ x \in \Omega^+ : |x| < \rho \}$ for ρ sufficiently large.

4.5 Theorem. If $\mathcal{P}$ is coercive on $H^1(\Omega^-)^m$ and on $H^1(\Omega_\rho^+)^m$, then R is coercive on $H^{1/2}(\Gamma)^m$, i.e.,

$$\mathrm{Re}(R\phi, \phi)_\Gamma \geq c\|\phi\|^2_{H^{1/2}(\Gamma)^m} - C\|\phi\|^2_{L_2(\Gamma)^m} \quad \text{for all } \phi \in H^{1/2}(\Gamma)^m.$$

For $\mathcal{P} = -\nabla^2$ it can be shown that

$$(R\psi, \phi)_\Gamma = \int_{\Omega^+ \cup \Omega^-} \nabla \,\mathrm{DL}\, \bar{\psi} \cdot \nabla \,\mathrm{DL}\, \phi \, dx,$$

but, if Γ is connected,

$$\begin{aligned} (R\psi, \psi)_\Gamma = 0 \quad &\Rightarrow \quad \nabla \,\mathrm{DL}\, \psi = 0 \text{ on } \Omega^\pm \\ &\Rightarrow \quad \psi = [\gamma \,\mathrm{DL}\, \psi]_\Gamma = \text{const.} \end{aligned}$$

Conversely, if $\psi = 1$ then it follows from the third Green identity that $\mathrm{DL}\, \psi = 1$ on Ω^- and 0 on Ω^+, so $R\psi = -\partial_\nu \,\mathrm{DL}\, \psi = 0$.

Mixed Boundary Conditions

Suppose we have a *Lipschitz dissection*

$$\Gamma = \Gamma_D \cup \Pi \cup \Gamma_N.$$

Given $f \in \widetilde{H}^{-1}(\Omega)^m$, $g_D \in H^{1/2}(\Gamma_D)^m$ and $g_N \in H^{-1/2}(\Gamma_N)^m$, we seek $u \in H^1(\Omega)^m$ satisfying

$$\mathcal{P}u = f \text{ in } \Omega, \qquad \gamma u = g_D \text{ on } \Gamma_D, \qquad \mathcal{B}_\nu u = g_N \text{ on } \Gamma_N. \tag{13}$$

To state the Fredholm alternative, we consider at the same time the homogeneous problem

$$\mathcal{P}v = 0 \text{ in } \Omega, \qquad \gamma v = 0 \text{ on } \Gamma_D, \qquad \mathcal{B}_\nu v = 0 \text{ on } \Gamma_N. \tag{14}$$

4.6 Theorem. If Φ is coercive on $H^1(\Omega)^m$, then either

> the mixed problem (13) has a unique solution $u \in H^1(\Omega)^m$ and the homogeneous problem (14) has only the trivial solution $v = 0$,

or else

> the homogeneous mixed problem (14) has p linearly independent solutions $v_1, \ldots, v_p \in H^1(\Omega)^m$ and the inhomogeneous mixed problem (13) is solvable iff f, g_D and g_N satisfy the compatibility conditions
>
> $$(f, v_j)_\Omega + (g_\mathrm{N}, \gamma v_j)_{\Gamma_\mathrm{N}} = (g_\mathrm{D}, \mathcal{B}_\nu v_j)_{\Gamma_\mathrm{D}} \quad \text{for } 1 \le j \le p.$$

Proof based on variational formulation. Let

$$H^1_{\mathsf{D}}(\Omega)^m = \{\, u : \gamma u = 0 \text{ on } \Gamma_{\mathsf{D}} \,\}.$$

If $u \in H^1(\Omega)^m$ is a solution of (13) and if $v \in H^1_{\mathsf{D}}(\Omega)^m$ then the first Green identity gives

$$\Phi(u,v) = (\mathcal{P}u, v)_\Omega + (\mathcal{B}_\nu u, \gamma v)_\Gamma = (f, v)_\Omega + (g_{\mathsf{N}}, \gamma v)_{\Gamma_{\mathsf{N}}}.$$

Introduce restricted boundary integral operators

$$S_{\mathsf{DD}}\psi = (S\psi)|_{\Gamma_{\mathsf{D}}} \quad \text{and} \quad T^*_{\mathsf{ND}}\psi = (T^*\psi)|_{\Gamma_{\mathsf{N}}} \quad \text{if } \operatorname{supp}\psi \subseteq \Gamma_{\mathsf{D}} \cup \Pi,$$

and

$$R_{\mathsf{NN}}\psi = (R\psi)|_{\Gamma_{\mathsf{N}}} \quad \text{and} \quad T_{\mathsf{DN}}\psi = (T\psi)|_{\Gamma_{\mathsf{D}}} \quad \text{if } \operatorname{supp}\psi \subseteq \Gamma_{\mathsf{N}} \cup \Pi.$$

Thus,

$$\left.\begin{aligned} S_{\mathsf{DD}}\psi(x) &= \int_{\Gamma_{\mathsf{D}}} G(x,y)\psi(y)\,d\sigma_y, \\ T_{\mathsf{DN}}\psi(x) &= \mathsf{pv}\int_{\Gamma_{\mathsf{N}}} \mathcal{B}_{\nu,y}G(x,y)\psi(y)\,d\sigma_y \end{aligned}\right\} \quad \text{for } x \in \Gamma_{\mathsf{D}},$$

and

$$\left.\begin{aligned} R_{\mathsf{NN}}\psi(x) &= \mathsf{fp}\int_{\Gamma_{\mathsf{N}}} \mathcal{B}_{\nu,x}\mathcal{B}_{\nu,y}G(x,y)\psi(y)\,d\sigma_y, \\ T^*_{\mathsf{ND}}\psi(x) &= \mathsf{pv}\int_{\Gamma_{\mathsf{D}}} \mathcal{B}_{\nu,x}G(x,y)\psi(y)\,d\sigma_y \end{aligned}\right\} \quad \text{for } x \in \Gamma_{\mathsf{N}}.$$

Mapping properties:

$$\begin{aligned} S_{\mathsf{DD}} &: \widetilde{H}^{-1/2}(\Gamma_{\mathsf{D}})^m \to H^{1/2}(\Gamma_{\mathsf{D}})^m, \\ T_{\mathsf{DN}} &: \widetilde{H}^{1/2}(\Gamma_{\mathsf{N}})^m \to H^{1/2}(\Gamma_{\mathsf{D}})^m, \\ R_{\mathsf{NN}} &: \widetilde{H}^{1/2}(\Gamma_{\mathsf{N}})^m \to H^{-1/2}(\Gamma_{\mathsf{N}})^m, \\ T^*_{\mathsf{ND}} &: \widetilde{H}^{-1/2}(\Gamma_{\mathsf{D}})^m \to H^{-1/2}(\Gamma_{\mathsf{N}}). \end{aligned}$$

4.7 Theorem. Let

$$f \in \widetilde{H}^{-1}(\Omega^-)^m, \quad g_\mathsf{D} \in H^{1/2}(\Gamma)^m, \quad g_\mathsf{N} \in H^{-1/2}(\Gamma)^m$$

and put

$$h_\mathsf{D} = \Big(\tfrac{1}{2}(g_\mathsf{D} + T g_\mathsf{D}) - S g_\mathsf{N} - \gamma \mathcal{G} f\Big)\Big|_{\Gamma_\mathsf{D}} \quad \in H^{1/2}(\Gamma_\mathsf{D})^m,$$
$$h_\mathsf{N} = \Big(\tfrac{1}{2}(g_\mathsf{N} - T^* g_\mathsf{N}) - R g_\mathsf{D} - \mathcal{B}_\nu \mathcal{G} f\Big)\Big|_{\Gamma_\mathsf{N}} \in H^{-1/2}(\Gamma)^m,$$

The function $u \in H^1(\Omega^-)^m$ is a solution of the mixed problem (13) iff the differences

$$\psi_\mathsf{D} = \mathcal{B}_\nu^- u - g_\mathsf{N} \ \in \widetilde{H}^{-1/2}(\Gamma_\mathsf{D})^m,$$
$$\psi_\mathsf{N} = \gamma^- u - g_\mathsf{D} \ \in \widetilde{H}^{1/2}(\Gamma_\mathsf{N})^m,$$

satisfy the 2×2 system of integral equations

$$\begin{bmatrix} S_\mathsf{DD} & -\tfrac{1}{2} T_\mathsf{DN} \\ \tfrac{1}{2} T^*_\mathsf{ND} & R_\mathsf{NN} \end{bmatrix} \begin{bmatrix} \psi_\mathsf{D} \\ \psi_\mathsf{N} \end{bmatrix} = \begin{bmatrix} h_\mathsf{D} \\ h_\mathsf{N} \end{bmatrix},$$

in which case u has the integral representation

$$u = \mathcal{G} f - \mathsf{DL}(\psi_\mathsf{N} + g_\mathsf{D}) + \mathsf{SL}(\psi_\mathsf{D} + g_\mathsf{N}) \quad \text{on } \Omega^-.$$

Can show that the Fredholm alternative holds for

$$A = \begin{bmatrix} S_{\mathsf{DD}} & -\frac{1}{2}T_{\mathsf{DN}} \\ \frac{1}{2}T^*_{\mathsf{ND}} & R_{\mathsf{NN}} \end{bmatrix}$$

as an operator

$$\widetilde{H}^{-1/2}(\Gamma_{\mathsf{D}})^m \times \widetilde{H}^{1/2}(\Gamma_{\mathsf{N}})^m \to H^{1/2}(\Gamma_{\mathsf{D}})^m \times H^{-1/2}(\Gamma_{\mathsf{N}})^m$$

by observing that

$$\begin{aligned}(A\psi, \psi)_{\Gamma_{\mathsf{D}}\times\Gamma_{\mathsf{N}}} &= \left(S_{\mathsf{DD}}\psi_{\mathsf{D}} - \tfrac{1}{2}T_{\mathsf{DN}}\psi_{\mathsf{N}}, \psi_{\mathsf{D}}\right)_{\Gamma_{\mathsf{D}}} \\ &\qquad + \left(\tfrac{1}{2}T^*_{\mathsf{ND}}\psi_{\mathsf{D}} + R_{\mathsf{NN}}\psi_{\mathsf{N}}, \psi_{\mathsf{N}}\right)_{\Gamma_{\mathsf{N}}}\end{aligned}$$

and

$$(T^*_{\mathsf{ND}}\psi_{\mathsf{D}}, \psi_{\mathsf{N}})_{\Gamma_{\mathsf{N}}} = (\psi_{\mathsf{D}}, T_{\mathsf{DN}}\psi_{\mathsf{N}})_{\Gamma_{\mathsf{D}}},$$

so

$$\operatorname{Re}(A\psi, \psi)_{\Gamma_{\mathsf{D}}\times\Gamma_{\mathsf{N}}} = (S_{\mathsf{DD}}\psi_{\mathsf{D}}, \psi_{\mathsf{D}})_{\Gamma_{\mathsf{D}}} + (R_{\mathsf{NN}}\psi_{\mathsf{N}}, \psi_{\mathsf{N}})_{\Gamma_{\mathsf{N}}}.$$