Multifractal wave functions on a class of one-dimensional quasicrystals: Exact $f(\alpha)$ curves and the limit of dilute quasiperiodic impurities

Mark Holzer
Laboratory of Atomic and Solid State Physics, Cornell University, Ithaca, New York 14853-2501

(Received 25 January 1991)

We calculate the exact multifractal scaling spectrum $f(\alpha)$ for the center-band wave function of an off-diagonal tight-binding Hamiltonian defined on the "precious-mean" (PM) lattices, i.e., the class of one-dimensional quasiperiodic lattices generated recursively by $(A,B) \to (A^nB,A)$. We find that, in the limit of dilute quasiperiodic "impurities," $n \to \infty$, the center-band wave function approaches a Bloch state for n even, whereas for n odd a limiting "critical" state is approached. This difference between even and odd n is explained in terms of the convergence properties of the spectrum of the same Hamiltonian defined on periodic extensions of finite-iteration approximants to the PM lattices. For both even and odd n, corrections to the $n=\infty$ limit go to zero like $1/\ln n$. The scaling properties of generic eigenstates are discussed.

I. INTRODUCTION

In recent years, there has been considerable interest in generalized Fibonacci lattices.[1-5] Studies have been concerned primarily with the associated renormalization-group equations, their trace maps, and trace-map invariants. The precise nature of the eigenstates corresponding to nonescaping trace-map orbits has been studied much less.

In this paper, we study the scaling properties of the electronic states of a tight-binding model defined on a class of generalized Fibonacci lattices. We focus on the class defined recursively by the inflation rule[3,5]

$$(A,B) \to (A^nB,A) \ , \tag{1}$$

where A^n denotes a string of n A's. The nth member of this class corresponds to the infinite iterate of (1) for fixed n. For $n=1$, 2, and 3, these lattices have incommensurabilities which are traditionally called the golden, silver, and bronze mean, respectively.[2,3,5] In keeping with this nomenclature, we call the lattices (1) "precious-mean" (PM) lattices.[3,5] The associated matrix renormalization group for 2×2 unimodular transfer matrices has been studied previously[3-5] and was found to possess a trace map and trace-map invariant manifold. A tight-binding model defined on the lattices (1) has a zero-Lebesgue-measure, multifractal spectrum and the corresponding eigenstates are "critical," i.e., intermediate between localized and extended.[1-3,6] Such wave functions have complex multifractal scaling properties and can be characterized by the so-called $f(\alpha)$ curve.[7,8]

Generally, $f(\alpha)$ can be computed only numerically[7] or by approximate analytical methods.[6,9,10] However, when an eigenstate corresponds to a cycle of the underlying renormalization-group matrix recursion equation, it is possible to compute $f(\alpha)$ *exactly* as was recently demonstrated by Fujiwara, Kohmoto, and Tokihiro[8] for the 6-cycle of an off-diagonal tight-binding model defined on the Fibonacci lattice ($n=1$). Here, we will extend their analysis to the center-band wave functions of an off-diagonal tight-binding model defined on the PM lattices for general n.

It is interesting to consider the limit $n \to \infty$. In this limit we may regard the PM lattices to be periodic lattices containing dilute quasiperiodic impurities (the B's). On one hand, we might naively expect the wave functions to become more like Bloch states as $n \to \infty$. On the other hand, for any finite n, we know that the wave functions must always be critical. The $n \to \infty$ limit turns out to be very subtle. However, for the center-band wave functions, the exact analytical solution for $f(\alpha)$ for general n allows us to determine the $n \to \infty$ limit unambiguously and to gain some insight into the nature of the eigenstates. We find that the $n \to \infty$ limit for n even is fundamentally different from the $n \to \infty$ limit for n odd. For n even, the center-band eigenstates do, indeed, approach a Bloch state. For n odd, it turns out to be impossible for the center-band eigenstate to converge to a Bloch state and a limiting critical state is approached instead. We will link the failure of the n=odd center-band wave function to approach a Bloch state to the following fact: If the Hamiltonian of interest is defined on *periodic extensions* of finite inflations of (1), then the center of the corresponding spectrum lies in a gap for every third inflation.

In Sec. II, we present an exact calculation of $f(\alpha)$ for general n, and give its large-n asymptotic form. In Sec. III, we explain why the $n \to \infty$ limit of $f(\alpha)$ for n=even is different from the $n \to \infty$ limit for n=odd. We discuss the relevance of our results to a generic eigenstate of the spectrum. In Sec. IV, we conclude.

II. CALCULATION OF $f(\alpha)$

Our starting point is the off-diagonal tight-binding Hamiltonian

$$\mathcal{H} = \sum_j t_{j+1}(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) \ , \tag{2}$$

where j labels the sites of a PM lattice, $t_j = t_A$ or t_B depending on whether the site j is of type A or B, and $c_j^\dagger$ creates an electron at site j. We will consider finite lattices corresponding to k iterations of (1) and take the limit $k \to \infty$. The number of A's in the kth iterate of (1) is F_k and the number of B's is F_{k-1}, where $F_k = nF_{k-1} + F_{k-2}$, with $F_0 = 1$ and $F_1 = n$. The incommensurability of the lattices is given by the precious means $\tau_n \equiv \lim_{k\to\infty}(F_k/F_{k-1}) = (n+\sqrt{n^2+4})/2$. The number of sites of the kth iteration PM lattice, $L_k = F_k + F_{k-1}$, thus goes like τ_n^k for large k. We may think of $1/\tau_n$ as the density of the "impurities" B, which goes to zero like $1/n$ for large n.

Following Refs. 7 and 8, we now define the scaling properties of the site wave functions ψ_j. First, the wave function is normalized so that $\sum_{j=1}^{L_k} |\psi_j|^2 = 1$. We then define the set of scaling exponents $\{\alpha_j\}$ through the relation $|\psi_j|^2 = L_k^{-\alpha_j}$. The quantity of interest is the distribution of the α_j's in the limit as $k \to \infty$. Let $\Omega_k(\alpha)$ denote the number of sites for which $\alpha_j \in (\alpha, \alpha + d\alpha)$. As $k \to \infty$, $\Omega_k(\alpha)$ scales with the number of sites, L_k, like $L_k^{f(\alpha)}$, so that $f(\alpha) = \lim_{k\to\infty}[\ln \Omega_k(\alpha)/\ln L_k]$.[11] $f(\alpha)$ may be found as the saddle point of the $k \to \infty$ limit of the partition function

$$Z_k(q) \equiv \left(\sum_{j=1}^{L_k} |\psi_j|^{2q}\right) \Bigg/ \left(\sum_{j=1}^{L_k} |\psi_j|^2\right)^q \ , \tag{3}$$

where the denominator serves to explicitly normalize the wave function and $q \in (-\infty, +\infty)$. One thus easily finds that $f(\alpha)$ is given parametrically by

$$f(\alpha) = g(q) + q\alpha(q), \quad \alpha(q) = -\frac{\partial}{\partial q} g(q) \ , \tag{4}$$

with

$$g(q) \equiv \lim_{k\to\infty} \frac{\ln Z_k(q)}{\ln L_k} \ . \tag{5}$$

For a Bloch state, $\psi_j \sim 1/\sqrt{L_k}$, and the $f(\alpha)$ curve reduces to the single point $(\alpha, f(\alpha)) = (1,1)$.

The Schrödinger equation corresponding to Hamiltonian (2) can be written in transfer-matrix form as

$$\Psi_{j+1} = T_{j+1,j}\,\Psi_j \ , \tag{6}$$

with

$$\Psi_j \equiv \begin{pmatrix} \psi_j \\ \psi_{j-1} \end{pmatrix} \ , \tag{7}$$

and

$$T_{j+1,j} = \begin{pmatrix} E/t_{j+1} & -t_j/t_{j+1} \\ 1 & 0 \end{pmatrix} \ , \tag{8}$$

for energy E. Thus, there are three types of transfer matrices T_{AA}, T_{AB}, and T_{BA} corresponding to the three possibilities for (t_{j+1}, t_j), i.e., (t_A, t_A), (t_A, t_B), and (t_B, t_A), respectively. In order to take advantage of the inflation symmetry (1), it is useful to have only two transfer matrices, one of type A and one of type B. A natural choice for such transfer matrices is

$$M_B = T_{AA}, \qquad M_A = T_{AB}T_{BA}T_{AA}^{n-1}. \tag{9}$$

M_A and M_B connect the subset $\mathcal{L}_k$ of the kth iteration PM lattice. (We will label the sites of $\mathcal{L}_k$ by capitalized letters.) The lattice $\mathcal{L}_k$ forms itself a PM lattice, so that under inflation the transfer matrices for $\mathcal{L}_k$ obey the renormalization-group recursion equation

$$M_{k+1} = M_{k-1}M_k^n, \tag{10}$$

with the initial condition $M_0 = M_B$ and $M_1 = M_A$. M_A and M_B are unimodular [i.e., $\mathrm{Det}(M_A) = \mathrm{Det}(M_B) = 1$] so that the trace-map analysis of Refs. 3–5, 12, and 13 applies. The condition for E to be an eigenvalue of Hamiltonian (2) with open boundary conditions is that $\mathrm{Tr}[M_k(E)]$ remain bounded as $k \to \infty$.

For $E = 0$ and n even, Eq. (10) has the 4-cycle

$$M_B \to M_A \to \sigma M_B \to \sigma M_A \to M_B \ , \tag{11}$$

and for $E = 0$ and n odd, Eq. (10) has the 6-cycle

$$M_B \to M_A \to \bar\sigma R(r^{-n}) \to -\bar\sigma R(r^{1-n}) \to -\bar\sigma \bar R^{-1} \to R(r) \to M_B \ , \tag{12}$$

with $\sigma \equiv (-1)^{n/2}$, $\bar\sigma \equiv (-1)^{(n+1)/2}$,

$$R(r) \equiv \begin{pmatrix} 0 & -r \\ 1/r & 0 \end{pmatrix}, \quad \text{and } \bar R \equiv \begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix}, \tag{13}$$

where $r \equiv t_B/t_A$. For $E = 0$, M_B and M_A are given by

$$M_B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{14}$$

and

$$M_A = \begin{cases} \sigma R(r) & \text{for } n \text{ even} \\ \bar\sigma \bar R & \text{for } n \text{ odd.} \end{cases} \tag{15}$$

For the cycles (11) and (12), $\mathrm{Tr}(M_k)$ is clearly bounded as $k \to \infty$ so that $E = 0$ belongs, indeed, to the spectrum of Hamiltonian (2). Because this spectrum is symmetric about $E = 0$, we refer to the $E = 0$ eigenstate as the center-band wave function. Since the cycles (11) and (12) appear to be the only cycles of the *matrix* renormalization-group equation for Hamiltonian (2), and since we will only be able to compute $f(\alpha)$ exactly for such cycles, we will from now on take $E = 0$. (The case of general E will be discussed at the end of this paper.) However, $E = 0$ is not the only eigenvalue whose wave function is dominated by the 4- and 6-cycles (11) and (12). In fact, it follows from the nonlinear dynamics of the associated trace maps, that there is an infinite hierarchical subset of the spectrum whose scaling properties are governed by the same cycles.[3,13] These energies lie at the centers of a hierarchy of clusters of allowed energies. All the corresponding wave functions are characterized by the same $f(\alpha)$ curve.

We can now compute $g(q)$ via a straightforward extension of the method of Ref. 8. For convenience, we consider open boundary conditions with the initial condition $\Psi_0 = \binom{1}{1}$. The simple form of the transfer matrices M_B and M_A then guarantees that Ψ_j has the form

$$\pm\begin{pmatrix} \pm r^{-s} \\ r^s \end{pmatrix}$$

for all j, with s an integer. Consider any two neighboring sites of $\mathcal{L}$, J and $J+1$. Under m inflations, $J \to J'$ and $(J+1) \to (J+1)'$, where now J' and $(J+1)'$ are no longer neighbors but are separated by the m-fold inflation of either A or B. The greatly simplifying consequence of the fact that the renormalization-group equation has an m-cycle ($m = 4$ or 6 here) is that $\Psi_J = \Psi_{J'}$. Because of this fact, we can compute the wave functions of $\mathcal{L}_{k+m}$ from the wave functions of $\mathcal{L}_k$ by using the transfer matrices M_A and M_B to "propagate" the wave function at J' over the finite number of new sites to $(J+1)'$. The simplicity of the form of the wave functions allows us to do this for arbitrary J. We are thus led to a recursion equation for a self-similar wave function. Denote by $N_A(s,k)$ and $N_B(s,k)$ the number of sites $J \in \mathcal{L}_k$ which are connected to the previous site $(J-1) \in \mathcal{L}_k$ by either M_A or M_B, respectively. Then we have

$$N_A(s,k+m) = \sum_{s'}[a_A^+(s')N_A(s-s',k) + a_B^+(s')N_B(s-s',k) + a_A^-(s')N_A(-s-s',k) + a_B^-(s')N_B(-s-s',k)] \quad (16)$$

and

$$N_B(s,k+m) = \sum_{s'}[b_A^+(s')N_A(s-s',k) + b_B^+(s')N_B(s-s',k) + b_A^-(s')N_A(-s-s',k) + b_B^-(s')N_B(-s-s',k)]. \quad (17)$$

In terms of $N_A(s,k)$ and $N_B(s,k)$, the sums of the partition function, Eq. (3), become

$$\sum_{j=1}^{L_k} |\psi_j|^{2q} = \sum_{s=-\infty}^{+\infty} \left\{ N_A(s,k)\left[r^{-2sq} + r^{2sq} + (\tfrac{n}{2})r^{-2(s+1)q} + (\tfrac{n}{2}-1)r^{2(s+1)q} \right] + N_B(s,k)r^{-2sq} \right\} \quad (18)$$

for n even, and

$$\sum_{j=1}^{L_k} |\psi_j|^{2q} = \sum_{s=-\infty}^{+\infty} \left\{ N_A(s,k)\left[r^{-2sq} + r^{2sq} + \left(\tfrac{n-1}{2}\right) r^{-2(s+1)q} + \left(\tfrac{n-1}{2}\right) r^{2(s+1)q} \right] + N_B(s,k)r^{-2sq} \right\} \quad (19)$$

for n odd. The coefficient of N_A is not simply r^{-2sq} because every M_A transfers over n sites of the original PM lattice. The recursion equations (16) and (17) are (block-) diagonal in the variables

$$n_A(x,k) \equiv \sum_{s=-\infty}^{+\infty} x^{-s} N_A(s,k) \quad \text{and} \quad n_B(x,k) \equiv \sum_{s=-\infty}^{+\infty} x^{-s} N_B(s,k). \quad (20)$$

If we define

$$|\nu(x,k)\rangle \equiv \begin{pmatrix} n_A(x,k) \\ n_A(1/x,k) \\ n_B(x,k) \\ n_B(1/x,k) \end{pmatrix}, \quad (21)$$

then the recursion equations (16) and (17) take the form

$$|\nu(x,k+m)\rangle = \Lambda(x)|\nu(x,k)\rangle, \quad (22)$$

where Λ is a 4×4 matrix of unit determinant. The sums (18) and (19) can be written in terms of $|\nu(x,k)\rangle$ as

$$\sum_{j=1}^{L_k} |\psi_j|^{2q} = \langle a(r^{2q})|\nu(r^{2q},k)\rangle, \quad (23)$$

so that the partition function becomes

$$Z_{k+ml}(q) = \frac{\langle a(r^{2q})|\Lambda^l(r^{2q})|\nu(r^{2q},k)\rangle}{\langle a(r^2)|\Lambda^l(r^2)|\nu(r^2,k)\rangle^q}, \quad (24)$$

where $\langle a(x)| = (g(x), f(x), 1, 0)$. For completeness, $g(x) = 1 + n/(2x)$, $f(x) = 1 + (n-2)x/2$ for n even, and $g(x) = f(1/x) = 1 + (n-1)/(2x)$ for n odd. However, the precise form of $\langle a(x)|$ is not important since it is clear from the form of (24) that, in the limit $k \to \infty$, the partition function $Z_k(q)$ is dominated by the largest eigenvalue, λ, of the matrix Λ, so that we finally obtain

$$g(q) = \frac{1}{m \ln \tau_n}\left[\ln \lambda(r^{2q}) - q \ln \lambda(r^2)\right]. \quad (25)$$

It immediately follows from Eq. (4) that

$$\alpha = \frac{1}{m \ln \tau_n}\left(\ln \lambda(r^2) - \frac{\partial}{\partial q}\ln \lambda(r^{2q})\right) \quad (26)$$

and

$$f(\alpha) = \frac{1}{m \ln \tau_n}\left(\ln \lambda(r^{2q}) - q\frac{\partial}{\partial q}\ln \lambda(r^{2q})\right). \quad (27)$$

These are basically the same formulas as those for $n = 1$

in Ref. 8, except that, of course, we must compute λ for the recursion equations of PM lattices for general n.

Since only the absolute values of the site wave functions enter the partition function $Z_k(q)$, we can treat the 4-cycle for n even as an effective 2-cycle and the matrix Λ is readily obtained. For n=odd, however, we are forced to deal with the full 6-cycle (12). (The simplification that the 6-cycle reduces to an effective 3-cycle for $n = 1$, which was used in Ref. 8, does not hold for odd $n > 1$.) To obtain the recursion equations (16) and (17) we must, therefore, compute the site wave functions for the sixfold inflations of A and B for general n, which correspond to $\simeq \tau_n^6(1 + \tau_n)/\sqrt{n^2+4}$ and $\simeq \tau_n^5(1 + \tau_n)/\sqrt{n^2+4}$ sites, respectively. Also, the number of nonzero coefficients of the recursion equations grows linearly with n for n odd, unlike in the n even case, where the number of nonzero coefficients does not depend on n. After a considerable amount of algebra, we eventually obtain the matrices $\Lambda(x)$ given in the Appendix. These matrices are of the form $\mathrm{diag}(\lambda(x), \lambda^{-1}(x), 1, 1)$ in their eigenbasis. The largest eigenvalue λ is of the form

$$\lambda = \chi + \sqrt{\chi^2 - 1}, \tag{28}$$

where

$$\chi \equiv \tfrac{1}{2}\mathrm{Tr}(\Lambda) - 1. \tag{29}$$

With $y \equiv \ln(\sqrt{x})$, we obtain for n even,

$$\mathrm{Tr}(\Lambda) = n^2 \cosh^2 y + 4, \tag{30}$$

and for n odd,

$$\begin{aligned}\mathrm{Tr}(\Lambda) = {} & [(n^4+1)\cosh^2 y + 2n^2]\left(\frac{\sinh(ny)}{\sinh y}\right)^2 \cosh^2 y \\ & + 2n(n^2+1)\left(\frac{\sinh(2ny)}{\sinh y}\right)\cosh^3 y \\ & + 2n^2[2 + 3\sinh^2(ny)]\cosh^2 y + 4. \end{aligned} \tag{31}$$

The corresponding $f(\alpha)$ curves are now obtained by substituting λ into Eqs. (26) and (27) with $m = 2$ for n even, and with $m = 6$ for n odd. The resulting $f(\alpha)$ curves for the first few n, and $r = 2$, are shown in Figs. (1) and (2) for even and odd n, respectively. The maximum of $f(\alpha)$ is the Hausdorff dimension of the support of the wave function[7,11] which is equal to unity here as it should be.

The large-n asymptotic form of $f(\alpha)$ is given by

$$\alpha = 1 + \frac{1}{\ln n}[\ln(\cosh\rho) - \rho\tanh(\rho q)], \tag{32}$$

$$f(\alpha) = 1 + \frac{1}{\ln n}\{\ln[\cosh(\rho q)] - \rho q\tanh(\rho q)\} \tag{33}$$

for n even, and by

$$\begin{aligned}\alpha = \frac{2}{3} + \frac{1}{3\ln n}\Bigg[& \ln\left(\frac{\cosh^2\rho}{2\sinh\rho}\right) + \rho[\coth(\rho q) - 2\tanh(\rho q)] \\ & + n\rho[1 - \coth(n\rho q)]\Bigg], \end{aligned} \tag{34}$$

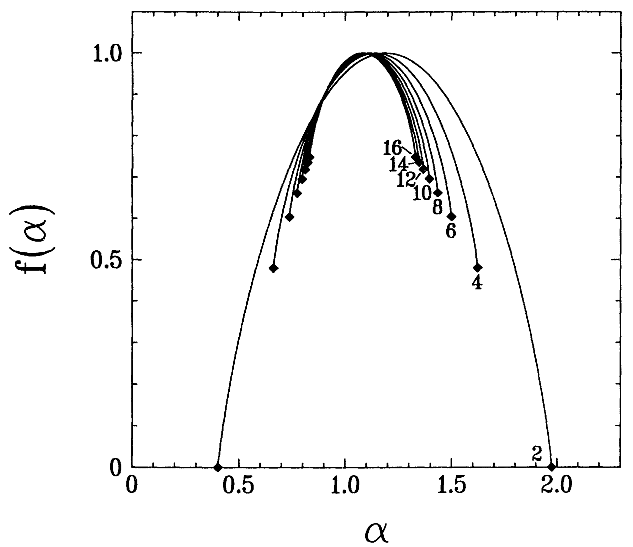

FIG. 1. $f(\alpha)$ vs α for the first few even n and $r = 2$. In the limit $n \to \infty$, the curves asymptotically collapse to the point (1,1).

$$\begin{aligned}f(\alpha) = \frac{2}{3} + \frac{1}{3\ln n}\Bigg[& \ln\left(\cosh^2(\rho q)\frac{\sinh(n\rho q)}{\sinh(\rho q)}\right) \\ & + \rho q[\coth(\rho q) - 2\tanh(\rho q)] \\ & - n\rho q\coth(n\rho q)\Bigg], \end{aligned} \tag{35}$$

for n odd, where $\rho \equiv |\ln r|$. In particular, the end points of the $f(\alpha)$ curve, $(\alpha_{\min}, f(\alpha_{\min}))$ and $(\alpha_{\max}, f(\alpha_{\max}))$, and the value of α for which $f(\alpha) = 1$, α_c, (corresponding

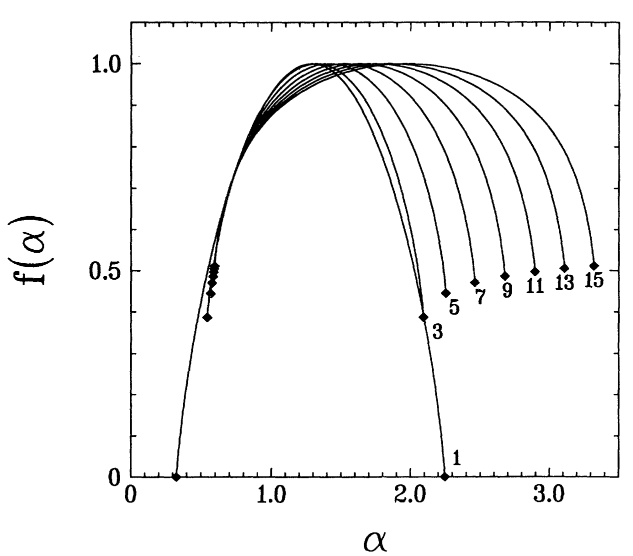

FIG. 2. $f(\alpha)$ vs α for the first few odd n and $r = 2$. In the limit $n \to \infty$, the left and right end points of the curves asymptotically approach the points $(\frac{2}{3}, \frac{2}{3})$ and $(\infty, \frac{2}{3})$, respectively.

to a value of q of $+\infty$, $-\infty$, and 0, respectively) are asymptotically given by

$$\alpha_{\min,\max} = 1 + \frac{1}{\ln n}\left[\ln(\cosh\rho) \mp \rho\right], \tag{36}$$

$$\alpha_c = 1 + \frac{\ln(\cosh\rho)}{\ln n}, \tag{37}$$

$$f(\alpha_{\min}) = 1 - \frac{\ln 2}{\ln n}, \tag{38}$$

$$f(\alpha_{\max}) = f(\alpha_{\min}) \tag{39}$$

for n even, and by

$$\alpha_{\min} = \frac{2}{3} + \frac{1}{3\ln n}\left[\ln\left(\frac{\cosh^2\rho}{2\sinh\rho}\right) - \rho\right], \tag{40}$$

$$\alpha_{\max} = \frac{2n\rho}{3\ln n}, \tag{41}$$

$$\alpha_c = \frac{n\rho}{3\ln n}, \tag{42}$$

$$f(\alpha_{\min}) = \frac{2}{3}\left(1 - \frac{\ln 2}{\ln n}\right), \tag{43}$$

$$f(\alpha_{\max}) = f(\alpha_{\min}) \tag{44}$$

for n odd.

III. DISCUSSION

From Eqs. (36)–(44) it is apparent that there is a fundamental difference between n even and n odd in the limit $n \to \infty$. The $f(\alpha)$ curves for n even shrink to the limit point $(\alpha, f(\alpha)) = (1,1)$ and, therefore, the $E = 0$ eigenstates approach a Bloch state. The $f(\alpha)$ curves for n odd do not shrink to the point (1,1); instead $(\alpha_{\min}, f(\alpha_{\min}))$ approaches the limit point $(\frac{2}{3}, \frac{2}{3})$ and $(\alpha_{\max}, f(\alpha_{\max}))$ approaches $(\infty, \frac{2}{3})$. Thus, for n odd, the $E = 0$ eigenstate approaches a limiting critical state and not a Bloch state. Note also that convergence to the limit points is exceedingly slow in both cases, with corrections to the $n = \infty$ limit going to zero like $1/\ln n$.[14]

The reason why Bloch states are approached for n even and not for n odd may be understood as follows. Recall that we are taking the limit $k \to \infty$ first and then the limit $n \to \infty$. Thus, as $n \to \infty$, the PM lattices reduce not simply to a periodic lattice of all A's but to a PM lattice containing *all* inflations of A. If the $E = 0$ eigenstate is to be a Bloch state, it must be a Bloch state for every kth inflation of A, A_k, with $k > k_0$ for some finite k_0. In the $n \to \infty$ limit, A_k becomes a periodic lattice of $n \to \infty$ unit cells, corresponding to n A_{k-1}'s, plus a single B_{k-1}. We are thus led to consider periodic approximations (PA's) to the PM lattices, where the kth PA is defined as the periodic lattice which has the kth iterate of the inflation rule (1) as its unit cell. The transfer matrix which takes us across the unit cell of the kth PA is M_k. For the kth PA to have an $E = 0$ eigenstate (which will automatically be a Bloch state), it no longer suffices for $\mathrm{Tr}(M_k)$ to be bounded, but we must have $|\mathrm{Tr}(M_k)| \le 2$. Therefore, it follows that the $E = 0$ eigenstate can be a Bloch state of the $n \to \infty$ PM lattice if and only if $|\mathrm{Tr}(M_k)| \le 2$ for all $k > k_0 - 1$. For the 4-cycle (11) of n=even PM lattices, $\mathrm{Tr}(M_k) = 0$ for every k and Bloch states are approached as $k \to \infty$. However, for the 6-cycles (12) of n=odd PM lattices, $\mathrm{Tr}(M_k) = 0$ only if $k \bmod 3 \ne 1$. If $k \bmod 3 = 1$, then $|\mathrm{Tr}(M_k)| = 2\cosh(\ln r) > 2$ for $r \ne 1$. In other words, for n odd, $E = 0$ lies in a gap of the spectrum of the periodic extension of A_k for every third k. Thus, in the $n \to \infty$ limit for n odd, it is not possible to sustain Bloch states over the entire PM lattice and the system must settle for a limiting critical state.

It is fair to ask how typical the large-n asymptotic behavior of the $f(\alpha)$ curves obtained here is for a generic part of the spectrum whose scaling properties are not governed by the matrix cycles (11) and (12). The argument given above for the difference between n even and n odd is general. We conjecture that the eigenstates corresponding to an energy E will converge in the $n \to \infty$ limit to Bloch states only if $|\mathrm{Tr}[M_k(E)]| \le 2$ for all $k > k_0$ and arbitrarily large n. This is clearly a very stringent condition which will generally be satisfied only for very special states. Numerically, we typically find the following behavior for finite k and n. If E lies in an allowed energy band of the kth PA, then whether or not E is also an allowed energy for the preceding PA's, labeled by $k' < k$, is typically a random function of k'.[15] Furthermore, the corresponding approximant $f_k(\alpha)$ to $f(\alpha)$ typically has $f_k(\alpha_{\max}) = 0$, where $f_k(\alpha)$ is defined through Eq. (4) but with $g(q)$ replaced with $g_k(q) \equiv \ln Z_k(q)/\ln L_k$ (no $k \to \infty$ limit). These findings support the statement that the 4-cycle eigenstates for n even are very special and that we can expect a typical state to approach a limiting critical state with $f(\alpha_{\max}) = 0$, and not a Bloch state, as $n \to \infty$.

IV. CONCLUSION

In conclusion, we have calculated the exact $f(\alpha)$ curves for the states of PM lattices dominated by the 4-cycle (11) for n even, and by the 6-cycle (12) for n odd, for general n. For n even, these states approach a Bloch state in the limit $n \to \infty$. The 6-cycle states for n odd cannot approach a Bloch state as $n \to \infty$ and converge to a limiting critical state instead. In both cases corrections to the $n \to \infty$ limit go to zero like $1/\ln n$. Numerical calculations suggest that the 4-cycle eigenstates for n even are very special and that Bloch states are typically not approached as $n \to \infty$.

Viewed differently, we have shown that dilute, n=even quasiperiodic impurities of PM type have a weak effect in forcing the $E = 0$ eigenstate of a periodic lattice of all A's into a critical state. Dilute, n=odd quasiperiodic impurities of PM type have a strong effect which persists in the limit of zero impurity density.

ACKNOWLEDGMENTS

I have benefited from conversations with J. P. Sethna. This work was supported in part by the Cornell Materials Science Center.

APPENDIX

For n even, the matrix $\Lambda(x)$ of the recursion equation (22) is given by

$$\Lambda(x) = \begin{pmatrix} \left(\frac{n}{2}\right)^2\left(1+\frac{1}{x}\right)+1 & \left(\frac{n}{2}\right)^2(x+x^2) & \left(\frac{n}{2}\right)x & \left(\frac{n}{2}\right) \\ \left(\frac{n}{2}\right)^2\left(\frac{1}{x}+\frac{1}{x^2}\right) & \left(\frac{n}{2}\right)^2(1+x)+1 & \left(\frac{n}{2}\right) & \left(\frac{n}{2}\right)\frac{1}{x} \\ \left(\frac{n}{2}\right)\frac{1}{x} & \left(\frac{n}{2}\right)x & 1 & 0 \\ \left(\frac{n}{2}\right)\frac{1}{x} & \left(\frac{n}{2}\right)x & 0 & 1 \end{pmatrix}, \tag{A1}$$

and for n odd, by

$$\Lambda(x) = \begin{pmatrix} P_{a1}(x) & x[P_{a1}(1/x)-1] & P_{a3}(x) & \frac{1}{x}P_{a3}(x) \\ \frac{1}{x}[P_{a1}(x)-1] & P_{a1}(1/x) & \frac{1}{x}P_{a3}(x) & \frac{1}{x^2}P_{a3}(x) \\ P_{b1}(x) & P_{b2}(x) & P_{b3}(x) & \frac{1}{x}[P_{b3}(x)-1] \\ P_{b2}(1/x) & P_{b1}(1/x) & x[P_{b3}(1/x)-1] & P_{b3}(1/x) \end{pmatrix}, \tag{A2}$$

where

$$\begin{aligned} P_{a1}(x) = {} & \left(\frac{n}{2}\right)^4 \frac{(x+1)^3}{x(x-1)^2}\left(\sqrt{x}^{\,n}-\sqrt{x}^{\,-n}\right)^2 + \left(\frac{n}{2}\right)^3 \frac{(x+1)^2}{x(x-1)}(x^n-2x^{-n}+1) \\ & + n^2\frac{(x+1)}{8x(x-1)^2}\left[(3-2x+3x^2)x^{-n}+4x^{n+1}+x^2-10x+1\right] \\ & + n\frac{1}{8x(1-x)}\left[2(1+x)^2x^{-n}+(1-6x+x^2)(1+x^n)\right] \\ & + \frac{1}{16x(x-1)^2}[(1+x)^3x^{-n}-(1-5x-5x^2+x^3)x^n+8x(1-5x+2x^2)], \end{aligned} \tag{A3}$$

$$\begin{aligned} P_{a3}(x) = {} & \frac{x}{8}\left(\frac{x+1}{x-1}\right)\left[n^3\left(\frac{x+1}{x-1}\right)\left(\sqrt{x}^{\,n}-\sqrt{x}^{\,-n}\right)^2+(3n^2+1)(x^n-x^{-n})\right] \\ & + n\frac{x}{8(x-1)^2}\left[(3-2x+3x^2)(x^n+x^{-n})+2(1-x)^2-8x\right], \end{aligned} \tag{A4}$$

$$\begin{aligned} P_{b1}(x) = {} & \left(\frac{n^4+1}{16}\right)\frac{(1+x)^3}{x^2(x-1)}(1-x^{-n})+\left(\frac{n(n^2+1)}{4}\right)(1+x)^2x^{-n-2} \\ & + \frac{n^2}{8}\frac{(x+1)}{x^2(1-x)}\left[(3-2x+3x^2)x^{-n}+x^2-6x+1\right], \end{aligned} \tag{A5}$$

$$P_{b2}(x) = \left(\frac{1-x^{-n}}{16}\right)\left(\frac{x+1}{x-1}\right)\left[n^4(1+x)^2+8n^2x-x^2+6x-1\right]+\left(\frac{1+x^{-n}}{8}\right)\left[n^3(x+1)^2-x^2+6x-1\right], \tag{A6}$$

$$\begin{aligned} P_{b3}(x) = \frac{1}{8}\Bigg(& n^3\frac{(x+1)^2}{(x-1)}(1-x^{-n})+n^2(1+x)(1+3x^{-n}) \\ & + \frac{n}{(1-x)}[(3-2x+3x^2)x^{-n}+x^2-6x+1]+(1+x)x^{-n}+7-x\Bigg). \end{aligned} \tag{A7}$$

For n odd, the matrix elements of Λ obey some identities which are useful in the calculation of the largest eigenvalue. In writing (A2) we have already made use of the fact that

$$xP_{a3}(1/x) = \frac{1}{x}P_{a3}(x). \tag{A8}$$

In addition, we find that

$$\mathrm{Tr}(\Lambda) = [\mathrm{P_{a1}(x)} + \mathrm{P_{a1}(1/x)}]\,[\mathrm{P_{b3}(x)} + \mathrm{P_{b3}(1/x)}] - \mathrm{P_{a3}(x)}\left(\mathrm{P_{b1}(x)} + \frac{1}{\mathrm{x}^2}\mathrm{P_{b1}(1/x)} + \frac{1}{\mathrm{x}}[\mathrm{P_{b2}(x)} + \mathrm{P_{b2}(1/x)}]\right). \qquad \text{(A9)}$$

[1] M. Holzer, Phys. Rev. B **38**, 1709 (1988).
[2] G. Gumbs and M. K. Ali, Phys. Rev. Lett. **60**, 1081 (1988).
[3] M. Holzer, Phys. Rev. B **38**, 5756 (1988).
[4] J. Q. You and Q. B. Yang, Phys. Rev. B **41**, 7073 (1990).
[5] M. Kolář and M. K. Ali, Phys. Rev. B **41**, 7108 (1990).
[6] Q. Niu and F. Nori, Phys. Rev. B **42**, 10 329 (1990).
[7] T. C. Halsey, M. H. Jensen, L. P. Kadanoff, I. Procaccia, and B. I. Shraiman, Phys. Rev. A **33**, 1141 (1986).
[8] T. Fujiwara, M. Kohmoto, and T. Tokihiro, Phys. Rev. B **40**, 7413 (1989).
[9] Q. Niu and F. Nori, Phys. Rev. Lett. **57**, 2057 (1986).
[10] W. M. Zheng, Phys. Rev. A **35**, 1467 (1987).
[11] In the language of the original formulation of the $f(\alpha)$ scaling spectrum (Ref. 7), the PM lattice is subdivided into L_k pieces of length $l_k = 1/L_k$. With the jth piece, l_k, we associate the probability measure $p_j = |\psi_j|^2$. $f(\alpha)$ may be regarded as the fractal dimension of the support of the subset of pieces l_k on which p_j scales like ${l_k}^{\alpha}$ in the limit $k \to \infty$.
[12] M. Kohmoto, L. P. Kadanoff, and C. Tang, Phys. Rev. Lett. **50**, 1870 (1983).
[13] M. Kohmoto and Y. Oono, Phys. Lett. **102A**, 145 (1984).
[14] The logarithmic form of the asymptotic n dependence of Eqs. (36)–(44) is the same as that obtained for the scaling exponents α associated with the multifractal *energy* spectrum of the PM lattices in Ref. 3.
[15] A special case is the band-edge energy, E_e, of a $(k \to \infty)$ PM lattice. E_e lies in a gap of the corresponding PA's for every other k. This appears to be true for any n and is consistent with a 2-cycle of the associated trace map dominating the scaling of the band edge.