LARGE SIEVES AND EXPONENTIAL SUMS

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ABSTRACT OF THE DISSERTATION

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The central theme that runs through this thesis is the large sieve type inequalities. We first expand the classical theory additive character to square moduli. One expects the result should be weaker than that of the classical inequalities, but, conjecturally at least, not by much. We also developed large sieve type inequalities for special Dirichlet characters to prime square moduli. Our result yields non-trivial result in certain ranges. We also developed some large sieve type inequalities for quadratic amplitudes. The thesis is concluded by considering the cancellations of Fourier coefficents at shifted primes, where we obtained a non-trivial upper bound for average of Fourier coefficients of cusp forms for the full modular group at primes twisted with an exponential function whose amplitude is the square root function.

Preface

"Age cannot wither her, nor custom stale
Her infinite variety*." Such is math.
But pardon you must, our experience pale,
This bending author's slow pursuits for truth.
Can we, our all unable pen be our foil,
Fence off the difficulties in our way?
Or can we cipher, burning mid-night oil,
Secrets of our trade, ere the break of day?
But sith our imperfection in season,
Make we up i'the abundance of passion,
Let's truly hope, 'mongst other reason,
Our work qualifies in graceful fashion.
Hence to all, let us our gratitude state.
Dear gentles, suffer our clumsy relate.

L. Z.

16 October 2001

* [29], *Antony and Cleopatra, II, ii, 240-241*

Acknowledgements

Indeed, as I have stated earlier, mathematics is something that "age cannot wither her, nor custom stale her infinite variety"([29], *Antony and Cleopatra, II, ii, 240-241*). I begin to study mathematics because I believe it is interesting and even hope that I can take part in making it beautiful. I believed that if I applied myself to it, I could be good at it. Such, among other things, are my only motives for studying mathematics.

I started my undergraduate studies wanting to become a physicist. It took very little to covert me into mathematics as I studied more of it. I was fortunate enough to see what G. H. Hardy [15] would call "real mathematics" as soon as my undergraduate studies started. This "real mathematics" was introduced to by in a honors calculus course taught by Professor Charles David Keys and I was hooked ever since then. I must thank him here for kindling my interest in mathematics. Moreover, I was first introduced to what I would like to call "real number theory" during the second of my undergraduate years. Professor Jacob Sturm of the Department of Mathematics and Computer Sciences of Rutgers University in Newark gave me my first course in number theory. From then on, the "damage" was done. I was forever hooked to the subject. Hence I would like to thank him here for introducing me into theory of numbers. Much of my interests in number theory was greatly enhanced by the lectures of Professor Jerrold Tunnell. I must thank him here for his lectures which I have found to be most informative and enlightening.

Of course, I could never complete my thesis without the guidance of my thesis adviser, Professor Henryk Iwaniec. He has always been so well-provided with ideas. In addition of being a brilliant mathematician, he is a true mentor and gentleman. I have found his lectures most enjoyable over the years. No one can sit in his class and not immediate realize his vast knowledge and passion for analytic number theory and mathematics at large. Indeed, he is a true romantic when it comes to lecturing. Also, his passion for analytic number theory has infected all around him. Hence, to him, I must express my gratitude for suggesting all the problems in my thesis and for his continual support and advise in solving them.

During my graduate career, I have enjoyed the companies of many of my friends, especially those of Misters Alfredo Rios and David Radnell. Both mathematical and non-mathematical discussions with them have broadened my horizons and made my stays at Rutgers more enjoyable. I thank them here for their friendship and company.

Finally, I would like to thank my loving parents, Chunxin Zhao and Qiuhua Li, for their gift of life and continual support throughout the years of my being. Their love and grace are well beyond the reaches of my poor abilities to ever fully repay.

Liangyi Zhao

15 May 2003

Dedication

Although it is improper in Chinese culture to use one's elder's proper name, I shall not dishonor my late paternal grandfather by leaving him nameless here. After all, it is to the memory of my grandfather, Yangtang Zhao, I would like to dedicate this thesis. Being born during the Second Year of Reign of Xüan Tong(Anno Domini 1910) in the Empire of China and passing on in Anno Domini 1992 in the People's Republic, he survived two states and countless number of governments in China, but his untimely demise has prevented him from seeing the completion of my degree. However, I am positive that his wisdom in life will illuminate my endeavors long after.

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List of Abbreviations

$\mathbb{C}$	Set of complex numbers.
$\Re z$	The real part of the complex number z.
$\Im z$	The imaginary parts of a complex number z.
$\mathbb{R}$	Set of real numbers.
$\mathbb{H}$	$\{z \in \mathbb{C} \mid \Im z > 0\}$, the upper half plane.
$\mathbb{Q}$	Set of rational numbers.
$\mathbb{Z}$	Set of integers.
$\mathbb{N}$	Set of natural numbers, *id est*, positive integers.
$\mathbb{P}$	Set of prime integers.
$e(z)$	$\exp(2\pi i z) = e^{2\pi i z}$, for $z \in \mathbb{C}$.
$f = O(g)$	Notation of E. Landau and means $\|f\| \leq cg$ for some unspecified positive constant c, not necessarily the same in each occurrence.
$f \ll g$	$f = O(g)$ and is the notation of I. M. Vinogradov.
$f \gg g$	$g \ll f$.
$f \asymp g$	$f \ll g$ and $g \ll f$.
$f(x) \sim g(x)$	$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$.
$f(x) = \Omega_+(g(x))$	There is a constant $c > 0$ and a sequence x_n tending to ∞ such that $f(x_n) > cg(x_n)$ for all n.
$f(x) = \Omega_-(g(x))$	There is a constant $c > 0$ and a sequence x_n tending to ∞ such that $f(x_n) < -cg(x_n)$ for all n.
$f(x) = \Omega_\pm(g(x))$	$f(x) = \Omega_+(g(x))$ and $f(x) = \Omega_-(g(x))$.
$\\|x\\|$	$\inf_k\{\|x-k\| \mid k \in \mathbb{Z}\}$ and donotes the distance of a real number x to its closest integer.
$\frac{\overline{a}}{c}$	$\frac{d}{c}$ where d is a solution of the congruence $ad \equiv 1 \pmod{c}$.
$\square$	The conclusion of a proof or that the proof is easy or well-known.

Chapter 1

Introduction

In this chapter, we introduce things *ab initio* and list some of the results that we intend to use for the remainder of the thesis.

1.1 Properties of Some Arithmetic Functions

An arithmetic function is a function defined on the natural numbers. One of the most basic arithmetic functions in number theory is the Möbius μ function.

$$\mu(n) : \mathbb{N} \longrightarrow \{\pm 1, 0\} : n \longrightarrow \begin{cases} 0, & \text{if } n \text{ is not square free,} \\ (-1)^{\omega(n)}, & \text{otherwise,} \end{cases}$$

where $\omega(n)$ is the number of distinct prime divisors of n.

One of the most important properties of the $\mu(n)$ is the following.

$$\delta(n) = \sum_{d|n} \mu(n) = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n > 1. \end{cases} . \tag{1.1.1}$$

Also we shall make use of the following Lemma in later chapters.

Lemma 1.1 (Ramanujan's Sums). *With $\mu(n)$ defined above, we have*

$$\sum_{\substack{a=1 \\ \gcd(a,q)=1}}^{q} e\left(\frac{ah}{q}\right) = \sum_{\substack{c|h \\ c|q}} c\mu\left(\frac{q}{c}\right). \tag{1.1.2}$$

Proof. As we know $\sum_{d|n} \mu(d) = 0$ unless $n = 1$. We have

$$\sum_{\substack{a=1 \\ \gcd(a,q)=1}}^{q} e\left(\frac{ah}{q}\right) = \sum_{a=1}^{q} e\left(\frac{ah}{q}\right) \sum_{\substack{d|a \\ d|q}} \mu(d) = \sum_{d|q} \mu(d) \sum_{b=1}^{q/d} e\left(\frac{bh}{q/d}\right),$$

where we set $a = bd$. The inner-most sum in the above is zero unless h is a multiple of q/d. Therefore, it becomes

$$\sum_{\substack{d|q \\ q/d|h}} \mu(d)\frac{q}{d} = \sum_{c|h}\sum_{cd=q} \mu(d) = \sum_{\substack{c|h \\ c|q}} c\mu\left(\frac{q}{c}\right).$$

Hence the proof is completed. □

We also have the von Mangoldt function, $\Lambda(n)$. It is supported on the prime powers and defined as

$$\Lambda : \mathbb{N} \longrightarrow \mathbb{R} : n \longrightarrow \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } \omega(n) > 1, \\ \log p, & \text{if } n = p^k \text{ for a prime number } p. \end{cases}$$

The von Mangoldt function is useful as the Dirichlet series that it generates is the negative of the logarithmic derivative of the Riemann Zeta function. Furthermore, it can be easily verified that the following formula holds.

$$\sum_{m|n} \Lambda(n) = \log n.$$

Also, we shall need the Euler φ function. $\varphi(n)$ is the number of units in the ring $\mathbb{Z}/n\mathbb{Z}$. In other word, it is the number of natural numbers not exceeding n and prime to n. It is given by the following formula

$$\varphi(n) = n\prod_{p|n}\left(1 - \frac{1}{p}\right) = \sum_{d|n}\frac{n}{d}\mu(d) = N(n) \star \mu(n),$$

where $N(n) = n$ is the identity arithmetic function on $\mathbb{N}$. Therefore, the Dirichlet series generated by $\varphi(n)$ is $\zeta(s-1)\zeta^{-1}(s)$ for $\Re s > 2$.

We shall certainly need the properties of the divisor function in this thesis. For a fixed $m \in \mathbb{N}$, let $\tau_m(n)$, the m-th divisor function, denote the number of ways to write n as product of m factors, where we do distinguish the order of multiplication. It is elementary to note that $\tau_m(n)$ is multiplicative, i.e. $\tau_m(n)\tau_m(r) = \tau_m(nr)$ for all n, r

with $\gcd(n,r) = 1$. Also, it is easy to observe that $\tau_m(n)$ is the generating series for $\zeta^m(s)$. For $\sigma = \Re(s) > 1$, we have

$$\zeta^m(s) = \left(\sum_{n=1}^{\infty} \frac{1}{n^s}\right)^m = \sum_{n=1}^{\infty} \frac{\tau_m(n)}{n^s}.$$

We now record the following easily but very important lemma.

Lemma 1.2. *Given m a positive integers, we have*

$$\tau_m(n) \ll n^{\epsilon}, \tag{1.1.3}$$

for all positive ϵ, with the implied constant depending only on ϵ and m.

Proof. We first observe that $\tau_m^l(p) = m^l$ and $\tau_{2^{ml+1}}(p) = 2^{ml+1}$. Moreover, for $k \geq 2$, we have $\tau_m^l(p^k) = k^{ml}$ and $\tau_{2^{ml+1}}(p^k) = k^{2^{ml+1}}$. Since we have already noted that $\tau_m(n)$ is multiplicative for all m. We comfortably have the following inequality.

$$\tau_{2^{ml+1}}(n) \geq \tau_m^l(n). \tag{1.1.4}$$

Now let $\epsilon > 0$ and m a positive integer be given, choose l such that $2l^{-1} < \epsilon$. It is clearly that

$$1 \gg \zeta^{2^{ml+1}}(2) = \sum_{n=1}^{\infty} \frac{\tau_{2^{ml+1}}(n)}{n^2} \geq \sum_{n=1}^{\infty} \frac{\tau_m^l(n)}{n^2} \geq \frac{\tau_m^l(h)}{h^2},$$

for any positive integer h and the first implied constant depends on m and l which depends on ϵ. The above gives the following inequality,

$$\tau_m(h) \ll n^{2m/l} \ll n^{\epsilon}.$$

where the implied constants depend on m and ϵ. This is precisely our desired result. $\square$

The above lemma is trivial in the sense that during its proof, we bound an individual term in a series of positive numbers by the whole series. This result is extremely important nevertheless, as we shall unfold it utility in later chapters.

Also, one can develop easily using the so-called "hyperbola method" the following average result for the divisor function.

Lemma 1.3. *With $\tau(n)$ denoting the divisor function, we have*

$$\sum_{n\le x}\tau(n) = x\log x + (2\gamma-1)x + O(\sqrt{x}), \tag{1.1.5}$$

where the implied constant is absolute.

Proof. The proof is easy and follows from the so-called hyperbola method of Dirichlet. □

The error term in the above lemma is obviously not the best possible, but it is good enough for all our enterprises. Of course, reducing the size of the error term in the above lemma is the famous Dirichlet divisor problem. In any event, the following should suffice for our purposes.

Lemma 1.4. *Give $k, l \in \mathbb{N}$, we have*

$$\sum_{n\le x}[\tau_k(n)]^l \ll x[\log(2x)]^{k^l-1}, \tag{1.1.6}$$

where the implied constant is absolute.

Proof. The proof is easy and standard. □

1.2 Dirichlet characters and Gauss Sums

A Dirichlet character of modulus q is a map from $\mathbb{Z}$ to $\mathbb{C}$ that is a homomorphism from $(\mathbb{Z}/q\mathbb{Z})^*$ under the canonical projection. We say that a Dirichlet character is primitive of modulus q if it is not a character to any modulus $q_1 < q$. We have the following easy lemma.

Lemma 1.5. *Let $\chi(n)$ be a Dirichlet character of modulus q and $q_1|q$ be its least period. Then $\chi(n)$ is induced by a primitive character of modulus q_1.*

Proof. We define $\chi_1(n) = \chi(n)$ if $\gcd(n,q) = 1$. But if $\gcd(n,q_1) = 1$ but $\gcd(n,q) > 1$, then we may choose an integer t such that $\gcd(n+tq_1,q) = 1$ and $\chi_1(n) = \chi(n+tq_1)$. Such a t clearly exists and our choice of it is immaterial. Lastly, the properties that $\chi_1(n)$ inherits through its definition makes it a primitive Dirichlet character of modulus q_1 and it induces $\chi(n)$. □

The Gauss sum is a sum in which a multiplicative character is twisted by an additive character. More precisely, we have

$$G(\chi) = \sum_{b \pmod m} \chi(b) e\left(\frac{b}{m}\right). \tag{1.2.1}$$

Indeed, the Gauss sums may be viewed as a type of Fourier transform. Therefore, one has the "inverse" Fourier transform as follows

$$\chi(n) = \frac{1}{G(\overline{\chi})} \sum_{a \bmod q} \overline{\chi}(a) e\left(\frac{an}{q}\right), \tag{1.2.2}$$

provided that χ is a primitive character modulo q. Moreover, the following result is well-known. We shall utilize this result later in the chapters on large sieves.

Lemma 1.6. *Suppose* $\chi \pmod m$ *is induced by the primitive character* $\chi^* \pmod{m^*}$. *Then*

$$G(\chi) = \mu\left(\frac{m}{m^*}\right) \chi^*\left(\frac{m}{m^*}\right) G(\chi^*). \tag{1.2.3}$$

If $\chi \pmod m$ *is primitive, then*

$$|G(\chi)| = \sqrt{m}. \tag{1.2.4}$$

Proof. Starting with the defintion of the Gauss sum, we have

$$G(\chi) = \sum_{\substack{a \bmod m \\ \gcd(a,m)=1}} \chi^*(a) e\left(\frac{a}{m}\right) = \sum_{d|m} \mu(d) \chi^*(d) \sum_{a \bmod \frac{m}{d}} \chi^*(a) e\left(\frac{ad}{m}\right).$$

The inner-most sum vanishes unless the modulus of the additive and mutliplication character match, *id est* $d = \dfrac{m}{m^*}$. This prove (1.2.3). To prove (1.2.4), we have

$$\begin{aligned} |\tau(\chi)|^2 &= \sum\sum_{a,b \bmod m} \chi(a)\overline{\chi}(b) e\left(\frac{a-b}{m}\right) \\ &= \sum_{a \bmod m} \chi(a) \sum_{\substack{b \bmod m \\ \gcd(b,m)=1}} e\left[\frac{(a-1)b}{m}\right]. \end{aligned}$$

The inner-most sum is the Ramanujan's sum in Lemma 1.1 and upon inserting (1.1.2) into the above expression, we have

$$|\tau(\chi)|^2 = \sum_{d|m} d\mu\left(\frac{m}{d}\right) \sum_{\substack{a \bmod m \\ a \equiv 1 \bmod d}} \chi(a).$$

Recall our assumption that χ is primitive modulo m and the inner-most sum of the above expression vanishes unless $d = m$ from which we may infer (1.2.4). □

1.3 The Hypothesis of Riemann

In his memoir, Georg Friedrich Bernhard Riemann made a number of remarkable deductions and conjectures about $\zeta(s)$ among which only one remains unproven today. Of course, the open problem is the famous or infamous hypothesis of Riemann: If $\zeta(s) = 0$ and $\Re s > 0$, then $\Re s = \frac{1}{2}$. However, Godfrey Harold Hardy proved that infinitely many zeros of the kind of our interest lie on the line $\Re s = \frac{1}{2}$ in 1914 and Atle Selberg proved that a positive proportion of all the zeros lie on the line.

The above conjecture admits analogues for the Dirichlet L-function, $L(s, \chi)$. In particular, the generalized Riemann hypothesis asserts that the real parts all zeros of the Dirichlet L-function are either non-positive or $\frac{1}{2}$.

The hypothesis of Riemann may be understood as the attempt to answer the question of how far one can extend the effect of the infinite series of $\zeta(s)$ or $L(s, \chi)$ to the left of the line $\Re s = 1$. As one can deduce easily, by partial summation, that following equivalent statement of the generalized hypothesis of Riemann.

$$\left| \sum_{1 \le n \le x} \chi(n)\mu(n) \right| \ll x^{\frac{1}{2}+\epsilon}, \tag{1.3.1}$$

where the implied constants depends on ϵ alone.

Moreover, the generalized Riemann hypothesis admits other interpretations which we shall also use later. More precisely, for a non-principle Dirichlet character $\chi(n)$ (mod q), we have that

$$\left| \sum_{n \le N} \chi(n)\Lambda(n) \right| \ll N \log^2(2qN), \tag{1.3.2}$$

is equivalent to the generalized Riemann hypothesis.

Of course, there are more romantic ways of interpreting the hypothesis of Riemann. The following is a quote from [16].

> The rational numbers form a countable set, dense in the real numbers. They are set in the real line like stars in the sky 'to illuminate the mystery of the continuum'(a remark attributed to Borel).
> The extent to which the rationals are uniformly distributed depends on the truth or falsity of the Riemann hypothesis; perhaps that is what the Riemann hypothesis really means.

Therefore, determining the truth of the Riemann hypothesis is the same as asking: how uniformly are the stars in the universe distributed?

1.4 The Conjecture of Montgomery

It was due to H. L. Montgomery, see [26], that the following conjecture was formulated. Let $D(s)$ be a Dirichlet polynomial

$$D(s) = \sum_{n=1}^{N} a_n n^{-s}. \tag{1.4.1}$$

As tradition dictates, we write $s = \sigma + it$. Then the powerful conjectures says.

Conjecture 1.1 (Montgomery). *With $D(s)$ defined as in (1.4.1), and $|a_n| \leq 1$ for all n, then we have*

$$\int_0^T |D(it)|^q \mathrm{d}t \ll \left(T + N^{\frac{q}{2}}\right) N^{\frac{q}{2}+\epsilon}, \tag{1.4.2}$$

uniformly for $2 \leq q \leq 4$.

This conjecture is known to be true is for the Dirichlet polynomial $D(s) = \sum_{n=1}^{N} n^{-s}$. Moreover, as a consequence of the large sieve inequalities, Theorem 6.3, (1.4.2) is also known to hold if one replaces q in the inequality by any even positive integer. In fact, if q is an even positive integer, then (1.4.2) holds for arbitrary complex numbers $\{a_n\}$ regardless of their modulus.

We make mention of this conjecture because its truth would greatly improve our results in the last chapter of this thesis.

1.5 Modular Forms and Hecke Operators

$\mathbb{H}$ denotes the upper half complex plane.

$$\mathbb{H} = \{z \in \mathbb{C} \mid \Im z > 0\}.$$

$SL_2(\mathbb{Z})$, some times referred to as the full modular group, is the group of 2×2 matrices with determinant 1, *id est*

$$SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \mathbb{Z}, \text{ and } ad - bc = 1 \right\}. \tag{1.5.1}$$

A modular function of weight k is a function that preserves the actions of $SL_2(\mathbb{Z})$ on $\mathbb{C}$. In that words, $f(z)$ is a modular function of weight k for the full modular group, if

$$f(z) = (cz + d)^{-k} f\left(\frac{az + b}{cz + d}\right), \text{ for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}). \tag{1.5.2}$$

If we take

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

in (1.5.2), we see that a modular function is periodic of period 1. Therefore, a modular function $f(z)$ has a Fourier series expansion, say,

$$f(z) = \sum_{n \in \mathbb{Z}} a_n e(nz). \tag{1.5.3}$$

Moreover, if we take

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

in (1.5.2), we see that if $f(z)$ is not identically zero and is a modular function of weight k, then k has to be even.

Now a function is called a modular form if it is holomorphic on the upper half plane $\mathbb{H}$. Modular forms contain much number theoretic information and their Fourier coefficients are generalizations of characters in many regards; and as in the case for characters, the oscillation of the Fourier coefficients of modular forms is a matter of great interest. We shall touch upon this issue in the last chapter of this thesis.

1.6 History and Significance of the Large Sieve Inequalities

It was in 1941 that Yuri Vladimirovich Linnik [23] first purposed the simple but fundamental idea of large sieve. Many have made contributions and refinements to the idea ever since then. However, what we refer to as large sieve is hardly a sieve of any kind.

When Linnik first purposed the idea of large sieve, he applied it to study the distribution of quadratic non-residues. A. Rényi was the first to extensively study the large sieve and made other applications to the Goldbach problem. Using large sieve, he was able to show that every sufficiently large even number is the sum a prime and a composite having at most k prime factors, with of course k an absolutely bounded integer constant. However, Rényi did not specify what k can be. M. B. Barban showed that k may be taken to be 4. E. Bombieri's mean value theorem may be used to show that k can be taken to be 3 and the world record in this direction was obtained by Jingren Chen with $k = 2$. Large sieve serves as a major tool in all of these proofs. Of course, the next refinement will give the solution to the Goldbach problem which conjectures that every even number greater than 4 is the sum of two primes, and such refinement is unlikely to be attained by any idea from sieve.

Other applications of the large sieve include the bound for the average of the least primitive root by Burgess and Elliott [7], the improvement by Joshi [21] concerning the size of $L(1, \chi)$. Also, results regarding the distribution of zeros of L-functions have been obtained using the large sieve.

Perhaps the most important application of the large sieve is the following. The average result of the famous Riemann Hypothesis was obtained using inequalities provided by large sieves. The follow theorem can be viewed as the truth of Riemann Hypothesis on average and was due to Bombieri [3] and A. I. Vinogradov [34] and [35]. Indeed, in many instances, the following theorem serves as a substitute for the Riemann hypothesis.

Theorem 1.1 (Bombieri-Vinogradov). *Let $A > 0$ be fixed. Then*

$$\sum_{q\le Q} \max_{y\le x} \max_{\substack{a \\ \gcd(a,q)=1}} \left| \psi(x; q, a) - \frac{x}{\varphi(q)} \right| \ll x^{\frac{1}{2}} Q (\log x)^5, \tag{1.6.1}$$

provided that

$$x^{\frac{1}{2}} (\log x)^{-A} \le Q \le x^{\frac{1}{2}}.$$

Proof. This is quoted from Chapter 28 of [8]. □

There are many more application of the large sieve upon which this section does not touch. The survey articles of Bombieri [4], Barban [2] and Montgomery [25] are good sources of such, not to mention the book by Montgomery [24].

A large portion of this thesis is devoted to expanding the classical theory of large sieve inequalities.

1.7 Exponential Sums

The studies of exponential sums lie at the center of analytic number theory and literature in the subject is abundant. Many problems reduces to the estimation, or evaluation if we are lucky, of the modulus of exponential sums of various types. We are very often interested in the sums like

$$\sum_{m=M}^{M_1} e[f(m)], \tag{1.7.1}$$

where $f(x)$ is some smooth function on $[M, M_1]$. We shall presently see the use of estimates of sums of this kind in this thesis. The basic techniques for estimating sums like that in (1.7.1) are as follows:

1. Weyl-Hardy-Littlewood method;

2. van der Corput method;

3. Vinogradov method.

Among the good reference for these methods, are [32], [14], [26] and [16].

In this thesis, exponential sums is another central theme, besides the large sieve inequalities. We shall make use the first two methods in various kinds of estimates.

Very often, we shall also encounter exponential sums whose range of summation runs over prime integers only. In this regard, we know of only one non-trivial method developed by Ivan Matveevič Vinogradov in 1937 [36]. In 1922, Godfrey Harold Hardy and John Edensor Littlewood showed that assuming the truth of the generalized Riemann hypothesis, the ternary Goldbach conjecture is true; *id est* every sufficiently large odd number is a sum of three primes, which is a trivial corollary of the binary Goldbach problem mentioned earlier. Indeed, the generalized Riemann hypothesis would give us non-trivial majorants for some exponential sums over primes from which we may infer the truth of the ternary Goldbach problem. I. M. Vinogradov, using his techniques for estimating sums over primes, was able to shock the whole world and resolve the aforementioned problem unconditionally. His method was later simplified by R. C. Vaughan [33]. In the last chapter of the thesis, we shall also estimate an exponential sum over primes, using of course Vinogradov's method. In the last chapter of this thesis, we will make use of this technique for summation of primes in the estimation of an oscillatory sums.

Chapter 2

Large Sieve Inequalities for Characters with Square Moduli

2.1 Introduction

Given $\frac{1}{2} > \delta > 0$, we say that, for a finite sequence of real numbers, $\{x_k\}$ is δ-spaced mod 1, if $\|x_k - x_l\| \geq \delta$ for all $x_k \neq x_l$. Note that by the pigeon-hole principle, the cardinality of any sequence of real numbers that are δ-spaced modulo 1 cannot exceed $\delta^{-1}+1$.

As we know that $\mathbb{R}/\mathbb{Z} = S^1$ the circle, for real number x and y, we can interpret $\|x-y\|$ as the length of the minor arc between the images of x and y on S^1 under the canonical projection. This interpretation is particularly enlightening on our proof of the following.

The large sieve inequality, which we henceforth refer to as the classical large sieve inequality, is stated as follows. Different proofs of the theorem can be found in [8], [11], [24], and [25]. The theorem, in the following form, was first introduced by Davenport and Halberstam, [9] and [10].

Theorem 2.1 (Large Sieve Inequality). *Let $\{a_n\}$ be an arbitrary sequence of complex numbers and $\{x_k\}$ be a sequence of real numbers that are δ-spaced modulo 1. Also suppose N is positive integers and M is an integer. Then we have*

$$(2.1.1) \qquad \sum_k \left| \sum_{n=M+1}^{M+N} a_n e(x_k n) \right|^2 \ll (\delta^{-1} + N) \sum_{n=M+1}^{M+N} |a_n|^2,$$

where the implied constant is absolute.

Save for the computation of the implied constant, the above inequality is the best possible. Moreover, Cohen and Selberg have shown independently that

$$\sum_k \left| \sum_{n=M+1}^{M+N} a_n e(x_k n) \right|^2 \le (\delta^{-1} - 1 + N) \sum_{n=M+1}^{M+N} |a_n|^2, \tag{2.1.2}$$

which is the absolute best possible, since Bombieri and Davenport [6] gave examples of $\{x_k\}$ and a_n with $\delta \longrightarrow 0$, $N \longrightarrow \infty$ such that $N\delta \longrightarrow \infty$ and

$$\sum_k \left| \sum_{n=M+1}^{M+N} a_n e(x_k n) \right|^2 = (\delta^{-1} - 1 + N) \sum_{n=M+1}^{M+N} |a_n|^2. \tag{2.1.3}$$

However, in our studies, we shall not be concerned with the implied constants.

The theorem just quoted is useful at many places in analytic number theory. Many proofs for the theorem has been given. The author counts P. X. Gallagher's proof in [11] as his favorite and quotes it here for completeness.

To complete the proof of the large sieve inequality, we need to following lemma of Sobolev type.

Lemma 2.1 (Sobolev-Gallagher). *Let $f(x) : [a,b] \to \mathbb{C}$ be a smooth function with continuous first derivative on (a,b). Then given $u \in [a,b]$, we have*

$$|f(u)| \le \frac{1}{b-a} \int_a^b |f(x)| \mathrm{d}x + \int_a^b |f'(x)| \mathrm{d}x, \tag{2.1.4}$$

and

$$\left| f\left(\frac{a+b}{2}\right) \right| \le \frac{1}{b-a} \int_a^b |f(x)| \mathrm{d}x + \frac{1}{2} \int_a^b |f'(x)| \mathrm{d}x. \tag{2.1.5}$$

Proof. By partial integration, we have the following.

$$\int_u^b \left(\frac{x-b}{b-a}\right) f'(x) \mathrm{d}x = -\left(\frac{u-b}{b-a}\right) f(u) - \frac{1}{b-a} \int_u^b f(x) \mathrm{d}x.$$

Similarly, we have

$$\int_a^u \left(\frac{x-a}{b-a}\right) f'(x) \mathrm{d}x = \left(\frac{u-a}{b-a}\right) f(u) - \frac{1}{b-a} \int_a^u f(x) \mathrm{d}x$$

Adding the two above equalities, we get

$$f(u) = \frac{1}{b-a} \int_a^b f(x) \mathrm{d}x + \int_u^b \left(\frac{x-b}{b-a}\right) f'(x) \mathrm{d}x + \int_a^u \left(\frac{x-a}{b-a}\right) f'(x) \mathrm{d}x,$$

which, when taken the absolute value, yields the following inequality.

$$\begin{aligned} |f(u)| &\le \frac{1}{b-a}\int_a^b |f(x)|\mathrm{d}x + \int_u^b \left|\left(\frac{x-b}{b-a}\right) f'(x)\right| \mathrm{d}x \\ &\quad + \int_a^u \left|\left(\frac{x-a}{b-a}\right) f'(x)\right| \mathrm{d}x \\ &\le \frac{1}{b-a}\int_a^b |f(x)|\mathrm{d}x + \int_a^b |f'(x)|\mathrm{d}x. \end{aligned}$$

The last inequality is valid because neither $\left|\frac{x-a}{b-a}\right|$ nor $\left|\frac{x-b}{b-a}\right|$ exceeds 1. This proves (2.1.4). Furthermore if $u = \frac{a+b}{2}$, then neither $\left|\frac{x-a}{b-a}\right|$ nor $\left|\frac{x-b}{b-a}\right|$ exceeds $\frac{1}{2}$ in the above inequality, from which we can infer (2.1.5). □

We are now ready to prove the large sieve inequality.

Proof of Theorem 2.1. It is clear that M does not play a role on the left-hand side of (2.1.1), since any change of linear variable will change M into any desired integer value. Therefore, we may assume that $M = 0$ and it suffices to prove our theorem in that case. We take our function to be $S(x) = \sum_{n=1}^{N} a_n e(xn)$. If we apply (2.1.5) to $S^2(x)$ on the interval $\left[x_k - \frac{\delta}{2}, x_k + \frac{\delta}{2}\right]$, then we get

$$|S(x_k)|^2 \le \frac{1}{\delta}\int_{x_k-\delta/2}^{x_k+\delta/2} |S(x)|^2\mathrm{d}x + \int_{x_k-\delta/2}^{x_k+\delta/2} |S(x)S'(x)|\mathrm{d}x.$$

We now sum the above inequality over all the x_k's and conclude that

$$\begin{aligned} \sum_k |S(x_k)|^2 &\le \sum_k \left[\frac{1}{\delta}\int_{x_k-\delta/2}^{x_k+\delta/2} |S(x)|^2\mathrm{d}x + \int_{x_k-\delta/2}^{x_k+\delta/2} |S(x)S'(x)|\mathrm{d}x\right] \\ &\le \frac{1}{\delta}\int_0^1 |S(x)|^2\mathrm{d}x + \int_0^1 |S(x)S'(x)|\mathrm{d}x, \end{aligned}$$

where the last inequality is valid because the integrands are non-negative and the interval of integration are non-overlapping, as we are assuming the x_k's are δ-spaced modulo 1. The first integral in the above expression yields $\delta^{-1}\sum_{n=1}^{N} |a_n|^2$ by Parseval identity.

We apply Hölder's inequality of the second integral and get

$$\begin{aligned}&\int_0^1 |S(x)S'(x)|\mathrm{d}x\\ \leq\ & \left(\int_0^1 |S(x)|^2\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_0^1 |S'(x)|^2\mathrm{d}x\right)^{\frac{1}{2}}\\ =\ & \left(\sum_{n=M+1}^{M+N}|a_n|^2\right)^{\frac{1}{2}}\left(\sum_{n=M+1}^{M+N}|2\pi i n a_n|^2\right)^{\frac{1}{2}}\\ \ll\ & N\sum_{n=M+1}^{M+N}|a_n|^2.\end{aligned}$$

Combining all estimates we have above gives use the large sieve inequality. □

The large sieve inequality admits two corollaries for additive and multiplicative characters. First it is certainly elementary to say that rational number in $(0,1]$ with height not exceeding Q are Q^{-2}-spaced. Therefore, applying the large sieve inequality, we get

Corollary 2.1. *Let $\{a_n\}$, M, N be defined as in Theorem 2.1 and $Q \in \mathbb{N}$. Then we have*

$$\sum_{q=1}^{Q}\sum_{\substack{a \pmod q\\ \gcd(a,q)=1}}\left|\sum_{n=M+1}^{M+N} a_n e\left(\frac{a}{q}n\right)\right|^2 \ll (Q^2+N)\sum_{n=M+1}^{M+N}|a_n|^2, \tag{2.1.6}$$

where the implied constant is absolute.

Proof. This is just proved before the statement of the corollary. □

Furthermore, we also have the following corollary.

Corollary 2.2. *Let $\{a_n\}$, Q, M, N be defined as in Corollary 2.1. Then we have*

$$\sum_{q=1}^{Q}\frac{q}{\varphi(q)}\sideset{}{^\star}\sum_{\chi \pmod q}\left|\sum_{n=M+1}^{M+N} a_n\chi(n)\right|^2 \ll (Q^2+N)\sum_{n=M+1}^{M+N}|a_n|^2, \tag{2.1.7}$$

where the implied constant is absolute and we shall henceforth denote the sum over primitive characters to modulus q as $\sideset{}{^\star}\sum_{\chi \pmod q}$.

Proof. We recall the Poisson summation formula for multiplicative and additive character. If χ is a primitive Dirichlet character modulo q, then from (1.2.2), we have

$$\chi(n) = \frac{1}{G(\overline{\chi})} \sum_{a=1}^{q} \overline{\chi}(a) e\left(\frac{an}{q}\right),$$

for all n, where $G(\chi)$ denotes the first Gauss sum. Upon multiply both sides by a_n, we get

$$\sum_{n=M+1}^{M+N} a_n \chi(n) = \frac{1}{G(\overline{\chi})} \sum_{a=1}^{q} \overline{\chi}(a) S\left(\frac{a}{q}\right),$$

where $S(x)$ is defined in the proof of Theorem 2.1. As we know, by Lemma 1.6, that $|G(\chi)| = \sqrt{q}$ if χ is primitive with conductor q, we conclude, upon summing over all primitive characters of the above, that

$$\sum_{\chi \bmod q}^{\star} \left| \sum_{n=M+1}^{M+N} a_n \chi(n) \right|^2 = \frac{1}{q} \sum_{\chi \bmod q}^{\star} \left| \sum_{a=1}^{q} \overline{\chi}(a) S\left(\frac{a}{q}\right) \right|^2.$$

The right-hand side of the above increases if we drop the restriction that χ is primitive modulo q. Therefore it is bounded by

$$\begin{aligned} \frac{1}{q} \sum_{\chi \pmod q} \left| \sum_{a=1}^{q} \overline{\chi}(a) S\left(\frac{a}{q}\right) \right|^2 &= \sum_{a=1}^{q} \sum_{a'=1}^{q} S\left(\frac{a}{q}\right) \overline{S\left(\frac{a}{q}\right)} \sum_{\chi \pmod q} \overline{\chi}(a) \chi(a') \\ &= \frac{\varphi(q)}{q} \sum_{\substack{a \pmod q \\ \gcd(a,q)=1}} \left| S\left(\frac{a}{q}\right) \right|^2. \end{aligned}$$

Upon multiplying by $\frac{q}{\varphi(q)}$, last expression in the above becomes the left-hand side of (2.1.6), without the sum over q. Therefore, our proof is completed if we sum over $1 \le q \le Q$ and then apply Corollary 2.1. □

In this chapter, we shall develop theorems touching upon large sieve inequality for additive characters in which the moduli are squares, *id est* q^2 rather than q. The problem reduces down to the spacing properties of rational numbers with square denominators. The key idea that we employ is resolving such a problem is the Weyl-Hardy-Littlewood method for exponential sums. The method is completely generalizable to higher power moduli. However, as Weyl's estimates for exponential sums weakens when the degree of the polynomial is of high degree our corresponding results also become weaker.

2.2 Preliminary Lemmas

We begin by quote the duality principle, which is present, though sometimes under a disguise, in many of the proofs of the classical large sieve inequality. Itsays, in essence, that the norm of a bounded linear operator in a Banach space is equal to that of its adjoint operator. More precisely, we state the following.

Lemma 2.2 (Duality Principle). *Let $T = [t_{mn}]$ be a square matrix with entries from the complex numbers. The following two statements are equivalent:*

1. *For any absolutely square summable sequence of complex numbers $\{a_n\}$, we have*

$$\sum_m \left| \sum_n a_n t_{mn} \right|^2 \leq D \sum_n |a_n|^2. \tag{2.2.1}$$

2. *For any absolutely square summable sequence of complex numbers $\{b_n\}$, we have*

$$\sum_n \left| \sum_m b_m t_{mn} \right|^2 \leq D \sum_m |b_m|^2. \tag{2.2.2}$$

Proof. By symmetry, we only need to prove that the first statement implies the second. Toward that end, we let $a_n = \sum_m b_m t_{mn}$, so the left-hand side of (2.2.2) becomes

$$\sum_n |a_n|^2 = \sum_n \overline{a}_n \sum_m b_m t_{mn} = \sum_m b_m \sum_n \overline{a}_n t_{mn}.$$

Applying Cauchy-Schwartz inequality and (2.2.1), we get

$$\left(\sum_n |a_n|^2 \right)^2 \leq \left(\sum_m |b_m|^2 \right) \sum_m \left| \sum_n \overline{a}_n t_{mn} \right|^2 \leq D \left(\sum_n |a_n|^2 \right) \left(\sum_m |b_m|^2 \right).$$

Hence we have our desired result if $\sum_n |a_n|^2 \neq 0$. Otherwise, there is nothing to be proved if the afore-mentioned sum vanishes. $\square$

We, as a matter of course, need to use the Poisson summation formula which asserts that if $f(x)$ is a reasonably well-behaved function, then summing $f(n)$ over all integers n is the same as summing the Fourier transform of $f(x)$ over all integers n. More precisely, we have

Lemma 2.3 (Poisson Summation Formula). *Let $f(x)$ be a function on the real numbers of rapid decay at $\pm\infty$ so that the series*

$$\sum_{n\in\mathbb{Z}} f(x+n)$$

converges absolutely. Then we have

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n), \tag{2.2.3}$$

where

$$\hat{f}(x) = \int_{-\infty}^{\infty} f(y)e(xy)\mathrm{d}y, \tag{2.2.4}$$

and is the Fourier transform of $f(x)$.

Proof. Let $F(x) = \sum_{n=-\infty}^{\infty} f(x+n)$. We easily see that $F(x)$ is well-defined by the virtue of the assumption that $f(x)$ is of rapid decay at $\pm\infty$. $F(x)$ is also periodic with period 1. Therefore, $F(x)$ has a Fourier series expansion, say,

$$F(x) = \sum_{m=-\infty}^{\infty} a_n e(mx),$$

where

$$a_m = \int_0^1 F(x)e(-mx)\mathrm{d}x = \int_0^1 \sum_{n=-\infty}^{\infty} f(x+n)e(-mx)\mathrm{d}x.$$

After swapping the order of summation and making change of variables, we see that the right hand side of the above yields

$$\sum_{n=-\infty}^{\infty} \int_0^1 f(x+n)e[m(x+n)]\mathrm{d}x = \int_{-\infty}^{\infty} f(x)e(-mx)\mathrm{d}x = \hat{f}(-m).$$

Finally, we have

$$\sum_{n=-\infty}^{\infty} f(n) = F(0) = \sum_{n=-\infty}^{\infty} a_n = \sum_{n=-\infty}^{\infty} \hat{f}(n),$$

as desired. □

Last, but certainly not least, we must quote Weyl shift as it provides the key idea in our proof. It comes from the simple idea that if we have $f(n)$ in the amplitude of an exponential sum with $f(n)$ is a polynomial of degree k in n, then $f(n+h) - f(n)$

is a polynomial in n of degree $k-1$. Iterating this process $k-1$ times, we shall obtain a linear polynomial which will turn our exponential sum into a geometric series from which we will obtain saving. More precisely, we have

Lemma 2.4 (Weyl Shift). *Let I be an interval of length N and $f(x)$ be a polynomial of degree $k \geq 2$ with real coefficients. Suppose that the leading coefficient, the coefficient of x^k, of $f(x)$ is α. Also set*

$$S = \sum_{n \in I} e(f(n)), \text{ and } \kappa = 2^{k-1}.$$

Then we have

$$|S|^{\kappa} \leq 2^{2\kappa} N^{\kappa-1} + 2^{\kappa} N^{\kappa-k} \sum_{r_1, \cdots, r_{k-1}} \min\left(N, \frac{1}{\|\alpha k! r_1 \cdots r_{k-1}\|}\right), \tag{2.2.5}$$

where each r_i runs from 1 to $N-1$.

Proof. The proof is done via induction on k. For $k=2$, we have

$$\left|\sum_{n=1}^{N}\sum_{m=1}^{N} e(\alpha n^2 + \beta n - \alpha m^2 - \beta m)\right|.$$

Setting $m = n - l$, the above becomes

$$\begin{aligned}
\sum_n \sum_l e(2\alpha n r - \alpha l^2 + \beta l) &\leq \sum_{l=-N+1}^{N-1} \left|\sum_m e(2\alpha n l)\right| \\
&\leq \sum_{l=-N+1}^{N-1} \min\left(N, \frac{2}{\|2\alpha l\|}\right) \\
&\leq N + 4 \sum_{1 \leq l < N} \min\left(N, \frac{1}{\|2\alpha l\|}\right).
\end{aligned}$$

Here we evaluate the geometric series and used the well-known inequalities

$$\left|\frac{1 - e[(b-a)\lambda]}{1 - e(\lambda)}\right| \leq \frac{1}{|\sin \pi\lambda|} \leq \frac{1}{2\|\lambda\|}.$$

Hence the proof is concluded for the case $k=2$. Now it suffices to assume that the statement of the Lemma hold for $k-1$ and prove also its validity for k. We first make the following observation.

$$\begin{aligned}
|S|^2 &= \sum_n \sum_m e[f(n) - f(m)] \\
&= \sum_n \sum_{r_1} e[f(n) - f(n-r_1)] \text{ with } m = n - r_1 \\
&\leq \sum_{r_1=-N+1}^{N-1} |S_1|,
\end{aligned}$$

where $S_1 = \sum_n e[f(n) - f(n-r_1)] = \sum_n e(\alpha k r_1 n^{k-1} + \cdots)$.

Therefore, by Hölder's inequality together with $\left(1 - \frac{2}{\kappa}\right) + \frac{2}{\kappa} = 1$, we have

$$\begin{aligned}
|S|^2 &\leq \left(\sum_{r_1=-N+1}^{N-1} 1\right)^{1-2/\kappa} \left(\sum_{r_1=-N+1}^{N-1} |S_1|^{\kappa/2}\right)^{2/\kappa} \\
&\leq (2N)^{1-2/\kappa} \left(N^{\kappa/2} + \sum_{\substack{r_1=-N+1 \\ r_1 \neq 0}}^{N-1} |S_1|^{\kappa/2}\right)^{2/\kappa}.
\end{aligned}$$

Taking the $\kappa/2$ power of the above, we get

$$|S|^\kappa \leq (2N)^{\kappa/2-1} \left(N^{\kappa/2} + \sum_{\substack{r_1=-N+1 \\ r_1 \neq 0}}^{N-1} |S_1|^{\kappa/2}\right).$$

Now the amplitude of the exponential sum S_1 is a polynomial of degree $k-1$. Therefore, by our induction hypothesis, we have

$$\begin{aligned}
|S_1|^{\kappa/2} \leq & 2^\kappa N^{\kappa/2-1} \\
& + 2^{\kappa/2} N^{\kappa/2-k+1} \sum_{r_2 \cdots r_{k-1}} \min\left(N, \|\alpha k r_1 (k-1)! r_2 \cdots r_{k-1}\|^{-1}\right).
\end{aligned}$$

Finally, we combine everything and obtain

$$\begin{aligned}
|S|^\kappa \leq & 2^{\kappa/2-1} N^{\kappa-1} + 2^{3\kappa/2} N^{\kappa-1} \\
& + 2^\kappa N^{\kappa-k} \sum_{r_1 \cdots r_{k-1}} \min\left(N, \|\alpha k! r_1 \cdots r_{k-1}\|^{-1}\right).
\end{aligned}$$

This is the statement of the theorem for k, and hence our induction is completed and the lemma is proved. □

2.3 Heuristics and "trivial" bounds

As stated earlier, we are interested in having an estimate of the following kind:

$$\sum_{q=1}^{Q} \sum_{\substack{a \bmod q^2 \\ \gcd(a,q)=1}} \left| \sum_{n=M+1}^{M+N} a_n e\left(\frac{a}{q^2} n\right) \right|^2 \ll \Delta \sum_{n=M+1}^{M+N} |a_n|^2. \tag{2.3.1}$$

Here Δ may be thought of as the norm of an operator and will depend on Q and N. It is a remark attributed to Borel that the rational numbers in the real line are like stars in the heavens "to illuminate the mystery of the continuum." Indeed, we must investigate the well-spacedness of some of these "stars." As it certainly suffices to consider only q's in dyadic intervals, we set

$$S_Q = \left\{ \frac{a}{q^2} \in \mathbb{Q} \,\middle|\, \gcd(a,q) = 1,\ 1 \le a < q^2,\ Q < q \le 2Q \right\}.$$

We easily see that if x and x' are two distinct elements of S_Q, then $\|x - x'\| \ge Q^{-4}$. Therefore, just from the classical large sieve inequality, we may take, essentially,

$$\Delta = Q^4 + N \tag{2.3.2}$$

in (2.3.1). On the other hand, all rational numbers $\frac{a}{q^2}$ in S_Q with a fixed denominator q^2 are clearly q^{-2}-spaced. Therefore, again by the virtue of the classical large sieve inequality, we may also take in (2.3.1) after summing over q, essentially,

$$\Delta = Q(Q^2 + N). \tag{2.3.3}$$

It is worthwhile to note that when $N \asymp Q^3$, both (2.3.2) and (2.3.3) can be interpreted as Q^4. However, neither (2.3.2) nor (2.3.3) exploits the fact that squares are so sparsely distributed among the integers. One easily deduces that there are $\asymp Q^3$ rational numbers between 0 and 1 with square denominators and height at most Q^2. Hence, these rational numbers are "on average" Q^{-3}-spaced. Therefore, we aim to exploit these "facts" and improve the estimates in (2.3.2) and (2.3.3). Toward that end, we "divide and conquer."

2.4 The main contention

Again, we are interested in having an estimate of the following kind:

$$\sum_{q=1}^{Q} \sum_{\substack{a \bmod q^2 \\ \gcd(a,q)=1}} \left| \sum_{n=M+1}^{M+N} a_n e\left(\frac{a}{q^2} n\right) \right|^2 \ll \Delta \sum_{n=M+1}^{M+N} |a_n|^2.$$

Before we state and prove our main contention of this section, we first prove the following lemma which is the central issue of our theorem. We must investigate the spacing properties of the elements in S_Q. Set

$$M(Q,N) = \max_{x' \in S_Q} \left| \left\{ x \in S_Q \,\middle|\, \|x - x'\| < \frac{1}{2N} \right\} \right|, \tag{2.4.1}$$

and we have

Lemma 2.5. *Given $\epsilon > 0$ and $N \in \mathbb{N}$, then we have*

$$M(Q,N) \ll \frac{Q^3}{N} + \left(\sqrt{Q} + \frac{Q^2}{\sqrt{N}} \right) N^{\epsilon}, \tag{2.4.2}$$

where the implied constant in (2.4.2) *depends on ϵ alone.*

Proof. The task before us is as follows. Let $x, x' \in S_Q$ with $x = \frac{a}{q^2}$, $x' = \frac{a_1}{q_1^2}$ and $\gcd(a,q) = \gcd(a_1,q_1) = 1$. Then we have $aq_1^2 - a_1q^2 \equiv b \pmod{q^2q_1^2}$ with $|b| \le \frac{1}{2}q^2q_1^2$.

We have

$$0 \le \left\| \frac{a}{q^2} - \frac{a_1}{q_1^2} \right\| = \frac{|b|}{q^2q_1^2} < \frac{1}{2N}. \tag{2.4.3}$$

This yields that $|b| \ll q^2Q^2N^{-1} = B$, say. We want to estimate, for each $\frac{a}{q^2}$, the number of fractions $\frac{a_1}{q_1^2} \in S_Q$ satisfying (2.4.3). Let $b = aq_1^2 - a_1q^2$. We have $|b| < B$ and

$$\begin{cases} q^2 \equiv b\overline{a}_1, & \pmod{q_1^2} \\ b \equiv aq_1^2, & \pmod{q^2} \end{cases}, \tag{2.4.4}$$

where $\overline{a}_1$ is the multiplicative inverse of a_1 modulo q_1^2.

We shall estimate the number of b's and q_1's that satisfy the second congruence relation in (2.4.4) with $q_1 < Q$ and $|b| < B$, which clearly majorizes the maximum that we need to estimate in (2.4.1).

First, we set $\phi(x) = \left(\frac{\sin \pi x}{2x}\right)^2$, a constant multiple of Féjer kernel. We note that $\phi(x)$ is non-negative, $\phi(x) \geq 1$ for $|x| \leq \frac{1}{2}$ and $\phi(0) = \pi^2/4$. Therefore

$$\sum_{n \equiv a q_1^2 \mod q^2} \phi\left(\frac{n}{2B}\right) \tag{2.4.5}$$

majorizes $M(Q)$. There is certainly no unique choice for this test function $\phi(x)$. However, as we shall presently apply Poisson summation formula, we find it most convenient to choose $\phi(x)$ this way, since its Fourier transform is a function of compact support, specifically $\hat{\phi}(s) = \frac{\pi^2}{4} \max(1 - |s|, 0)$.

Now we apply Poisson summation, Lemma 2.3 with a linear change of variable, to (2.4.5) and sum q_1 over dyadic intervals, we get

$$\frac{2B}{q^2} \sum_{Q < q_1 \leq 2Q} \sum_j e\left(\frac{ajq_1^2}{q^2}\right) \hat{\phi}\left(\frac{2jB}{q^2}\right). \tag{2.4.6}$$

More precisely, the above is

$$\begin{aligned} & \frac{\pi^2 B}{2q^2} \sum_{|j| < \frac{q^2}{2B}} \sum_{Q < q_1 \leq 2Q} \left(1 - \frac{2|j|B}{q^2}\right) e\left(\frac{ajq_1^2}{q^2}\right) \\ = \ & \frac{\pi^2 Q^2}{N} \sum_{|k| < \frac{N}{4Q^2}} \sum_{Q < q_1 \leq 2Q} \left(1 - \frac{4|j|Q^2}{N}\right) e\left(\frac{ajq_1^2}{q^2}\right) \\ \leq \ & \frac{\pi^2 Q^3}{N} + \frac{2\pi^2 Q^2}{N} \sum_{0 < k < \frac{N}{4Q^2}} \left| \sum_{Q < q_1 \leq 2Q} e\left(\frac{ajq_1^2}{q^2}\right) \right|, \end{aligned}$$

where the first term above corresponds to the contribution of $k = 0$. Applying Cauchy-Schwartz inequality, we see that the above expression is bounded by

$$\ll \frac{Q^6}{N^2} + \frac{Q^2}{N} \sum_{0 < k < \frac{N}{4Q^2}} \left| \sum_{Q < q_1 \leq 2Q} e\left(\frac{ajq_1^2}{q^2}\right) \right|^2.$$

Applying Weyl Shift, Lemma 2.4 to the inner-most sum of the second term, we see that the double sum of the second term is

$$\begin{aligned} &\ll \sum_k Q + \sum_k \sum_{0<l<Q} \min\left\{Q, \left\|\frac{2ajl}{q^2}\right\|^{-1}\right\} \\ &\ll \frac{N}{Q} + \sum_{0<m<NQ^{-1}} \tau(m) \min\left\{Q, \left\|\frac{am}{q^2}\right\|^{-1}\right\}, \end{aligned}$$

where $\tau(m)$ is the divisor function, is $O(m^\epsilon)$ and estimates the multiplicity of representations of $m = 2jl$. The inequalities go in the correct direction by the virtue of positivity.

What still remains is to estimate the sum over m. We have, with $am \equiv d \pmod{q^2}$ and $|d| \leq \frac{1}{2}q^2$,

$$\begin{aligned} \sum_m &\leq \sum_{|d|<q^2} \min\left\{Q, \frac{q^2}{|d|}\right\} \sum_{0<m<NQ^{-1}} \tau(m) \\ &\ll q^{-2}N^{1+\epsilon} + \sum_{0<d<q^2} \frac{q^2}{d}\left(q^{-2}\frac{N}{Q}+1\right)\left(\frac{N}{Q}\right)^\epsilon \\ &\ll q^{-2}N^{1+\epsilon} + \left(\frac{N}{Q} + q^2\right) N^\epsilon \\ &\ll \left(\frac{N}{Q} + Q^2\right) N^\epsilon. \end{aligned}$$

Recall that we are only considering the q's in the dyadic interval $Q < q \leq 2Q$. Combining everything and taking the square root, we infer that for every $x \in S_Q$,

$$\left|\left\{x' \in S_Q \,\middle|\, \|x - x'\| < \frac{1}{2N}\right\}\right| \ll \frac{Q^3}{N} + \left(\sqrt{Q} + \frac{Q^2}{\sqrt{N}}\right) N^\epsilon, \tag{2.4.7}$$

where the implied constant depends on ϵ alone. Since what we have proved is for valid any $x \in S_Q$, we infer the result of our lemma from (2.4.7). □

Now we are finally able to state and prove our main contention of the chapter. The beginning of the proof will go very much like that of the classical large sieve inequalities.

Theorem 2.2. *With $\{a_n\}$, Q, M, and N defined as before, we have*

$$\begin{aligned} &\sum_{q=1}^{Q} \sum_{\substack{a \bmod q^2 \\ \gcd(a,q)=1}} \left|\sum_{n=M+1}^{M+N} a_n e\left(\frac{a}{q^2}n\right)\right|^2 \\ &\qquad \ll \log(2Q)\left[Q^3 + (N\sqrt{Q} + \sqrt{N}Q^2)N^\epsilon\right] \sum_{n=M+1}^{M+N} |a_n|^2, \end{aligned} \tag{2.4.8}$$

where the implied constant depends on ϵ alone.

Proof. It is easily observed that the theorem, after breaking the summation over q into dyadic intervals with $Q < q \leq 2Q$ and the duality principle Lemma 2.2, and by assuming $M = 0$ via the shift, $n \longrightarrow n - M$, is equivalent to

$$\sum_{0<n\leq N} \left| \sum_{x\in S_Q} b_x e(xn) \right|^2 \ll \left[Q^3 + (N\sqrt{Q} + \sqrt{N}Q^2)N^\epsilon \right] \sum_{x\in S_Q} |b_x|^2, \tag{2.4.9}$$

for any sequence of complex numbers $\{b_x\}$.

As before, we take $\phi(x) = \left(\frac{\sin \pi x}{2x}\right)^2$. By positivity, the left-hand side of (2.4.9) is majorized by

$$\sum_{n=-\infty}^{\infty} \phi\left(\frac{n}{2N}\right) \left| \sum_{x\in S_Q} b_x e(nx) \right|^2 = \sum_x \sum_{x'} b_x \bar{b}_{x'} V(x - x'),$$

where $V(y) = \sum_n \phi\left(\frac{n}{2N}\right) e(ny)$.

Apply Poisson summations formula and a change of variables, we see that

$$\begin{aligned} V(y) &= 2N \sum_m \hat{\phi}[2N(m+y)] \\ &= \frac{\pi^2 N}{2} \sum_{|m+y|<(2N)^{-1}} (1 - 2N|m+y|) \\ &= \frac{\pi^2 N}{2}(1 - 2N\|y\|), \end{aligned}$$

if $\|y\| < (2N)^{-1}$ and $V(y) = 0$ otherwise.

Hence the left-hand side of (2.4.9) is majorized by

$$\frac{\pi^2 N}{2} \sum_x \sum_{\substack{x' \\ ||x-x'||<(2N)^{-1}}} |b_x b_{x'}| \leq \frac{\pi^2 N}{2} \sum_x |b_x|^2 M(Q, N),$$

where $M(Q, N)$ is as defined in Lemma 2.5. We insert the result of the afore-mentioned lemma, the theorem is proved. Note that we lose one logorithm after summing over all the dyadic intervals. □

It is obvious that the implied constant in our theorem can be made absolute by keeping track of the powers of logarithms in our estimates. However, we have not been too concerned with that.

The greatest strength of our result lies in the range where $N \asymp Q^3$. There, our result gives the majorant of $O(Q^{\frac{7}{2}+\epsilon})$ while both (2.3.2) and (2.3.3) give the majorant of $O(Q^4)$.

In the same spirit that the classical large sieve inequality for additive characters implies that of the multiplicative characters, we have the following easy corollary.

Corollary 2.3. *For any sequence of complex numbers $\{a_n\}$, we have*

$$\sum_{q=1}^{Q} \frac{q}{\varphi(q)} \sum_{\chi \bmod q^2}^{\star} \left| \sum_{n=M+1}^{M+N} a_n \chi(n) \right|^2 \ll \log(2Q) \left[Q^3 + (N\sqrt{Q} + \sqrt{N} Q^2) N^{\epsilon} \right] \sum_{n=M+1}^{M+N} |a_n|^2, \tag{2.4.10}$$

where the implied constant depends on ϵ alone.

Proof. Using Gauss sums $G(\chi)$, we have

$$\chi(n) = \frac{1}{G(\overline{\chi})} \sum_{a \bmod q^2} \overline{\chi}(a) e\left(\frac{an}{q^2}\right).$$

It is an elementary fact that the modulus of the Gauss sum $G(\chi)$ is the square root of its modulus, q in our case. Hence we have,

$$\begin{aligned} \sum_{\chi \bmod q^2}^{\star} \left| \sum_{n=M+1}^{M+N} a_n \chi(n) \right|^2 &\le \frac{1}{q^2} \sum_{\chi \bmod q^2} \left| \sum_{a \bmod q^2} \overline{\chi}(a) \sum_n a_n e\left(\frac{an}{q^2}\right) \right|^2 \\ &= \frac{\varphi(q)}{q} \sum_{\substack{a \bmod q^2 \\ \gcd(a,q)=1}} \left| \sum_n a_n e\left(\frac{an}{q^2}\right) \right|^2. \end{aligned}$$

The last equality is obtained by opening the modulus square and applying the orthogonality of Dirichlet characters. Note also that $\varphi(q^2) = q\varphi(q)$. Our the corollary now follows from Theorem 2.2. □

2.5 Notes

We do not believe (2.4.8) is the best possible result. As we have related in the heuristics in section 2.3, there are approximately Q^3 elements in S_Q and they are "on average" Q^{-3}-spaced. Consequently, if the last sentence has any virtue at all, the majorant in (2.4.8) should be, essentially,

$$(Q^3 + N) \sum_{n=M+1}^{M+N} |a_n|^2. \tag{2.5.1}$$

In fact, we shall make conjectures based on some computational evidences in Appendix A, which yields a stronger result.

Moreover, we doubt that any improvement upon our present result is attainable via the means estimating of incomplete Gauss sums, as we have done. Indeed, if one is a true believer of the so-called "principle of square rooting," then we should believe that the saving attained by any estimation of an exponential sum cannot be good enough to yield the majorant in (2.5.1).

What stays us from obtaining better result is precisely the want of better means to estimate $M(Q, N)$ as defined in Lemma 2.5. As mentioned before, we believe that any improvement upon our present result must come from a completely different avenue.

Chapter 3

Large Sieve Inequalities for Characters with Powerful Moduli

3.1 Introduction and Heuristics

As mentioned in the previous chapter, the ideas that we used to establish the theorems of large sieve type inequalities with square moduli are perfectly generalizable to higher power moduli. We will treat such cases in this chapter. Of course, as the result of Weyl shift weakens as the polynomial in the amplitude is of high degree, we expect that our corresponding result also weakens.

We are interested in having an estimate of the following kind:

$$\sum_{q=1}^{Q} \sum_{\substack{a \bmod q^k \\ \gcd(a,q)=1}} \left| \sum_{n=M+1}^{M+N} a_n e\left(\frac{a}{q^k} n\right) \right|^2 \ll \Delta \sum_{n=M+1}^{M+N} |a_n|^2. \tag{3.1.1}$$

As before, the problem reduces to the well-spacing properties of special fractions.

Let $S_{Q,k} = \left\{ \frac{a}{q^k} \in \mathbb{Q} \middle| \gcd(a,q) = 1,\ 1 \le a < q^k \le Q^k,\ Q < q \le 2Q \right\}$. We easily see that if x and x' are two distinct elements of S_Q, then $\|x - x'\| \ge \frac{1}{Q^{2k}}$. Therefore, just from the classical large sieve inequality, we may take

$$\Delta = Q^{2k} + N. \tag{3.1.2}$$

On the other hand, all rational numbers $\frac{a}{q^k}$ between 0 and 1 with a fixed denominator q^k are clearly q^{-k}-spaced. Therefore, again by the virtue of the classical large sieve inequality, we may also take

$$\Delta = Q(Q^k + N). \tag{3.1.3}$$

It is worthwhile to note that when $N \asymp Q^{k+1}$, both (3.1.2) and (3.1.3) can be interpreted as Q^{2k}. However, neither (3.1.2) nor (3.1.3) exploited the fact that squares are so sparsely distributed among the integers. As one easily deduces that there are $\asymp Q^{k+1}$ rational numbers between 0 and 1 with with square denominators and height at most Q^k. Hence, the rational numbers are "on average" $Q^{-(k+1)}$-spaced. Therefore, we aim to exploit these "facts" and improve the expression in (3.1.2) and (3.1.3). Toward that end, we "divide and conquer," as before.

3.2 The Main Contention

Again, we are interested in having an estimate of the following kind:

$$\sum_{q=1}^{Q} \sum_{\substack{a \mod q^k \\ \gcd(a,q)=1}} \left| \sum_{n=M+1}^{M+N} a_n e\left(\frac{a}{q^k} n\right) \right|^2 \ll \Delta \sum_{n=M+1}^{M+N} |a_n|^2.$$

Before we state and prove our main contention of this section, we first prove the following lemma which is the central issue of our theorem. As before, we must investigate the spacing properties of the elements in $S_{Q,k}$. If we set

$$M_k(Q,N) = \max_{x' \in S_{Q,k}} \left| \left\{ x \in S_{Q,k} \middle| \|x - x'\| < \frac{1}{2N} \right\} \right|, \tag{3.2.1}$$

then we have

Lemma 3.1. *Given $\epsilon > 0$ and $N \in \mathbb{N}$, then we have*

$$M_k(Q,N) \ll \frac{Q^{k+1}}{N} + \left(Q^{\frac{\kappa-1}{\kappa}} + \frac{Q^{\frac{\kappa+k}{\kappa}}}{N^{\frac{1}{\kappa}}} \right) N^{\epsilon}, \tag{3.2.2}$$

where the implied constant in (3.2.2) *depends on ϵ and k.*

Proof. The task before us is as follows. Let $x = \dfrac{a}{q^k}$ and $x' = \dfrac{a_1}{q_1^k}$, with $\gcd(a,q) = \gcd(a_1, q_1) = 1$. Let $aq_1^k - a_1 q^k \equiv b \pmod{q^k q_1^k}$ with $|b| \leq \dfrac{1}{2} q^k q_1^k$.

We have

$$0 \leq \left\| \frac{a}{q^k} - \frac{a_1}{q_1^k} \right\| = \frac{|b|}{q^k q_1^k} < \frac{1}{2N}. \tag{3.2.3}$$

This yields that $|b| \ll q^k Q^k N^{-1} = B$, say. We want to estimate, for each $\frac{a}{q^k}$, the number of fractions $\frac{a_1}{q_1^k} \in S_{Q,k}$ satisfying (3.2.3). Let $b = aq_1^k - a_1 q^k$. We have $|b| < B$ and

(3.2.4)
$$\begin{cases} q^k \equiv b\overline{a}_1, & \pmod{q_1^k} \\ b \equiv aq_1^k, & \pmod{q^k} \end{cases},$$

where $\overline{a}_1$ is the multiplicative inverse of a_1 modulo q_1^k.

We shall estimate the number of b's and q_1's that satisfy the second congruence relation in (3.2.4) with $q_1 < Q$ and $|b| < B$, which clearly majorizes the maximum that we need to estimate in (3.2.1).

First, we set $\phi(x) = \left(\frac{\sin \pi x}{2x}\right)^2$, a constant multiple of Féjer kernel. We note that $\phi(x)$ is non-negative, $\phi(x) \geq 1$ for $|x| \leq \frac{1}{2}$ and $\phi(0) = \pi^2/4$. Therefore

(3.2.5)
$$\sum_{n \equiv aq_1^k \mod q^k} \phi\left(\frac{n}{2B}\right)$$

majorizes $M_k(Q, N)$. There is certainly no unique choice for this test function $\phi(x)$. However, as we shall presently apply Poisson summation formula, we find it most convenient to choose $\phi(x)$ this way, since its Fourier transform is a function of compact support, specifically $\hat{\phi}(s) = \frac{\pi^2}{4}\max(1 - |s|, 0)$.

Now we apply Poisson summation, Lemma 2.3 with a linear change of variable, to (3.2.5) and sum q_1 over dyadic intervals, we get

(3.2.6)
$$\frac{2B}{q^k} \sum_{Q < q_1 \leq 2Q} \sum_j e\left(\frac{ajq_1^k}{q^k}\right) \hat{\phi}\left(\frac{2jB}{q^k}\right).$$

More precisely, the above is

$$\begin{aligned}
& \frac{\pi^2 B}{2q^k} \sum_{|j|<\frac{q^k}{2B}} \sum_{Q<q_1\le 2Q} \left(1-\frac{2|j|B}{q^k}\right) e\left(\frac{ajq_1^k}{q^k}\right) \\
= \;& \frac{\pi^2 Q^k}{N} \sum_{|j|<\frac{N}{4Q^k}} \sum_{Q<q_1\le 2Q} \left(1-\frac{4|j|Q^k}{N}\right) e\left(\frac{ajq_1^k}{q^k}\right) \\
\le \;& \frac{\pi^2 Q^{k+1}}{N} + \frac{2\pi^2 Q^k}{N} \sum_{0<j<\frac{N}{4Q^k}} \left| \sum_{Q<q_1\le 2Q} e\left(\frac{ajq_1^k}{q^k}\right)\right|,
\end{aligned}$$

where the first term above corresponds to the contribution of $k=0$. Applying Cauchy-Schwartz inequality $k-1$ times, we see that the κ-th power of the above expression is bounded by

$$\ll \frac{Q^{\kappa(k+1)}}{N^\kappa} + \frac{Q^k}{N} \sum_{0<j<\frac{N}{4Q^k}} \left| \sum_{Q<q_1\le 2Q} e\left(\frac{ajq_1^k}{q^k}\right)\right|^\kappa . \tag{3.2.7}$$

Applying Weyl Shift, Lemma 2.4 to the inner-most sum of the second term, we see that the double sum of the second term is

$$\begin{aligned}
\ll \;& \sum_j Q^{\kappa-1} + Q^{\kappa-k} \sum_j \sum_{r_1,r_2,\cdots,r_{k-1}} \min\left\{Q,\ \left\|\frac{2ajr_1r_2\cdots r_{k-1}}{q^k}\right\|^{-1}\right\} \\
\ll \;& \frac{N}{Q}^{\kappa-k-1} + Q^{\kappa-k} \sum_{0<m<NQ^{-1}} \tau_k(m)\min\left\{Q,\ \left\|\frac{am}{q^k}\right\|^{-1}\right\},
\end{aligned}$$

where $\tau_k(m)$ is the k-th divisor function, is $O(m^\epsilon)$ and estimates the multiplicity of representations of $m=2jl$. The inequalities go in the correct direction by the virtue of positivity.

What still remains is to estimate the sum over m. We have, with $am \equiv d \pmod{q^k}$ and $|d| \le \frac{1}{2}q^k$,

$$\begin{aligned}
\sum_m \tau_k(m)\min\left\{Q,\ \left\|\frac{am}{q^k}\right\|^{-1}\right\} \le \;& \sum_{|d|<q^k} \min\left\{Q,\ \frac{q^k}{|d|}\right\} \sum_{0<m<NQ^{-1}} \tau_k(m) \\
\ll \;& q^{-k}N^{1+\epsilon} + \sum_{0<d<q^k} \frac{q^k}{d}\left(q^{-k}\frac{N}{Q}+1\right)\left(\frac{N}{Q}\right)^\epsilon \\
\ll \;& q^{-k}N^{1+\epsilon} + \left(\frac{N}{Q}+q^k\right)N^\epsilon \\
\ll \;& \left(\frac{N}{Q}+Q^k\right)N^\epsilon.
\end{aligned}$$

Recall that we are only considering the q's in the dyadic interval $Q < q \le 2Q$. Combining things, we see that (3.2.7) is

$$\begin{aligned}
&\ll \frac{Q^{\kappa(k+1)}}{N^\kappa} + \frac{Q^k}{N} N^\epsilon \left[NQ^{\kappa-k-1} + Q^{\kappa-k}\left(\frac{N}{Q} + Q^k\right)\right] \\
&= \frac{Q^{\kappa(k+1)}}{N^\kappa} + \frac{Q^k}{N} N^\epsilon \left(NQ^{\kappa-k-1} + NQ^{\kappa-k-1} + Q^\kappa\right) \\
&\ll \frac{Q^{\kappa(k+1)}}{N^\kappa} + Q^{k-1} + \frac{Q^{\kappa+k}}{N}.
\end{aligned}$$

Taking the κ-th root, we infer that for every $x \in S_{Q,k}$,

$$\left| \left\{ x' \in S_{Q,k} \middle| \|x - x'\| < \frac{1}{2N} \right\} \right| \ll \frac{Q^{k+1}}{N} + \left(Q^{\frac{\kappa-1}{\kappa}} + \frac{Q^{\kappa+k}}{N^{\frac{1}{\kappa}}} \right) N^\epsilon.$$

Of course, the implied constant will depend on both k and ϵ. Now we infer the lemma from the above expression. □

Now we are finally able to state and prove our main contention of the paper. The beginning of the proof will go very much like that of the classical large sieve inequalities. As far as that part is concerned, we are following the proof given in [18].

Theorem 3.1. *With $\{a_n\}$, Q, M, and N defined as before, we have*

$$\sum_{q=1}^{Q} \sum_{\substack{a \bmod q^k \\ \gcd(a,q)=1}} \left| \sum_{n=M+1}^{M+N} a_n e\left(\frac{a}{q^k} n\right) \right|^2 \tag{3.2.8}$$
$$\ll \log(2Q) \left[Q^{k+1} + (NQ^{\frac{\kappa-1}{\kappa}} + N^{1-\frac{1}{\kappa}} Q^{\frac{\kappa+k}{\kappa}}) N^\epsilon \right] \sum_{n=M+1}^{M+N} |a_n|^2,$$

where the implied constant depends on ϵ and k.

Proof. It is easily observed that the theorem, after breaking the summation over q into dyadic intervals with $Q < q \le 2Q$ and the duality principle Lemma 2.2, and by assuming $M = 0$ via the shift, $n \longrightarrow n - M$, is equivalent to

$$\sum_{0<n\le N} \left| \sum_{x \in S_Q} b_x e(xn) \right|^2 \ll \left[Q^{k+1} + (NQ^{\frac{\kappa-1}{\kappa}} + N^{1-\frac{1}{\kappa}} Q^{\frac{\kappa+k}{\kappa}}) N^\epsilon \right] \sum_{x \in S_Q} |b_x|^2, \tag{3.2.9}$$

for any sequence of complex numbers $\{b_x\}$.

As before, we take $\phi(x) = \left(\frac{\sin \pi x}{2x}\right)^2$. By positivity, the left-hand side of (3.2.9) is majorized by

$$\sum_{n=-\infty}^{\infty} \phi\left(\frac{n}{2N}\right) \left| \sum_{x \in S_Q} b_x e(nx) \right|^2 = \sum_x \sum_{x'} b_x \bar{b}_{x'} V(x - x'),$$

where $V(y) = \sum_n \phi\left(\frac{n}{2N}\right) e(ny)$.

Apply Poisson summations formula and a change of variables, we see that

$$\begin{aligned} V(y) &= 2N \sum_m \hat{\phi}[2N(m+y)] \\ &= \frac{\pi^2 N}{2} \sum_{|m+y|<(2N)^{-1}} (1 - 2N|m+y|) \\ &= \frac{\pi^2 N}{2}(1 - 2N\|y\|), \end{aligned}$$

if $\|y\| < (2N)^{-1}$ and $V(y) = 0$ otherwise.

Hence the left-hand side of (3.2.9) is majorized by

$$\frac{\pi^2 N}{2} \sum_x \sum_{\substack{x' \\ \|x-x'\|<(2N)^{-1}}} |b_x b_{x'}| \le \frac{\pi^2 N}{2} \sum_x |b_x|^2 M_k(Q, N),$$

where $M_k(Q, N)$ is as defined in Lemma 3.1. We insert the result of the afore-mentioned lemma and sum over all the dyadic intervals, the theorem is proved. □

It is obvious that the implied constant in our theorem can be made absolute by keeping track of the powers of logarithms in our estimates. However, we have not been too concerned with that.

The greatest strength of our result lies in the range where $N \asymp Q^{k+1}$. There, our result gives the majorant of $O(Q^{k+1+\frac{\kappa-1}{\kappa}+\epsilon})$ while (3.1.2) gives the extremely poor majorant of $O(Q^{2k})$ and (3.1.3) give the majorant of $O(Q^{k+2})$.

In the same spirit that the classical large sieve inequality for additive characters implies that of the multiplicative characters, we have the following easy corollary.

Corollary 3.1. *For any sequence of complex numbers $\{a_n\}$, we have*

(3.2.10)
$$\sum_{q=1}^{Q}\frac{q}{\varphi(q)}\sum_{\chi \bmod q^k}^{\star}\left|\sum_{n=M+1}^{M+N}a_n\chi(n)\right|^2 \ll \log(2Q)\left[Q^{k+1}+(NQ^{\frac{\kappa-1}{\kappa}}+N^{1-\frac{1}{\kappa}}Q^{\frac{\kappa+k}{\kappa}})N^{\epsilon}\right]\sum_{n=M+1}^{M+N}|a_n|^2,$$

where the implied constant depends on ϵ and k.

Proof. Using Gauss sums $G(\chi)$, we have
$$\chi(n)=\frac{1}{G(\overline{\chi})}\sum_{a \bmod q^k}\overline{\chi}(a)e\left(\frac{an}{q^k}\right).$$

It is an elementary fact that the modulus of the Gauss sum $G(\chi)$ is the square root of its modulus, q in our case. Hence we have,

$$\begin{aligned}\sum_{\chi \bmod q^k}^{\star}\left|\sum_{n=M+1}^{M+N}a_n\chi(n)\right|^2 &\le \frac{1}{q^k}\sum_{\chi \bmod q^k}\left|\sum_{a \bmod q^k}\overline{\chi}(a)\sum_{n}a_ne\left(\frac{an}{q^k}\right)\right|^2\\ &= \frac{\varphi(q)}{q}\sum_{\substack{a \bmod q^k\\ \gcd(a,q)=1}}\left|\sum_{n}a_ne\left(\frac{an}{q^k}\right)\right|^2.\end{aligned}$$

The last equality is obtained by opening the modulus square and applying the orthogonality of Dirichlet characters. Note also that $\varphi(q^k)=q^{k-1}\varphi(q)$. Our the corollary now follows from Theorem 3.1. □

3.3 Notes

Again, we do not believe (3.2.8) is the best possible result. As we have related in the heuristics at the beginning of the chapter, there are approximately Q^{k+1} elements in $S_{Q,k}$ and they are "on average" Q^{-k-1}-spaced. Consequently, if the last sentence has any virtue at all, the majorant in (2.4.8) should be, essentially,

(3.3.1)
$$\left(Q^{k+1}+N\right)\sum_{n=M+1}^{M+N}|a_n|^2.$$

Moreover, as before, we doubt that any improvement upon our present result is attainable via the means estimating of incomplete Gauss sums, as we have done. Indeed, what more saving can we possibly hope for in an incomplete Gauss sum than the square-root of the length of summation, as we have been able to achieve. Consequently, if one believes the validity of the majorant in (3.3.1) at all, such an estimate has to comes from a completely different avenue as we have used thus far.

Chapter 4

Large Sieve Inequality for Special Characters to Prime Square Moduli

4.1 Introduction and History

In this chapter, we still again consider a large sieve type inequality for characters to square moduli, but we shall restrict to some special characters, not necessarily primitive. With such restrictions, we can obtain some non-trivial results. More precisely, we aim to have a result of the following kind.

$$\sum_{q=1}^{Q} \frac{1}{\varphi(q)} \sideset{}{'}\sum_{\chi \bmod q^2} \left| \sum_{n=M+1}^{M+N} a_n \chi(n) \right|^2 \ll \Delta \sum_{n=M+1}^{M+N} |a_n|^2, \tag{4.1.1}$$

where the $\sum'$ runs over some special Dirichlet characters to the specified modulus, and as usual Δ will be in terms of Q and N.

In all of our investigations of this chapter, we shall restrict our attention to prime square moduli only. The result can be generalized, but restricting the prime square moduli gives us great convenience in estimates, *id est* two prime squares are not co-prime if and only if they are the same.

The idea of studying characters sums to prime-power moduli was started in a paper by A. G. Postnikov [28]. He gave formulas on the decomposition of groups of characters of powerful moduli. P. X. Gallagher [12] also studied characters of this type. Iwaniec [17] expanded Postnikov's ideas to composite moduli.

4.2 Heuristics

First we note that there are $\varphi(q^2) = q\varphi(q)$ Dirichlet characters modulo q^2 which is the same as the number of primitive residue classes modulo q^2 and the groups $\left(\mathbb{Z}/q^2\mathbb{Z}\right)^*$. The group $G = \left(\mathbb{Z}/q^2\mathbb{Z}\right)^*$ contains the following subgroup

$$H = \{x \in G \mid x \equiv 1 \pmod{q}\},$$

which is isomorphic to the additive group $\mathbb{Z}/q\mathbb{Z}$. The isomorphism is given by

$$H \longrightarrow \mathbb{Z}/q\mathbb{Z} : x \longrightarrow \frac{x-1}{q}.$$

Let $\xi \pmod{q^2}$ be a character on G. Then the induced character on H is an additive character, so we have

$$\xi(x) = e\left[\frac{a(x-1)}{q^2}\right], \tag{4.2.1}$$

for some $a \pmod{q}$ and $x \in H$. If $\xi' \pmod{q^2}$ is another character satisfying (4.2.1) with the same $a \pmod{q}$. Then $\xi'\xi^{-1}$ is a character on G that is trivial on H. Therefore, $\xi' = \xi\chi$ where χ is induced by a character on $(\mathbb{Z}/q\mathbb{Z})^*$. Let G_a be the set of characters $\xi \pmod{q^2}$ satisfying (4.2.1). We see that G_a has $\varphi(q)$ elements, and that every element of G_a is obtained in a unique way by multiplying a fixed character $\xi \in G_a$ by a character $\chi \pmod{q}$ on $(\mathbb{Z}/q\mathbb{Z})^*$.

Therefore, the context in which we shall consider the sum in the left-hand side of (4.1.1) is with $a = 1$ in (4.2.1); *Id est*, we shall study the following sum.

$$\sum_{q=1}^{Q} \frac{1}{\varphi(q)} \sum_{\xi \in G_1} \left| \sum_{n=M+1}^{M+N} a_n \xi(n) \right|^2. \tag{4.2.2}$$

The more general case in which one considers characters in G_a is similar to the investigation for G_1. Therefore, we restrict our attention to only G_1. Also, we note that the characters being summed are not necessarily primitive, a feature that the classical large sieve inequalities, Corollary 2.2, do not possess.

Next, we observe the following.

$$\frac{1}{\varphi(q)}\sum_{\xi\in G_1}\left|\sum_{n=M+1}^{M+N}a_n\xi(n)\right|^2$$
$$= \frac{1}{\varphi(q)}\sum_{n}\sum_{n'}a_n\overline{a}_{n'}\xi(n\overline{n}')\sum_{\chi}\chi(n\overline{n'}),$$

where ξ is a fixed character in G_1 and χ runs over Dirichlet character of $(\mathbb{Z}/q\mathbb{Z})^*$. The inner-most sum vanishes unless $n \equiv n' \pmod q$, in which case it yields $\varphi(q)$ and $\xi(n\overline{n}') = e\left(\frac{nn'-1}{q}\right) = e\left(\frac{\overline{n}}{q}\frac{n'-n}{q}\right)$ by (4.2.1). Note that $\frac{n'-n}{q}$ is an integer. Hence, we have

$$\frac{1}{\varphi(q)}\sum_{\xi \bmod q^2}^{(1)}\left|\sum_{n=M+1}^{M+N}a_n\xi(n)\right|^2 = \sum_{\substack{n \quad n' \\ n\equiv n' \bmod q}}\sum a_n\overline{a}_{n'}e\left(\frac{\overline{n}}{q}\frac{n'-n}{q}\right), \tag{4.2.3}$$

where henceforth $\sum\limits_{\xi \bmod q^2}^{(1)}$ denotes sum over characters $\xi \in G_1$. Trivially estimating the contribution of the above sum gives

$$\frac{1}{\varphi(q)}\sum_{\xi \bmod q^2}^{(1)}\left|\sum_{n=M+1}^{M+N}a_n\xi(n)\right|^2 = \left[1+O\left(\frac{N}{q}\right)\right]\sum_{\gcd(n,q)=1}|a_n|^2. \tag{4.2.4}$$

Estimating thus gives that

$$\sum_{q=1}^{Q}\frac{1}{\varphi(q)}\sum_{\xi \bmod q^2}^{(1)}\left|\sum_{n=M+1}^{M+N}a_n\xi(n)\right|^2 = [Q+O(N)]\sum_{n}|a_n|^2. \tag{4.2.5}$$

Although (4.2.5) is obtained trivially, it is an asymptotic formula rather than simply an upper bound, a main feature of the classical large sieve inequalities. (4.2.5) is of interest when $N \ll Q$. Moreover, from (4.2.5), we see that any improvement upon the exponent of Q is not possible and any improvement upon this trivial bound has to come from that of N. However, we hope that we can do better in some ranges. Toward that end, we must use the results in the following section.

4.3 Preliminaries Lemmas

As before, we write down, in this section, the results that we shall utilize later in the chapter. First, we shall need the following famous estimate for the classical Kloosterman Sums.

Theorem 4.1 (Weil). *For any $c \geq 1$, m and n, we have*

$$\left| \sum_{ad \equiv 1 \bmod c} e\left(\frac{ma+nd}{c}\right) \right| \leq [\gcd(m,n,c)]^{\frac{1}{2}} c^{\frac{1}{2}} \tau(c), \tag{4.3.1}$$

where $\tau(c)$ is the divisor function.

Proof. This is quoted from [19] and is deduced from the celebrated Riemann hypothesis for curves over finite fields proved by A. Weil in 1948. □

Next we shall also need the following estimate for Ramanujan's Sums.

Lemma 4.1.

$$\left| \sum_{\substack{a \pmod q \\ \gcd(a,q)=1}} e\left(\frac{an}{q}\right) \right| \leq \gcd(n,q). \tag{4.3.2}$$

Proof. Ramanujan's Sum is the right-hand side of (1.1.2). Estimate for such sums are well-known. We already showed in Lemma 1.1 that

$$\sum_{\substack{a \pmod q \\ \gcd(a,q)=1}} e\left(\frac{an}{q}\right) = \sum_{d \mid \gcd(q,n)} d\mu\left(\frac{m}{d}\right). \tag{4.3.3}$$

Hence, we have

$$\sum_{\substack{a \pmod q \\ \gcd(a,q)=1}} e\left(\frac{an}{q}\right) = \mu\left(\frac{n}{\gcd(n,q)}\right) \frac{\varphi(n)}{\varphi\left(\frac{n}{\gcd(n,q)}\right)}, \tag{4.3.4}$$

from which we have the following estimate which is useful for us later. □

The inequality in (4.3.2) is the best possible, since one can easily find examples in which equality holds in (4.3.2).

4.4 Main Contention

The following is a quote from [16]:

> An analytic number theorist, it is said, is someone who is very good at using Cauchy's inequality.

Indeed, we must demonstrate our worth in that regard. In this section, we shall restrict our attention only to moduli that are prime squares. The advantage of restricting to prime square moduli is that two moduli are not co-prime only if they are the same. This will provide convenience in our analysis. We now state and prove the main result of the chapter.

Theorem 4.2. *Let $\epsilon > 0$ be given. Suppose $Q, N \in \mathbb{N}$, $M \in \mathbb{Z}$ and $\{a_n\}$ be a sequence of complex numbers. We have*

$$\sum_{\substack{q=1\\ q\in\mathbb{P}}}^{Q} \frac{1}{\varphi(q)} \sum_{\xi \bmod q^2}^{(1)} \left| \sum_{n=M+1}^{M+N} a_n \xi(n) \right|^2 \ll Q^{\epsilon} \left(NQ^{-\frac{1}{4}} + N^{\frac{1}{4}}Q + \sqrt{N}Q^{\frac{1}{4}} + N^{\frac{3}{4}}Q^{\frac{3}{8}} \right) \sum_{n=M+1}^{M+N} |a_n|^2, \tag{4.4.1}$$

where the implied constant depends on ϵ alone and the meaning of $\sum^{(1)}$ is the same as used throughout the chapter.

Proof. From (4.2.3), we have

$$\begin{aligned} &\sum_{\substack{q=1\\ q\in\mathbb{P}}}^{Q} \frac{1}{\varphi(q)} \sum_{\xi \bmod q^2}^{(1)} \left| \sum_{n=M+1}^{M+N} a_n \xi(n) \right|^2 \\ = \quad & \sum_{q} \sum_{\substack{n \quad n' \\ n\equiv n' \bmod q}}\sum a_n \overline{a}_{n'} e\left(\frac{\overline{n'}(nn'-1)}{q^2} \right) \\ = \quad & \sum_{n}\sum_{n'} \sum_{q|(n-n')} a_n \overline{a}_{n'} e\left(\frac{\overline{n'}(nn'-1)}{q^2} \right) \\ = \quad & \sum_{|l|\le N/Q} \sum_{q}\sum_{n}\sum_{n'} a_n \overline{a}_{n'} e\left(\frac{\overline{n}l}{q} \right). \end{aligned}$$

If we set

$$T(N,Q) = \sum_{\substack{q=1\\ q\in\mathbb{P}}}^{Q} \frac{1}{\varphi(q)} \sum_{\xi \bmod q^2}^{(1)} \left| \sum_{n=M+1}^{M+N} a_n \xi(n) \right|^2,$$

we have, after applying the Cauchy-Schwartz inequality and opening the modulus square,

$$\begin{aligned} T^2(N,Q) &\leq \left(\sum_{n'} |a_{n'}|^2\right) \left[\sum_{n'} \left|\sum\sum\sum_{n-n'=lq} a_n e\left(\frac{\overline{n}l}{q}\right)\right|^2\right] \\ &= \left(\sum_{n'} |a_{n'}|^2\right) \left[\sum\sum\sum\sum\sum\sum_{n_1-n_2=l_1q_1-l_2q_2} a_{n_1}\overline{a}_{n_2} e\left(\frac{\overline{n}_1 l_1}{q_1} - \frac{\overline{n}_2 l}{q_2}\right)\right]. \end{aligned}$$

From the above, we get that

$$\begin{aligned} T^2(N,Q) \leq \left(\sum_{n} |a_n|^2\right) \sum_{n_1}\sum_{n_2} |a_{n_1} a_{n_2}| \\ \times \sum_{q_1} \left| \sum\sum\sum_{l_1q_1-l_2q_2=n_1-n_2} e\left(\frac{\overline{(n_2-l_2q_2)}l_1}{q_1} - \frac{\overline{n}_2 l_2}{q_2}\right)\right|. \end{aligned} \tag{4.4.2}$$

Here, we apply once again, the Cauchy-Schwartz inequality, we get

$$T^4(N,Q) \leq \left(\sum_{n} |a_n|^2\right)^4 Q \sum_{n_1}\sum_{n_2}\sum_{q} \left|\sum_{q_1}\sum_{l_1}\sum_{l} e\left(\frac{\overline{(n-lq)}l}{q_1} - \frac{\overline{n}_2 l}{q}\right)\right|^2. \tag{4.4.3}$$

Opening the square modulus, we get

$$T^4(N,Q) \leq \left(\sum_{n} |a_n|^2\right)^4 Q \sum_{q}\sum_{q_1}\sum_{q_2}\sum_{l_1}\sum_{l_2}\sum_{l'}\sum_{l''}_{l_1q_1-l_2q_2=q(l'-l'')} \left\{\sum_{n}\right\}, \tag{4.4.4}$$

where the inner-most sum is

$$\sum_{n} e\left[\frac{\overline{(n-l'q)}l_1}{q_1} - \frac{\overline{n}l'}{q} - \frac{\overline{(n-l''q)}l_2}{q_2} + \frac{\overline{n}l''}{q}\right], \tag{4.4.5}$$

which is an incomplete Kloosterman type sum and may be completed by Fourier techniques thus. Note that we only need one of the sums over n_1 and n_2 in (4.4.4) as the other one will be determined once q, q_1, q_2, l_1, l_2, l', l'' and n are determined.

It suffices to estimate a sum of the following form

$$\sum_{N\leq n\leq N_1} e\left[\frac{\overline{(n-l'q)}l_1}{q_1} - \frac{\overline{n}l}{q} - \frac{\overline{(n-l''q)}l_2}{q_2} + \frac{\overline{n}l}{q}\right], \tag{4.4.6}$$

where $N < N_1 \leq 2N$. Our sum may be written as

$$\frac{1}{q_1q_2q} \sum_{a \bmod q_1q_2q} \sum_{n} e\left(\frac{an}{q_1q_2q}\right) \sum_{x \bmod q_1q_2q} e\left[f(x) - \frac{ax}{q_1q_2q}\right], \tag{4.4.7}$$

where $f(x)$ is the amplitude of the exponential sum in (4.4.6) and the range of summation for n is the same as before. The main contribution will come from the part where $a \equiv 0 \pmod{q_1q_2q}$. That part in (4.4.7) is the following

$$\frac{N}{q_1q_2q} \sum_{x \bmod q_1q_2q} e\left[f(x)\right]. \tag{4.4.8}$$

The other parts will have small contributions, but they are nevertheless

$$\frac{1}{q_1q_2q} \sum_{\substack{a \bmod q_1q_2q \\ a\not\equiv 0 \bmod q_1q_2q}} \sum_{n} e\left(\frac{an}{q_1q_2q}\right) \sum_{x \bmod q_1q_2q} e\left[f(x) - \frac{ax}{q_1q_2q}\right] \tag{4.4.9}$$

$$\ll \sum_{0<a\leq q_1q_2q/2} \frac{1}{a}\left|\sum_{x \bmod q_1q_2q} e\left[f(x) - \frac{ax}{q_1q_2q}\right]\right|, \tag{4.4.10}$$

where we have applied the bound for the geometric series over n. The sum in (4.4.8) is a Ramanujan type sum while the one in (4.4.10) may be estimated via Weil's bound for Kloosterman sums, Theorem 4.1.

Making the change of variable from x into $x + l''q$, the inner-most sum of (4.4.10), it becomes

$$e\left(\frac{al''}{qq_1q_2}\right) \sum_{x \bmod q_1q_2q} e\left[\left(\frac{\overline{(x+lq)}l_1}{q_1} - \frac{\overline{x}l_2}{q_2} + \frac{\overline{x}l}{q}\right) - \frac{ax}{q_1q_2q}\right], \tag{4.4.11}$$

where $l = l'' - l'$. The above is, after a change of variables, a Kloosterman sum of modulus qq_1q_2. Hence we apply the renowned Weil's bound for Kloosterman sums and the above is

$$\ll \gcd(a, q_1q_2q)^{\frac{1}{2}} Q^{\frac{3}{2}+\epsilon}. \tag{4.4.12}$$

Recall that we are still assuming that q_1, q_2 and q are primes. Summing over a, we get that the sum in (4.4.10) is

$$\ll Q^{\frac{3}{2}+\epsilon}. \tag{4.4.13}$$

Here we have used the following bound.

$$\sum_{a \bmod Q} \frac{\sqrt{\gcd(a,Q)}}{a} \leq \sum_{d|Q} d^{-\frac{1}{2}} \sum_{k_d=1}^{Q/d-1} \frac{1}{k_d} \ll Q^{\epsilon}.$$

The estimates we have thus far is better than the one in (4.2.5) if $N \gg Q^{\frac{3}{2}+\epsilon}$ equivalently $Q \ll N^{\frac{2}{3}-\epsilon}$.

It still remains to estimate the sum in (4.4.8). The sum simplifies to

$$\sum_{x \bmod q_1 q_2 q} e\left[\frac{\overline{(x+lq)}l_1}{q_1} - \frac{\overline{x}l_2}{q_2} + \frac{\overline{x}l}{q}\right]. \tag{4.4.14}$$

If q_1, q_2 and q_3 are pair-wise co-prime, *id est* distinct since we are only considering primes, we write

$$x = x_1\overline{q_2 q}q_2 q + x_2\overline{q_1 q}q_1 q + y\overline{q_1 q_2}q_1 q_2, \tag{4.4.15}$$

where $x_1 \pmod{q_1}$, $x_2 \pmod{q_2}$ and $y \pmod{q}$. The sum of our interest in (4.4.14) factors into the product of three Ramanujan type sums thus

$$R\left(\frac{l_1}{q_1}\right) R\left(\frac{l_2}{q_2}\right) R\left(\frac{l}{q}\right), \tag{4.4.16}$$

where

$$R\left(\frac{l}{q}\right) = \sum_{\substack{n \bmod q \\ \gcd(n,q)=1}} e\left(\frac{nl}{q}\right). \tag{4.4.17}$$

We already know that

$$\left|R\left(\frac{l}{q}\right)\right| \leq \gcd(l,q)$$

by the virtue of (4.3.2) and we also note the following almost trivial estimate.

$$\sum_{d \bmod q} \gcd(d,q) \leq \sum_{l|q} \frac{q}{l} l = q\tau(q) \ll q^{1+\epsilon}.$$

Hence, taking the sums over all relevant variables, we have the contribution of (4.4.8) to the majorant of $T^4(N,Q)$ is

$$QN\left(\frac{N}{Q}+Q\right)^3 \left(\sum_n |a_n|^2\right)^4. \tag{4.4.18}$$

If some of the q's are not pair-wise distinct, then the estimate will essentially go the same way as before, but the contribution to the majorant will be different. If two of the q's are the same, then the contribution of the majorant is

$$Q^\epsilon\left(N^4Q^{-1} + N^2Q\right)\left(\sum_n |a_n|^2\right)^4, \tag{4.4.19}$$

and similarly if all q's are the same, the contribution is

$$Q^{\epsilon}\left(N^4Q^{-3}+N^3Q^{-2}\right)\left(\sum_n |a_n|^2\right)^4. \tag{4.4.20}$$

Combining everything and taking the fourth root, we get

$$T(N,Q) \ll Q^{\epsilon}\left(NQ^{-\frac{1}{4}}+N^{\frac{1}{4}}Q+\sqrt{N}Q^{\frac{1}{4}}+N^{\frac{3}{4}}Q^{\frac{3}{8}}\right)\sum_n |a_n|^2. \tag{4.4.21}$$

Again, the result is better than (4.2.5) when $N \gg Q^{\frac{3}{2}+\epsilon}$. □

4.5 Notes

The asymptotic formula of (4.2.5) is useful when the error term does not exceed the main term or when $N \ll Q$. Our Theorem 4.2 is non-trivial when $Q^{\frac{3}{2}+\epsilon} \ll N$. We would certainly hope that our result is good whenever (4.2.5) is not, *id est* whenever $Q \ll N$. However, since we already have to resort to the strength of Weil bound for our present result, any desire for improvement upon it is perhaps too greedy, at least for the time being.

Chapter 5

Large Sieve Inequalities with Quadratic Amplitude

In this chapter, we shall be interested in estimating the sum of the following kind.

$$\sum_{k=1}^{K}\left|\sum_{n=M+1}^{M+N} a_n e\left[x_k f(n)\right]\right|^2 \ll \Delta \sum_{n=M+1}^{M+N} |a_n|^2,$$

where $\{x_k\}$ are some well-spaced real number, and $f(x) = \alpha x^2 + \beta x + \gamma$ with α, β, $\gamma \in \mathbb{R}$ and $\alpha > 0$. Δ, as before, is the norm of an operator and will depend on the length of the outer and inner sums.

It is certainly elementary to write $f(x) = \alpha\left(x + \frac{\beta}{2\alpha}\right)^2 + \frac{\beta^2}{4\alpha} + \gamma$, and hence it suffices to consider the sum

$$\sum_{k=1}^{K}\left|\sum_{n=M+1}^{M+N} a_n e\left[\alpha x_k\left(n + \frac{\beta}{2\alpha}\right)^2\right]\right|^2.$$

We start out with the Weyl-Hardy-Littlewood method which will give us not so attractive results. Then we will restrict our attention to some special cases of the above sums using what is called double large sieve in [16], a Lemma that plays an important role in Bombieri and Iwaniec's [5].

5.1 Preliminary lemmas

First we need a generalized Bessel's inequality *á la* Rényi.

Lemma 5.1 (Bessel's Inequality). *Let ϕ_1, ϕ_2, $\cdots$, ϕ_R and ξ be arbitrary vectors in an inner product space V over $\mathbb{C}$, then*

$$\sum_{r=1}^{R} |(\xi,\ \phi_r)|^2 \le \max_r \left\{\sum_{s=1}^{R} |(\phi_r,\ \phi_s)|\right\} \|\xi\|_{l_2}^2, \tag{5.1.1}$$

where $||\xi||_{l_2}$ denotes the l_2 norm of ξ.

Proof. We begin by the parallelogram rule $|u_r\overline{u}_s| \le \frac{1}{2}|u_r|^2 + \frac{1}{2}|u_s|^2$ for an arbitrary sequence of complex numbers $u_1, u_2, \cdots, u_R$. Then we have

$$\begin{aligned}
\sum_{\substack{1\le r\le R\\ 1\le s\le R}} u_r\overline{u}_s\,(\phi_r,\ \phi_s) &\le \sum_{r,\,s}\left(\frac{1}{2}|u_r|^2+\frac{1}{2}|u_s|^2\right)|(\phi_r,\ \phi_s)|\\
&= \sum_{r}|u_r|^2\sum_{s=1}^{R}|(\phi_r\ \phi_s)|\\
&\le \left(\max_r\sum_{s=1}^{R}|(\phi_r,\ \phi_s)|\right)\sum_{r=1}^{R}|u_r|^2.
\end{aligned}$$

Now let $A = \max_r \sum_{s=1}^{R} |(\phi_r,\ \phi_s)|$. We have

$$\begin{aligned}
0 &\le \left\|\xi - \sum_{r=1}^{R} u_r\phi_r\right\|_{l_2}^2 = \|\xi\|_{l_2}^2 - 2\Re\sum_{r=1}^{R}\overline{u}_r(\xi,\ \phi_r) + \sum_{r,\,s} u_r\overline{u}_s(\phi_r,\ \phi_s)\\
&\le \|\xi\|_{l_2}^2 - 2\Re\sum_{r=1}^{R}\overline{u}_r(\xi,\ \phi_r) + A\sum_{r=1}^{R}|u_r|^2.
\end{aligned}$$

Upon taking $u_r = \dfrac{(\xi,\ \phi_r)}{A}$, the above becomes

$$0 \le \|\xi\|_{l_2}^2 - \frac{1}{A}\sum_{r=1}^{R}|(\xi,\ \phi_r)|^2.$$

Hence we have the desired result. ☐

As mentioned before, we need the double large sieve.

Lemma 5.2 (double large sieve). *Suppose $\vec{x}^{(1)}, \cdots, \vec{x}^{(M)}$ and $\vec{y}^{(1)}, \cdots, \vec{y}^{(N)}$ are real vectors in k dimensions with*

$$-\frac{X_1}{2} \le x_1^{(m)} \le \frac{X_i}{2},\ -\frac{Y_i}{2} \le y_i^{(m)} \le \frac{Y_i}{2},$$

for $i = 1, \cdots, k$, $m = 1, \cdots, M$, $n = 1, \cdots, N$. Put $\epsilon_i = X_i^{-1}$, $\delta_i = \dfrac{X_i}{X_iY_i+1}$, and $\Lambda(x) = \max(1-|x|, 0)$. Then we have

$$\left|\sum_{m=1}^{M}\sum_{n=1}^{N} a_m b_n e\left[\vec{x}^{(m)}\vec{y}^{(n)}\right]\right|^2 \le \left(\frac{\pi}{2}\right)^{4k} A(\delta)B(\epsilon)\prod_{i=1}^{k}(X_iY_i+1), \tag{5.1.2}$$

where

$$A(\delta) = \sum_{m=1}^{M}\sum_{r=1}^{M}|a_m||a_r|\prod_{i=1}^{K}\Lambda\left(\frac{x_i^{(m)}-x_i^{(r)}}{\delta_i}\right),\ \textit{and}$$

$$B(\epsilon) = \sum_{n=1}^{N}\sum_{r=1}^{N} b_n\overline{b}_r \prod_{i=1}^{k} \Lambda\left(\frac{y_1^{(n)} - y_i^{(r)}}{\epsilon_i}\right).$$

Proof. The k-dimensional case goes similarly as that of one dimension. Hence, we begin with

$$\int_{y-\epsilon/2}^{y+\epsilon/2} e(sx)\mathrm{d}s = e(xy)\frac{e(\epsilon x/2) - e(-\epsilon x/2)}{2\pi i x} = \epsilon e(xy)\frac{\sin(\pi\epsilon x)}{\pi\epsilon x}. \tag{5.1.3}$$

Then we have

$$\begin{aligned}\sum_m\sum_n a_m b_n e(x_m y_n) &= \sum_m\sum_n \frac{a_m b_n \pi\epsilon x}{\epsilon\sin(\pi\epsilon x_m)}\int_{y_n-\epsilon/2}^{y_n+\epsilon/2} e(sx_m)\mathrm{d}s \\ &= \int_{-(Y+\epsilon)/2}^{(Y+\epsilon)/2} \sum_m A_m e(sx_m) \sum_{|y_n - s|\le\epsilon/2} b_n \mathrm{d}s,\end{aligned}$$

where

$$A_m = \frac{a_m \pi\epsilon x}{\epsilon\sin(\pi\epsilon x_m)}, \text{ and } |A_m| \le \frac{\pi}{2\epsilon}|a_m|.$$

Applying the Cauchy-Schwartz inequality, we get

$$\left|\sum_m\sum_n a_m b_n e(x_m y_n)\right|^2 \le I_1 I_2,$$

where I_1 and I_2 are the following integrals.

$$\begin{aligned} I_2 &= \int_{-(Y+\epsilon)/2}^{(Y+\epsilon)/2}\left|\sum_{|y_n-s|\le\epsilon/2} b_n\right|^2 \mathrm{d}s \\ &= \sum_n\sum_r b_n\overline{b}_r \max(\epsilon - |y_n - y_r|,\ 0) \\ &= \epsilon B(\epsilon); \\ I_1 &= \int_{-(Y+\epsilon)/2}^{(Y+\epsilon)/2}\left|\sum_m A_m e(sx_m)\right|^2 \mathrm{d}s.\end{aligned}$$

It still remains to estimate I_1. Toward that end, we have

$$\begin{aligned} I_1 &\le \int_{-\infty}^{\infty}\frac{\pi^2}{4}\left(\frac{\sin(\pi\delta s)}{\pi\delta s}\right)^2\left|\sum_m A_m e(sx_m)\right|^2 \mathrm{d}s \\ &= \sum_m\sum_k A_m\overline{A}_k\frac{\pi^2}{4\delta}\Lambda\left(\frac{x_k - x_m}{\delta}\right) \\ &\le \sum_m\sum_k |A_m A_k|\frac{\pi^2}{4\delta}\max\left(1 - \frac{|x_k - x_m|}{\delta},\ 0\right) \\ &\le \frac{\pi^4}{16}\frac{1}{\delta\epsilon^2}A(\delta) \\ &= \frac{\pi^4}{16}\frac{XY+1}{\epsilon}A(\delta).\end{aligned}$$

Hence, we have completed the proof for the one dimensional case. For higher dimensions, we apply the Cauchy-Schwartz inequality inside a k-fold integral. □

5.2 Trivial bounds and Heuristics

We are interested in estimating the size of the following expression.

$$\sum_{k=1}^{K}\left|\sum_{n=M+1}^{M+N} a_n e\left[x_k f(x)\right]\right|^2 . \tag{5.2.1}$$

If we estimate *really* trivially, we first apply Cauchy-Schwartz inequality to the innermost sum on the left hand side of (5.2.1) and then sum over the x_k's. We have the extremely poor upper bound of

$$\sum_{k=1}^{K} N \sum_{n=M+1}^{M+N} |a_n|^2 \leq (\delta^{-1}+1)N \sum_{n=M+1}^{M+N} |a_n|^2, \tag{5.2.2}$$

where it is certainly an elementary fact that the cardinality of any sequence of real numbers that are δ-spaced modulo 1 does not exceed $\delta^{-1}+1$. One should certainly hope that we can do better than *this.*

By the virtue of the duality principle, Lemma 2.2, it is equivalent to estimate for the following sum.

$$\sum_{n=M+1}^{M+N}\left|\sum_{k=1}^{K} c_k e\left(x_k f(n)\right)\right|^2 , \tag{5.2.3}$$

where $\{c_k\}$ is an arbitrary sequence of complex numbers, as $\{a_n\}$. Therefore, if $f(n) = n^2$, then the virtue of the regular large sieve inequality, in its dual form, would majorize (5.2.3) as follows.

$$\sum_{n=M+1}^{M+N}\left|\sum_{k=1}^{K} c_k e\left(x_k n^2\right)\right|^2 \ll (\delta^{-1}+N^2)\sum_{k=1}^{K}|c_k|^2, \tag{5.2.4}$$

where the N^2 in the (5.2.4) comes from changing the variables and then filling in all the non-squares n's. Consequently, we have the corresponding result.

$$\sum_{k=1}^{K}\left|\sum_{n=M+1}^{M+N} a_n e(x_k n^2)\right|^2 \ll (\delta^{-1}+N^2)\sum_{n=M+1}^{M+N}|a_n|^2. \tag{5.2.5}$$

In the light of how we arrived at (5.2.4), we have added a lot more terms into our upper bounded than there were originally. Thus One may tend to think that the waste in the estimate of (5.2.4) is rather great. Hence, one should expect, (5.2.4) can be improved, and perhaps we should have the same estimate that the regular large sieve inequality has. Namely, the upper bound would be, essentially,

$$\Delta = \delta^{-1} + N. \tag{5.2.6}$$

However, the bound in (5.2.6) is not to be, as we shall presently give a counter example. Let $\{x_k\}$ be the Farey sequence of order Q, which we shall henceforth denote as $F(Q)$, i.e. the rational number in $(0, 1]$ of height at most Q. These numbers are Q^{-2}-spaced modulo 1. Therefore, $\delta^{-1} = Q^2$ in this case. Moreover, take $Q = p^2$ be a prime square and a_n be p if n is a multiple of p and zero otherwise. Also take $M = 0$. Therefore, we have

$$\sum_{q=1}^{Q} \sum_{\substack{a \pmod q \\ \gcd(a,q)=1}} \left| \sum_{n=M+1}^{M+N} a_n e\left(\frac{a}{q} n^2\right) \right|^2 \gg \sum_{\substack{a \pmod{Q=p^2} \\ \gcd(a,Q)=1}} \left| p \frac{N}{p} \right|^2 \gg QN^2. \tag{5.2.7}$$

where we bound the sum over q from below by the single term corresponding to $q = Q$. But on the other hand,

$$(\delta^{-1} + N) \sum_{n=M+1}^{M+N} |a_n|^2 \asymp (Q^2 + N) \frac{N}{p} p^2 = Q^{\frac{5}{2}} + Q^{\frac{1}{2}} N^2. \tag{5.2.8}$$

Now if we take $N = Q$, then we have the lower bound of $\gg Q^3$ in (5.2.7) but an upper bound of only $O(Q^{5/2})$ in (5.2.8). Thus we have a contradiction and the result in (5.2.6) is not attainable in general.

Furthermore, if one consider the general case of $f(n) = \alpha n^2 + \beta n + \gamma$, a quadratic polynomial with real coefficients, one could mimic Gallagher's proof of the regular large sieve inequality in [11]. More precisely, we apply the Sobolev-Gallagher's Lemma, Lemma 2.1, to $\left\{ \sum_{n=M+1}^{M+N} a_n e[f(n)] \right\}^2$ on the intervals $[x_k - \delta, x_k + \delta]$ and then sum over the x_k's. We then get an upper bound for the sum of our interest,

$$\delta^{-1}\int_0^1\left|\sum_{n=M+1}^{M+N} a_n e[xf(n)]\right|^2 \mathrm{d}x$$
$$+\, 2\pi\int_0^1\left|\sum_{n=M+1}^{M+N} a_n e[xf(n)]\right|\left|\sum_{n=M+1}^{M+N} a_n f(n) e[xf(n)]\right| \mathrm{d}x.$$

Applying Parseval's identity to the first integral, we infer that it is precisely

$$\sum_{n=M+1}^{M+N} |a_n|^2.$$

By the virtues of Hölder's inequality to the second integral and then Parseval's inequality, we see that the second integral is majorized by

$$\leq \left(\int_0^1\left|\sum_{n=M+1}^{M+N} a_n e[xf(n)]\right|^2 \mathrm{d}x\right)^{\frac{1}{2}}\left(\sum_{n=M+1}^{M+N} |a_n f(n)|^2\right)^{\frac{1}{2}}.$$

Applying Parseval's identity again to the first factor, we get that above is bounded by

$$[\delta^{-1} + |\alpha|(M+N)^2]\sum_{n=M+1}^{M+N} |a_n|^2. \tag{5.2.9}$$

During this proof, there is hardly any waste at all in the estimates. The only inequalities that we applied are those of Hölder, and Sobolev-Gallagher, neither of which admits great waste. Therefore, although the result in (5.2.9) is rather bad, one must admit that it has certain virtues to it.

5.3 Weyl-Hardy-Littlewood Method

Our contention in this section is to prove a different form of the theorem stated at the beginning of the chapter. It will certainly be better than (5.2.2), but it will not always be better than (5.2.4). Here and after, let $F(Q)$ denote the Farey fractions of level Q in $(0, 1]$. We first state our theorem.

Theorem 5.1. *Let $\{a_n\}$ be an arbitrary sequence of complex numbers, N and M positive integers, $f(x) = \alpha x^2 + \beta x + \gamma$ a quadratic polynomial with real coefficients and*

$\alpha \neq 0$, and $\alpha = \frac{s}{t}$ with $\gcd(s,t) = 1$. Then

$$\sum_{x \in F(Q)} \left| \sum_{n=M+1}^{M+N} a_n e\left(x f(n)\right) \right|^2 \ll \Delta(Q,t,N,\epsilon) \sum_{n=M+1}^{M+N} |a_n|^2, \tag{5.3.1}$$

where $\Delta(Q,t,N,\epsilon)$ may be taken to be

$$\Delta(Q,t,N,\epsilon) = \sqrt{s} Q^{\frac{3}{2}+\epsilon} N + Q^2 \sqrt{N \log(2Qt)}, \tag{5.3.2}$$

and the implied constant depends on ϵ alone.

In fact, the theorem follows from the following proposition which is slightly more general, and we shall only give the proof of the proposition.

Proposition 5.1. *Let $\{a_n\}$ be an arbitrary sequence of complex numbers, N and M positive integers, $f(x) = \alpha x^2 + \beta x + \gamma$ a quadratic polynomial with real coefficients and $\alpha \neq 0$. Furthermore, let us assume, that there is a rational number, $\frac{s}{t}$, in lowest terms with the property*

$$\left| \alpha - \frac{s}{t} \right| \leq \frac{1}{4tQ^2 N}. \tag{5.3.3}$$

Then

$$\sum_{x \in F(Q)} \left| \sum_{n=M+1}^{M+N} a_n e\left(x f(n)\right) \right|^2 \ll \Delta(Q,t,N,\epsilon) \sum_{n=M+1}^{M+N} |a_n|^2, \tag{5.3.4}$$

where $\Delta(Q,t,N,\epsilon)$ may be taken to be

$$\Delta(Q,t,N,\epsilon) = \sqrt{s} Q^{\frac{3}{2}+\epsilon} N + Q^2 \sqrt{N \log(2Qt)}, \tag{5.3.5}$$

and the implied constant depends on ϵ alone.

As one familiar with the Weyl-Hardy-Littlewood method would know, the Diophantine nature of α should play a role in the final estimation. Furthermore, the condition (5.3.3) is not a great restriction at all, by the virtue of Dirichlet approximation. Also, without loss of generality, we may just assume that $\alpha > 0$ since we may multiply $f(x)$ by -1 if needed.

Proof. We begin by applying the generalized Bessel's inequality, Lemma 5.1. We take

$$\xi = \left\{a_n b_n^{-\frac{1}{2}}\right\}_{M+1}^{M+N}, \text{ and } \phi_x = \left\{b_n^{\frac{1}{2}} e[-f(n)x]\right\}_{-\infty}^{\infty}.$$

Moreover, we take

$$b_n = \begin{cases} 1, & M+1 \leq n \leq M+N, \\ 0, & \text{otherwise.} \end{cases}$$

These are not the best possible choices for b_n, since we may be able to eliminate a factor of $\log Q$ if we choose the b_k's more cleverly. However, it will make little difference for us, since we shall eventually have Q^ϵ anyway. Therefore, by Bessel's inequality, the left hand side of (5.3.4) is majorized by,

$$\max_x \sum_{x'} \left| \sum_{n=M+1}^{M+N} e[(x-x')f(n)] \right| \sum_{n=M+1}^{M+N} |a_n|^2.$$

Thus it suffices to bound $\sum_{x'} \left| \sum_{n=M+1}^{M+N} e[(x-x')f(n)] \right|$ which we accomplish thus.

$$\sum_{x'} \left| \sum_{n=M+1}^{M+N} e[(x-x')f(n)] \right| \tag{5.3.6}$$

$$= N + \sum_{x \neq x'} \left| \sum_{n=M+1}^{M+N} e[(x-x')f(n)] \right| \tag{5.3.7}$$

$$\leq N + \left(\sum_{x' \neq x} 1 \right)^{\frac{1}{2}} \left(\sum_{x \neq x'} \left| \sum_{n=M+1}^{M+N} e[(x-x')f(n)] \right|^2 \right)^{\frac{1}{2}} \tag{5.3.8}$$

$$\leq N + Q \left\{ \sum_{x \neq x'} \left[N + \sum_{l=1}^{N} \min \left(N, \frac{1}{\|\alpha(x-x')l\|} \right) \right] \right\}^{\frac{1}{2}} \tag{5.3.9}$$

$$\ll N + Q^2\sqrt{N} + Q \left[\sum_{x \neq x'} \sum_{1 \leq l \leq N} \min \left(N, \frac{1}{\|\alpha(x-x')l\|} \right) \right]^{\frac{1}{2}}. \tag{5.3.10}$$

We have applied the Cauchy-Schwartz inequality to go from (5.3.8) to (5.3.9). To go from (5.3.9) to (5.3.10), we employ Weyl-Hardy-Littlewood method, Lemma 2.4. Lastly, the concavity of the square root function gives us (5.3.10). So far, all implied constants are absolute, although we shall not be too concerned with them.

It now remains to estimate the sum in (5.3.10). As in (5.3.10), we need to estimate, with $\frac{a}{q} \in F(Q)$ fixed,

$$\sum_{\substack{\frac{a_1}{q_1} \neq \frac{a}{q} \\ \frac{a_1}{q_1} \in F(Q)}} \sum_{l=1}^{N} \min\left[N, \left\| \alpha\left(\frac{a}{q} - \frac{a_1}{q_1}\right) l \right\|^{-1}\right].$$

If we let $d = \gcd(q_1, q)$, then

$$\frac{a}{q} - \frac{a_1}{q_1} = \frac{aq_1 - a_1 q}{qq_1} = \frac{aq_1/d - a_1 q/d}{qq_1/d}.$$

Therefore, we easily see that any prime, p, dividing $\frac{q_1}{d}$ has $p \nmid \frac{q}{d}$ and $p \nmid a_1$ as $\gcd(a_1, q_1) = \gcd(q_1, q/d) = 1$. Therefore, if $p | q_1/d$, we must have $p \nmid (aq_1/d - a_1 q/d)$. By the same token, if $p | q/d$, then $p \nmid (aq_1/d - q_1 q/d)$. In other words, when reduced to lowest terms, the primes in q_1/d and q/d will survive the cancellation. Therefore, when written in lowest terms, the denominator of $\frac{a}{q} - \frac{a_1}{q_1}$ is bounded from below by $\frac{\max(q_1, q)}{\gcd(q_1, q)}$. Now suppose that $\alpha = \frac{u}{v}$, a rational number in lowest terms. Then the denominator of $\alpha\left(\frac{a}{q} - \frac{a_1}{q_1}\right)$ is no less than $\frac{1}{u}\frac{\max(q_1, q)}{\gcd(q_1, q)}$. Now let

$$\delta : \mathbb{Q} \longrightarrow \mathbb{N} : \frac{a}{b} \longrightarrow b,$$

with $b > 0$ and $\gcd(a, b) = 1$. Hence, what we have now is

$$\delta\left[\alpha\left(\frac{a}{q} - \frac{a_1}{q_1}\right)\right] \geq \frac{1}{u}\frac{\max(q_1, q)}{\gcd(q_1, q)}. \tag{5.3.11}$$

We can now estimate the following.

$$\begin{aligned} &\sum_{\frac{a_1}{q_1} \neq \frac{a}{q}} \sum_{l=1}^{N} \min\left[N, \left\| \frac{u}{v}\left(\frac{a}{q} - \frac{a_1}{q_1}\right) \right\|^{-1}\right] \\ &\ll \sum_{\frac{a_1}{q_1} \neq \frac{a}{q}} N \frac{N}{\delta\left[\frac{u}{v}\left(\frac{a}{q} - \frac{a_1}{q_1}\right)\right]} + \\ &\sum_{\frac{a_1}{q_1} \neq \frac{a}{q}} \frac{N}{\delta\left[\frac{u}{v}\left(\frac{a}{q} - \frac{a_1}{q_1}\right)\right]} \sum_{l \pmod{\delta\left[\frac{u}{v}\left(\frac{a}{q} - \frac{a_1}{q_1}\right)\right]}} \delta\left[\frac{u}{v}\left(\frac{a}{q} - \frac{a_1}{q_1}\right)\right] l^{-1}, \end{aligned} \tag{5.3.12}$$

where the first sum on the right-hand side of (5.3.12) comes from the l's that are possibly divisible by $\delta\left[\frac{u}{v}\left(\frac{a}{q} - \frac{a_1}{q_1}\right)\right]$ while the second sum are the remaining terms.

The second sum in (5.3.12) is disposed easily and is

$$\ll N \sum_{\frac{a}{q} \neq \frac{a_1}{q_1}} \log \left\{ \delta \left[\frac{u}{v} \left(\frac{a}{q} - \frac{a_1}{q_1} \right) \right] \right\} \leq Q^2 N \log(vQ^2) \ll Q^2 N \log(Qv).$$

We now concentrate on the first sum which is

$$\begin{aligned} & N^2 \sum_{\frac{a}{q} \neq \frac{a_1}{q_1}} \left\{ \delta \left[\frac{u}{v} \left(\frac{a}{q} - \frac{a_1}{q_1} \right) \right] \right\}^{-1} \\ \leq \quad & uN^2 \sum_{\frac{a}{q} \neq \frac{a_1}{q_1}} \left[\frac{\max(q_1, q)}{\gcd(q_1, q)} \right]^{-1} \\ = \quad & uN^2 \sum_{q_1=1}^{Q} \varphi(q_1) \left[\frac{\gcd(q_1, q)}{\max(q_1, q)} \right] \\ \leq \quad & uN^2 \sum_{q_1=1}^{Q} \gcd(q_1, q) \\ \ll \quad & uN^2 \frac{Q}{q} \sum_{q_1=1}^{q} \gcd(q_1, q) \\ \ll \quad & uN^2 Q^{1+\epsilon}. \end{aligned}$$

Obviously, all implied constants above depend on ϵ only. Moreover, we have utilized the following rather trivial estimate

$$\sum_{a \pmod Q} \gcd(a, Q) \leq \sum_{d|Q} d\frac{Q}{d} = Q \sum_{d|Q} 1 \ll Q^{1+\epsilon}.$$

Combining everything, we see that we may take

$$\begin{aligned} \Delta &= N + Q^2\sqrt{N} Q^2 \sqrt{N \log(2Qv)} + Q^{\frac{3}{2}+\epsilon} N \sqrt{u} \\ &\ll \sqrt{u} Q^{\frac{3}{2}+\epsilon} N + Q^2 \sqrt{N \log(2Qv)}. \end{aligned}$$

We have already proved the theorem at this point. Now let us assume that α is an irrational number. Moreover, by the virtue of Dirichlet approximation, we take a rational number $\frac{s}{t}$, in lowest terms, such that

$$\left| \alpha - \frac{s}{t} \right| < \frac{1}{4tQ^2N}.$$

We then conclude that

$$\left| \alpha \left(\frac{a}{q} - \frac{a_1}{q_1} \right) l - \frac{s}{t} \left(\frac{a}{q} - \frac{a_1}{q_1} \right) l \right| < \frac{l}{4tQ^2N} \left| \frac{a}{q} - \frac{a_1}{q_1} \right| < \frac{1}{4tQ^2}.$$

Note that we have $\delta\left[\frac{s}{t}\left(\frac{a}{q}-\frac{a_1}{q_1}\right)\right] < tQ^2$. Therefore, we comfortably have

$$\left\|\alpha\left(\frac{a}{q}-\frac{a_1}{q_1}\right)l\right\| \gg \left\|\frac{s}{t}\left(\frac{a}{q}-\frac{a_1}{q_1}\right)l\right\|,$$

with an absolute implied constant.

Therefore, the sum of our interest is

$$\sum_{\frac{a}{q}\neq\frac{a_1}{q_1}}\sum_{l=1}^{N}\min\left[N,\left\|\alpha\left(\frac{a}{q}-\frac{a_1}{q_1}\right)l\right\|^{-1}\right]$$
$$\ll \sum_{\frac{a}{q}\neq\frac{a_1}{q_1}}\sum_{l=1}^{N}\min\left[N,\left\|\frac{s}{t}\left(\frac{a}{q}-\frac{a_1}{q_1}\right)l\right\|^{-1}\right].$$

The last sum has been previous computed and is

$$\ll sN^2Q^{1+\epsilon}+Q^2N\log(2Qt),$$

where the implied constant here depends also on ϵ alone. Therefore, we have proved the theorem. □

We also wish to, though "it is most retrograde to our desire" ([29] *Hamlet, I, ii, 114*), remark that our result of the section (5.3.5) is not always better than one of the trivial bounds, (5.2.4). It is worse than the trivial bound if $Q^2 \asymp N^2$ is much larger than N. Therefore, we in fact have two contenting results and that gives us further reason to believe that neither (5.3.5) nor (5.2.4) is the best possible.

5.4 Application of Double Large Sieve

The long and clumsy estimates of the previous section gives us unattractive results. In this section, we apply the double large sieve, Lemma 5.2, used by Bombieri and Iwaniec in the renowned [5] to obtain a better result.

We need not the full strength of the double large sieve. To avoid notational confusion, we express the sum of our interest thusly.

$$\sum_{x\in F(Q)}\left|\sum_{s=M+1}^{M+N}c_s e[xf(s)]\right|^2, \tag{5.4.1}$$

where $\{c_s\}$ is an arbitrary sequence of complex numbers, and $f(x) = \alpha x^2 + \beta x + \gamma$, with α, β, $\gamma \in \mathbb{R}$ and $\alpha > 0$.

In this section, we want to restrict our attention to the case where $f(x) = \alpha \left(x + \frac{\beta}{2\alpha} \right)^2$. There is no loss of generality as mentioned at the beginning of the chapter. Then (5.4.1) is precisely, after opening up the modulus square,

$$\sum_{x \in F(Q)} \sum_s \sum_t c_s \overline{c}_t e[\alpha x (s-t)(s+t+\beta/\alpha)].$$

Before stating and proving our contention of the section, we need the following lemma concerning the difference of the difference of quadratic polynomials.

Lemma 5.3. *Let* $S = [M+1, M+N] \cap \mathbb{Z}$, *and* $\epsilon > 0$ *be given,* $g(x,y) = (x-y)(x+y+s/t)$, *and* m, $n \in S$ *with* $|g(m,n)| > 1$. *Also* s, $t \in \mathbb{Z}$ *with* $t > 0$ *and* $\gcd(a,b) = 1$. *Let* T *denote the number of pairs of* m', $n' \in S$ *with*

$$|g(m,n) - g(m',n')| \leq \frac{1}{2}. \tag{5.4.2}$$

Then we have

$$T \ll t\,[t(M+N) + |s|]^{\epsilon}, \tag{5.4.3}$$

where the implied constant depends only on ϵ.

Proof. First given a rational number $\frac{a}{b}$, we have $|g(m,n) - g(m',n')| \leq 1/2$ if and only if

$$|(m-n)(bm+bn+a) - (m'-n')(bm'+bn'+a)| \leq b/2. \tag{5.4.4}$$

Note now that the difference inside the absolute value sign is that of two integers. For a fixed pair m, $n \in S$, there are $b+1$ or less non-zero integers that are at most $b/2$ away from $(m-n)(bm+bn+a)$. Thus if m', n' satisfies (5.4.4), then $(m'-n')(bm'+bn'+a)$ is one of the afore-mentioned integers, *id est*

$$\begin{aligned} (m-n)(bm+bn+a) - \frac{b}{2} &\leq (m'-n')(bm'+bn'+a) \\ &\leq (m-n)(bm+bn+a) + \frac{b}{2}. \end{aligned}$$

Note that the cases $g(m', n') = 0$ is already ruled out by some of the conditions of the Lemma, *id est* $|g(m, n)| > 1$ and $|g(m, n) - g(m', n')| \leq 1/2$.

For each of such non-zero integers, say w, if $k = (m' - n')(tm' + tn' + s)$, then let $u = m' - n'$ and $v = tm' + tn' + s$. We have $k = uv$. Conversely, given $uv = k$ and $tu + v - s = 2tm'$ and $-tu + v - s = 2tn'$ are solvable for m' and n' in S, then $k = (m' - n')(tm' + tn' + s)$. Thus we have proved that k is representable by $(m' - n')(tm' + tn' + s)$ with m', $n' \in S$ if and only if there are u and $v \in \mathbb{Z}$ such that $k = uv$ and

$$\text{(5.4.5)} \qquad \begin{cases} tu + v - s &= 2tm' \\ -tu + v - s &= 2tn' \end{cases}$$

are solvable for m' and n' in S.

Therefore, given $u|k$, then v is determined as $v = k/u$ and thus m' and n' are determined by the equations of (5.4.5). Hence of the number of ways to write k as $(m' - n')(tm' + tn' + s)$ with m' and $n' \in S$ is $\ll \tau(|k|) \ll |k|^\epsilon$. Of course, k can be as large as $2Nt(M + N) + |s| + t/2$. Therefore, the total number of pairs of m' and $n' \in S$ satisfying (5.4.4) is $\ll t[tN(M + N) + |s|]^\epsilon$ and we have completed the proof of the lemma. □

Now we are ready to state and prove our contention for the section.

Theorem 5.2. *Under the usual notations, and* $f(x) = \alpha x^2 + \beta x + \gamma$ *with* $\beta/\alpha = u/v \in \mathbb{Q}$. *We have*

$$\text{(5.4.6)} \qquad \begin{aligned} &\sum_{x \in F(Q)} \left| \sum_{n=M+1}^{M+N} a_n e[x f(n)] \right|^2 \\ &\ll (\alpha^{-1} + 1)(Q^2 + QN) v [v(M + N) + |u|]^\epsilon \sum_{n=M+1}^{M+N} |a_n|^2, \end{aligned}$$

where the implied constant depends on ϵ *alone.*

Proof. Using the notations of Lemma 5.2, we take

$$k = 1, \ \vec{x} = \alpha x, \ \vec{y} = g(s, t), \ X = 2\alpha, \ Y = 2MN + N^2,$$

$$a_m = 1,\ \forall\ m;\ b_n = c_s\overline{c}_t,$$

$$\epsilon = \frac{1}{X} = \frac{1}{2\alpha},\ \delta = \frac{X}{XY+1} = \frac{2}{2(2MN+N^2)+1}.$$

With these notations, we have

$$\begin{aligned} A(\delta) &= \sum_x \sum_{x'} \Lambda\left[\frac{4MN+2N^2+1}{2}\alpha(x-x')\right] \\ &= Q^2 + \sum_x \sum_{x' \neq x} \Lambda\left[\frac{4MN+2N^2+1}{2}\alpha(x-x')\right]. \end{aligned}$$

From the definition of $\Lambda(s)$, we wee that $0 \leq \Lambda(x) \leq 1$ and the terms in the above double sum is non-zero if and only if

$$\begin{aligned} & 0 < \alpha\left|\frac{4MN+2N^2+1}{2}(x-x')\right| < 1 \\ \iff & 0 < |x-x'| < \frac{2}{\alpha(4MN+2N^2+1)}. \end{aligned}$$

If I is an interval of length Δ, then as elements of $F(Q)$ are at least Q^{-2} apart, the number of elements in $F(Q) \cap I$ is $\Delta Q^2 + 1$ or less. Therefore, for a fixed $x \in F(Q)$, the number of $x' \in F(Q)$ in the interval $\left[x - \frac{2}{\alpha(4MN+2N^2+1)}, x + \frac{2}{\alpha(4MN+2N^2+1)}\right]$ is

$$\ll \frac{2Q^2}{\alpha(4MN+2N^2+1)} + 1 \leq \frac{Q^2}{\alpha(2MN+N^2)} + 1.$$

Now we estimate $A(\delta)$ almost trivially, as we may by the positivity of $\Lambda(x)$ and get that

$$A(\delta) \ll Q^2 + \frac{Q^4}{\alpha(2MN+N^2)}.$$

It still remains to estimate $B(\epsilon)$. By definition, we have

$$B(\epsilon) = \sum_{s,t} \sum_{s',t'} c_s\overline{c}_t\overline{c}_{s'}c_{t'}\Lambda\{2\alpha[g(s,t) - g(s',t')]\}.$$

As before, the Λ function in the above sum is non-zero if and only if $|g(s,t) - g(s,t)| < (2\alpha)^{-1}$. Now Lemma 5.3 becomes useful. We need to estimate, given fixed s, $t \in [M+1, M+N]$ how many pairs of s' and t' there are in the same interval satisfies $|g(s,t) - g(s,t)| < (2\alpha)^{-1}$. First, for m fixed, the number of n, m', n' with $|g(m,n)| < 1$ and $|g(m,n) - g(m',n')| < (2\alpha)^{-1}$ is $O(1)$. So we divide the conquer and get

$$B(\epsilon) \ll \left(\sum_s |c_s|^2\right)^2 + \sum_{s,t} \sum_{s',t'} c_s\overline{c}_t\overline{c}_{s'}c_{t'},$$

where the first sum is contributed by the $|g(s,t)| < 1$ and $|g(s,t) - g(s',t')| < (2\alpha)^{-1}$, and the inner sum of the second term of the above run only over the s' and t' that satisfy the appropriate conditions of Lemma 5.3. By parallelogram rule, the second term of the above is majorized by

$$\ll \sum_{s,t}\sum_{s',t'}\left(|c_s\overline{c}_t|^2 + |\overline{c}_{s'}c_{t'}|^2\right) = \sum_{s,t}\sum_{s',t'}|c_s\overline{c}_t|^2 + \sum_{s\neq t}\sum_{s',t'}|\overline{c}_{s'}c_{t'}|^2.$$

After changing the order of summation of the second sum of the above and regrouping like terms, we see that it is the same as the first sum. Consequently, the above is $O(\sum_{s,t}\sum_{s',t'}|c_s\overline{c}_t|^2)$ which is

$$\begin{aligned} &\ll \quad (\alpha^{-1}+1)t[t(M+N)+|s|]^\eta \sum_{s,t}|c_s|^2|c_t|^2 \\ &\leq \quad (\alpha^{-1}+1)t[t(M+N)+|s|]^\eta \left(\sum_s |c_s|^2\right)^2. \end{aligned}$$

Combining everything, we get that

$$B(\epsilon) \ll (\alpha^{-1}+1)t[t(M+N)+|s|]^\eta \left(\sum_s |c_s|^2\right)^2,$$

where the implied constant depends on η alone.

Now by the virtue of Lemma 5.2 and take square roots of both sides, we arrive at

$$\sum_{x\in F(Q)}\left|\sum_{s=M+1}^{M+N} c_s e(x\alpha s^2)\right|^2 \ll (\alpha^{-1}+1)(Q^2+QN)t[t(M+N)+|s|]^\eta \sum_s |c_s|^2, \tag{5.4.7}$$

where the implied constant depends on η only. □

The above theorem may be extended to the case $\beta/\alpha \in \mathbb{R}$ by approximating β/α with a rational number and the proof works almost exactly the same. More precisely, we have

Proposition 5.2. *The result of Theorem 5.2 still hold if the condition* $\frac{\beta}{\alpha} \in \mathbb{Q}$ *is replaced by* $\frac{\beta}{\alpha} \in \mathbb{R}$ *and there exists* $\frac{u}{v} \in \mathbb{Q}$ *with* $\gcd(u,v) = 1$ *and*

$$\left|\frac{\beta}{\alpha} - \frac{u}{v}\right| < \frac{1}{4vN}. \tag{5.4.8}$$

Proof. The proof goes precisely the same as that of Theorem 5.2. □

5.5 Double Large Sieve with Duality

Our proof of the previous section depends on the fact that the number of divisors of a non-zero integer is smaller. However, we lose such luxury if we go to the general case in which β/α is a general real number. We want to investigate that case further in this section. Toward that end, we evoke the duality principle, Lemma 2.2, and apply Lemma 5.2 to the dual sum of the sum of our interest. This has the advantage that we need not to worry about that difference of the difference of quadratic polynomials, but rather the difference of differences of Farey fractions which, in the author humble opinion, is easier to handle.

Toward that end, we must estimate the following.

$$\sum_{n=M+1}^{M+N} \left| \sum_{x\in F(Q)} c_x e[\alpha x(n+\beta/(2\alpha))^2] \right|^2 \tag{5.5.1}$$

$$= \sum_{n=M+1}^{M+N} \sum_{x} \sum_{x'} c_x \overline{c}_{x'} e[\alpha(x-x')(n+\beta/(2\alpha))^2]. \tag{5.5.2}$$

As "brevity is the soul of wit," ([29], *Hamlet*) we shall be brief, as much of the estimates go parallel with those of the previous section. We have

$$A(\delta) = \sum_{x,x'} \sum_{y,y'} c_x \overline{c}_{x'} \overline{c}_y c_{y'} \Lambda \left\{ \frac{2\alpha\left(M+N+\frac{\beta}{2\alpha}\right)+1}{2} [(x-x')-(y-y')] \right\}.$$

As before, the Λ function gives non-zero values only if

$$0 \le \left| \frac{2\alpha\left(M+N+\frac{\beta}{2\alpha}\right)+1}{2} [(x-x')-(y-y')] \right| < 1 \tag{5.5.3}$$

$$\iff \quad 0 \le |[(x-x')-(y-y')]| < \left[\alpha\left(M+N+\frac{\beta}{2\alpha}\right)\right]^{-2}. \tag{5.5.4}$$

So given x, x', $y \in F(Q)$, there are

$$\ll Q^2 \left[\alpha\left(M+N+\frac{\beta}{2\alpha}\right)\right]^{-2}$$

$y' \in F(Q)$ that satisfies (5.5.4). Therefore, the number of pairs y and $y' \in F(Q)$ satisfying (5.5.4) are

$$\ll Q^2 \left[\alpha\left(M+N+\frac{\beta}{2\alpha}\right)\right]^{-2}.$$

With the same argument as used in the previous section, we have.

$$A(\delta) \ll \left[\frac{Q^4}{\left(M+N+\frac{\beta}{2\alpha}\right)^2}+1\right]\left(\sum_x |c_x|^2\right)^2.$$

We still have

$$\begin{aligned} B(\epsilon) &= \sum_n \sum_m \Lambda\left[2\alpha(n-m)(m+n+\beta/\alpha)\right] \\ &= N + \sum_n \sum_{m\neq n} \Lambda\left[2\alpha(n-m)(m+n+\beta/\alpha)\right]. \end{aligned}$$

We need to estimate the number of pairs m and n with $m \neq n$ that satisfy

$$0 \leq |(m-n)(m+n+\beta/\alpha)| < \frac{1}{2\alpha},$$

which is easily seen to be $O(1+\alpha^{-1})$. Now upon combining everything and taking the square root, we have the following.

Theorem 5.3. *Under the usual notations of this chapter, we have*

$$\sum_{x\in F(Q)} \left|\sum_{n=M+1}^{M+N} a_n e[xf(n)]\right|^2 \ll \sqrt{N(1+\alpha^{-1})}(Q^2+N) \sum_{n=M+1}^{M+N} |a_n|^2. \tag{5.5.5}$$

where the implied constant is absolute.

Proof. This is already proved before the statement of the theorem. □

5.6 Notes

With the counter example given in Section 5.1, the author suspects the result with $\Delta = Q^2 + QN$ is essentially the best possible. The author also believes that some of the methods used in the chapter may be used to deal with large sieve inequality of higher power frequencies, especially what is used in Section 5.4. Furthermore, we have only been restricting our attention to Farey fraction. However, the result of double large sieve should enable us to consider any set of well-spaced real numbers.

As mentioned in Section 5.1, we cannot possibly attain a result that is analogous to the classical inequalities. We can see from the counter example given in Section 5.1, that the last term in the summation over q is enough to break the bound of $Q^2 + N$. Hence attempting such a feat will only be met by defeat.

Chapter 6

Cancellations of Hecke Eigenvalues at Primes

6.1 History and Introduction

In this chapter, we are interested in estimating an exponential sum over primes with square root amplitude twisted with Hecke eigenvalues. More precisely, we want to have an estimate for the following sum

$$\sum_{p\le N}\lambda(p)e(\alpha\sqrt{p}). \tag{6.1.1}$$

It is elementary to see, by partial summation that estimation of (6.1.1) is equivalent to estimate

$$S(N)=\sum_{n\le N}\lambda(n)\Lambda(n)e(\alpha\sqrt{n}). \tag{6.1.2}$$

Here and throughout the chapter, $\lambda(n)$ are the normalized Fourier coefficients of a cusp form $f(z)$ of weight $k\ge 12$ for the full modular group $\Gamma = SL_2(\mathbb{Z})$, and $f(z)$ is an eigenform of all the Hecke operators. *Id est*,

$$f(z)=(cz+d)^{-k}f\left(\frac{az+b}{cz+d}\right)\text{ for all }\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma,$$

and $T_nf=\lambda(n)f$ for all $n\ge 1$, where T_n is the n-th Hecke operator. These Hecke eigenvalues, $\lambda(n)$'s, agree with the coefficients in the Fourier series expansion of $f(z)$

$$f(z)=\sum_{n=1}^{\infty}\lambda(n)n^{\frac{k-1}{2}}e(nz).$$

$\Lambda(n)$, as usual, denotes the von Mangoldt function and α is a non-zero real number. It is well-known that up to some normalization factors, the Fourier coefficients $\lambda(n)$ and the Hecke eigenvalue are the same. References on the subject are abundant. Among my favorites are [19] and [30].

Estimation of $S(N)$ is interesting from two points of view. First, the sum

$$\sum_{n \leq N} \Lambda(n) e(\alpha\sqrt{n})$$

has been an object of interest ever since the method of I. M. Vinogradov was first developed and it was Vinogradov himself [36] who showed the afore-mentioned sum is $O(N^{\frac{7}{8}+\epsilon})$ with the implied constant depending on α and ϵ.

Second, the size and oscillations of the Hecke eigenvalues themselves are object of great interest. First, one can prove that

$$\sum_{n \leq N} |\lambda(n)|^2 \ll N, \tag{6.1.3}$$

with the implied constant depending on the form $f(z)$. Moreover, the above estimate is actually the best possible, as by Rankin-Selberg method, one can achieve a more precise asymptotic

$$\sum_{n \leq N} |\lambda(n)|^2 \sim cN, \tag{6.1.4}$$

as N tends to ∞ and c here is a positive constant that depends on $f(z)$.

The above results concerned the size of the Hecke eigenvalues are but average results regarding what is known as the Ramanujan conjecture which says that what is true on average in (6.1.3) and (6.1.4) is also true for the individual terms in the sums. More precisely, we have the following.

Theorem 6.1 (Ramanujan Conjecture). *With $\lambda(n)$ denoting the n-th Hecke eigenvalue of a cusp form, $f(z)$, for the full modular group, we have*

$$|\lambda(n)| \leq \tau(n) \ll n^{\epsilon}, \tag{6.1.5}$$

where $\tau(n)$ is the divisor function.

Proof. This famous result was of course, proved by P. Deligne in 1971, and this result we shall appeal to in later sections. □

Regarding the sign changes of the Hecke eigenvalues, it was due to Hardy and Ramanujan, and Good [13], respectively that

$$\sum_{n\leq N} \lambda(n)e(\alpha n) \ll N^{\frac{1}{2}}\log(2N), \text{ and} \sum_{n\leq N}\lambda(n) \ll N^{\frac{1}{3}+\epsilon}.$$

Moreover, there have also been Ω results concerning the sum of the prime-th eigenvalues of Hecke operators, due to M. R. Murty [27]. He conjectured that

$$\sum_{p\leq N}\lambda(p) = \Omega_{\pm}\left(\frac{\sqrt{N}\log\log\log N}{\log N}\right).$$

and succeeded in proving it provided some L-function has no real zero between $\frac{1}{2}$ and 1. Also, S. D. Adhikari [1] proved essentially the same result for cusp forms for the group $\Gamma_0(N)$.

The method that we employ in estimating $S(N)$ was first developed by Vinogradov, [36]. As one familiar with the method knows, the best possible results that technology can be proved would be

$$S(N) \ll N^{\Theta+\epsilon}, \tag{6.1.6}$$

with $\Theta = 3/4$. However, if one is to believe the so-called "principle of square-rooting," then one may also be led to believe that

$$S(N) \ll N^{\frac{1}{2}+\epsilon}$$

should hold. However, Iwaniec, Luo and Sarnak in [20], after assuming the truth of some hypotheses whose strength supersede even that of the Riemann hypothesis, gave surprising heuristics that $\Theta = \frac{3}{4}$ is where truth actually lies. What they were able to show, more precisely, was that under the assumption of the afore-mentioned hypotheses, we have

$$S(N) = ZN^{\frac{3}{4}} + O(N^{\frac{5}{8}+\epsilon}), \tag{6.1.7}$$

where Z is some constant, non-zero of course, that depends on the cusp form. In this chapter, we shall show the following.

Theorem 6.2. *With $S(N)$ defined as in (6.1.2), we have*

$$S(N) \ll N^{\frac{5}{6}}[\log(3N)]^{36}, \tag{6.1.8}$$

where the implied constant depends on α in the definition of $S(N)$ and the cusp form $f(z)$.

Proof. $S(N)$ will be decomposed and each component will be estimated individually in the following sections. □

6.2 Preliminaries

In this section, we quote all the results that we shall need in this chapter. First, we shall certainly also make good use of the multiplicity of Hecke eigenvalues.

Lemma 6.1. *Hecke eigenvalues are multiplicative, id est, $\lambda(mn) = \lambda(m)\lambda(n)$ if $\gcd(m, n) = 1$. More generally, they satisfy the following relation.*

$$\lambda(mn) = \sum_{d|\gcd(m,n)} \mu(d)\lambda\left(\frac{m}{d}\right)\lambda\left(\frac{n}{d}\right). \tag{6.2.1}$$

Proof. This Lemma follows easily by applying the Möbius inversion formula to the product formula for the Hecke eigenvalues. □

In our proof, we shall utilize the method of van der Corput for exponential sums. More precisely, we have the following series of lemmas.

Lemma 6.2 (Truncated Poisson Summation Formula). *Let $f(x)$ be a real function with continuous and steadily decreasing derivative $f'(x)$ in (a, b), and let $f'(b) = \alpha$ and $f'(b) = \beta$. Then we have*

$$\sum_{a<n\leq b} e[f(n)] = \sum_{\alpha-\eta<\nu<\beta+\eta} \int_a^b e[f(x) - \nu x]\mathrm{d}x + O[\log(\beta - \alpha + 2)], \tag{6.2.2}$$

where η is any positive constant less than 1.

Proof. Without loss of generality, we assume that $\eta - 1 < \alpha \le \eta$, so that $\nu > 0$. Such is true as we can also take k and integer such that $\eta - 1 < \alpha - k \le \eta$ and

$$h(x) = f(x) - kx.$$

Then the formula we shall desire to prove is

$$\sum_{a<n\le b} e[h(n)] = \sum_{\alpha-\eta<\nu-k<\beta+\eta} \int_a^b e[f(x) - (\nu - k)x]\mathrm{d}x + O[\log(\beta' - \alpha' + 2)],$$

where $\alpha' = \alpha - k$ and $\beta' = \beta - k$, *id est* the same formula for $h(x)$.

By partial integration, one easily arrives at

$$\sum_{a<n\le b} e[f(n)] = \int_a^b e[f(x)]dx + 2\pi i \int_a^b \left(x - [x] - \frac{1}{2}\right) f'(x)e[f(x)]\mathrm{d}x + O(1). \tag{6.2.3}$$

We now insert the Fourier series for $x - [x] - \frac{1}{2}$. If $x \notin \mathbb{Z}$, we have

$$x - [x] - \frac{1}{2} = -\frac{1}{\pi}\sum_{\nu=1}^{\infty} \frac{\sin(2\nu\pi x)}{\nu}.$$

Since the above series converges boundedly, we may swap the integration and summation and the second term in (6.2.3) becomes

$$-2i\sum_{\nu=1}^{\infty} \int_a^b \frac{\sin(2\nu\pi x)}{\nu} e[f(x)]f'(x)\mathrm{d}x = \sum_{\nu=1}^{\infty} \frac{1}{\nu}\int_a^b [e(-\nu x) - e(\nu x)]e[f(x)]f'(x)\mathrm{d}x.$$

The integral in the above expression is

$$\frac{1}{2\pi i}\int_a^b \frac{f'(x)}{f'(x) - \nu}\mathrm{d}\{e[f(x) - \nu x]\} - \frac{1}{2\pi i}\int_a^b \frac{f'(x)}{f'(x) + \nu}\mathrm{d}\{e[f(x) + \nu x].\} \tag{6.2.4}$$

Since $\frac{f'(x)}{f'(x) + \nu}$ is steadily decreasingly, the second term is term in the above is $O\left(\frac{\beta}{\beta + \nu}\right)$. Therefore, the contribution of these terms are

$$\ll \sum_{\nu=1}^{\infty} \frac{\beta}{\nu(\beta + \nu)} \ll \sum_{\nu\le\beta} \frac{1}{\nu} + \sum_{\nu>\beta} \frac{\beta}{\nu^2} \ll \log(\beta + 2) + 1.$$

Similarly, the first term is (6.2.4) is $O\left(\frac{\beta}{\nu - \beta}\right)$ if $\nu \ge \beta + \eta$, and the contribution of these terms is

$$\ll \sum_{\nu\ge\beta+\eta} \frac{\beta}{\nu(\nu - \beta)} \ll \sum_{\beta+\eta\le\nu<2\beta} \frac{1}{\nu - \beta} + \sum_{\nu\ge 2\beta} \frac{\beta}{\nu^2} \ll \log(\beta + 2) + 1.$$

Lastly, for the remaining terms, we have

$$\sum_{\nu=1}^{\beta+\eta}\frac{1}{\nu}\int_a^b e[f(x)-\nu x]f'(x)\mathrm{d}x=\sum_{\nu=1}^{\beta+\eta}\left\{\frac{e[f(x)-\nu x]}{2\pi i\nu}\right\}_a^b+\sum_{\nu=1}^{\beta+\eta}\int_a^b e[f(x)-\nu x]\mathrm{d}x.$$

The first term in the above expression is easily seen to be $O\left[\log(\beta+2)\right]$. Hence we have our desired result. □

A special case of the above Lemma 6.2 yields the following.

Lemma 6.3. *Let $f(x)$ be a real-valued function with $|f'(x)|\leq 1-\theta$ and $f''(x)\neq 0$ on $[a,b]$. We have*

$$\sum_{a<n<b} e[f(n)]=\int_a^b e[f(x)]\mathrm{d}x+O(\theta^{-1}), \tag{6.2.5}$$

where the implied constant is absolute.

Proof. As mentioned before, this is but a special case of Lemma 6.2. Indeed, if the conditions of this lemma are satisfied, the sum on the right-hand side of (6.2.2) has only on term and the lemma follows. □

Next we have these estimates for exponential integrals.

Lemma 6.4. *Let $r(x)$ and $\theta(x)$ be real-valued functions on $[a,b]$ such that $r(x)$ and $\theta'(x)$ are continuous. Suppose that $\dfrac{\theta'(x)}{r(x)}$ are positive and monotonically increasing on this interval. If $0<\lambda_1\leq\dfrac{\theta'(a)}{r(a)}$ then*

$$\left|\int_a^b r(x)e[\theta(x)]\mathrm{d}x\right|\leq\frac{1}{\pi\lambda_1}. \tag{6.2.6}$$

Proof. We may write the integral of our interest as a Riemann-Stieltjes integral thus,

$$\frac{1}{2\pi i}\int_a^b\frac{r(x)}{\theta'(x)}\mathrm{d}e[\theta(x)],$$

and partial integration gives that it is

$$=\left\{\frac{r(x)e[\theta(x)]}{2\pi i\theta'(x)}\right\}_a^b-\frac{1}{2\pi i}\int_a^b e[\theta(x)]\mathrm{d}\left[\frac{r(x)}{\theta'(x)}\right].$$

Applying the triangle inequality, we get

$$\left|\int_a^b r(x)e[\theta(x)]\mathrm{d}x\right|\leq\frac{1}{2\pi}\left[\frac{r(a)}{\theta'(a)}+\frac{r(b)}{\theta'(b)}\right]+\frac{1}{2\pi}\int_a^b\left|\mathrm{d}\left[\frac{r(x)}{\theta'(x)}\right]\right|.$$

Since $\dfrac{r(x)}{\theta'(x)}$ is monotonically decreasing, the absolute value sign may be brought out of the integrand and the integral is $\dfrac{r(a)}{\theta'(a)} - \dfrac{r(b)}{\theta'(b)}$. Now our desired result follows from the bound $0 < \lambda_1 \leq \dfrac{\theta'(a)}{r(a)}$. □

We shall also need a result known as stationary phase which is from the method of van der Corput. We arrive at stationary phase through the following lemma.

Lemma 6.5. *Let $h(x)$ be a real function with two continuous derivatives on $[0, X]$ such that*

$$h(0) = 1,\ h(x) \gg 1,\ [xh(x)]' \gg 1,\ h'(x) \ll \frac{1}{X},\ \text{and } h''(x) \ll \frac{1}{X^2}. \tag{6.2.7}$$

Then for $\alpha > 0$, we have

$$\int_0^X e[\alpha x^2 h(x)]\mathrm{d}x = \frac{8}{\sqrt{\alpha}} e\left(\frac{1}{8}\right) + O\left(\frac{1}{\alpha X}\right), \tag{6.2.8}$$

where the implied constant in the above expression depends on the implied constants in (6.2.7).

Proof. We write $h(x) = g^2(x)$. $g(x)$ satisfies the same condition in (6.2.7) as $h(x)$. Making a change of variable and set $t = x^2 g^2(x)$, we have

$$\int_0^X e[\alpha x^2 h(x)]\mathrm{d}x = \int_0^T e(\alpha t)\mathrm{d}\sqrt{t} - \int_0^T e(\alpha t) f(x)\mathrm{d}t,$$

where $T = X^2 g^2(X)$ and $f(x) = \dfrac{[xg(x)]' - 1}{2xg(x)[xg(x)]'}$. Using the conditions in (6.2.7) for $g(x)$ and the Taylor expansion of $g(x) = 1 + xg'(0) + O\left(\dfrac{x^2}{X^2}\right)$ and $g'(x) = g'(0) + O\left(\dfrac{x}{X^2}\right)$, we can get the following bounds for $f(x)$:

$$f(x) \ll \frac{1}{X} \text{ and } f'(x) \ll \frac{1}{X^2}.$$

Applying partial integration, we have

$$\begin{aligned}
\int_0^T e(\alpha t) f(x)\mathrm{d}t &= \frac{1}{2\pi i\alpha}\int_0^T f(x)\mathrm{d}e(\alpha t) \\
&= \frac{1}{2\pi i\alpha}\left[e(\alpha T) f(X) - f(0) - \int_0^T e(\alpha t) f'(x)\mathrm{d}x(t)\right] \\
&\ll \frac{1}{\alpha X} + \frac{1}{\alpha}\int_0^X \left|f'(x)\right| \mathrm{d}x \\
&\ll \frac{1}{\alpha X}.
\end{aligned}$$

Note that $x'(t)$ is positive. Now by Dirichlet evaluation, we know that

$$\int_0^\infty e(\alpha t)\mathrm{d}\sqrt{t} = \frac{8}{\sqrt{\alpha}} e\left(\frac{1}{8}\right)$$

which gives the main term of our formula and we again estimate the excess via partial integration thus

$$\begin{aligned}\int_T^\infty e(\alpha t)\mathrm{d}\sqrt{t} &= \frac{1}{2\pi i\alpha}\left[\frac{-e(\alpha T)}{\sqrt{T}} - \int_T^\infty e(\alpha t)\mathrm{d}t^{-\frac{1}{2}}\right] \\ &\ll \frac{1}{\alpha\sqrt{T}} \\ &\ll \frac{1}{\alpha X},\end{aligned}$$

because of the definition of T and the lower bound for $g(x)$. Combining everything, we get our desired result. □

Lemma 6.6 (Stationary Phase). *Let $f(x)$ be a real valued function with four continuous derivatives on $[a, b]$ such that*

(6.2.9) $$f''(x) \geq \Lambda,\ |f'''(x)| \leq \Lambda X^{-1},\ \textit{and}\ |f^{(4)}(x)| \leq \Lambda X^{-2},$$

for some $\Lambda > 0$ and $X > 0$. Also suppose $f'(c) = 0$ at some point $c \in (a, b)$. Then we have

(6.2.10) $$\int_a^b e[f(x)]\mathrm{d}x = e\left[f(c) + \frac{1}{8}\right] f''(c)^{-\frac{1}{2}} + O\left[\frac{1}{\Lambda}\left(\frac{1}{b-c} + \frac{1}{c-a} + \frac{1}{X}\right)\right],$$

where the implied constant is absolute.

Proof. Making two changes of variables, we split our integral as follows.

$$\int_a^b e[f(x)]\mathrm{d}x = \int_0^{b-c} e[f(c+x)]\mathrm{d}x + \int_0^{c-a} e[f(c-x)]\mathrm{d}x.$$

We use the Taylor expansion of $f(c+x) = f(c) + \alpha x^2 h(x)$, where $\alpha = \frac{1}{2}f''(c)$ and $h(x)$ is a function that satisfies the following conditions which follow from (6.2.9).

$$h(x) \geq 1 - \frac{x}{3X},\ [xh(x)]' \geq 1 - \frac{2x}{3X} - \frac{x^2}{12X^2},\ h'(x) \ll \frac{1}{X},\ \text{and}\ h''(x) \ll \frac{1}{X^2}.$$

Now the conditions of Lemma 6.5 are satisfied for $0 \leq x \leq Y = \min(b-c, X)$, and we have

$$\int_0^Y e[f(c+x)]\mathrm{d}x = e\left[f(c) + \frac{1}{8}\right]\frac{1}{2\sqrt{f''(c)}} + O\left(\frac{1}{\Lambda Y}\right).$$

The remaining part of the integral is easily disposed with partial integration and we have

$$\begin{aligned}\int_Y^{b-c} e[f(c+x)]\mathrm{d}x &= \frac{1}{2\pi i}\left\{\frac{e[f(b)]}{f'(b)} - \frac{e[f(c+Y)]}{f'(c+Y)} - \int_Y^{b-c} e[f(c+x)]\mathrm{d}\left[\frac{1}{f'(c+x)}\right]\right\} \\ &\ll \frac{1}{|f'(c+Y)|} + \frac{1}{|f'(b)|} \\ &\ll \frac{1}{\Lambda Y}.\end{aligned}$$

Combining what we have thus far, we arrive at

$$\int_c^b e[f(x)]\mathrm{d}x = e\left[f(c) + \frac{1}{8}\right]\frac{1}{2\sqrt{f''(c)}} + O\left[\frac{1}{\Lambda}\left(\frac{1}{b-c} + \frac{1}{X}\right)\right].$$

The integral over $[a, c]$ is estimated completely analogously and leads to a similar result. Of course, combining the estimates for the two integrals will gives us the result of the lemma. □

We shall also need the following Perron-type formula which is used to approximate Dirichlet polynomials.

Lemma 6.7 (Perron's Formula). *Let $f(s)$ be defined by the Dirichlet series*

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \text{ for } \Re s > 1.$$

where $a_n \ll \psi(n)$ for some non-decreasing function $\psi(n)$, and

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = O\left[\frac{1}{(\sigma-1)^\alpha}\right],$$

as σ tends to 1. Then if $c > 0$, $\sigma + c > 1$, x not an integer, we have

$$\sum_{n<x} \frac{a_n}{n^s} = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} f(s+w)\frac{x^w}{w}\mathrm{d}w + O\left[\frac{\psi(N)x^{1-\sigma}}{T\|x\|}\right] + O\left[\frac{x^c}{T(\sigma+c-1)^\alpha} + \frac{\psi(c)x^{1-\sigma}\log x}{T}\right]; \tag{6.2.11}$$

and if $x \in \mathbb{N}$, then

$$\sum_{n=1}^{x-1} \frac{a_n}{n^s} + \frac{a_x}{2x^s} = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} f(s+w)\frac{x^w}{w}\mathrm{d}w + O\left[\frac{x^c}{T(\sigma+c-1)^\alpha}\right] O\left[\frac{\psi(2x)x^{1-\sigma}\log x}{T} + \frac{\psi(x)x^{-\sigma}}{T}\right]. \tag{6.2.12}$$

Proof. Suppose that x is not an integer. If $n < x$, then Cauchy integral formula gives that

$$\int_{-\infty-iT}^{c-iT}\left(\frac{x}{n}\right)^w\frac{\mathrm{d}w}{w}+\int_{c-iT}^{c+iT}\left(\frac{x}{n}\right)^w\frac{\mathrm{d}w}{w}+\int_{c+iT}^{-\infty+iT}\left(\frac{x}{n}\right)^w\frac{\mathrm{d}w}{w}=2\pi i.$$

Applying integration by parts for the last integral, we have

$$\begin{aligned}\int_{-\infty+iT}^{c+iT}\left(\frac{x}{n}\right)^w\frac{\mathrm{d}w}{w} &= \left[\left(\frac{x}{n}\right)^w\frac{1}{w\log\left(\frac{x}{n}\right)}\right]_{-\infty+iT}^{c+iT}+\frac{1}{\log\left(\frac{x}{n}\right)}\int_{-\infty+iT}^{c+iT}\left(\frac{x}{n}\right)^w\frac{\mathrm{d}w}{w^2}\\ &\ll \left(\frac{x}{n}\right)^c\frac{1}{T\log\left(\frac{x}{n}\right)}+\left(\frac{x}{n}\right)^c\frac{1}{\log\left(\frac{x}{n}\right)}\int_{-\infty}^{\infty}\frac{\mathrm{d}u}{u^2+T^2}\\ &\ll \left(\frac{x}{n}\right)^c\frac{1}{T\log\left(\frac{x}{n}\right)}.\end{aligned}$$

The similar estimation also applies for the integral over $(-\infty-iT, c-iT)$. Therefore, we have

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\left(\frac{x}{n}\right)^w\frac{\mathrm{d}w}{w}=1+O\left[\left(\frac{x}{n}\right)^c\frac{1}{T\log\left(\frac{x}{n}\right)}\right].$$

If $n > x$, we have the similar argument with $-\infty$ replaced by $+\infty$, and we have no residue term. Therefore, we obtain a formula as the above without the main term. Now we multiply everything by $a_n n^{-s}$ and sum over all n, we have

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT}f(s+w)\frac{x^w}{w}\mathrm{d}w=\sum_{n<x}\frac{a_n}{n^s}+O\left[\frac{x^c}{T}\sum_{n=1}^{\infty}\frac{|a_n|}{n^{\sigma+c}\left|\log\left(\frac{x}{n}\right)\right|}\right].$$

If $n \notin \left[\frac{x}{2}, 2x\right]$, then $\left|\log\left(\frac{x}{n}\right)\right|$ is minorized by an absolute constant, and these parts of the sum are

$$\ll\sum_{n=1}^{\infty}\frac{|a_n|}{n^{\sigma+c}}\ll\frac{1}{(\sigma+c-1)^{\alpha}}.$$

If $N < n \leq 2x$, let $n = N + r$. We have

$$\log\left(\frac{n}{x}\right)\geq\log\left(\frac{N+r}{N+\frac{1}{2}}\right)>\frac{Ar}{N}>\frac{Ar}{x}.$$

The contribution of this part of the sum is

$$\ll\psi(2x)x^{1-\sigma-c}\sum_{1\leq r\leq x}\frac{1}{r}\ll\psi(2x)x^{1-\sigma-c}\log x.$$

We argue similarly for the contribution of the terms with $\frac{x}{2}\leq n<N$. Finally, we have

$$\frac{|a_N|}{N^{\sigma+c}\left|\log\left(\frac{x}{N}\right)\right|}\ll\frac{\psi(N)}{N^{\sigma+c}\log\left(1+\frac{x-N}{N}\right)}\ll\frac{\psi(N)x^{1-\sigma-c}}{|x-N|}.$$

Combining everything we have thus far, we arrive at (6.2.11). Toward the end of proving (6.2.12), everything is the same except we have the estimate

$$\frac{a_x}{2\pi i x^s}\int_{c-iT}^{c+iT}\frac{\mathrm{d}w}{w} = \frac{a_x}{2\pi i x^s}\log\left(\frac{c+iT}{c-iT}\right) = \frac{a_x}{2\pi i x^s}\left[i\pi + O\left(\frac{1}{T}\right)\right],$$

from which we may infer (6.2.12). □

Finally, we shall need an integral form of the large sieve inequality which will help us in the estimation of the mean values of Dirichlet polynomials.

Theorem 6.3 (Large Sieve Inequality). *Suppose that $\lambda_1, \cdots \lambda_N$ are distinct real numbers, and supposed that $\delta > 0$ is chosen so that*

$$|\lambda_m - \lambda_n| \geq \delta \tag{6.2.13}$$

whenever $m \neq n$. Then for any complex coefficients a_n, and any $T > 0$, we have

$$\int_0^T \left|\sum_{n=1}^{N} a_n e(\lambda_n t)\right|^2 \mathrm{d}t = \left(T + \frac{\theta}{\delta}\right)\sum_{n=1}^{N}|a_n|^2 \tag{6.2.14}$$

for some θ with $-1 \leq \theta \leq 1$.

Proof. We shall not need the full strength of (6.3) and replacing the equal sign in (6.3) by $\ll$ should suffice for our enterprise. This is quoted from [26] which arrive after using the Selberg majorant and minorant functions for indicator functions. □

As mentioned before, we shall use the above theorem toward the end of estimating the mean value of Dirichlet polynomials. Hence our λ_n is $\alpha \log n$ with n in some interval $[1, L]$ and α some real number. Hence if $m, n \in [1, L]$ and $m > n$ with $m = n + l$, we have

$$|\lambda_m - \lambda_n| = \alpha\left(\log(n+l) - \log n\right) = \alpha\left[\log\left(\frac{n+l}{n}\right) - \log 1\right] = \alpha\frac{l}{nc}, \tag{6.2.15}$$

with some $c \in \left[1, \dfrac{n+l}{n}\right]$ by the virtue of mean value theorem in differential calculus. c^{-1} is clearly at least $\dfrac{n}{n+l}$. Hence we have

$$|\lambda_m - \lambda_n| \geq \frac{\alpha l}{n+l} \geq \frac{\alpha}{2L}, \tag{6.2.16}$$

if $m,n \in [1,L]$ and $m \neq n$ and $\lambda_n = \alpha \log n$. Thus we have verified that Theorem 6.3 is applicable for Dirichlet polynomials with $\delta = \dfrac{\alpha}{2L}$. In fact, we have the asymptotic formula

$$\int_0^T \left| \sum_{n=1}^{N} a_n n^{-it} \right|^2 = [T + O(N)] \sum_{n=1}^{N} |a_n|^2, \tag{6.2.17}$$

where the implied constant is absolute.

6.3 Partition of the von Mangoldt Function

We begin with the following identity.

Lemma 6.8 (Vaughan). *Suppose $y \geq 2$ is a real positive number, then if $n > y$, we have*

$$\Lambda(n) = \sum_{\substack{ab=n \\ b \leq y}} \mu(b) \log a - \sum_{\substack{abc=n \\ b,c \leq y}} \mu(b)\Lambda(c) + \sum_{\substack{abc=n \\ b,c > y}} \mu(b)\Lambda(c). \tag{6.3.1}$$

Proof. This is the well-known identity in [33]. □

We shall decompose the second sum of the right-hand side of (6.3.1) further. Take $N < n \leq 2N$ in dyadic intervals, $y = N^{\frac{1}{3}}$ and $z = \sqrt{2N}$, we have

$$\sum_{\substack{abc=n \\ b,c \leq y}} \mu(b)\Lambda(c) = \sum_{\substack{abc=n \\ b,c \leq y \\ a \geq z}} \mu(b)\Lambda(c) + \sum_{\substack{abc=n \\ b,c \leq y \\ y < a < z}} \mu(b)\Lambda(c). \tag{6.3.2}$$

The partition according to the dichotomy of either $a \geq z$ or $a < z$ is obvious. Moreover, the extra condition in the second term of the right-hand side of (6.3.2) that $y > a$ is also obvious as

$$b, c \leq y \Longrightarrow bc \leq y^2 = N^{\frac{2}{3}},$$

together with $abc = n \leq 2N$, we have $a > N^{\frac{1}{3}} = y$.

The third sum of the right-hand side of (6.3.1) is similarly decomposed further. We have

$$\sum_{\substack{abc=n \\ b,c > y}} \mu(b)\Lambda(c) = \sum_{\substack{abc=n \\ c > y \\ y < b < z}} \mu(b)\Lambda(c) + \sum_{\substack{abc=n \\ b \geq z \\ y < c \leq z}} \mu(b)\Lambda(c). \tag{6.3.3}$$

Again, the dichotomy of $b < z$ or $b \geq z$ in the right-hand side of (6.3.3) is obvious. Furthermore, the extra condition in the second sum of the right-hand side of (6.3.3) is apparent as

$$abc = n \leq 2N \text{ and } b \geq z = \sqrt{2N} \Longrightarrow ac \leq \sqrt{2N} = z \Longrightarrow c \leq z.$$

Therefore, combining Lemma 6.3.1 and (6.3.2) and (6.3.3), we have the following lemma.

Lemma 6.9. *For $N < n \leq 2N$, $n \geq y = N^{\frac{1}{3}}$ and $z = \sqrt{2N}$, we have*

$$\Lambda(n) = \Lambda_1(n) + \Lambda_2(n) + \Lambda_3(n) + \Lambda_4(n) + \Lambda_5(n), \tag{6.3.4}$$

where

$$\Lambda_1(n) = \sum_{\substack{ab=n\\ b\leq y}} \mu(b)\log a,$$

$$\Lambda_2(n) = -\sum_{\substack{abc=n\\ b,c\leq y\\ a\geq z}} \mu(b)\Lambda(c),$$

$$\Lambda_3(n) = -\sum_{\substack{abc=n\\ b,c\leq y\\ y<a<z}} \mu(b)\Lambda(c),$$

$$\Lambda_4(n) = \sum_{\substack{abc=n\\ c>y\\ y<b<z}} \mu(b)\Lambda(c), \text{ and}$$

$$\Lambda_5(n) = \sum_{\substack{abc=n\\ b\geq z\\ y<c\leq z}} \mu(b)\Lambda(c).$$

Proof. This is already proved before the statement of the lemma. □

Thus the sum of our interest in (6.1.2) is decomposed and it suffices to estimate each component. We have

$$\left|\sum_{N<n\leq 2N} \Lambda(n)\lambda(n)e(\alpha\sqrt{n})\right| \leq |S_1(N)|+|S_2(N)|+|S_3(N)|+|S_4(N)|+|S_5(N)|, \tag{6.3.5}$$

where

$$S_i(N) = \sum_{N<n\leq 2N} \Lambda_i(n)\lambda(n)e(\alpha\sqrt{n})$$

with the $\Lambda_i(n)$'s defined as in Lemma 6.9. We shall estimate the size of each of the $S_i(N)$'s individually.

6.4 Bilinear Forms Treatment

The last three sums of (6.3.5) are similar and can be treated using the technique of bilinear forms. Toward that end, it suffices to estimate sums of the following form.

$$\sum_{m} \beta(m) \left| \sum_{y<l\leq z} \gamma(l) g(lm) \right|,$$

where $g(n)$ is an arithmetic function whose support is a subset of $[N, 2N]$, and $\beta(n)$ and $\gamma(n)$ are arithmetic functions. Of course, we shall take $g(n) = \lambda(n)e(\alpha\sqrt{n})$ for $n \in [N, 2N]$ and $g(n) = 0$ otherwise. Therefore, inserting this information into the above expression and break the summation over m into dyadic intervals, and it suffices to estimate

$$\sum_{M<m\leq 2M} \beta(m) \left| \sum_{L<l\leq 2L} \gamma(l)\lambda(lm)e(\alpha\sqrt{lm}) \right|, \tag{6.4.1}$$

with $N^{\frac{1}{2}} \leq M \leq N^{\frac{2}{3}}$, $N^{\frac{1}{3}} \leq L \leq N^{\frac{1}{2}}$ and $ML = N$.

We now apply the multiplicity of Hecke eigenvalues, Lemma 6.1, and (6.4.1) becomes the following.

$$\begin{aligned} &\sum_{M<m\leq 2M} \beta(m) \left| \sum_{s|m} \mu(s)\lambda\left(\frac{m}{s}\right) \sum_{\frac{L}{s}<t\leq \frac{2L}{s}} \gamma(st)\lambda(t)e(\alpha\sqrt{stm}) \right| \\ \leq &\sum_{M<m\leq 2M} \beta(m) \sum_{s|m} \left|\lambda\left(\frac{m}{s}\right)\right| \left| \sum_{\frac{L}{s}<t\leq \frac{2L}{s}} \gamma(st)\lambda(t)e(\alpha\sqrt{stm}) \right|. \end{aligned}$$

We divide the range of summation of the inner-most sum of the above even further and estimate sums of the following kind.

$$\sum_{M<m\leq 2M} \beta(m) \sum_{s|m} \left|\lambda\left(\frac{m}{s}\right)\right| \left| \sum_{\frac{L}{s}<t\leq \frac{L+L_0}{s}} \gamma(st)\lambda(t)e(\alpha\sqrt{stm}) \right|, \tag{6.4.2}$$

where $L_0 \leq L$ will be chosen later. We note that the number of sums like the above is $O\left[\frac{L}{L_0}(\log N)^2\right]$. We apply the Cauchy-Schwartz inequality, we see that the square of the sum in (6.4.2) is majorized by

(6.4.3)
$$\left(\sum_{M<m<2M} \tau(m)^4 \left|\beta(m)\right|^2\right) \left(\sum_{s\leq 2M} \sum_{\frac{M}{s}<m\leq\frac{2M}{s}} \left|\sum_{\frac{L}{s}<t\leq\frac{L+L_0}{s}} \gamma(st)\lambda(st)e(\alpha s\sqrt{tm})\right|^2\right).$$

There is no cancellation in the sum of the first factor of (6.4.3). We open up the complex modulus square in the second factor and swap the order of summation. We see that the afore-mentioned sum is majorized by

$$\text{(6.4.4)} \qquad \sum_{s\leq 2M} \sum\sum_{\frac{L}{s}<t,t'\leq\frac{L+L_0}{s}} \gamma(st)\overline{\gamma}(st')\lambda(st)\overline{\lambda}(st') \sum_{\frac{M}{s}<m\leq\frac{2M}{s}} e[\alpha s(\sqrt{t}-\sqrt{t'})\sqrt{m}].$$

The contribution of the diagonal terms in (6.4.4), *id est* the terms with $t=t'$, is

$$\ll M \sum_{s\leq 2M} s^{-1} \sum_{\frac{L}{s}<t\leq\frac{L+L_0}{s}} \left|\gamma(st)\lambda(st)\right|^2.$$

For the terms with $t\neq t'$, we note that if $f(m)=\alpha s(\sqrt{t}-\sqrt{t'})$, then

$$f'(m)=\frac{\alpha s(\sqrt{t}-\sqrt{t'})}{2\sqrt{m}}=\frac{\alpha s(t-t')}{2\sqrt{m}(\sqrt{t}+\sqrt{t'})}\leq\frac{\alpha\sqrt{s}L_0}{2\sqrt{N}}.$$

Therefore, by choosing $L_0=\dfrac{7L}{4\alpha\sqrt{s}}$ and recalling that $L_0\leq L\leq\sqrt{N}$, we can ensure that $f'(m)\leq\dfrac{7}{8}$ for all m's of interest. then the modulus of the inner-most sum of (6.4.4) is well-approximated by that of its corresponding integral in a special case of the Truncated Poisson Summation formula. For the modulus of the inner-most sum in question, we have the following

$$\begin{aligned} &= \int_{\frac{M}{s}}^{\frac{2M}{s}} e\left[\alpha s(\sqrt{t}-\sqrt{t'})\sqrt{x}\right]\mathrm{d}x+O(1) \\ &\ll \frac{\sqrt{M}}{s^{\frac{3}{2}}(\sqrt{t}-\sqrt{t'})}+1, \end{aligned}$$

where the implied constant in the last $\ll$ depends on α and the last inequality comes from Lemma 6.4. Now we observe that $\beta(m)$ is

$$\text{(6.4.5)} \qquad \sum_{\substack{bc=m\\ b,c\leq y}} \mu(b)\Lambda(c),\ \sum_{\substack{c|m\\ c>y}} \Lambda(c) \text{ and } \sum_{\substack{b|m\\ b\geq z}} \mu(b)$$

for $S_3(N)$, $S_4(N)$ and $S_5(N)$ respectively. The moduli of the first and second sum in (6.4.5) are majorized by $\log 2m$ while that of the third is majorized by the divisor function $\tau(m)$. In all three cases, we have, by Lemma 1.3,

$$\sum_{M<m\leq 2M} \tau^4(m)\,|\beta(m)|^2 \leq \sum_{M<m\leq 2M} \tau^6(m) \ll M(\log 3M)^{63}, \tag{6.4.6}$$

with implied constant absolute, by the virtue of Lemma 1.4.

Similarly, we have $\gamma(l)$ is 1, $\mu(l)$ and $\Lambda(l)$ for $S_3(N)$, $S_4(N)$ and $S_5(N)$ respectively. Again, in all three cases, we have

$$|\gamma(l)| \leq \log(3l). \tag{6.4.7}$$

Consequently, the modulus of the sum in (6.4.4) is bounded from above by

$$\begin{aligned} &\ll M^2[\log 3M]^{63} \sum_{s\leq 2M} s^{-1} \sum_{\frac{L}{s}<t\leq\frac{L+L_0}{s}} |\lambda(st)|^2 \\ &\qquad + M[\log(3M)]^{63} \sum_{s\leq 2M} s^{-\frac{3}{2}} \mathop{\sum\sum}_{\substack{\frac{L}{s}<t,t'\leq\frac{L+L_0}{s} \\ t\neq t'}} \left[\frac{\sqrt{M}}{|\sqrt{t}-\sqrt{t'}|}+1\right]. \end{aligned} \tag{6.4.8}$$

We shall need the majorant for the Hecke eigenvalues given by the Ramanujan conjecture, Theorem 6.1, *id est*

$$\lambda(n) \ll \tau(n).$$

Inserting this bound into (6.4.8), we see that it is bounded above by

$$\begin{aligned} &\ll \left[M^2L + M^{\frac{1}{2}}L^{\frac{3}{2}} + ML^2\right](\log 3M)^{66} \\ &\ll N^2\left[\frac{1}{L}+\frac{1}{N^{\frac{1}{2}}M}+\frac{1}{M}\right](\log 3M)^{66}. \end{aligned}$$

Now recall that $N^{\frac{1}{2}} \leq M \leq N^{\frac{2}{3}}$, $N^{\frac{1}{3}} \leq L \leq N^{\frac{1}{2}}$ and $ML = N$. We see that the contribution of the diagonal term will dominate over the others. We have the following lemma, after taking the square root and then add up the sums over all the dyadic intervals.

Lemma 6.10. *With $S_i(N)$ defined as before, we have*

$$|S_3(N)| + |S_4(N)| + |S_5(N)| \ll N^{\frac{5}{6}}(\log 3N)^{35},$$

where the implied constant depends on α and the cusp form $f(z)$.

Proof. This is just proved. □

6.5 Type I Sums

It still remains to estimate the other terms in (6.3.5). Since the terms that are yet to be estimated in (6.3.5) are all similar, we again disposed of them simultaneously. Toward that end, we must resort to large sieve inequalities, Theorem 6.3 and the mean value theorem for automorphic L-functions which can be obtained by similar means as the mean value theorems for the Zeta-fiction of Riemann.

We take $f(t) = \frac{t}{2\pi}\log\left(\frac{t}{ex}\right)$. It is clear that $f(t)$ satisfies the conditions of Lemma 6.6 with $\Lambda = (2\pi T)^{-1}$ and $c = x$. Therefore, by the virtue of the aforementioned lemma, we have

$$\int_T^{4T} e[f(t)]\mathrm{d}t = e\left(\frac{-x}{2\pi}+\frac{1}{8}\right)\sqrt{2\pi x} + O\left[T\left(\frac{1}{4T-x}+\frac{1}{x-T}+\frac{1}{T}\right)\right], \tag{6.5.1}$$

where T will of course be chosen later appropriately in terms of N and toward the end of making the error term in (6.5.1) $O(1)$. From (6.5.1), we have

$$e\left(-\frac{1}{8}\right)\sqrt{\frac{1}{2\pi x}}\int_T^{4T} e[f(t)]\mathrm{d}t = e\left(\frac{-x}{2\pi}\right) + O\left[x^{-\frac{1}{2}}\left(\frac{T}{4T-x}+\frac{T}{x-T}+1\right)\right]. \tag{6.5.2}$$

We now take $x = 2\pi\alpha\sqrt{n}$, with $N < n \leq 2N$; also we take $T = \pi\alpha\sqrt{N}$. Note that $2T \leq x \leq 3T$. Now (6.5.2) yields

$$e(-\alpha\sqrt{n}) = e\left(-\frac{1}{8}\right)(2\pi\sqrt{\alpha})^{-1}n^{-\frac{1}{4}}\int_T^{4T} e[f(t)]\mathrm{d}t + O\left(N^{-\frac{1}{4}}\right). \tag{6.5.3}$$

We dispose of $S_1(N)$ with the following Lemma.

Lemma 6.11. *With $S_1(N)$ defined as before and for any $\epsilon > 0$ given, we have*

$$S_1(N) \ll N^{\frac{3}{4}+\epsilon},$$

where the implied constant depends on α, the cusp form $f(z)$ and ϵ.

Proof. Recall that

$$S_1(N) = \sum_{N<n\leq 2N} \Lambda_1(n)\lambda(n)e(\alpha\sqrt{n}).$$

We insert (6.5.3) into the conjugate of $S_1(N)$, we have

$$S_1(N) = e\left(-\frac{1}{8}\right)(2\pi\sqrt{\alpha})^{-1}\int_T^{4T}\sum_{N<n\le 2N}\Lambda_1(n)\lambda(n)n^{-\frac{1}{4}}\left(\frac{t}{2\pi e\alpha\sqrt{n}}\right)^{it}\mathrm{d}t + O\left[N^{\frac{3}{4}}(\log 2N)^4\right]. \tag{6.5.4}$$

After applying partial summation to the sum inside the integrand of (6.5.4), it suffices, toward the end of majorizing the modulus of $S_1(N)$, to estimate, for $N < M \le 2N$, the size of the following.

$$N^{-\frac{1}{4}}\int_T^{4T}\left|\sum_{N<n\le M}\Lambda_1(n)\lambda(n)n^{-\frac{it}{4}}\right|\mathrm{d}t \tag{6.5.5}$$

$$= \frac{1}{4}N^{-\frac{1}{4}}\int_{\frac{T}{4}}^{T}\left|\sum_{N<n\le M}\sum_{\substack{ab=n\\ b\le y}}\mu(b)\log a\lambda(ab)(ab)^{-it}\right|\mathrm{d}t. \tag{6.5.6}$$

If we applying the multiplicity of Hecke eigenvalues and swap the order of summation of the integrand of (6.5.6), we must estimate the size of the following.

$$\left|\sum_{d\le y}\frac{\mu(d)}{d^{2it}}\sum_{l\le\frac{y}{d}}\frac{\lambda(l)\mu(ld)}{l^{it}}\sum_{\frac{N}{ld^2}<h\le\frac{M}{ld^2}}\frac{\lambda(h)(\log h+\log d)}{h^{it}}\right| \tag{6.5.7}$$

Therefore, in the light of (6.5.7), the Dirichlet series that is under our consideration is that of

$$[(\log d)L(s,f) - L'(s,f)]\sum_{l\le\frac{y}{d}}\frac{\mu(ld)\lambda(l)}{l^s},$$

where $L(s,f)$ is the L-function for the cusp form $f(z)$. We first consider the sum with $L'(s,f)$ and the sum with $L(s,f)$ is treated in the same way and yields the same majorant. By the virtue of Perron's formula, Lemma 6.7, we have

(6.5.8)

$$\sum_{l\le\frac{y}{d}}\frac{\mu(ld)\lambda(l)}{l^{it}}\sum_{\frac{N}{ld^2}<h\le\frac{M}{ld^2}}\frac{\lambda(h)\log h}{h^{it}} = \frac{-1}{2\pi i}\int_{\sigma-iV}^{\sigma+iV}L'(w+it,f)\left[\sum_{l\le\frac{y}{d}}\frac{\mu(ld)\lambda(l)}{l^{w+it}}\right]\frac{M^w-N^w}{d^{2w}w}\mathrm{d}w + O\left(\frac{N^{1+\epsilon}}{V}\right),$$

where, in the notations of Lemma 6.7, we take $\sigma = 1+\frac{1}{\log N}$, $s = it$ and $\psi(x) \ll x^{\epsilon}$ can be chosen. We shall choose V later according to need. If we insert (6.5.8) into (6.5.7), and the part corresponding to $L'(s,f)$ is

(6.5.9)
$$\frac{1}{2\pi i}\sum_{d\le y}\frac{\mu(d)}{d^{2it}}\int_{1+\frac{1}{\log N}-iV}^{1+\frac{1}{\log N}+iV}L'(w+it,f)\left(\sum_{l\le\frac{y}{d}}\frac{\mu(ld)\lambda(l)}{l^{w+it}}\right)\frac{M^w-N^w}{d^{2w}w}\mathrm{d}w + O\left(\frac{N^{1+\epsilon}}{V}\right).$$

As in Figure 6.1, we now move the line of integration from the line $\Re s = 1+\dfrac{1}{\log N}$ to the line $\Re s = \dfrac{1}{2}$, we see that (6.5.9) is well-approximated by

(6.5.10)
$$\frac{1}{2\pi i}\sum_{d\le y}\frac{\mu(d)}{d^{2it}}\int_{\frac{1}{2}-iV}^{\frac{1}{2}+iV}L'(w+it,f)\left(\sum_{l\le\frac{y}{d}}\frac{\mu(ld)\lambda(l)}{l^{w+it}}\right)\frac{M^w-N^w}{d^{2w}w}\mathrm{d}w + O\left(\frac{\sqrt{V+T}}{V}\sqrt{N}y^{\frac{1}{2}+\epsilon}\right),$$

where the error term in (6.5.10) comes from bounds by convexity of $L'(s,f)$ on horizontal lines, and trivial bounds over the other factors.

We now insert everything into (6.5.6), apply Hölder's inequality and choose $V = T = \pi\alpha\sqrt{N}$. We get that $S_1(N)$ is majorized

(6.5.11)
$$\ll N^{\frac{1}{4}}\sum_{d\le y}\frac{1}{d}\int_{-iV}^{iV}\left[\int_{\frac{T}{4}}^{T}\left|L'(\frac{1}{2}+it,f)\right|^2\mathrm{d}t\right]^{\frac{1}{2}}\left[\int_{\frac{T}{4}}^{T}\left|\sum_{l\le\frac{y}{d}}\frac{\mu(ld)\lambda(l)}{l^{\frac{1}{2}+it}}\right|^2\mathrm{d}t\right]^{\frac{1}{2}}\frac{\mathrm{d}w}{1+2|w|} + N^{\frac{3}{4}+\epsilon}.$$

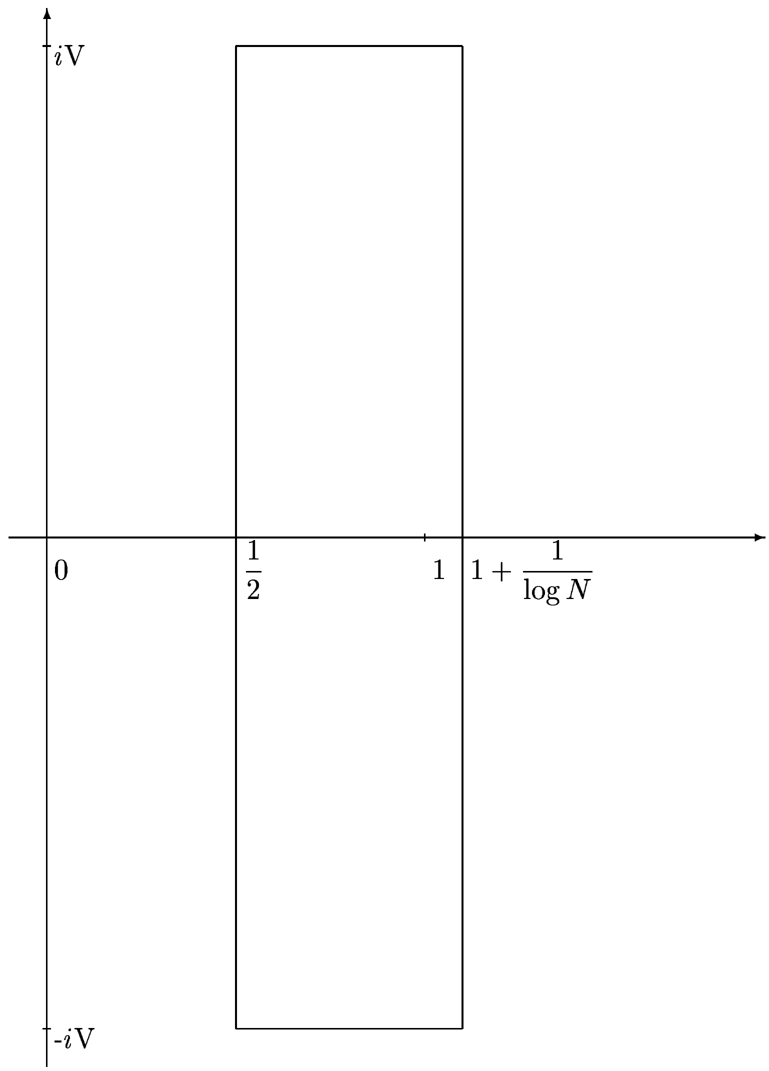
We apply the large sieve inequality, Theorem 6.3, for the second inner integral and it is

$$\ll \left(T+\frac{y}{d}\right)\left(\frac{y}{d}\right)^{\epsilon}.$$

We utilize the mean value theorem for automorphic L-functions for the first inner integral. The mean value theorems for automorphic L-fictions can be obtained using similar means as the analogous result for the Riemann Zeta function. We see that the afore-mentioned integral is

$$\ll T^{1+\epsilon}.$$

Figure 6.1: Contour of Integration under Consideration



Recall that $T = \pi\alpha\sqrt{N}$ and $y = N^{\frac{1}{3}}$. The contribution in (6.5.7) of $L(s,f)$ is treated precisely the same way as the treatment of $L'(s,f)$. Insert those bounds, we have the result of the lemma. □

We still need estimate the size of $S_2(N)$. We dispose of that task with the following lemma, which is proved via similar means as Lemma 6.11.

Lemma 6.12. *With $S_2(N)$ defined as before and for $\epsilon > 0$ given, we have*

$$S_2(N) \ll N^{\frac{3}{4}+\epsilon},$$

where the implied constant depends on α, the cusp form $f(z)$ and ϵ.

Proof. Recall that $\Lambda_2(n) = -\sum_{\substack{abc=n \\ b,c\leq y,\ a\geq z}} \mu(b)\Lambda(c)$. We decompose this component of the von Mongoldt function even further. Define a type of truncated Möbius convolution thus

$$\nu(d) = \sum_{\substack{bc=d \\ b,c\leq y}} \mu(b)\Lambda(c).$$

It is obvious, from the definition of $\nu(d)$ that we have the following inequality.

$$|\nu(d)| \leq \log 2d. \tag{6.5.12}$$

Now we can decompose $\Lambda_2(n)$ as follows.

$$\Lambda_2(n) = -\sum_{\substack{ad=n \\ z>d}} \nu(d) + \sum_{\substack{ad=n \\ a,d<z}} \nu(d).$$

Therefore, we have also correspondingly decomposed $S_2(N)$ further and we have the following inequality.

$$\begin{aligned} |S_2(N)| \leq & \left| \sum_{N<n\leq 2N} \sum_{\substack{ad=n \\ z>d}} \nu(d)\lambda(ad)e(\alpha\sqrt{ad}) \right| \\ & + \left| \sum_{N<n\leq 2N} \sum_{\substack{ad=n \\ a,d<z}} \nu(d)\lambda(ad)e(\alpha\sqrt{ad}) \right|. \end{aligned} \tag{6.5.13}$$

For convenience, we shall call the first sum in (6.5.13) $S_{21}(N)$ and the second $S_{22}(N)$. If we insert (6.5.3) into the definition of $S_{21}(N)$ and apply partial summation to the integrand. We arrive at an expression that is similar to (6.5.13) and it suffices to estimate, for $N < M \leq 2N$, the following.

$$N^{-\frac{1}{4}} \int_{\frac{T}{4}}^{T} \left| \sum_{N<n\leq M} \sum_{\substack{ad=n \\ z>d}} \nu(d)\lambda(ad)(ad)^{-it} \right| \mathrm{d}t + N^{\frac{3}{4}+\epsilon} \tag{6.5.14}$$

We now apply the multiplicity of Hecke eigenvalues, Lemma 6.1, and the integrand on the right-hand side of (6.5.14) becomes

$$\sum_{m<z} \frac{\mu(m)}{m^{2it}} \sum_{l<\frac{z}{m}} \nu(ml)\frac{\lambda(l)}{l^{it}} \sum_{\frac{N}{m^2 l}<k\leq\frac{M}{m^2 l}} \frac{\lambda(k)}{k^{it}}. \tag{6.5.15}$$

Similarly, to estimate the modulus of $S_{22}(N)$, it suffices to estimate the following sum.

$$N^{-\frac{1}{4}} \int_{\frac{T}{4}}^{T} \left| \sum_{N<n\leq M} \sum_{\substack{ad=n \\ a,d<z}} (\mu\star\Lambda)(d)\lambda(ad)(ad)^{-it} \right| \mathrm{d}t + N^{\frac{3}{4}+\epsilon}. \tag{6.5.16}$$

After applying the multiplicity of Hecke eigenvalues, Lemma 6.1, we arrive at something that is complete analogous to (6.5.14). The integrand of (6.5.16) becomes

$$\sum_{m<z} \frac{\mu(m)}{m^{2it}} \sum_{l<\frac{z}{m}} \nu(ml)\frac{\lambda(l)}{l^{it}} \sum_{\substack{\frac{N}{m^2 l}<k\leq\frac{M}{m^2 l} \\ \frac{z}{2m}<k<\frac{z}{m}}} \frac{\lambda(k)}{k^{it}}. \tag{6.5.17}$$

Note that the only difference between (6.5.15) and (6.5.17) is the additional restriction on the range of summation of the inner-most sum of (6.5.17).

Now our proof of the lemma will go exactly the same was in that of Lemma 6.11. We first insert Perron's formula, Lemma 6.7, into (6.5.15) and (6.5.17), and then move the lines of integration to the critical line and then applying the large sieve inequalities and mean value theorems of automorphic L-functions. Combining everything, we have our desired result of the lemma. □

Combining all the lemmas of the previous two sections and sum up all the dyadic intervals, we have proved Theorem 6.2. Note that we were not concerned with the powers of logarithms that appear in the estimates of $S_1(N)$ and $S_2(N)$, but instead wrote only N^ϵ in both cases, because neither is the dominating term in the final result.

6.6 Notes

In I. M. Vinogradov's original work of estimating the exponential sum over primes

$$\sum_{p\le N} e(\alpha\sqrt{p}),$$

he used the exponent pair $\left(\frac{1}{2},\frac{1}{2}\right)$ in the treatment of the bilinear forms. That led to the majorant of $N^{\frac{7}{8}}$, essentially, rather than the $N^{\frac{5}{6}}$ that we have here.

Moreover, instead of what we have done in the previous section, we could have also applied the result of M. Jutila, Theorem 6.4 for the so-called Type I sums. We quote the his result as the following.

Theorem 6.4 (Jutila). *Let $2 \le M < M' \le 2M$, and let f be a holomorphic function in the domain*

$$D = \left\{z\big||z-x| < cM \text{ for some } x \in [M, M']\right\},$$

where c is a positive constant. Suppose that $f(x)$ is real for $M \le x \le M'$, and that either

$$f(z) = Bz^{\beta}\left(1 + O(F^{-\frac{1}{3}})\right) \text{ for } z \in D$$

where $\beta \neq 0,\ 1$ is a fixed real number, and

$$F = |B|M^{\beta},$$

or

$$f(x) = B\log z\left(1 + O(F^{-\frac{1}{3}})\right) \text{ for } z \in D,$$

where

$$F = |B|.$$

Let g be continuously differentiable function on $[M, M']$ and for $x \in [M, M']$, we have

$$|g(x)| \ll G, \text{ and } |g'(x)| \ll G'.$$

Suppose also that

$$M^{\frac{3}{4}} \ll F \ll M^{\frac{3}{2}}.$$

Then we have

$$\left|\sum_{M\le m\le M'} b(m)g(m)e[f(m)]\right| \ll (G+MG')\sqrt{M}F^{\frac{1}{3}+\epsilon},$$

where $b(m)$ is may be taken to be either the divisor function $\tau(m)$ or the normalized Fourier coefficients of cusp forms $\lambda(m)$.

Proof. This is Theorem 4.6 in [22]. □

In short, Jutila's result will give that

$$S_1(N)+S_2(N) \ll N^{\frac{5}{6}+\epsilon}.$$

Indeed, this will not affect our final result of Theorem 6.2, but our results of the previous section does give better estimates.

Moreover, we can now see that where the obstacle lies in trying to attain the upper bound of $N^{\frac{3}{4}}$. What stays us is the want of better means to estimate the bilinear forms in Section 6.4. We could, in principle, do the similar thing that we did in Section 6.5, transform the bilinear forms in Section 6.4 into integrals using Lemma 6.6 and then use the mean value theorems of L-function and Dirichlet series to estimate the resulting integrals. However, the known results only give essentially the same estimates that we have in section 6.4.

However, since we shall need to estimate the size of some Dirichlet polynomials in the method mentioned in the previous paragraph, if one assumes the truth of the Montgomery conjecture, Conjecture 1.1 and see [26], then we do obtain the expected majorant of $\Theta = \dfrac{3}{4}$, with Θ as in (6.1.6).

Also, it is note-worthy that in the treatment of the bilinear forms mentioned in the last paragraph, if one applies the appropriate version of the Lindelöf hypothesis for automorphic L-functions, we would also arrive at the expected majorant with $\Theta = \dfrac{3}{4}$. This certainly is not a surprising revelation given the relation between L-functions and

Dirichlet polynomials in the critical strip via the approximate functional equations.

Of course, even if the truth of either the Montgomery Conjecture or Lindelöf Hypothesis is known, we still have a long way to go to get to the asymptotic formula of (6.1.7).

Appendix A

Distance between Sepcial Fractions

The task that presents itself in Chapter 2 was to estimate, given a rational number of height not exceeding Q^k with a k-power denominator between 0 and 1, the number of rational number, different from the given one, of the same kind as the given one and close to the given one. In this appendix, we give some conjectures and some evidences for them.

A.1 Conjectures based on heuristics and empirical evidences

If we take $N = Q^3$ in Lemma 2.5, we have the following

$$M(Q) \overset{\text{def}}{=} \max_{x' \in S_Q} \left| \{ x \in S_Q | 2 \| x - x' \| < Q^{-3} \} \right| \ll Q^{\frac{1}{2}+\epsilon}, \tag{A.1.1}$$

and the implied constant in (A.1.1) depends on ϵ alone. With this, we can arrive at something a slightly different.

We shall need the following lemma, touching theta series in the proof of our contention here. It is worthwhile to mention that in my humble opinion, Lemma A.1 is of interest because the left hand side of (A.1.2) is oscillatory, while all terms on the right hand side are positive, a key property that we shall not hesitate to exploit. Indeed, positivity provides a great convenience in estimations and is a luxury that we cannot always afford.

Lemma A.1. *For $x, y \in \mathbb{R}$, and $y > 0$, we have*

$$\sum_{m=-\infty}^{\infty} e^{-\pi m^2 y + 2\pi i m x} = y^{-\frac{1}{2}} \sum_{n=-\infty}^{\infty} e^{-\pi (n+x)^2 y^{-1}}. \tag{A.1.2}$$

Proof. By Poisson summation formula, Lemma 2.3, the left-hand side of (A.1.2) can be re-written as the following sum

$$\begin{aligned} & \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left[-\pi t^2 y + 2\pi i t(n+x)\right] \mathrm{d}t \\ = & \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left[-\pi y \left(t - i\frac{n+x}{y}\right)^2 - \pi \frac{(n+x)^2}{y}\right] \mathrm{d}t \\ = & \sum_{n=-\infty}^{\infty} \exp\left(-\pi \frac{(n+x)^2}{y}\right) \int_{-\infty}^{\infty} \exp(-\pi y t^2) \mathrm{d}t. \end{aligned}$$

The last integral is that of the well-known Gaussian curve, which yields $y^{-\frac{1}{2}}$ and gives our contention. Indeed, we must apply substitution to the following

$$\int_{-\infty}^{\infty} e^{-x^2} \mathrm{d}x = \sqrt{\pi}, \tag{A.1.3}$$

and as Lord Kelvin have bluntly, and perhaps arrogantly, pointed out, as quoted in [31],

> A mathematician is one to whom *that*(A.1.3) is as obvious as that twice two makes four is to you. Liouville was a mathematician.

□

Now we can state and prove the contention of this appendix.

Proposition A.1. *With* $\{a_n\}$, Q, M, *and* N *defined as before, we have*

$$\sum_{q=1}^{Q} \sum_{\substack{a \bmod q^2 \\ \gcd(a,q)=1}} \left| \sum_{n=M+1}^{M+N} a_n e\left(\frac{a}{q^2} n\right) \right|^2 \ll Q^{\frac{1}{2}+\epsilon}(Q^3+N) \sum_{n=M+1}^{M+N} |a_n|^2, \tag{A.1.4}$$

where the implied constant depends on ϵ *alone.*

Proof. It is easily observed that the theorem, by the duality principle Lemma 2.2, and by assuming $M=0$ via the shift, $n \longrightarrow n-M$, is equivalent to

$$\sum_{0<n\leq N} \left| \sum_{x\in S_Q} b_x e(xn) \right|^2 \ll Q^{\frac{1}{2}+\epsilon}(Q^3+N) \sum_{x\in S_Q} |b_x|^2, \tag{A.1.5}$$

for any sequence of complex numbers $\{b_x\}$.

The left hand side of (A.1.5) increases when we introduce a "smoothing" function and sum over all integers n instead of only between 1 and N. Hence we have the majorant

$$\text{(A.1.6)} \qquad \frac{e^{\pi}}{2}\sum_{n=-\infty}^{\infty} e^{-\frac{\pi n^2}{N^2}}\left|\sum_{x\in S_Q} b_x e(nx)\right|^2 \ll \sum_{x\in S_Q}\sum_{x'\in S_Q} b_x\bar{b}_{x'}\sum_{n=-\infty}^{\infty} e^{-\frac{\pi n^2}{N^2}} e[n(x-x')].$$

We split the sum in (A.1.6) and get

$$\text{(A.1.7)} \qquad \sum_{x\in S_Q}|b_x|^2\sum_{n=-\infty}^{\infty} e^{-\frac{\pi n^2}{N^2}} + \sum\sum_{x\neq x'} b_x\bar{b}_{x'}\sum_{n=-\infty}^{\infty} e^{-\frac{\pi n^2}{N^2}} e[n(x-x')].$$

The first sum in (A.1.7), the diagonal terms, is easily disposed as we apply the following well-known result

$$\text{(A.1.8)} \qquad \sum_{m=-\infty}^{\infty} e^{-\pi(cm)^2} \le 1 + \int_{-\infty}^{\infty} e^{-\pi(ct)^2}\mathrm{d}t = 1 + \frac{1}{|c|}$$

for any real number $c \neq 0$, and see that it is

$$\text{(A.1.9)} \qquad \ll N\sum_{x\in S_Q}|b_x|^2.$$

Applying Lemma A.1, and the parallelogram rule

$$2|b_x\bar{b}_{x'}| \le |b_x|^2 + |b_{x'}|^2,$$

we find that the second sum in (A.1.7) is majorized by

$$\text{(A.1.10)} \qquad N\sum_{x\in S_Q}|b_x|^2\sum_{n=-\infty}^{\infty}\sum_{\substack{x'\in S_Q\\ x\neq x'}} e^{-\pi(n+x-x')^2N^2}.$$

Note that the summands of the inner-most sum are all positive.

Now consider the partition of $[0,1) = \bigcup_{l=0}^{2Q^3-1} K_l$, where $K_l = \left[\frac{l}{2Q^3}, \frac{l+1}{2Q^3}\right)$. By the virtue of (A.1.1), the cardinality of $K_l \cap S_Q$ is $O(Q^{\frac{1}{2}+\epsilon})$ for each l. Now let T_1 be the a set satisfying for each $0 \le l < 2Q^3$ and l odd, $|T_1 \cap (K_l \cap S_Q)| \le 1$ and $T_1 \cap (K_l \cap S_Q)$ is empty only if $K_l \cap S_Q$ is empty. In other words, T_1 picks out representatives from the sets $K_l \cap S_Q$ if there are representatives at all. Let T_3 be a set satisfying for each

$0 \le l < 2Q^3$ and l odd, $|T_3 \cap [(K_l \cap S_Q) - T_1]| \le 1$ and $T_3 \cap [(K_l \cap S_Q) - T_1]$ is empty only if $(K_l \cap S_Q) - T_1$ is empty. Continuing thus, for l odd with $0 < l < 2Q^3$, T_l is a set satisfying that $|T_3 \cap [(K_l \cap S_Q) - T_1 - T_3 - \cdots - T_{l-2}]| \le 1$ and $T_l \cap [(K_l \cap S_Q) - T_1 - T_3 - \cdots - T_{l-2}]$ is empty only if $(K_l \cap S_Q) - T_1 - T_3 - \cdots - T_{l-2}$ is empty. Of course, we cannot continue indefinitely and still get non-empty T_l's, the process will terminate after $O(Q^{\frac{1}{2}+\epsilon})$ steps. Of course, for even l's, we define T_l's similarly, and exhaust the set S_Q after $O(Q^{\frac{1}{2}+\epsilon})$ steps. If you will, we are sifting the rational numbers in S_Q with a "sieve" whose holes are of size Q^{-3}.

There are $O(Q^{\frac{1}{2}+\epsilon})$ of T_l's and in each T_l the elements are $(2Q^3)^{-1}$ spaced. Hence, we majorize (A.1.10) thus:

$$\ll N \sum_{x \in S_Q} |b_x|^2 \sum_l \sum_{n=-\infty}^{\infty} \sum_{\substack{x' \in T_l \\ x \neq x'}} e^{-\pi(n+x-x')^2 N^2}. \tag{A.1.11}$$

The inner double sum over n and x' is majorized by

$$\sum_{m=-\infty}^{\infty} \exp[-\pi(mQ^{-3}N/2)^2] \le 1 + \frac{2Q^3}{N},$$

again by (A.1.8). Note further that we can combine the two inner sums of (A.1.11) into one sum by the virtue of positivity. Hence the second term in (A.1.7) is

$$\ll Q^{\frac{1}{2}+\epsilon}(N + Q^3) \sum_{x \in S_Q} |b_x|^2. \tag{A.1.12}$$

Combining the two estimates in (A.1.9) and (A.1.12) that we have, we get our desired result. □

We gave the complete proof of the above proposition for the reason that it is easy to give conjectures based on empirical result along that line of the above proposition. Of course, it is possible to formulate proposition along the line of Proposition A.1 for high power moduli. In that direction, we record the following.

If we take $N = Q^{k+1}$ in Lemma 3.1, we have the following

$$M_k(Q) \stackrel{\text{def}}{=} \max_{x' \in S_{Q,k}} \left| \left\{ x \in S_{Q,k} \middle| 2\|x - x'\| < Q^{-k-1} \right\} \right| \ll Q^{\frac{\kappa-1}{\kappa}+\epsilon}, \tag{A.1.13}$$

where $\kappa = 2^{k-1}$ and the implied constant in (A.1.13) depends on ϵ and k. With this, we can arrive at the following.

Proposition A.2. *With $\{a_n\}$, Q, M, and N defined as before, we have*

$$\sum_{q=1}^{Q} \sum_{\substack{a \mod q^k \\ \gcd(a,q)=1}} \left| \sum_{n=M+1}^{M+N} a_n e\left(\frac{a}{q^k} n\right) \right|^2 \ll Q^{\frac{\kappa-1}{\kappa}+\epsilon}(Q^{k+1} + N) \sum_{n=M+1}^{M+N} |a_n|^2, \tag{A.1.14}$$

where the implied constant depends on ϵ and k.

Proof. The proof of the present proposition goes analogously as that of the previous proposition. □

Moreover, both Propositions above admit obvious corollaries to multiplicative characters.

A.2 Computational evidences

This Table A.1 below on page 94 lists some values of $M(Q)$, as defined in (A.1.1), for some small values of Q. This table was constructed using a C++ program written with the gracious help of Mr. Waldeck Schützer, a friend who is also enduring the Ph.D. program with me, and the author thanks him very much. As stated in Chapter 2, we see that $M(Q)$ increases rather slowly and it is our strong belief that $M(Q) = O(Q^\epsilon)$ with the implied constant depending on ϵ alone. Also, as mentioned in Chapter 2, the computer calculation becomes rather time-consuming quite fast.

We believe the growth of $M(Q)$ is of independent interest. Therefore, we record the following conjecture.

Conjecture A.1. *Let denote the rational numbers between 0 and 1 with square denominator and height not exceeding Q^2. In other words,*

$$S_Q = \left\{ \frac{a}{q^2} \in \mathbb{Q} \middle| \gcd(a,q) = 1,\ 1 \leq a < q^2,\ Q < q \leq 2Q \right\}.$$

Then we have

$$\max_{x \in S_Q} \left| \left\{ x' \in S_Q \middle| \|x - x'\| < Q^{-3} \right\} \right| \ll Q^\epsilon, \tag{A.2.1}$$

where the implied constant depends on ϵ alone.

In the same spirit, we also believe that analogous conjectures could be made for higher power moduli.

Conjecture A.2. *Let $S_{k,Q}$ denotes the rational numbers between 0 and 1 with k-th power denominator and height not exceeding Q^k. In other words,*

$$S_{k,Q} = \left\{ \frac{a}{q^k} \in \mathbb{Q} \middle| \gcd(a,q) = 1,\ 1 \leq a < q^k,\ Q < q \leq 2Q \right\}.$$

Then we have

$$\max_{x \in S_{k,Q}} \left| \left\{ x' \in S_{k,Q} \middle| \|x - x'\| < Q^{-k-1} \right\} \right| \ll Q^{\epsilon}, \tag{A.2.2}$$

where the implied constant depends only on k and ϵ.

If we assume the truth of Conjecture A.2, then we would have the following.

Conjecture A.3. *Let $\{a_n\}$ be an arbitrary sequence of complex numbers, Q, $N \in \mathbb{N}$ and $M \in \mathbb{Z}$. We have*

$$\sum_{q=1}^{Q} \sum_{\substack{a \bmod q^k \\ \gcd(a,q)=1}} \left| \sum_{n=M+1}^{M+N} a_n e\left(\frac{a}{q^k} n\right) \right|^2 \ll Q^{\epsilon} \left(Q^{k+1} + N \right) \sum_{n=M+1}^{M+N} |a_n|^2, \tag{A.2.3}$$

where the implied constant depends on ϵ and k.

Of course, it goes without saying that Conjecture A.3 admits corollary for multiplicative characters, assuming its truth.

A.3 Evidences for some special cases

In search for the truth, perhaps no theorist of numbers can resist using the hypothesis of Riemann get prove things conditionally. That is what we shall do in this section.

We start out with the assumption that $a_n \neq 0$ only if n is co-prime with q. Therefore,

Table A.1: $M(Q)$ for some small $Q's$.

Q	$M(Q)$	Q	$M(Q)$	Q	$M(Q)$	Q	$M(Q)$	Q	$M(Q)$
1	0	2	0	3	1	4	1	5	2
6	1	7	1	8	2	9	2	10	2
11	2	12	2	13	2	14	2	15	2
16	2	17	2	18	2	19	2	20	2
21	2	22	2	23	2	24	3	25	2
26	2	27	3	28	2	29	2	30	2
31	2	32	2	33	2	34	2	35	2
36	2	37	3	38	3	39	3	40	3
41	3	42	3	43	3	44	3	45	2
46	3	47	3	48	3	49	3	50	3
51	3	52	3	53	4	54	3	55	3
56	3	57	3	58	3	59	3	60	3
61	3	62	3	63	3	64	3	65	3
66	3	67	3	68	3	69	3	70	3
71	3	72	3	73	3	74	3	75	3
76	3	77	3	78	3	79	3	80	3
81	3	82	3	83	4	84	4	85	3
86	3	87	3	88	3	89	4	90	4
91	4	92	4	93	4	94	3	95	3
96	3	97	4	98	4	99	4	100	4

we have

$$\text{(A.3.1)} \qquad \sum_{\substack{a=1 \\ \gcd(a,q^2)=1}}^{q^2} \left| \sum_{n=M+1}^{M+N} a_n e\left(\frac{a}{q^2}n\right) \right|^2$$

$$\text{(A.3.2)} \qquad = q \sum_{\substack{a=1 \\ \gcd(a,q)=1}}^{q} \sum_{n \equiv n' \mod q} a_n \overline{a}_{n'} e\left(\frac{a}{q}\frac{n-n'}{q}\right)$$

$$\text{(A.3.3)} \qquad = q \sum_{rs=q} r\mu(s) \sum_{n \equiv n' \mod qr} a_n \overline{a}_{n'}$$

$$\text{(A.3.4)} \qquad = q \sum_{rs=q} \frac{r\mu(s)}{\varphi(qr)} \sum_{\chi \mod qr} \left| \sum_{n=M+1}^{M+N} a_n \chi(n) \right|^2$$

$$\text{(A.3.5)} \qquad = \frac{q}{\varphi(q)} \sum_{rs=q} \mu(s) \sum_{k|qr} \sideset{}{^\star}\sum_{\chi \mod k} \left| \sum_{n=M+1}^{M+N} a_n \chi(n) \right|^2$$

$$\text{(A.3.6)} \qquad = \frac{q}{\varphi(q)} \sum_{k|q^2} \left(\sum_{\substack{rs=q \\ \frac{k}{\gcd(k,q)}|r}} \mu(s) \right) \sideset{}{^\star}\sum_{\chi \mod k} \left| \sum_{n=M+1}^{M+N} a_n \chi(n) \right|^2 .$$

All implications in the above are obvious save for that from (A.3.2) to (A.3.3) where we have use the Ramanujan sum, Lemma 1.1. Also the fact that all characters modulo q is induced by a primitive character of modulus dividing q, and *vice versa* enables us to go from (A.3.4) to (A.3.5). Now we note that the sum over $rs = q$ in (A.3.6) vanishes unless $q = \dfrac{k}{\gcd(k,q)}$ or $k = q^2$. Therefore, we conclude that

$$\text{(A.3.7)} \qquad \sum_{\substack{a=1 \\ \gcd(a,q^2)=1}}^{q^2} \left| \sum_{n=M+1}^{M+N} a_n e\left(\frac{a}{q^2}n\right) \right|^2 = \frac{q}{\varphi(q)} \sideset{}{^\star}\sum_{\chi \mod q^2} \left| \sum_{n=M+1}^{M+N} a_n \chi(n) \right|^2 .$$

If we take $a_n = \Lambda(n)$, where $\Lambda(n)$ is von Mongoldt function, then (A.3.7) is $O(q(qN)^{1+\epsilon})$ by the generalized Riemann hypothesis, (1.3.2). Moreover, if $a_n = \left(\dfrac{n}{q}\right) \Lambda(n)$, where $\left(\dfrac{n}{q}\right)$ is the Legendre symbol, then again by the generalized Riemann hypothesis (1.3.2), (A.3.7) is also $O(q(qN)^{1+\epsilon})$. Therefore, in both of those cases, with the help of the powerful generalized Riemann hypothesis, not only should Conjecture A.3 be true, but in fact the corresponding bounds for each individual terms should also be valid.

Of course, one should not be surprised at the result of this section. Large sieve is in so many way, Riemann Hypothesis "on average." Therefore, assuming the truth of the Riemann Hypothesis, one should be able to make some interesting statements regarding large sieves as well.

Epilogue

The author must, alas, end this thesis.
Progressing only after pain and smart,
We cipher our trade in broken pieces,
With much help from a master of the art.
As God disposes and men merely purpose,
His wit hardly sustains his ambition.
Somewhere must he find his soul's true repose;
Be it here or what other profession.
Hence ending thus with my golden words,
Thank I the world with naught but empty purse,
My strength and heart must henceforth find accords,
And my ambition be not my sinew's curse,
Which oft a man must do; and for my sake,
"In your fair minds let this acceptance take"*.

L.Z.

11 March 2003

*[29], *King Henry V, Epilogue, 14.*

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