Digital lattice rules
Multivariate integration and discrepancy estimates

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Abstract

High dimensional integrals appear in many applications, notably in financial mathematics and statistics. Those integrals are often not analytically tractable and hence one relies on numerical approximation methods. Theoretical investigations are frequently restricted to functions defined on the unit cube, as many functions defined on more general domains can be transformed to functions over the unit cube. Monte Carlo and quasi-Monte Carlo algorithms approximate the integral of a function over the unit cube by taking the average of $n$ function values. For Monte Carlo those sample points are chosen randomly, whereas for quasi-Monte Carlo one chooses the nodes deterministically. In the latter the aim is to select a well distributed point set. Several quality measures for the irregularity of distribution of a point set in the unit cube have been defined. Some are based on geometrical properties of the point set and some are based on the worst-case error for integration in certain function spaces.

Two main construction methods for deterministic point sets are commonly used. Those are lattice rules and digital nets. Lattice rules have long been known to work well for integrating periodic functions, but have recently also been shown to work well for integration of functions from certain Sobolev spaces. In the latter case one uses randomly shifted lattice rules rather than lattice rules. In many cases reproducing kernel Hilbert spaces have been considered, as this allows us to develop an elegant theory for the integration problem. Furthermore, computers can be employed to search for point sets yielding a small worst-case error for integration in such spaces. Digital nets on the other hand have been shown to yield well distributed point sets in a geometrical sense and to work well for approximating the integral of functions from function spaces which are based on Haar or Walsh functions.

In this thesis we develop an analogous theory for the integration problem in certain function spaces based on Walsh functions using digital nets to the elegant theory already known for the integration problem in Korobov spaces using lattice rules. The starting point of this theory is the introduction of a suitably defined reproducing kernel Hilbert space based on Walsh functions. The crux of this definition is that the worst-case error for integration in such a space can be calculated for a given set of sample points. If one uses digital nets the worst-case error is a direct function of the generating matrices of the digital net. (This resembles the worst-case error for integration in weighted Korobov spaces using lattice rules to a great extent, as in this case the worst-
case error depends on the generating vector of the lattice rule.) Step by step we then unfold analogous results for the integration problem in our newly defined Hilbert spaces to the known results for the integration problem in Korobov spaces using lattice rules.

Furthermore we extend our results to the approximation of integrals of functions from certain Sobolev spaces. This has previously been done for lattice rules by exploiting a connection between the integration problem in Korobov spaces using lattice rules and the integration problem in Sobolev spaces using shifted lattice rules. Similarly, such a connection is also established here. Correspondingly we apply a digital shift to the digital nets, and we show that the worst-case error for integration in those Sobolev spaces using digitally shifted digital nets is similar to the worst-case error for integration in the Hilbert spaces based on Walsh functions. The integration problem in those Sobolev spaces is the common ground where the theory for lattice rules and the theory developed here meet. The results established here show that digital nets perform just as well as lattice rules for integrating functions from those spaces. Those results lay the groundwork for further developments, some of which are established here.

With the previously developed theory we are then able to introduce, for the first time, construction algorithms for polynomial lattice rules based on the worst-case error. Polynomial lattice rules are quasi-Monte Carlo rules where the underlying point set stems from a special class of digital nets. The construction of those point sets is based on polynomials over finite fields and is similar to the construction of the underlying point set of lattice rules, hence the name polynomial lattice rules. Again, comparable results for integrating functions from Sobolev spaces are obtained for polynomial lattice rules to the known results for lattice rules. The results for integrating functions from function spaces based on Walsh functions using polynomial lattice rules are of course new. The construction algorithms for polynomial lattice rules include a component-by-component construction and a Korobov type construction. Using these construction algorithms we show that good polynomial lattice rules can be found by computer search, thus making them also valuable in practical applications where one has to approximate integrals of functions from such function spaces. We present numerical results comparing the performance of polynomial lattice rules with the performance of lattice rules.

The last part of this thesis deals with the expected value of the $L_2$ discrepancy of randomly digitally shifted digital nets over $\mathbb{Z}_2$. The $L_2$ discrepancy is a measure for the irregularity of distribution of a point set in the unit cube and is at the same time related to the worst-case error of integration for functions from certain Sobolev spaces. Here we use a generalized version of the digital shift, which we call a digital shift of depth $m$. We obtain upper bounds on the mean square $L_2$ discrepancy. Those upper bounds are stated in terms of the quality parameter $t$ of the digital net. Those bounds are especially interesting as it allows us, by using Niederreiter-Xing constructions of digital nets, to show that the lower bound on the $L_2$ discrepancy established by Roth in 1954
has not only the best possible dependence on the number of points, but also essentially the best possible dependence on the dimension.
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Chapter 1

Introduction

In many applications, for example deriving from financial mathematics or
statistics, one has to calculate an \( s \) dimensional integral over the unit cube.
Quite often the dimension \( s \) is rather large, possibly in the hundreds or thou-
sands. Furthermore the integral might not even be analytically tractable, so
that one has to rely on numerical methods. Frequently quasi-Monte Carlo
methods are used for such problems. Hereby the integral over the unit cube
is approximated by calculating the average of the values \( f(x_h) \) at some deter-
mministically chosen sample points \( x_1, \ldots, x_n \in [0,1)^s \).

There are two main ways to choose those point sets. The first one we discuss
here are lattice rules. Those rules originated independently by Hlawka [22] and
Korobov [23] and have been studied extensively in recent years by Sloan and
his collaborators (see for example [18, 60, 61, 64]). An integer vector \( z \), the
generating vector of the lattice rule, is used to generate the \( n \) points by \( \{hz/n\} \)
for \( h = 0, \ldots, n-1 \). The braces indicate that we take the fractional part of each
component. In a series of papers the worst-case error of multivariate integration
in Korobov and Sobolev spaces has been analyzed, see for example [26, 61, 64].
The Korobov space is a Hilbert space of periodic functions with absolutely
convergent Fourier series. The worst-case error here means the supremum of
the integration error over all functions in the unit ball of this function space.
Lattice rules have long been known to perform well for integrating periodic
functions. The theory which has been developed (see [60]) exploits properties
of lattice rules and Fourier series. This combination yielded many valuable
results. Construction methods for good generating vectors have been studied
based on the worst-case error of certain function classes and spaces. It was
found that lattice rules also yield good results for integration in non-periodic
function spaces, so-called Sobolev spaces (see [26, 61, 62]). In this case one uses
shifted lattice rules, that is, the point set is given by \( \{hz/n+\Delta\} \), with the shift
\( \Delta \in [0,1)^s \) (see [5]). The mean square worst-case error, that is, the average
of the worst-case error of a shifted point set over all shifts, for integration in
Sobolev spaces is similar to the worst-case error for integration in Korobov
spaces. This relationship was exploited to get many results for the integration
problem in Sobolev spaces. In randomized quasi-Monte Carlo algorithms the
shift is chosen randomly. This also allows us to obtain a probabilistic estimation of the error. On the other hand the bounds on the worst-case error are only in a probabilistic sense.

The second main class of quadrature points are \((t, m, s)\)-nets in base \(b\). First examples of \((t, m, s)\)-nets were given by Sobol [66] and later by Faure [14]. Niederreiter [40] has given a detailed introduction and investigation of the general concept, and in a series of papers he established several powerful construction methods (see [42] for a survey of this theory). Quality measures for these point sets are based on geometrical properties, especially the so-called star discrepancy has been studied (for the definition of the star discrepancy see for example [13, 25, 42]). A construction method introduced by Niederreiter [40], so-called digital \((t, m, s)\)-nets, is based on algebraical ideas. A survey of recent constructions based on algebraic geometry can be found in [45]. The worst-case error for integration is obtained via the so-called Köckma-Hlawka inequality, see for example [13, 25, 42]. This inequality states that the error of integration of a function \(f\), with bounded variation in the sense of Hardy and Krause, is bounded by the product of the variation and the star discrepancy. As in practice the function whose integral one wants to approximate is normally given by the task at hand, the analysis of these type of point sets focused mainly on the star discrepancy and related concepts.

But there are also other developments, where function classes were introduced and the worst-case error in these function classes using \((t, m, s)\)-nets was analyzed, notably function classes based on Walsh functions and Haar functions. Such functions appear for example in image and signal processing, see for example [39]. In [66, 67], Sobol gave first results for the numerical integration of Haar series. Further, Owen [49] (see also [50]) introduced a sophisticated randomization method for \((t, m, s)\)-nets and analyzed the mean square worst-case error of integration using Haar functions. This approach was further developed in many articles, see for example [15, 16, 21].

In our investigations here we use Walsh functions. As it turns out, Walsh functions can be used in a similar fashion for our analysis as Fourier functions are used for the analysis of lattice rules. The idea of using Walsh functions stems from Larcher [27], see also Larcher and Traunfellner [33] and the survey [32]. Larcher [28] referred to quasi-Monte Carlo rules using digital \((t, m, s)\)-nets as digital lattice rules. We are able to use some ideas used for the analysis of lattice rules for our approach here, as often those two lines of research are very similar. (It appears that the name ‘digital lattice rules’ is indeed very appropriate.) We introduce the Hilbert space \(H_{\text{wal},\gamma}\) (see section 2.2.2), which is based on Walsh functions. As formerly done for Korobov and Sobolev spaces, we introduce a sequence \(\gamma = (\gamma_1, \gamma_2, \ldots)\) of weights \(\gamma_j\), which are assumed to be positive and non-increasing. The idea of using weights stems from Sloan and Woźniakowski [63]. According to their argumentation, it may be useful to order the coordinates \(x_1, \ldots, x_s\) in such a way that \(x_1\) is the most important one, \(x_2\) the next, and so on; and to quantify this by associating non-increasing weights \(\gamma_1, \ldots, \gamma_s\) to the successive coordinate directions. Such a function
space is commonly called a weighted function space. We analyze the worst-case error for integration in the weighted Hilbert space $H_{\text{wal}, s, \gamma}$. Those readers familiar with lattice rules and the theory behind them, will discover many similarities between those ideas and our concepts here. In fact, both approaches are based on the fact that the weighted Korobov space as well as $H_{\text{wal}, s, \gamma}$ are reproducing kernel Hilbert spaces (for more information about reproducing kernels see [1]). Reproducing kernel Hilbert spaces were used in many papers studying numerical integration, and many results have been established based on the reproducing kernel (see for example [20, 63]). This approach allows us to develop an elegant theory of the integration problem, as it is already known for the integration problem using lattice rules for integration of functions from weighted Korobov spaces.

Our theory here and the theory for lattice rules meet when we analyze randomized quasi-Monte Carlo rules for the integration problem in weighted Sobolev spaces. Our randomization method uses a ‘digital shift’ in base $b \geq 2$. More precisely, let $\sigma = \sigma_1 b + \sigma_2 b^2 + \ldots$ be the base $b$ representation of $\sigma \in [0, 1)$, which is called the ‘digital shift’, and let a point $x \in [0, 1)$ be represented in base $b$ by $x = x_1 b + x_2 b^2 + \ldots$. Then the digitally shifted point $y$ is given by $y = y_1 b + y_2 b^2 + \ldots$, where $y_i \equiv x_i + \sigma_i \pmod{b}$. This randomization method is much simpler than the scrambling introduced by Owen [49], nevertheless, as shown here, it proves to be as effective in terms of the worst-case error for numerical integration in weighted Sobolev spaces. (Scrambled nets have been shown, though, to yield an improved rate of convergence if one assumes more smoothness of the integrand, see for example [19, 21].) Similarly to Hickernell in [18], we introduce a digital shift invariant kernel associated to the reproducing kernel. We are able to show that the mean square worst-case error using a digitally shifted point set in a reproducing kernel Hilbert space is the same as the worst-case error of the previous (unshifted) point set in a reproducing kernel Hilbert space with the kernel given by the associated digital shift invariant kernel. (For the shift used for lattice rules as explained above, this result was shown by Hickernell in [18].) This result is the starting point of our analysis of the mean square worst-case error of integration in the weighted Sobolev space. We calculate the digital shift invariant kernel of our weighted Sobolev space. Surprisingly enough, this digital shift invariant kernel is of a very simple form. On the other hand, the representation of the digital shift invariant kernel in terms of Walsh functions is very similar to the reproducing kernel of $H_{\text{wal}, s, \gamma}$. This allows us to obtain the results previously shown for $H_{\text{wal}, s, \gamma}$ also for the weighted Sobolev space.

The results for the integration problem in weighted Sobolev spaces using randomly digitally shifted digital $(t, m, s)$-nets are, apart from a minor difference in the constant, the same as those for randomly shifted lattice rules. Results for scrambled $(t, m, s)$-nets and sequences for the integration problem in the weighted Sobolev space (see [21]) are of a similar form. The big advantage here is the simplicity of our randomization method.

These results are presented in Chapter 2 and are from taken from [9], which is a joint work with F. Pillichshammer. My main contribution includes the gen-
eral idea, the definition of the weighted Hilbert space based on Walsh functions,
the simplification of the reproducing kernel, parts of the worst-case error anal-
ysis and the section on the weighted Sobolev space including the appendices.

In Chapter 3 we study construction algorithms for polynomial lattice rules. Polynomi-
also lattice rules are a special type of digital lattice rules in which the
underlying point set stems from a special family of digital nets. The construc-
tion method for this type of point set was formulated by Niederreiter and uses polynomi-
als over finite fields (see [42]). (Hence QMC rules using those point
sets are called polynomial lattice rules, see [34]). Until now useful polynomials
for polynomial lattice rules can only be found by computer search. In [59] such
a search was done based on the quality parameter $t$, that is, the aim was to find
polynomials which minimize the $t$-value. Though a small $t$-value yields good
distribution properties of the point set it lacks some flexibility in adjusting the
point set to integration problems in weighted function spaces.

Using weighted function spaces allows one to adjust point sets more accu-
rately to certain problems by choosing appropriate weights (see [11, 12, 71, 72])
and therefore also to improve the performance of the underlying QMC algo-
rithm. Thus also the construction of the underlying point set should be flexible
each to adjust them to the weights. Such construction methods for lattice
rules are known, namely the construction by Korobov [23, 24, 73] and by
Sloan and his collaborators [7, 26, 61, 62], and it has been shown that they
yield good results. The construction method is based on an explicit formula
for the worst-case error and the generating vector is found by minimizing the
worst-case error in some sense. Until now no equivalent method was known
for polynomial lattice rules.

Here we introduce an analogous construction for polynomial lattice rules
to the known constructions for lattice rules. This is now possible as we have
an explicit formula for the worst-case error for integration in weighted Hilbert
spaces based on Walsh functions using digital nets. Using these results we
also obtain results for the special case where one uses polynomial lattice rules.
Further we show that for given polynomials determining a polynomial lattice
rule, the formula for the worst-case error can be computed in $O(ns)$ operations,
where $n$ is the number of points and $s$ is the dimension. Therefore we are
equipped with the basic ingredients which are also used for the construction of
lattice rules for integration in weighted Korobov spaces. In analogy to lattice
rules, we establish a component-by-component construction and a Korobov
construction of polynomial lattice rules for integration in weighted Hilbert
spaces based on Walsh functions. We also prove upper bounds on the worst-
case error showing the usefulness of those construction algorithms.

Further we establish the connection between the worst-case error of integra-
tion in weighted Hilbert spaces based on Walsh functions and the mean square
worst-case error of integration in weighted Sobolev spaces also for polynomial
lattice rules. Hence we also obtain construction algorithms for polynomial
lattice rules for integration in weighted Sobolev spaces. Furthermore, upper
bounds on the mean square worst-case error for randomly digitally shifted
polynomial lattice rules constructed by our algorithms can be obtained from the previous work. These upper bounds are comparable to those on the mean square worst-case error for randomly shifted lattice rules for integration in weighted Sobolev spaces. Also the construction costs for the algorithms presented here is of the same order as for the construction algorithms for randomly shifted lattice rules for integration in weighted Sobolev spaces. Numerical results are present at the end of Chapter 3.

Chapter 3 is based on [8]. The idea for this paper evolved through discussions with F. Pillichshammer. My central contribution comprises parts of the worst-case error analysis, Subsection 3.3.1 and making the construction algorithms computationally feasible.

Sloan and Woźniakowski [63] showed that the worst-case error of integration in a certain weighted Sobolev space (which is very similar to the one considered here) is the same as the $L_2$ discrepancy. The $L_2$ discrepancy is a measure of the uniformity of distribution of a point set in the unit cube and is based on geometrical ideas. In the last part, Chapter 4, of this thesis we use the notion of discrepancy rather than worst-case error. We start with proving three results on the mean square $L_2$ discrepancy of randomized digital nets. The randomization method used is a generalized shift, in the sense that we now shift only the first $m$ digits of each point of each coordinate with the same shift, whereas the remaining digits are shifted independently for each point. We call such a shift a ‘digital shift of depth $m$’. The first result is a formula for the mean square weighted $L_2$ discrepancy of randomized digital nets. The formula is exact and is a function of the generating matrices of the digital net. We use this result then to derive the exact formula of the mean square weighted $L_2$ discrepancy of randomized digital (0, $m$, $s$)-nets in dimension 2 and 3, which is in this case independent of the generating matrices. The convergence order is best possible and we compare the constant of the leading term with the lower bound by Roth [56]. The third result is an upper bound on the mean square weighted $L_2$ discrepancy of randomized digital nets in dimension $s > 3$. The difference between $s > 3$ and $s = 2, 3$ is that $s > 3$ implies that $t > 0$. Therefore the exact value of the mean square weighted $L_2$ discrepancy depends on the generating matrices. Still, we can obtain an upper bound for this case which is independent of the generating matrices and only depends on the weights, the $t$-value, the number of points and the dimension $s$. This bound is of a simple form and easily computable. Again, the convergence order is best possible.

Further, we deal also with the classical $L_2$ discrepancy, meaning that the weights corresponding to lower dimensional projections are assumed to be zero and the remaining weight is set to be one. In 1954 Roth [56] proved a lower bound on the classical $L_2$ discrepancy for arbitrary point sets in $[0, 1]^s$. He showed that the classical $L_2$ discrepancy of any point set in $[0, 1]^s$ consisting of $N$ elements is at least $c_1(s)(\log N)^{(s-1)/2}N^{-1}$, with a constant $c_1(s) = 2^{-2s-4}((s - 1)!)^{-1/2}$. In 1980 Roth [57] also showed that there exists a point set in $[0, 1]^s$ consisting of $N$ elements with an $L_2$ discrepancy of at most $c_2(s)(\log N)^{(s-1)/2}N^{-1}$, with a constant $c_2(s)$ depending only on the di-
mension $s$. (See also [2] for variations of Roth’s result. For dimension $s = 2$ this was proven by Davenport [6] already in 1956. Quite recently, Chen and Skriganov [3] gave concrete examples - not only existence results as Roth did - of point sets in arbitrary dimensions which achieve the minimal order of the $L_2$ discrepancy.) Therefore the exact dependence on $N$ is known. Here we are interested in the constant $c_1(s)$ of the lower bound of Roth. In a first result we extract the constant of the leading term from the previous calculations in Section 4.3. By a construction of Niederreiter-Xing [46] we know that for any $m$ and $s$ there always exists a digital $(5s, m, s)$-net over $\mathbb{Z}_2$. With an appropriate shift we obtain that such point sets achieve an $L_2$ discrepancy of at most

$$\frac{(\log N)^{(s-1)/2}}{N} \frac{22^s}{(\log 2)^{(s-1)/2}((s - 1)!)^{1/2}} + \mathcal{O}\left(\frac{(\log N)^{(s-2)/2}}{N}\right).$$

The constant $22^s(\log 2)^{-(s-1)/2}((s - 1)!)^{-1/2}$ improves a result by Hickernell [16] considerably and seems to be the best until now known constant of this kind.

Secondly we prove an upper bound on the classical $L_2$ discrepancy of shifted Niederreiter-Xing nets (see [47]). We consider a sequence of shifted digital nets, where the number of points $N$ is relatively small compared to the dimension. For this sequence of shifted digital nets we show an upper bound which even improves upon the upper bound previously obtained for digital sequences for certain choices of $m$ and $s$.

Chapter 4 stems from [10]. The idea for this paper emerged through discussions with F. Pillichshammer during which also the main parts of the analysis were carried out. The section on asymptotics was to a large extent my contribution.

A discussion of the results obtained in this thesis is given in Chapter 5. We discuss the underlying principles of the analysis of lattice rules and polynomial lattice rules and at the same time point out their differences. This may give raise to future developments of QMC methods.
Chapter 2

Multivariate integration in weighted Hilbert spaces based on Walsh functions and weighted Sobolev spaces

2.1 Introduction

We want to approximate an $s$-dimensional integral

$$I_s(f) := \int_{(0,1)^s} f(x) \, dx$$

of functions from a normed linear space $H_s$ by a quasi-Monte Carlo rule

$$Q_{n,s}(f) := \frac{1}{n} \sum_{h=1}^{n} f(x_h),$$

with some deterministically chosen sample points $x_1, \ldots, x_n$. We assume that function evaluation is well defined for functions in $H_s$.

We analyze the worst case error $e_{n,s}$ of integration of functions from $H_s$ using a QMC algorithm $Q_{n,s}$, that is,

$$e_{n,s}(Q_{n,s}) := \sup_{f \in H_s \atop \|f\| \leq 1} |I_s(f) - Q_{n,s}(f)|,$$

where $\| \cdot \|$ denotes the norm in $H_s$. For comparison we also define the initial error

$$e_{0,s}(Q_{0,s}) := \sup_{f \in H_s \atop \|f\| \leq 1} |I_s(f)|.$$

The aim is to reduce the initial error by a factor of $\varepsilon$, where $\varepsilon \in (0,1)$. Let

$$n_{\min}(\varepsilon, s) = \min\{n : \exists Q_{n,s} \text{ such that } e_{n,s}(Q_{n,s}) \leq \varepsilon e_{0,s}(Q_{0,s})\}. \quad (2.1.1)$$
Definition 2.1.1  1. We say that multivariate integration in the space $H_s$ is QMC-tractable if there exist nonnegative $c, p$ and $q$ such that

$$n_{\min}(\varepsilon, s) \leq c s^q \varepsilon^{-p}$$

holds for all dimensions $s = 1, 2, \ldots$ and for all $\varepsilon \in (0, 1)$. The numbers $p$ and $q$ are called $\varepsilon$- and $s$-exponents of QMC-tractability; we stress that they are not defined uniquely.

2. We say that multivariate integration in the space $H_s$ is strongly QMC-tractable if the inequality above holds with $q = 0$. The infimum of $p$ is called the $\varepsilon$-exponent of strong QMC-tractability.

We note that strong tractability holds iff there is a QMC algorithm $Q_{n,s}$ such that $c_{n,s}(Q_{n,s})$ decays polynomially in $n$ and is bounded above independently of the dimension, and tractability holds iff this fraction satisfies an upper bound which decays polynomially in $n$ and increases at most polynomially in the dimension.

This is the setting in which we analyze the performance of QMC algorithms with the worst-case error as the figure of merit. Different function spaces can yield different worst-case errors, and therefore there are various measures for the performance of QMC algorithms. Throughout the literature various function spaces have been considered, and it is an interesting task to investigate the performance of QMC methods for new function spaces and/or new norms. An example is given in this chapter, where we introduce a tensor product reproducing kernel Hilbert space which is based on Walsh functions. Before we introduce this space we give some background on reproducing kernel Hilbert spaces.

2.1.1 Reproducing kernel Hilbert spaces

A reproducing kernel Hilbert space $H_s$ of functions on $[0, 1]^s$ is a Hilbert space in which point evaluation

$$T_y(f) = f(y) \quad \text{for all } y \in [0, 1]^s$$

is a bounded linear functional on $H_s$. Let $\langle \cdot, \cdot \rangle_s$ denote the inner product and $\| \cdot \|_s$ denote the norm in $H_s$. By the Riesz representation theorem there exists a unique function $K_s(\cdot, y) \in H_s$ such that for all $y \in [0, 1]^s$ we have

$$T_y(f) = \langle f, K_s(\cdot, y) \rangle_s \quad \text{for all } f \in H_s. \quad (2.1.2)$$

The function $K_s$, the representer of $T_y$, is called the reproducing kernel of $H_s$. For any other bounded linear functional $T$ on $H_s$ the representer $\tilde{T}$ satisfying $T(f) = \langle f, \tilde{T} \rangle_s$ is given by

$$\tilde{T}(y) = \langle \tilde{T}, K_s(\cdot, y) \rangle_s = T(K_s(\cdot, y)). \quad (2.1.3)$$
Any real valued reproducing kernel $K_s$ has the symmetry property

$$K_s(x, y) = K_s(y, x) \quad \text{for all } x, y \in [0, 1)^s.$$ 

This follows from (2.1.2) and the symmetry of inner products, more precisely,

$$K_s(x, y) = (K_s(\cdot, y), K_s(\cdot, x))_s = (K_s(\cdot, x), K_s(\cdot, y))_s = K_s(y, x)$$

for all $x, y \in [0, 1)^s$. (Note that for complex kernels we have $K_s(x, y) = K_s(y, x)$ for all $x, y \in [0, 1)^s$, where for a complex number $a$ we denote the complex conjugate by $\overline{a}$.) More details about reproducing kernels can be found in [1].

### 2.1.2 Tensor product spaces

The function spaces considered in this thesis are tensor product spaces. Subsequently we introduce tensor product spaces. We follow [74] in our approach.

A tensor product space $H_s = H_{1,1} \otimes \cdots \otimes H_{1,s}$ of $s$ one-dimensional Hilbert spaces $H_{1,1}, \ldots, H_{1,s}$ is the completion of linear combinations of tensor products $f_1 \otimes \cdots \otimes f_s$, with $f_j \in H_{1,j}$ for each $1 \leq j \leq s$. That is, $H_s$ consists of functions of the form

$$f(x) = \sum_{h_1=1}^{\infty} \cdots \sum_{h_s=1}^{\infty} \left( c_{h_1,\ldots,h_s} \prod_{j=1}^{s} f_{j,h_j}(x_j) \right),$$

where $c_{h_1,\ldots,h_s}$ are real coefficients such that

$$\sum_{h_1=1}^{\infty} \cdots \sum_{h_s=1}^{\infty} c_{h_1,\ldots,h_s}^2 < \infty,$$

and for each $1 \leq j \leq s$, $\{f_{j,h}\}_{h=1}^{\infty}$ forms an orthonormal basis for $H_{1,j}$.

If each $H_{1,j}$ is a reproducing kernel Hilbert space, then the tensor product space $H_s$ is again a reproducing kernel Hilbert space, provided that the inner product is suitably defined (see [48]). We summarize this result in the following lemma.

**Lemma 2.1.2** Let $H_s = H_{1,1} \otimes \cdots \otimes H_{1,s}$, where for each $1 \leq j \leq s$ the reproducing kernel Hilbert space $H_{1,j}$ has reproducing kernel $K_{1,j}$ and inner product $\langle \cdot, \cdot \rangle_{1,j}$. Suppose the inner product of $H_s$ is defined for $f(x) = \prod_{j=1}^{s} f_j(x_j)$ and $g(x) = \prod_{j=1}^{s} g_j(x_j)$ with $f_j, g_j \in H_{1,j}$ as

$$\langle f, g \rangle_s := \prod_{j=1}^{s} \langle f_j, g_j \rangle_{1,j}.$$ 

Then the reproducing kernel for $H_s$ is

$$K_s(x, y) = \prod_{j=1}^{s} K_{1,j}(x_j, y_j).$$
Proof. For any \( f \in H_s \) with
\[
f(x) = \sum_{h_1=1}^{\infty} \cdots \sum_{h_s=1}^{\infty} \left( c_{h_1,\ldots,h_s} \prod_{j=1}^{s} f_{j,h_j}(x_j) \right),
\]
by the linearity of the inner product and (2.1.2), we have
\[
\langle f, K_s(\cdot, y) \rangle_s = \sum_{h_1=1}^{\infty} \cdots \sum_{h_s=1}^{\infty} \left( c_{h_1,\ldots,h_s} \prod_{j=1}^{s} \langle f_{j,h_j}, K_{1,j}(\cdot, y_j) \rangle_{1,j} \right)
\]
\[
= \sum_{h_1=1}^{\infty} \cdots \sum_{h_s=1}^{\infty} \left( c_{h_1,\ldots,h_s} \prod_{j=1}^{s} f_{j,h_j}(y_j) \right)
\]
\[
= f(y).
\]
Thus \( K_s \) is the reproducing kernel of \( H_s \) as claimed. \( \Box \)

2.1.3 Worst-case error in reproducing kernel Hilbert spaces

Let \( f \) be any function in the Hilbert space \( H_s \) with real valued reproducing kernel \( K_s(x, y) \). We approximate the \( s \)-dimensional integral \( I_s(f) \) with an \( n \) point quasi-Monte Carlo rule \( Q_{n,s} \). Clearly, \( Q_{n,s} \) is a bounded linear functional on \( H_s \) (since point evaluation is a bounded linear functional on \( H_s \)). We will assume that \( I_s(f) \) is also a bounded linear functional on \( H_s \). It follows from (2.1.3) that the representer of \( I_s \) is given by
\[
h_s(y) = I_s(K_s(\cdot, y)) = \int_{[0,1]^s} K_s(x, y) \, dx
\]
and the representer of \( Q_{n,s} \) is given by
\[
\varsigma_{n,s}(y) = Q_{n,s}(K_s(\cdot, y)) = \frac{1}{n} \sum_{h=1}^{n} K_s(x_h, y),
\]
where \( x_1, \ldots, x_n \) are the quadrature points. The representer of the quadrature error \( I_s(f) - Q_{n,s}(f) \) is then given by
\[
\xi(y) = h_s(y) - \varsigma_{n,s}(y),
\]
and hence
\[
I_s(f) - Q_{n,s}(f) = \langle f, \xi_{n,s} \rangle_s. \tag{2.1.4}
\]
By the Cauchy-Schwarz inequality we obtain
\[
|I_s(f) - Q_{n,s}(f)| = |\langle f, \xi_{n,s} \rangle_s| \leq \|f\|_s \|\xi_{n,s}\|_s,
\]
with equality being achieved when \( f \) is a multiple of \( \xi_{n,s} \). Here \( \|f\|_s := \langle f, f \rangle_s^{1/2} \).
Let now \( e_{n,s}(P_{n,s}, K_s) \) denote the worst-case error of integration in a reproducing kernel Hilbert space with kernel \( K_s \) using a QMC rule which employs the point set \( P_{n,s} = \{x_1, \ldots, x_n\} \). We conclude from the definition of the worst-case error that
\[
e^2_{n,s}(P_{n,s}, K_s) = \langle \xi_{n,s}, \xi_{n,s} \rangle_s = \langle h_s - s_{n,s}, h_s - s_{n,s} \rangle_s
= \langle h_s, h_s \rangle_s - 2\langle s_{n,s}, h_s \rangle_s + \langle s_{n,s}, s_{n,s} \rangle_s.
\]
As \( h_s \) is the representer of \( I_s \) it follows that
\[
\langle h_s, h_s \rangle_s = \int_{[0,1]^s} K_s(x, y) \, dx \, dy.
\]
Further we have
\[
\langle s_{n,s}, h_s \rangle_s = \frac{1}{n} \sum_{h=1}^{n} \int_{[0,1]^s} K_s(x_h, x) \, dx
\]
and
\[
\langle s_{n,s}, s_{n,s} \rangle_s = \frac{1}{n^2} \sum_{h,i=1}^{n} K_s(x_h, x_i)
\]
and therefore
\[
e^2_{n,s}(P_{n,s}, K_s)
= \int_{[0,1]^{2s}} K_s(x, y) \, dx \, dy - \frac{2}{n} \sum_{h=1}^{n} \int_{[0,1]^s} K_s(x_h, x) \, dx + \frac{1}{n^2} \sum_{h,i=1}^{n} K_s(x_h, x_i).
\]
The initial error is given by
\[
e^2_{0,s} = \langle h_s, h_s \rangle_s = \int_{[0,1]^{2s}} K_s(x, y) \, dx \, dy.
\]

2.2 Walsh functions and the Hilbert space \( H_{\text{wal},s,\gamma} \)

In this section we first recall the definition of Walsh functions and we state some of their basic properties used throughout the thesis. Subsequently we introduce the weighted Hilbert space \( H_{\text{wal},s,\gamma} \) and show that this is a reproducing kernel Hilbert space.

2.2.1 Walsh functions

In the following we define Walsh functions in base \( b \). These functions are piecewise constant, as can be seen from the definition below. For more information on Walsh functions see for example [4, 53, 55, 68]. Let \( \mathbb{N}_0 \) denote the set of non-negative integers.
Definition 2.2.1  Let $b \geq 2$ be an integer. For a non-negative integer $k$ with base $b$ representation

$$k = \kappa_{a-1}b^{a-1} + \ldots + \kappa_1b + \kappa_0,$$

with $\kappa_i \in \{0, \ldots, b-1\}$, we define the Walsh function $b_{\text{wal}}_k : [0, 1) \to \mathbb{C}$ by

$$b_{\text{wal}}_k(x) := e^{2\pi i (x_1\kappa_0 + \ldots + x_{a-1}\kappa_{a-1})/b},$$

for $x \in [0, 1)$ with base $b$ representation $x = \sum \frac{x_i}{b^i} + \ldots$ (unique in the sense that infinitely many of the $x_i$ must be different from $b-1$). If it is clear which base $b$ is chosen we simply write $\text{wal}_k$.

Definition 2.2.2  For dimension $s \geq 2$, $x_1, \ldots, x_s \in [0, 1)$ and $k_1, \ldots, k_s \in \mathbb{N}_0$ we define $b_{\text{wal}}_{k_1, \ldots, k_s} : [0, 1)^s \to \mathbb{C}$ by

$$b_{\text{wal}}_{k_1, \ldots, k_s}(x_1, \ldots, x_s) := \prod_{j=1}^s b_{\text{wal}}_{k_j}(x_j).$$

For vectors $k = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ and $x = (x_1, \ldots, x_s) \in [0, 1)^s$ we write

$$b_{\text{wal}}_k(x) := b_{\text{wal}}_{k_1, \ldots, k_s}(x_1, \ldots, x_s).$$

Again, if it is clear which base we mean we simply write $\text{wal}_k(x)$.

We introduce some notation. By $\oplus$ we denote the digit-wise addition modulo $b$, i.e., for $x = \sum_{i=w}^{\infty} \frac{z_i}{b^i}$ and $y = \sum_{i=w}^{\infty} \frac{y_i}{b^i}$ we have

$$x \oplus y := \sum_{i=w}^{\infty} \frac{z_i + y_i \pmod{b}}{b^i},$$

where $z_i := x_i + y_i \pmod{b}$,

and by $\ominus$ we denote the digit-wise subtraction modulo $b$, i.e.,

$$x \ominus y := \sum_{i=w}^{\infty} \frac{z_i - y_i \pmod{b}}{b^i},$$

where $z_i := x_i - y_i \pmod{b}$.

In the following proposition we summarize some basic properties of Walsh functions, see also [4, 39, 53].

Proposition 2.2.3  Let $b \geq 2$ be an integer.

1. For all $k, l \in \mathbb{N}_0$ and all $x, y \in [0, 1)$ we have

$$\text{wal}_k(x) \cdot \text{wal}_l(x) = \text{wal}_{k \oplus l}(x), \quad \text{wal}_k(x) \cdot \text{wal}_k(y) = \text{wal}_k(x \ominus y)$$

and

$$\text{wal}_k(x) \cdot \overline{\text{wal}_l(x)} = \text{wal}_{k \ominus l}(x), \quad \text{wal}_k(x) \cdot \overline{\text{wal}_k(y)} = \text{wal}_k(x \ominus y).$$
2. We have
\[ \int_0^1 \text{wal}_0(x) \, dx = 1 \quad \text{and} \quad \int_0^1 \text{wal}_k(x) \, dx = 0 \quad \text{if} \ k > 0. \]

3. For all \( k, l \in \mathbb{N}_0^s \) we have the following orthogonality properties:
\[ \int_{[0,1]^s} \text{wal}_k(x) \text{wal}_l(x) \, dx = \begin{cases} 1, & \text{if} \ k = l, \\ 0, & \text{otherwise}. \end{cases} \]

4. For any \( f \in L_2([0,1]^s) \) and any \( \sigma \in [0,1]^s \) we have
\[ \int_{[0,1]^s} f(x) \, dx = \int_{[0,1]^s} f(x \oplus \sigma) \, dx. \]

5. For any integer \( s \geq 1 \) the system \( \{ \text{wal}_{k_1,\ldots,k_s} : k_1, \ldots, k_s \geq 0 \} \) is a complete orthonormal system in \( L_2([0,1]^s) \).

Proof. The first part can be shown by direct calculation. We have, for example,
\[ \text{wal}_k(x) \cdot \text{wal}_k(y) = e^{2\pi i(x_1\kappa_0 + \ldots + x_a\kappa_{a-1})/b} e^{2\pi i(y_1\kappa_0 + \ldots + y_a\kappa_{a-1})/b} = e^{2\pi i((x_1+y_1)\kappa_0 + \ldots + (x_a+y_a)\kappa_{a-1})/b} = \text{wal}_k(x \oplus y). \]

The remaining equalities can be shown in an analogous fashion.

For the second part note that \( \int_0^1 \text{wal}_0(x) \, dx = 1 \) and therefore \( \int_0^1 \text{wal}_0(x) \, dx = 1. \) Further, for \( k = \kappa_{a-1}b^{a-1} + \cdots + \kappa_1 b + \kappa_0 \) with \( \kappa_{a-1} \neq 0, \) we have
\[ \int_0^1 \text{wal}_k(x) \, dx = \sum_{x_1=0}^{b-1} \cdots \sum_{x_a=0}^{b-1} e^{2\pi i(x_1\kappa_0 + \ldots + x_a\kappa_{a-1})/b} \int_{\frac{1}{b} + \cdots + \frac{1}{b}} 1 \, dx \]
\[ = \frac{1}{b^a} \sum_{x_1=0}^{b-1} e^{2\pi i x_1 \kappa_0 / b} \cdots \sum_{x_a=0}^{b-1} e^{2\pi i x_a \kappa_{a-1} / b} = 0, \]
as \( \sum_{x_a=0}^{b-1} e^{2\pi i x_a \kappa_{a-1} / b} = 0. \) Hence the second part is shown.

We have \( \text{wal}_k(x) \text{wal}_k(x) = 1 \) for all \( x \in [0,1]^s \) and therefore
\[ \int_{[0,1]^s} \text{wal}_k(x) \text{wal}_k(x) \, dx = 1. \]

For \( k \neq l \) there is an index \( 1 \leq j \leq s \) such that \( k_j \neq l_j. \) Hence
\[ \int_0^1 \text{wal}_{k_j}(x_j) \text{wal}_{l_j}(x_j) \, dx_j = \int_0^1 \text{wal}_{k \oplus l}(x_j) \, dx_j = 0, \]
which follows from the second part as \( k_j \oplus l_j \neq 0 \). Thus also the third part follows.

In order to prove item 4 we show that the Lebesgue measure \( \lambda \) has the following property: let \( x \in [0,1) \) and \( M \subseteq [0,1) \), \( M \) an Lebesgue measurable set, then

\[
\lambda(M) = \lambda(M \oplus x),
\]

(2.2.7)

where \( M \oplus x = \{ y \oplus x : y \in M \} \). Note that the number of \( y \in M \) for which \( y \oplus x \) is not defined is countable (\( y \oplus x \) is not defined if \( y \oplus x = \frac{b+1}{b} + \cdots + \frac{b+1}{b} + \frac{b+1}{b+2} + \cdots \)).

Now it is easy to see that the assertion holds for elementary intervals, that is, \( M \) is of the form \([kb^{-m}, lb^{-m})\), with \( 0 \leq k < l \leq b^m \). As every open interval can be written as a countable union of such intervals, it follows that (2.2.7) also holds for all open intervals and hence (2.2.7) also holds for all Lebesgue measurable sets.

For the one-dimensional case let now \( g(x) := f(x \oplus \sigma) \). Then we have for all \( M \subseteq f([0,1)) := \{ f(x) : x \in [0,1) \} \) that \( g^{-1}(M) = f^{-1}(M) \oplus \sigma \), where \( g^{-1}(M) := \{ x \in [0,1) : g(x) \in M \} \). Hence it follows that \( \lambda(g^{-1}(M)) = \lambda(f^{-1}(M)) \) and by the definition of the Lebesgue integral the fourth part follows for \( s = 1 \). For \( s > 1 \) we use the argument for each variable separately, that is,

\[
\int_{(0,1)^s} f(x \oplus \sigma) \, dx = \int_{0}^{1} \cdots \int_{0}^{1} f(x_1 \oplus \sigma_1, \ldots, x_s \oplus \sigma_s) \, dx_1 \cdots dx_s
\]

\[
= \int_{0}^{1} \cdots \int_{0}^{1} f(x_1, \ldots, x_s) \, dx_1 \cdots dx_s
\]

\[
= \int_{(0,1)^s} f(x) \, dx
\]

and the fourth part follows.

For the last part we observe that the orthogonality of the Walsh functions was already shown in the third part. In order to show the remaining part we first introduce the Dirichlet kernel \( D_k \), which is defined as

\[
D_k(x) := \sum_{u_1=0}^{k_1-1} \cdots \sum_{u_s=0}^{k_s-1} \text{wal}_u(x),
\]

where \( k = (k_1, \ldots, k_s) \) and \( u = (u_1, \ldots, u_s) \). For \( k = 1 = (1, \ldots, 1) \) we obtain

\[
D_1(x) = \sum_{u_1=0}^{0} \cdots \sum_{u_s=0}^{0} \text{wal}_u(x) = \text{wal}_0(x) = 1 \quad \text{for all } x \in [0,1)^s.
\]

Further we have

\[
\sum_{u_j=0}^{b^m-1} \text{wal}_{u_j}(x_j) = \sum_{u_{j,0}=0}^{b-1} e^{2\pi i u_{j,0} x_{j,1}/b} \cdots \sum_{u_{j,m-1}=0}^{b-1} e^{2\pi i u_{j,m-1} x_{j,m}/b}.
\]
As $\sum_{u_{j,i}=0}^{b-1} e^{2\pi i u_{j,i} x_{j,i+1}/b} = b$ if $x_{j,i+1} = 0$ and 0 otherwise we obtain

$$\sum_{u_{j}=0}^{b-1} \text{wal}_{u_{j}}(x_{j}) = b^{m} 1_{[0,b^{-m})}(x_{j}),$$

where $1_{[0,b^{-m})}(x_{j}) = 1$ if $x_{j} \in [0,b^{-m})$ and 0 otherwise. Hence, for $k = b^{m} := (b^{m}, \ldots, b^{m})$, we obtain

$$D_{b^{m}}(x) = \sum_{u_{1}=0}^{b^{m}-1} \text{wal}_{u_{1}}(x_{1}) \cdots \sum_{u_{s}=0}^{b^{m}-1} \text{wal}_{u_{s}}(x_{s}) = b^{ms} 1_{[0,b^{-m})^{s}}(x).$$

For $f \in L_{2}([0,1)^{s})$ we define

$$S_{k}(x, f) := \sum_{u_{1}=0}^{k_{1}-1} \cdots \sum_{u_{s}=0}^{k_{s}-1} \hat{f}_{u} \text{wal}_{u}(x),$$

where

$$\hat{f}_{u} := \int_{[0,1)^{s}} f(y) \text{wal}_{u}(y) \, dy.$$

We have

$$S_{k}(x, f) = \sum_{u_{1}=0}^{k_{1}-1} \cdots \sum_{u_{s}=0}^{k_{s}-1} \int_{[0,1)^{s}} f(y) \text{wal}_{u}(y \oplus x) \, dy$$

$$= \int_{[0,1)^{s}} f(y) \sum_{u_{1}=0}^{k_{1}-1} \cdots \sum_{u_{s}=0}^{k_{s}-1} \text{wal}_{u}(y \oplus x) \, dy$$

$$= \int_{[0,1)^{s}} f(y) \overline{D_{k}(y \oplus x)} \, dy$$

$$= \int_{[0,1)^{s}} f(y \oplus x) \overline{D_{k}(y)} \, dy.$$

Hence we obtain

$$|S_{b^{m}}(x, f) - f(x)| = \left| \int_{[0,1)^{s}} f(y \oplus x) \overline{D_{k}(y)} \, dy - b^{ms} \int_{[0,b^{-m})^{s}} f(x) \, dy \right|$$

$$= b^{ms} \left| \int_{[0,b^{-m})^{s}} f(y \oplus x) \, dy - \int_{[0,b^{-m})^{s}} f(x) \, dy \right|$$

$$= b^{ms} \left| \int_{x \oplus [0,b^{-m})^{s}} (f(y) - f(x)) \, dy \right|.$$
Let now $f \in C([0, 1]^s)$, then for all $\varepsilon > 0$ there is an $m_0$ such that for all $m > m_0$ and all $x, y \in [0, 1]^s$ such that $\|x - y\|_\infty < b^{-ms}$ it follows that $|f(y) - f(x)| < \varepsilon$. In this case we obtain

$$\|S_b^m(x, f) - f(x)\| < b^{ms}\lambda(x \oplus [0, b^{-m})^s) \sup_{x \oplus [0, b^{-m})^s} |f(y) - f(x)| < \varepsilon$$

for all $m > m_0$, as $x \oplus [0, b^{-m})^s \subseteq \prod_{j=1}^s \left[\frac{x_j}{b} + \cdots + \frac{x_j}{b^m} + \frac{1}{b^m}\right]$, where $x = (x_1, \ldots, x_s)$ and $x_j = \frac{x_j}{b} + \frac{x_j}{b^2} + \cdots$. As the bound above holds for all $x \in [0, 1)^s$ it follows that

$$\|S_b^m(\cdot, f) - f(\cdot)\|_\infty < \varepsilon \text{ for all } m > m_0.$$

Hence the set of all Walsh-polynomials is dense in $(C([0, 1]^s), \| \cdot \|_\infty)$, which is in turn dense in $(L_2([0, 1]^s), \| \cdot \|_2)$ and therefore the Walsh-polynomials are dense in $(L_2([0, 1]^s), \| \cdot \|_2)$. The result follows.

2.2.2 The Hilbert space $H_{\text{wal}, s, \gamma}$

In the following we define the weighted Hilbert space $H_{\text{wal}, s, \gamma}$ in base $b \geq 2$. This space is based on Walsh functions. First we consider the one-dimensional case. The $s$-dimensional space will then be defined as the tensor product of those one-dimensional spaces (see also [64]).

For a natural number $k = \kappa_a b^a + \cdots + \kappa_1 b + \kappa_0$, with $\kappa_a \neq 0$, let $\psi_b(k) = a$. For $\alpha > 1$ we define

$$r_b(\alpha, \gamma, k) = \begin{cases} 1, & \text{if } k = 0, \\ \gamma b^{-\alpha \psi_b(k)}, & \text{if } k \neq 0. \end{cases}$$

We define the inner product of two functions $f$ and $g$ as

$$\langle f, g \rangle_{\text{wal}, h, \gamma} := \sum_{k \in \mathbb{N}_0} r_b(\alpha, \gamma, k)^{-1} \hat{f}_{\text{wal}}(k) \hat{g}_{\text{wal}}(k),$$

where

$$\hat{f}_{\text{wal}}(k) = \int_0^1 f(x) \overline{\psi_{b,k}(x)} \, dx.$$ 

The norm is given by $\|f\|_{\text{wal}, \gamma} := \langle f, f \rangle_{\text{wal}, \gamma}^{1/2}$. Note that any function $f \in L_2([0, 1])$ can be written as

$$f(x) = \sum_{k=0}^{\infty} \hat{f}_{\text{wal}}(k) \psi_{b,k}(x).$$

The weighted Hilbert space $H_{\text{wal}, \gamma}$ is now given by all functions with finite norm, that is,

$$H_{\text{wal}, \gamma} := \{ f : \| f \|_{\text{wal}, \gamma} < \infty \}.$$
In the following we consider a fixed base \( b \geq 2 \) and therefore we write shortly \( \text{wal} \) and \( r \) instead of \( \text{wal}_b \) and \( r_b \) if appropriate. We now proceed by showing that the function \( K_{\text{wal},\gamma} \) defined by

\[
K_{\text{wal},\gamma}(x, y) := \sum_{k=0}^{\infty} r(\alpha, \gamma, k) \text{wal}_k(x) \overline{\text{wal}_k(y)}
\]

is the reproducing kernel of \( H_{\text{wal},\gamma} \).

First we show that \( K_{\text{wal},\gamma} \) is a real function: as \( r(\alpha, \gamma, k) = r(\alpha, \gamma, \ominus k) \), where \( \ominus k = 0 \ominus k \), we have

\[
K_{\text{wal},\gamma}(x, y) = \sum_{k=0}^{\infty} r(\alpha, \gamma, k) \text{wal}_k(x) \overline{\text{wal}_k(y)} = \sum_{k=0}^{\infty} r(\alpha, \gamma, \ominus k) \text{wal}_{\ominus k}(x) \overline{\text{wal}_{\ominus k}(y)}
\]

Further it follows from the above equation that \( K_{\text{wal},\gamma}(x, y) = \overline{K_{\text{wal},\gamma}(x, y)} = K_{\text{wal},\gamma}(y, x) \). For \( \alpha > 1 \) we define

\[
\mu_b(\alpha) := \sum_{k=1}^{\infty} b^{-\alpha \psi_b(k)}
\]

and hence we have

\[
\mu_b(\alpha) = \sum_{a=0}^{\infty} b^{-\alpha a}(b - 1)b^a = \frac{b^\alpha(b - 1)}{b^\alpha - b}.
\]

Again we will write \( \mu \) instead of \( \mu_b \) if it is clear which base \( b \) is meant. Observe that

\[
\sum_{k=0}^{\infty} r(\alpha, \gamma, k) = 1 + \gamma \mu(\alpha).
\]

We have \( K_{\text{wal},\gamma}(\cdot, y) \in H_{\text{wal},\gamma} \) as

\[
\|K_{\text{wal},\gamma}(\cdot, y)\|^2_{\text{wal},\gamma} = \sum_{k=0}^{\infty} r(\alpha, \gamma, k) = 1 + \gamma \mu(\alpha) < \infty.
\]

Further we have

\[
\langle f, K_{\text{wal},\gamma}(\cdot, y) \rangle_{\text{wal},\gamma} = \hat{f}_{\text{wal}}(0) + \sum_{k=1}^{\infty} \hat{f}_{\text{wal}}(k) \text{wal}_k(y) = f(y).
\]

Therefore \( K_{\text{wal},\gamma} \) is the reproducing kernel of the space \( H_{\text{wal},\gamma} \).

A very useful property of this kernel is that it can be simplified further. We have

\[
\sum_{k=1}^{\infty} b^{-\alpha \psi_b(k)} \text{wal}_k(x) \overline{\text{wal}_k(y)} = \sum_{a=0}^{\infty} b^{-\alpha a} \sum_{k=b^a}^{b^{a+1}-1} \text{wal}_k(x \ominus y)
\]

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and
\[ \sum_{k=b}^{b^{\alpha + 1} - 1} \text{wal}_k (x \ominus y) \]
\[ = \sum_{k=b}^{b^{\alpha + 1} - 1} e^{2\pi i ((x_1 - y_1)\kappa_0 + (x_{\alpha + 1} - y_{\alpha + 1})\kappa_a)/b} \]
\[ = \sum_{\kappa_a=1}^{b-1} e^{2\pi i (x_{\alpha + 1} - y_{\alpha + 1})\kappa_a/b} \sum_{\kappa_{a-1}=0}^{b-1} e^{2\pi i (x_a - y_a)\kappa_{a-1}/b} \ldots \sum_{\kappa_0=0}^{b-1} e^{2\pi i (x_1 - y_1)\kappa_0/b}. \]

As \( \sum_{\kappa=0}^{b-1} e^{2\pi i (x_i - y_i)\kappa/b} \) is 0 if \( x_i \neq y_i \) and \( b \) if \( x_i = y_i \), we have
\[ \sum_{k=b}^{b^{\alpha + 1} - 1} \text{wal}_k (x \ominus y) = 0, \]
if there is an \( i \in \{1, \ldots, a\} \) such that \( x_i \neq y_i \). Now let us assume \( x_i = y_i \) for \( i \in \{1, \ldots, a\} \). If now \( x_{\alpha + 1} = y_{\alpha + 1} \) then
\[ \sum_{k=b}^{b^{\alpha + 1} - 1} \text{wal}_k (x \ominus y) = (b - 1)b^\alpha, \]
and if \( x_{\alpha + 1} \neq y_{\alpha + 1} \) we have
\[ \sum_{k=b}^{b^{\alpha + 1} - 1} \text{wal}_k (x \ominus y) = b^\alpha \sum_{\kappa_a=1}^{b-1} e^{2\pi i (x_{\alpha + 1} - y_{\alpha + 1})\kappa_a/b} = -b^\alpha. \]

Let now
\[ D_a (x, y) := \sum_{k=b}^{b^{\alpha + 1} - 1} \text{wal}_k (x \ominus y). \]

Then we have shown
\[ D_a (x, y) = \begin{cases} 0, & \text{if } x_i \neq y_i \text{ for an } i \in \{1, \ldots, a\}, \\ (b - 1)b^\alpha, & \text{if } x_i = y_i \text{ for all } i \in \{1, \ldots, a + 1\}, \\ -b^\alpha, & \text{otherwise}, \end{cases} \]
giving
\[ \sum_{k=1}^{\infty} b^{-\alpha \psi_h(k)} \text{wal}_k (x) \frac{\text{wal}_k (y)}{\text{wal}_k (y)} = \sum_{a=0}^{\infty} b^{-\alpha a} D_a (x, y). \]

If \( x = y \) we have
\[ \sum_{a=0}^{\infty} b^{-\alpha a} D_a (x, y) = \sum_{a=0}^{\infty} b^{-\alpha a} (b - 1)b^\alpha = \frac{b^\alpha (b - 1)}{b^\alpha - b} = \mu(\alpha) \]
and if \( x \neq y \), more precisely, if \( x_i = y_i \) for \( i = 1, \ldots, i_0 - 1 \) and \( x_{i_0} \neq y_{i_0} \), we have

\[
\sum_{a=0}^{\infty} b^{-a\alpha} D_a(x, y) = \sum_{a=0}^{i_0-2} b^{-a\alpha} (b-1)b^a - b^{-a(i_0-1)}b^{i_0-1} = \mu(\alpha) - b^{(i_0-1)(1-\alpha)}(\mu(\alpha) + 1).
\]

We define

\[
\phi_{\text{wal}, \alpha}(x, y) = \begin{cases} 
\mu(\alpha), & \text{if } x = y, \\
\mu(\alpha) - b^{(i_0-1)(1-\alpha)}(\mu(\alpha) + 1), & \text{if } x_{i_0} \neq y_{i_0} \text{ and } \quad x_i = y_i \text{ for } i = 1, \ldots, i_0 - 1,
\end{cases}
\]

(2.2.9)

where \( \mu \) is given by (2.2.8).

Note that the function \( \phi_{\text{wal}, \alpha} \) can be computed for any \( \alpha > 1 \) and therefore also the reproducing kernel \( K_{\text{wal}, \gamma} \) can be computed.

We now turn to the \( s \)-dimensional case. For a sequence of non-increasing weights \( \gamma = (\gamma_1, \ldots, \gamma_s) \), \( \gamma_j > 0 \), we define the \( s \)-dimensional weighted Hilbert space \( H_{\text{wal}, s, \gamma} \) as a tensor product space, that is,

\[ H_{\text{wal}, s, \gamma} = H_{\text{wal}, \gamma_1} \otimes \cdots \otimes H_{\text{wal}, \gamma_s}. \]

Let \( \mathbf{k} = (k_1, \ldots, k_s) \) and \( \mathbf{x} \) and \( \mathbf{y} \) be defined analogously. The space \( H_{\text{wal}, s, \gamma} \) is again a reproducing kernel Hilbert space with reproducing kernel given by (see Lemma 2.1.2)

\[
K_{\text{wal}, s, \gamma}(\mathbf{x}, \mathbf{y}) = \prod_{j=1}^{s} K_{\text{wal}, \gamma_j}(x_j, y_j)
\]

\[
= \prod_{j=1}^{s} (1 + \gamma_j \phi_{\text{wal}, \alpha}(x_j, y_j))
\]

\[
= \prod_{j=1}^{s} \sum_{k_j=0}^{\infty} r(\alpha, \gamma_j, k_j)_{\text{wal}}(x_j)_{\text{wal}}(y_j)
\]

\[
= \sum_{\mathbf{k} \in \mathbb{N}_0^s} r(\alpha, \gamma, \mathbf{k})_{\text{wal}}(\mathbf{x})_{\text{wal}}(\mathbf{y}),
\]

(2.2.10)

where \( r(\alpha, \gamma, \mathbf{k}) = \prod_{j=1}^{s} r(\alpha, \gamma_j, k_j) \) and the inner product is given by

\[
\langle f, g \rangle_{\text{wal}, s, \gamma} = \sum_{\mathbf{k} \in \mathbb{N}_0^s} r(\alpha, \gamma, \mathbf{k})^{-1} \tilde{f}_{\text{wal}}(\mathbf{k}) \tilde{g}_{\text{wal}}(\mathbf{k}),
\]

with \( \mathbf{k} = (k_1, \ldots, k_s) \) and

\[
\tilde{f}_{\text{wal}}(\mathbf{k}) := \int_{[0,1]^s} f(\mathbf{x})_{\text{wal}}(\mathbf{x}) \, d\mathbf{x}.
\]

Again, from (2.2.10) we see that \( K_{\text{wal}, s, \gamma} \) can be computed.
2.3 \((t,m,s)\)-nets in base \(b\)

In this section we recall the definition of (digital) \((t,m,s)\)-nets in base \(b\) and we prove some very useful properties of such point sets.

A detailed theory of \((t,m,s)\)-nets was developed in Niederreiter [40] (see also [42, Chapter 4] for a survey of this theory). \((t,m,s)\)-nets in a base \(b\) provide sets of \(b^m\) points in the \(s\)-dimensional unit cube \([0,1]^s\), which are extremely well distributed if the quality parameter \(t\) is ‘small’.

Definition 2.3.1 Let \(b \geq 2\), \(s \geq 1\) and \(0 \leq t \leq m\) be integers. Then a point set \(P\) consisting of \(b^m\) points in \([0,1)^s\) forms a \((t,m,s)\)-net in base \(b\), if every subinterval \(J = \prod_{j=1}^{s} [a_j b^{-d_j}, (a_j + 1) b^{-d_j})\) of \([0,1]^s\), with integers \(d_j \geq 0\) and integers \(0 \leq a_j < b^{d_j}\) for \(1 \leq j \leq s\) and of volume \(b^{t-m}\), contains exactly \(b^t\) points of \(P\).

In practice, all concrete constructions of \((t,m,s)\)-nets in base \(b\) are based on the general construction scheme of digital nets.

Definition 2.3.2 Let \(b \geq 2\) be a given base. Let \(R := \{0, \ldots, b-1\}\) be an arbitrary ring with \(b\) elements and with zero element \(0\). Let \(C_j = (c_{j,k,l})\), \(k,l = 1, \ldots, m, j = 1, \ldots, s\), be \(s\) given \(m \times m\) matrices over \(R\) and let \(\varphi : \{0,1,\ldots,b-1\} \rightarrow R\) be an arbitrary but fixed bijection. Now we construct \(b^m\) points in \([0,1)^s\) represent \(h\), \(0 \leq h \leq b^m-1\), in base \(b\), \(h = h_0 + h_1 b + \ldots + h_{m-1} b^{m-1}\) and identify \(h\) with the vector \(\vec{h} = (\varphi(h_0), \ldots, \varphi(h_{m-1}))^T \in R^m\), where \(T\) means the transpose of the vector. For \(1 \leq j \leq s\) multiply the matrix \(C_j\) by \(\vec{h}\),

\[C_j \vec{h} = (\vec{y}_{j,1}, \ldots, \vec{y}_{j,m})^T \in R^m,\]

and set

\[x_{h,j} := \frac{\varphi^{-1}(\vec{y}_{j,1})}{b} + \ldots + \frac{\varphi^{-1}(\vec{y}_{j,m})}{b^m}.\]

If for some integer \(t\) with \(0 \leq t \leq m\) the point set consisting of the points

\[x_h = (x_{h,1}, \ldots, x_{h,s})\]

for \(0 \leq h < b^m\), is a \((t,m,s)\)-net in base \(b\), then it is called a digital \((t,m,s)\)-net in base \(b\) (or over \(R\)), or shortly a digital net (over \(R\)). Further we call a QMC rule using a digital net as sample points a digital lattice rule.

Concerning the determination of the quality parameter \(t\) of digital nets we refer to Niederreiter [42, Theorem 4.28].

Though many of the following results are true for arbitrary commutative rings with identity, for simplicity we restrict \(R\) in the following to the finite field \(\mathbb{Z}_b\), the least residue ring modulo \(b\), where \(b\) is prime, and we choose \(\varphi\) to be the identity map. Moreover, if there is no risk of confusion and if it is clear whether the digit or the corresponding element of the field is used, we always
omit the bar over elements of \( \mathbb{Z}_b \). Further from now on we index the points of a digital net always from 1, \ldots, \( b^m \) instead of 0, \ldots, \( b^m - 1 \).

Let \( \{x_1, \ldots, x_{b^m}\} \) be a digital net over \( \mathbb{Z}_b \) generated by the \( m \times m \) matrices \( C_1, \ldots, C_s \) over \( \mathbb{Z}_b \). For \( x_h = (x_{h,1}, \ldots, x_{h,s}) \) and \( x_{h,j} = \frac{x_{h,j,1}}{b} + \ldots + \frac{x_{h,j,m}}{b^m} \), \( 1 \leq j \leq s, 1 \leq h \leq b^m \), we identify \( x_h \) with \( (x_{h,1,1}, \ldots, x_{h,1,m}, \ldots, x_{h,s,1}, \ldots, x_{h,s,m}) \in \mathbb{Z}_{b^{ms}} \) and we define

\[
x_h \oplus x_i := (x_{h,1,1} + x_{i,1,1}, \ldots, x_{h,s,m} + x_{i,s,m}) \in \mathbb{Z}_{b^{ms}}.
\]

The subsequent Lemma is a direct consequence from the construction of digital nets.

**Lemma 2.3.3** Any digital net \( \{x_1, \ldots, x_{b^m}\} \) over \( \mathbb{Z}_b \) is a subgroup of \( (\mathbb{Z}_{b^{ms}}, \oplus) \).

Now we have

\[
b_{\text{wal}}_{k_1, \ldots, k_s}(x_h \oplus x_i) = b_{\text{wal}}_{k_1, \ldots, k_s}(x_h) b_{\text{wal}}_{k_1, \ldots, k_s}(x_i)
\]

and hence \( b_{\text{wal}}_{k_1, \ldots, k_s} \) is a character on \( (\mathbb{Z}_{b^{ms}}, \oplus) \). Together with Lemma 2.3.3 we get

\[
\sum_{h=1}^{b^m} b_{\text{wal}}_{k_1, \ldots, k_s}(x_h) = \begin{cases} 
  b^m, & \text{if } b_{\text{wal}}_{k_1, \ldots, k_s}(x_h) = 1 \ \forall h = 1, \ldots, b^m, \\
  0, & \text{otherwise}.
\end{cases}
\]

For more information see [29, 31].

**Lemma 2.3.4** Let \( \{x_1, \ldots, x_{b^m}\} \) be a digital net over \( \mathbb{Z}_b \) generated by the \( m \times m \) matrices \( C_1, \ldots, C_s \) over \( \mathbb{Z}_b \). Then for all integers \( 0 \leq k_1, \ldots, k_s < b^m \) we have

\[
\sum_{h=1}^{b^m} b_{\text{wal}}_{k_1, \ldots, k_s}(x_h) = \begin{cases} 
  b^m, & \text{if } C_1^T \vec{k}_1 + \ldots + C_s^T \vec{k}_s = \vec{0}, \\
  0, & \text{otherwise},
\end{cases}
\]

where for \( 0 \leq k < b^m \) with \( k = \kappa_0 + \kappa_1 b + \ldots + \kappa_{m-1} b^{m-1} \) we write \( \vec{k} = (\kappa_0, \ldots, \kappa_{m-1})^T \in \mathbb{Z}_b^{m} \) and \( \vec{0} \) denotes the zero vector in \( \mathbb{Z}_b^{m} \).

**Proof.** We have \( b_{\text{wal}}_{k_1, \ldots, k_s}(x_h) = 1 \) for all \( h = 1, \ldots, b^m \) iff

\[
\sum_{j=1}^{s} \vec{k}_j^T \vec{x}_{h,j} = 0 \ \forall h = 1, \ldots, b^m.
\]

This means by the definition of the net that

\[
\sum_{j=1}^{s} \vec{k}_j^T C_j \vec{h} = 0 \ \forall h = 0, \ldots, b^m - 1
\]

and this is satisfied iff

\[
C_1^T \vec{k}_1 + \ldots + C_s^T \vec{k}_s = \vec{0},
\]

as claimed. \( \square \)
Let $x = \sum_{i=1}^{\infty} \frac{x_i}{b^i} \in [0,1)$ and let $\sigma = \sum_{i=1}^{\infty} \frac{\sigma_i}{b^i} \in [0,1)$, where $x_i, \sigma_i \in \{0, \ldots, b-1\}$. As in Proposition 2.2.3, we define the digital $b$-adic shifted point $y$ by

$$y = x \oplus \sigma = \sum_{i=1}^{\infty} \frac{y_i}{b^i},$$

where $y_i = x_i + \sigma_i \in \mathbb{Z}_b$. For points $x \in [0,1)^s$ and $\sigma \in [0,1)^s$ the digital $b$-adic shift $x \oplus \sigma$ is defined component wise.

We note that if Walsh functions, digital shifts or $(t, m, s)$-nets are used in conjunction with each other they are always in the same base $b$. Therefore, if it is clear with respect to which base $b$ a point is shifted, we write of a digitally shifted point instead of a digitally $b$-adic shifted point.

**Lemma 2.3.5** Let $\{x_1, \ldots, x_{b^m}\}$ be a $(t, m, s)$-net in base $b$, where $x_h = (x_{h,1}, \ldots, x_{h,s})$, $1 \leq h \leq b^m$, and let $\sigma = (\sigma_1, \ldots, \sigma_s) \in [0,1)^s$. Then the digitally $b$-adic shifted point set $y_h = x_h \oplus \sigma$, $1 \leq h \leq b^m$, is again a $(t, m, s)$-net in base $b$ with probability 1 with respect to the Lebesgue measure of $\sigma$’s.

**Proof.** First we note that for any $x \in [0,1)$ the set of all $\sigma \in [0,1)$, for which the $b$-adic expansion of $x \oplus \sigma$ has only finitely many digits different from $b-1$, is countable. In fact, if $x_i$ resp. $\sigma_i$ denote the digits in the $b$-adic expansion of $x$ resp. $\sigma$, then $x \oplus \sigma$ has only finitely many digits different from $b-1$ iff there is an index $i_0$ such that for all $i \geq i_0$ we have $x_i + \sigma_i = b-1 \in \mathbb{Z}_b$ and this holds if and only if $\sigma_i = b-1 - x_i \in \mathbb{Z}_b$ for all $i \geq i_0$. Thus the Lebesgue measure of this set is 0 and the probability that this case occurs is 0 as well.

For $1 \leq j \leq s$ let $\sigma_j = \frac{\sigma_{j,1}}{b} + \frac{\sigma_{j,2}}{b^2} + \ldots$. Further for $1 \leq h \leq b^m$, $1 \leq j \leq s$ let $x_{h,j} = \frac{x_{h,j,1}}{b} + \frac{x_{h,j,2}}{b^2} + \ldots$ and $y_{h,j} = \frac{y_{h,j,1}}{b} + \frac{y_{h,j,2}}{b^2} + \ldots$ where $y_{h,j,k} = x_{h,j,k} + \sigma_{j,k} \in \mathbb{Z}_b$ for $k \geq 1$.

In the following we assume that infinitely many of the $y_{h,j,1}, y_{h,j,2}, \ldots$ are different from $b-1$. As shown above this occurs with probability 1. Let

$$J = \prod_{j=1}^{s} \left[ \frac{A_j}{b^{d_j}}, \frac{A_j + 1}{b^{d_j}} \right]$$

be an elementary interval of volume $b^{-m}$, i.e., $d_1 + \ldots + d_s = m - t$, and let

$$\frac{A_j}{b^{d_j}} = \frac{A_{j,1}}{b} + \ldots + \frac{A_{j,d_j}}{b^{d_j}}.$$

Then the point $y_h$ is contained in $J$ if and only if

$$y_{h,j,k} = A_{j,k} \quad \text{for all } k = 1, \ldots, d_j \text{ and } j = 1, \ldots, s,$$

and this is true iff

$$x_{h,j,k} = A_{j,k} - \sigma_{j,k} \in \mathbb{Z}_b \quad \text{for all } k = 1, \ldots, d_j \text{ and } j = 1, \ldots, s. \ (2.3.11)$$
Let now $B_{j,k} \in \{0, \ldots, b - 1\}$ such that

$$B_{j,k} = A_{j,k} - \sigma_{j,k} \in \mathbb{Z}_b,$$

and let

$$\frac{B_j}{b^{d_j}} = \frac{B_{j1}}{b^{d_j}} + \ldots + \frac{B_{jd_j}}{b^{d_j}}.$$

Then (2.3.11) is equivalent to

$$x_h \in M := \prod_{j=1}^{s} \left[ \frac{B_j}{b^{d_j}} + \frac{B_j + 1}{b^{d_j}} \right].$$

Now $M$ is again an elementary interval of volume $b^{t-m}$ and since $x_1, \ldots, x_{b^m}$ forms a $(t, m, s)$-net in base $b$, it follows that $M$ contains exactly $b^t$ points of $\{x_1, \ldots, x_{b^m}\}$. Therefore $J$ contains exactly $b^t$ points of $\{y_1, \ldots, y_{b^m}\}$ and we are done.

We may note that together with Proposition 2.2.3, Lemmas 2.3.3 and 2.3.4 and the definition of the weighted Hilbert space $H_{\text{wal}, s, \gamma}$ we are provided with a set of tools similar to those used for the investigation of the worst-case error in weighted Korobov spaces using lattice rules. Thus we are ready to analyze the integration problem in $H_{\text{wal}, s, \gamma}$.

### 2.4 Multivariate integration in $H_{\text{wal}, s, \gamma}$

In this section we are interested in approximating integrals of functions $f$ from $H_{\text{wal}, s, \gamma}$,

$$I_s(f) = \int_{[0,1)^s} f(x) \, dx.$$

Clearly

$$I_s(f) = \hat{f}_{\text{wal}}(0) = \langle f, 1 \rangle_{\text{wal}, s, \gamma},$$

thus the representer of the functional $I_s(f)$ in the reproducing kernel Hilbert space $H_{\text{wal}, s, \gamma}$ is the function 1. From $I_s(f) = \langle f, 1 \rangle_{\text{wal}, s, \gamma}$ it follows immediately that the initial error is given by

$$e_{0,s} = \|1\|_{\text{wal}, s, \gamma} = 1.$$

From (2.1.4) it follows that

$$I_s(f) - Q_{n,s}(f) = \left\langle f, 1 - \frac{1}{n} \sum_{h=1}^{n} K_{\text{wal}, s, \gamma}(\cdot, x_h) \right\rangle_{\text{wal}, s, \gamma}.$$
Now we proceed as in [64] and find that for \( H_{\text{wal},s,\gamma} \) the worst-case error for integration with a point set \( P = \{x_1, \ldots, x_n\} \) is given by, see (2.1.5),

\[
e_{n,s}^2 = \langle 1 - \frac{1}{n} \sum_{h=1}^{n} K_{\text{wal},s,\gamma}(\cdot, x_h), 1 - \frac{1}{n} \sum_{h=1}^{n} K_{\text{wal},s,\gamma}(\cdot, x_h) \rangle_{\text{wal},s,\gamma}
= -1 + \frac{1}{n^2} \sum_{h,i=1}^{n} K_{\text{wal},s,\gamma}(x_h, x_i)
= -1 + \frac{1}{n^2} \sum_{h,i=1}^{n} \prod_{j=1}^{s} (1 + \gamma_j \phi_{\text{wal},\alpha}(x_{h,j}, x_{i,j}))
= -1 + \frac{1}{n^2} \sum_{h=i}^{n} \sum_{k \in \mathbb{N}_0^s} r(\alpha, \gamma, k) b_{\text{wal}}(x_h) b_{\text{wal}}(x_i).
\]

From (2.4.13) we see that the worst-case error can be computed with a cost of \( O(n^2s) \). Later on we will see that this cost can be reduced for digital nets.

In the following we show a result which is a direct analogue to a result of Hickernell [18] for the weighted Korobov space.

Define

\[
\tilde{e}_{n,s}^2 := \int_{[0,1]^s} e_{n,s}^2(x_1, \ldots, x_n) \, dx_1 \ldots dx_n,
\]

where \( e_{n,s}(x_1, \ldots, x_n) \) denotes the worst-case error for the QMC algorithm with sample points \( x_1, \ldots, x_n \).

**Theorem 2.4.1** We have

\[
\tilde{e}_{n,s} \leq \frac{1}{n^{1/2}} \exp \left( \frac{\mu(\alpha)}{2} \sum_{j=1}^{s} \gamma_j \right),
\]

where \( \mu \) is given by (2.2.8).

This result can be shown in the same way as Hickernell did in [18] (see also [64]) for the weighted Korobov space, but for completeness and as a warm-up in working with Walsh functions we provide the proof.

**Proof.** From equality (2.4.12) we get

\[
\tilde{e}_{n,s}^2
= -1 + \frac{1}{n^2} \sum_{h=1}^{n} K_{\text{wal},s,\gamma}(x_h, x_h) \, dx_h + \frac{1}{n^2} \sum_{h,i=1}^{n} K_{\text{wal},s,\gamma}(x_h, x_i) \, dx_h \, dx_i.
\]

Now we have

\[
\int_{[0,1]^2} K_{\text{wal},s,\gamma}(x, y) \, dx \, dy = \sum_{k \in \mathbb{N}_0^s} r(\alpha, \gamma, k) \int_{[0,1]^2} b_{\text{wal}}(x) b_{\text{wal}}(y) \, dx \, dy = 1,
\]

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by Proposition 2.2.3, and
\[
\int_{[0,1]^s} K_{wal,s,\gamma}(x, x) \, dx = \sum_{k \in \mathbb{N}_0} r(\alpha, \gamma, k) \int_{[0,1]^s} wal_k(x) wal_k(x) \, dx
\]
\[
= \sum_{k \in \mathbb{N}_0} r(\alpha, \gamma, k),
\]
where we used Proposition 2.2.3 again. Therefore we obtain
\[
\tilde{e}^2_{n,s} = -\frac{1}{n} + \frac{1}{n} \sum_{k \in \mathbb{N}_0} r(\alpha, \gamma, k)
\]
where \(D_{\text{net}} = \{ k \in \mathbb{N}_0 \setminus \{0\} : C_1^T \text{tr}_m(\vec{k}_1) + \ldots + C_s^T \text{tr}_m(\vec{k}_s) = \vec{0} \} \).

Remark 2.4.2 It follows from Theorem 2.4.1 that there exists a point set \(\{x_1, \ldots, x_n\}\) such that
\[
e_{n,s}(x_1, \ldots, x_n) \leq \frac{1}{n^{1/2}} \exp \left( \frac{\mu(\alpha) \sum_{j=1}^s \gamma_j}{2} \right).
\]
Therefore, if \(\sum_{j=1}^\infty \gamma_j < \infty\) then the integration problem in the weighted Hilbert space \(H_{wal,s,\gamma}\) is strongly QMC-tractable with an \(\varepsilon\)-exponent of at most 2.

Now we use digital \((t, m, s)\)-nets over \(\mathbb{Z}_b\) with generating matrices \(C_1, \ldots, C_s\). In this case we write for the worst-case error \(e_{n,s} = e_{b^m,s}(C_1, \ldots, C_s)\). First we need some notation: for a non-negative integer \(k\) with base \(b\) representation \(k = \sum_{i=0}^\infty \kappa_i b^i\) we write
\[
\text{tr}_m(k) := \kappa_0 + \kappa_1 b + \ldots + \kappa_{m-1} b^{m-1},
\]
and
\[
\text{tr}_m(\vec{k}) := (\kappa_0, \ldots, \kappa_{m-1})^T \in \mathbb{Z}_b^m.
\]

Theorem 2.4.3 Let \(\{x_1, \ldots, x_{b^m}\}\) be a digital \((t, m, s)\)-net over \(\mathbb{Z}_b\) generated by the matrices \(C_1, \ldots, C_s\).

1. Then the square worst-case error for integration in the weighted Hilbert space \(H_{wal,s,\gamma}\) is given by
\[
e_{b^m,s}^2(C_1, \ldots, C_s) = \sum_{k \in D_{\text{net}}} r(\alpha, \gamma, k),
\]
where
\[
D_{\text{net}} = \{ k \in \mathbb{N}_0 \setminus \{0\} : C_1^T \text{tr}_m(\vec{k}_1) + \ldots + C_s^T \text{tr}_m(\vec{k}_s) = \vec{0} \}.
\]
2. Let \( x_h = (x_{h,1}, \ldots, x_{h,s}) \) for \( h = 1, \ldots, b^m \), then we have

\[
e_{b^m,s}(C_1, \ldots, C_s) = -1 + \frac{1}{b^m} \sum_{h=1}^{b^m} \prod_{j=1}^{s} \left( 1 + \gamma_j \phi_{\text{wal},\alpha}(x_{h,j}, 0) \right),
\]

where \( \phi_{\text{wal},\alpha} \) is given by (2.2.9).

**Remark 2.4.4**

1. Compare the above theorem with [64, formula (15)], which states that the squared worst-case error \( \bar{e}_{n,s}^2 \) for integration in the weighted Korobov space using an \( n \) point lattice rule with generating vector \( z \) is given by

\[
\bar{e}_{n,s}^2(z) = \sum_{k \in \mathcal{L}} \bar{r}(\alpha, \gamma, k),
\]

where \( \mathcal{L} = \{ k \in \mathbb{Z}^s \setminus \{0\} : k \cdot z \equiv 0 \pmod{n} \} \) and \( \bar{r}(\alpha, \gamma, k) = \prod_{j=1}^{s} \bar{r}(\alpha, \gamma_j, k_j) \) with \( \bar{r}(\alpha, \gamma, 0) = 1 \) and \( \bar{r}(\alpha, \gamma, k) = \gamma |k|^{-\alpha} \) for \( k \neq 0 \).

The set \( \mathcal{L} \) is called the dual lattice (see [60]), accordingly we call \( \mathcal{D}_{\text{net}} \) the dual digital net.

2. We note that it follows from item 2 of the above theorem that the worst-case error of a digital net can be calculated in \( O(b^m s) = O(ns) \) operations, compared to \( O(n^2 s) \) operations for the general case.

**Proof.** From (2.4.12) we get

\[
e_{b^m,s}^2(C_1, \ldots, C_s) = -1 + \frac{1}{b^m} \sum_{h=1}^{b^m} \sum_{k \in \mathbb{N}_0^s} K_{\text{wal},\gamma}(x_h, x_i) \bar{r}(\alpha, \gamma, k) \phi_{\text{wal},\alpha}(x_{h,j}, 0)
\]

Due to the group structure of a digital net, see Lemma 2.3.3, each term in the sum over \( i \) has the same value. Therefore

\[
e_{b^m,s}^2(C_1, \ldots, C_s) = -1 + \frac{1}{b^m} \sum_{h=1}^{b^m} \sum_{k \in \mathbb{N}_0^s} r(\alpha, \gamma, k) \phi_{\text{wal},\alpha}(x_{h,j}, 0)
\]

Now apply Lemma 2 and the first part of the result follows.

The second part follows from (2.4.13), \( \phi_{\text{wal},\alpha}(x_{h,j}, x_{i,j}) = \phi_{\text{wal},\alpha}(x_{h,j} \ominus x_{i,j}, 0) \) and the group structure of a digital net. \( \square \)
As a benchmark, we define the average of the square worst-case error over all generating matrices for digital nets. Let $M_{b,m}$ be the set of all $m \times m$ matrices with entries in $\{0, \ldots, b-1\}$ and let $\mathcal{C}_b := \{(C_1, \ldots, C_s) : C_j \in M_{b,m} \text{ for } j = 1, \ldots, s\}$. Then we define

$$A_{b,m,s}(\alpha) := \frac{1}{b^{ms}} \sum_{(C_1, \ldots, C_s) \in \mathcal{C}_b} \epsilon^2_{b,m,s}(C_1, \ldots, C_s). \quad (2.4.14)$$

We have the following lemma.

**Lemma 2.4.5** Let $\mu$ be given by (2.2.8).

1. For $A_{b,m,s}(\alpha)$ defined by (2.4.14) we have

$$A_{b,m,s}(\alpha) = -1 + \frac{1}{b^m} \prod_{j=1}^s (1 + \gamma_j \mu(\alpha)) + \left(1 - \frac{1}{b^m}\right) \prod_{j=1}^s \left(1 + \gamma_j \frac{\mu(\alpha)}{b^m}\right).$$

2. Further we have

$$A_{b,m,s}(\alpha) \leq \frac{2}{b^m} \prod_{j=1}^s (1 + \gamma_j \mu(\alpha)).$$

**Proof.** For the first part, we obtain from Theorem 2.4.3 that

$$A_{b,m,s}(\alpha) = \frac{1}{b^{ms}} \sum_{(C_1, \ldots, C_s) \in \mathcal{C}_b} \sum_{\mathbf{k} \in \mathcal{D}_{\text{net}}} r(\alpha, \gamma, \mathbf{k})$$

$$= \frac{1}{b^{ms}} \sum_{\mathbf{k} \in \mathbb{N}_0^s \setminus \{0\}} \sum_{(C_1, \ldots, C_s) \in \mathcal{C}_b} r(\alpha, \gamma, \mathbf{k}) \prod_{k \neq 0} 1.$$

For $\mathbf{k} \in \mathbb{N}_0^s$, $\mathbf{k} \neq 0$, we have to consider two cases:

1. Assume $\mathbf{k} = b^m \mathbf{l}$ with $\mathbf{l} \in \mathbb{N}_0^s$, $\mathbf{l} \neq 0$. In this case we have $\text{tr}_m(k_j) = 0$ for $1 \leq j \leq s$ and the condition

$$C^T_1 \text{tr}_m(\mathbf{k}_1) + \ldots + C^T_s \text{tr}_m(\mathbf{k}_s) = \mathbf{0}$$

is trivially fulfilled for any choice of $(C_1, \ldots, C_s) \in \mathcal{C}_b$.

2. Assume $\mathbf{k} = \mathbf{k}^* + b^m \mathbf{l}$ with $\mathbf{l} \in \mathbb{N}_0^s$, $\mathbf{k}^* = (k^*_1, \ldots, k^*_s) \neq 0$ and $0 \leq k^*_j \leq b^m - 1$ for all $1 \leq j \leq s$. In this case we have $\text{tr}_m(k_j) = k^*_j$ for all $1 \leq j \leq s$ and our condition becomes

$$C^T_1 k^*_1 + \ldots + C^T_s k^*_s = \mathbf{0}. \quad (2.4.15)$$

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Let $C_{j,i}$ denote the $i$-th row vector, $1 \leq i \leq m$, of the matrix $C_j$, $1 \leq j \leq s$. Then condition (2.4.15) becomes

$$
\sum_{j=1}^{s} \sum_{i=0}^{m-1} C_{j,i+1}^T \kappa_{j,i}^* = 0, \tag{2.4.16}
$$

where $k_j^* = \kappa_{j,0}^* + \kappa_{j,1}^* b + \ldots + \kappa_{j,m-1}^* b^{m-1}$. Since at least one $k_j^* \neq 0$ it follows that there is an $\kappa_{j,i}^* \neq 0$. First assume that $\kappa_{1,0}^* \neq 0$. Then for any choice of $C_{1,2}, \ldots, C_{1,m}, C_{2,1}, \ldots, C_{2,m}, \ldots, C_{s,1}, \ldots, C_{s,m}$ we can find exactly one vector $C_{1,1}$ such that condition (2.4.16) is fulfilled. The same argument holds with $\kappa_{1,0}^*$ replaced by $\kappa_{j,i}^*$ and $C_{1,1}$ replaced by $C_{j,i+1}$.

Now we have

$$
A_{b^m,s}(\alpha) = \frac{1}{b^{m} s} \sum_{l \in \mathbb{N}_0} r(\alpha, \gamma, b^m l) b^{m^2} + \frac{1}{b^{m s}} \sum_{l \in \mathbb{N}_0} \sum_{k^* \in \mathbb{N}_0 \setminus \{0\}} \sum_{|k^*| < b^m} r(\alpha, \gamma, k^* + b^m l) b^{m^2 s - m} \nonumber
$$

$$
= \frac{1}{b^{m}} \sum_{l \in \mathbb{N}_0} r(\alpha, \gamma, b^m l) + \frac{1}{b^{m s}} \sum_{l \in \mathbb{N}_0} \sum_{k^* \in \mathbb{N}_0 \setminus \{0\}} r(\alpha, \gamma, k^* + b^m l) \nonumber
$$

$$
= -1 + \frac{1}{b^{m}} \sum_{l \in \mathbb{N}_0} r(\alpha, \gamma, b^m l) + \frac{1}{b^{m s}} \sum_{l \in \mathbb{N}_0} r(\alpha, \gamma, l), \nonumber
$$

where we used the fact that $r(\alpha, \gamma, 0) = 1$. The sums in the expression above can be simplified as follows:

$$
\sum_{l \in \mathbb{N}_0} r(\alpha, \gamma, b^m l) = \prod_{j=1}^{s} \left( \sum_{l_j=0}^{\infty} r(\alpha, \gamma_j, b^m l_j) \right) = \prod_{j=1}^{s} \left( 1 + \gamma_j \frac{\mu(\alpha)}{b^{m \alpha}} \right)
$$

and

$$
\sum_{l \in \mathbb{N}_0} r(\alpha, \gamma, l) = \prod_{j=1}^{s} \left( \sum_{l_j=0}^{\infty} r(\alpha, \gamma_j, l_j) \right) = \prod_{j=1}^{s} (1 + \gamma_j \mu(\alpha)).
$$

The formula for the average follows.

The inequality in the second part can be derived by using

$$
-1 + \prod_{j=1}^{s} \left( 1 + \gamma_j \frac{\mu(\alpha)}{b^{m \alpha}} \right) \leq \frac{1}{b^{m s}} \prod_{j=1}^{s} (1 + \gamma_j \mu(\alpha)).
$$

\[\square\]

**Theorem 2.4.6** 1. There exists a digital $(t, m, s)$-net over $\mathbb{Z}_b$ such that the square worst-case error for multivariate integration of functions in the weighted Hilbert space $H_{\text{val}, s, \gamma}$ is bounded by

$$
eq_{b^m,s}^2 \leq c_{s, \gamma, \lambda, \alpha} b^{-m/\lambda},$$

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where
\[
c_{s,\gamma,\lambda,\alpha} = 2^{1/\lambda} \prod_{j=1}^{s} (1 + \gamma_j^\lambda \mu(\alpha \lambda))^{1/\lambda}
\]
for any \(1/\alpha < \lambda \leq 1\) and where \(\mu\) is given by (2.2.8).

2. For some \(\lambda \in (1/\alpha, 1]\) assume
\[
\sum_{j=1}^{\infty} \gamma_j^\lambda < \infty.
\] (2.4.17)

Then \(c_{s,\gamma,\lambda,\alpha} \leq c_{\infty,\gamma,\lambda,\alpha} < \infty\) and we have
\[
e^2_{b^m,s} \leq c_{\infty,\gamma,\lambda,\alpha} b^{-m/\lambda}
\]
for all \(s \geq 1\).

Thus, assuming (2.4.17), there exists a digital \((t,m,s)\)-net over \(\mathbb{Z}_b\) such that the worst-case error is bounded independently of the dimension. Let \(\lambda_0\) be the infimum over all \(\lambda\) which satisfy (2.4.17). Then the \(\varepsilon\)-exponent of strong QMC-tractability lies in the interval \([2/\alpha, 2\lambda_0]\).

3. Under the assumption
\[
A := \limsup_{s \to \infty} \frac{\sum_{j=1}^{s} \gamma_j^\lambda}{\log s} < \infty \tag{2.4.18}
\]
we obtain \(c_{s,\gamma,1,\alpha} \leq \tilde{c}_s s^{\mu(\alpha)(A+\delta)}\) and therefore
\[
e^2_{b^m,s} \leq \tilde{c}_s s^{\mu(\alpha)(A+\delta)} b^{-m}
\]
for any \(\delta > 0\), where the constant \(\tilde{c}_s\) depends only on \(\delta\). Thus, assuming (2.4.18), there exists a digital \((t,m,s)\)-net over \(\mathbb{Z}_b\) such that the worst-case error satisfies a bound which depends only polynomially on the dimension. Hence the \(\varepsilon\)-exponent is at most 2 and \(s\)-exponent at most \(\mu(\alpha)A\).

4. Let \(\nu\) be the equiprobable measure on the set \(\mathcal{C}_b\), i.e., \(\mu(C_1, \ldots, C_s) = b^{-m^2 s}\) for all \((C_1, \ldots, C_s) \in \mathcal{C}_b\). For \(c > 1\) and \(1/\alpha < \lambda \leq 1\), define the set
\[
\mathcal{C}_b(c,\lambda) = \left\{ (C_1, \ldots, C_s) \in \mathcal{C}_b : e_{b^m,s}(C_1, \ldots, C_s) \leq c^{1/\lambda} \sqrt{c_{s,\gamma,\lambda,\alpha} b^{-m/\lambda}} \right\}.
\]

Then
\[
\nu(\mathcal{C}_b(c,\lambda)) > 1 - c^{-2}.
\]

Remark 2.4.7 Note that the average \(A_{b^m,s}(\alpha)\) includes many \(s\)-tuples of generating matrices \((C_1, \ldots, C_s)\) which are not useful in practice. From item 4 in Theorem 2.4.6 we obtain for example by choosing \(c = 10\), that 99% of
the s-tuples of generating matrices satisfy an upper bound which is at most \(c^{1/\lambda} \leq c^\alpha = 10^\alpha\) times larger than the bound in item 1 of the theorem above. It appears likely that the known generating matrices (for example Sobol’, Faure, Niederreiter, Niederreiter-Xing) belong to the set \(\mathcal{C}_b(10, \lambda)\) and therefore satisfy a strong tractability error bound if (2.4.17) is satisfied. Work remains to be done to investigate if this is indeed the case.

In the following we will use Jensen’s inequality, which states that for a sequence \((a_k)\) of non-negative real numbers we have
\[
(\sum a_k)^\lambda \leq \sum a_k^\lambda,
\]
for any \(0 < \lambda \leq 1\).

**Proof of Theorem 2.4.6.** Let \(1/\alpha < \lambda \leq 1\). Here, for the square worst-case error we write \(e_{2b,m,s}^2(\alpha, \gamma)\). From Theorem 2.4.3 and by applying Jensen’s inequality we get
\[
e_{2b,m,s}^2(\alpha, \gamma) \leq \left( \sum_{k \in \mathcal{D}_{net}} r(\alpha, \gamma, k)^\lambda \right)^{1/\lambda} = \left( \sum_{k \in \mathcal{D}_{net}} r(\alpha \lambda, \gamma^\lambda, k) \right)^{1/\lambda} = (e_{b,m,s}^2(\alpha \lambda, \gamma^\lambda))^{1/\lambda}, \tag{2.4.19}
\]
since \(r(\alpha, \gamma, k)^\lambda = r(\alpha \lambda, \gamma^\lambda, k)\). Here \(\gamma^\lambda\) denotes the sequence \((\gamma^\lambda_1, \gamma^\lambda_2, \ldots)\).

From Lemma 2.4.5 we find that there exists a digital net such that
\[
e_{b,m,s}^2(\alpha \lambda, \gamma^\lambda) \leq \frac{2}{b^m} \prod_{j=1}^s \left( 1 + \gamma^\lambda_j \mu(\alpha \lambda) \right).
\]

The first part of the theorem follows. (Note that \(\mu(\alpha \lambda)\) is only defined for \(\lambda > 1/\alpha\), see (2.2.8).)

For the second part of the theorem we have
\[
c_{\infty,\gamma,\lambda,\alpha} = 2^{1/\lambda} \prod_{j=1}^\infty \left( 1 + \gamma^\lambda_j \mu(\alpha \lambda) \right)^{1/\lambda}
\]
\[
= 2^{1/\lambda} \exp \left( \frac{1}{\lambda} \sum_{j=1}^\infty \log \left( 1 + \gamma^\lambda_j \mu(\alpha \lambda) \right) \right)
\]
\[
\leq 2^{1/\lambda} \exp \left( \frac{\mu(\alpha \lambda)}{\lambda} \sum_{j=1}^\infty \gamma^\lambda_j \right)
\]
\[
< \infty,
\]
provided that \(\sum_{j=1}^\infty \gamma^\lambda_j < \infty\).
For the third part of the theorem observe that \( A < \infty \) and therefore for any positive \( \delta \) there exists a positive \( s_\delta \) such that

\[
\sum_{j=1}^{s} \gamma_j \leq (A + \delta) \log s \quad \forall s \geq s_\delta.
\]

Hence

\[
c_{s, \gamma, 1, \alpha} = 2 \prod_{j=1}^{s} (1 + \gamma_j \mu(\alpha)) = 2 s \sum_{j=1}^{s} \frac{\log(1 + \gamma_j \mu(\alpha))}{\log s} \leq 2 s^{\mu(\alpha)} \sum_{j=1}^{s} \gamma_j / \log s \leq 2 s^{\mu(\alpha)}(A + \delta),
\]

for any \( \delta > 0 \) and all \( s \geq s_\delta \). Thus there is a constant \( \tilde{c}_\delta \) such that

\[
c_{s, \gamma, 1, \alpha} \leq \tilde{c}_\delta s^{\mu(\alpha)(A + \delta)}.
\]

For the fourth part of the theorem we write \( e_{b_m,s}(\alpha, \gamma, (C_1, \ldots, C_s)) \) for the worst-case error of integration in the weighted Hilbert space \( H_{\text{wal},s,\gamma} \) with parameter \( \alpha \) and weights \( \gamma \) using a digital net with generating matrices \( C_1, \ldots, C_s \).

From Chebyshev’s inequality applied to (2.4.14) (where the parameter is now \( \alpha \lambda \) and the weights are \( \gamma_j^\lambda \)), it follows that the set (where \( c > 1 \))

\[
\tilde{C}_b(c) = \left\{ (C_1, \ldots, C_s) \in C_b : e_{b_m,s}(\alpha \lambda, \gamma^\lambda, (C_1, \ldots, C_s)) \leq c \frac{\sqrt{2}}{b_m/2} \prod_{j=1}^{s} (1 + \gamma_j^\lambda \mu(\alpha \lambda))^{1/2} \right\}
\]

satisfies

\[
\nu(\tilde{C}_b(c)) > 1 - c^{-2}.
\]

From (2.4.19) we obtain that for any \((C_1, \ldots, C_s) \in C_b\) we have

\[
e_{b_m,s}(\alpha, \gamma, (C_1, \ldots, C_s)) \leq e_{b_m,s}(\alpha \lambda, \gamma^\lambda, (C_1, \ldots, C_s))^{1/\lambda}.
\]

Therefore \( \tilde{C}_b(c) \subseteq C_b(c, \lambda) \) and the result follows.

In the following we will also obtain a lower bound. We have

\[
K_{\text{wal}, s, \gamma}(x, y) = \prod_{j=1}^{s} (1 + \gamma_j \phi_{\text{wal}, \alpha}(x_j, y_j)),
\]

with the function \( \phi_{\text{wal}, \alpha} \) given by (see (2.2.9)),

\[
\phi_{\text{wal}, \alpha}(x, y) = \begin{cases} 
\mu(\alpha), & \text{if } x = y, \\
\mu(\alpha) - b(i_0 - 1)(1-\alpha)(\mu(\alpha) + 1), & \text{if } x_{i_0} \neq y_{i_0}, \text{ and } x_i = y_i \text{ for } 1 \leq i < i_0,
\end{cases}
\]
where \( \mu(\alpha) = \frac{b^\alpha(b-1)}{b^\alpha - b} \). As \( b \geq 2 \) and \( \alpha > 1 \) we have \( \mu(\alpha) > 1 \). Further we have \( i_0 \geq 1 \) and therefore \( \mu(\alpha) - b^{(i_0-1)(1-\alpha)}(\mu(\alpha) + 1) \geq \mu(\alpha)(1 - b^{(i_0-1)(1-\alpha)}) - b^{(i_0-1)(1-\alpha)} \geq -1 \). Hence, as long as \( \gamma \leq 1 \), we have \( 1 + \gamma \phi_{\text{wal}, \alpha}(x, y) \geq 0 \) for any \( x, y \).

Let now \( \gamma' = \min(\gamma_j, 1) \) and \( \gamma' = (\gamma'_j)_{j \geq 1} \). Then we have \( \| f \|_{\text{wal}, \gamma'} \geq \| f \|_{\text{wal}, \gamma} \), hence \( \{ f \in H_{\text{wal}, \gamma'} \mid \| f \|_{\text{wal}, \gamma'} \leq 1 \} \subseteq \{ f \in H_{\text{wal}, \gamma} \mid \| f \|_{\text{wal}, \gamma} \leq 1 \} \). From this it follows that integration in \( H_{\text{wal}, \gamma'} \) is no harder than integration in \( H_{\text{wal}, \gamma} \), that is, \( e_{n,s}(H_{\text{wal}, \gamma'}, Q_{n,s}) \leq e_{n,s}(H_{\text{wal}, \gamma}, Q_{n,s}) \). Further we have \( K_{\text{wal}, s, \gamma'} \) is non-negative.

Now we estimate \( e(H_{\text{wal}, \gamma}, Q_{n,s}) \) from below by using (2.4.12). We have

\[
\begin{align*}
\epsilon_{n,s}^2(H_{\text{wal}, \gamma}, Q_{n,s}) & \geq e_{n,s}^2(H_{\text{wal}, \gamma'}, Q_{n,s}) \\
& = -1 + \frac{1}{n^2} \sum_{h=1}^{n} K_{\text{wal}, s, \gamma'}(x_h, x_i) \\
& \geq -1 + \frac{1}{n^2} \sum_{h=1}^{n} K_{\text{wal}, s, \gamma'}(x_h, x_h) \\
& = -1 + \frac{1}{n^2} \sum_{h=1}^{n} \sum_{k \in \mathbb{N}_0} r(\alpha, \gamma', k) \text{wal}_k(x_h) \overline{\text{wal}_k(x_h)} \\
& = -1 + \frac{1}{n} \sum_{k \in \mathbb{N}_0} r(\alpha, \gamma', k) \\
& = -1 + \frac{1}{n} \prod_{j=1}^{s} \left( 1 + \sum_{k=1}^{\infty} r(\alpha, \gamma'_j, k) \right) \\
& = -1 + \frac{1}{n} \prod_{j=1}^{s} \left( 1 + \gamma'_j \mu(\alpha) \right). \quad (2.4.20)
\end{align*}
\]

We summarize the result in the following theorem.

**Theorem 2.4.8** For any QMC algorithm \( Q_{n,s} \) for the worst-case error for integration in the weighted Hilbert space \( H_{\text{wal}, s, \gamma} \) we have

\[
\epsilon_{n,s}^2(H_{\text{wal}, \gamma}, Q_{n,s}) \geq -1 + \frac{1}{n} \prod_{j=1}^{s} \left( 1 + \min(\gamma_j, 1) \mu(\alpha) \right).
\]

From Theorem 2.4.8 we immediately find that

\[
\eta_{\text{min}}(s, \varepsilon) \geq \frac{\prod_{j=1}^{s} \left( 1 + \min(\gamma_j, 1) \mu(\alpha) \right)}{1 + \varepsilon^2}.
\]

Now we can proceed as Sloan and Woźniakowski did in [64, Section 3] to obtain the following corollary.
Corollary 2.4.9 1. A necessary and sufficient condition for multivariate integration in the weighted Hilbert space $H_{\text{wal},s,\gamma}$ to be strongly QMC-tractable is

$$\sum_{j=1}^{\infty} \gamma_j < \infty.$$ 

2. A necessary and sufficient condition for multivariate integration in the weighted Hilbert space $H_{\text{wal},s,\gamma}$ to be QMC-tractable is

$$\limsup_{s \to \infty} \frac{\sum_{j=1}^{s} \gamma_j}{\log s} < \infty.$$ 

In the following we consider the case where the parameter $\alpha$ approaches 1. In [12] it has been shown that the worst-case error for integration in the weighted Korobov space tends to infinity as $\alpha$ goes to 1. We show an analogous result for the space $H_{\text{wal},s,\gamma}$. In the following we write $H_{\text{wal},\alpha,s,\gamma}$ instead of $H_{\text{wal},s,\gamma}$ to stress the dependency on $\alpha$.

Theorem 2.4.10 Let $e(Q_{n,s}, H_{\text{wal},\alpha,s,\gamma})$ be the worst-case error of integration in the space $H_{\text{wal},\alpha,s,\gamma}$ using a QMC algorithm $Q_{n,s}$. For arbitrary $n$ and arbitrary QMC algorithms $Q_{n,s}$ which use sample points $x_j$ that may depend on $\alpha$ we have

$$\lim_{\alpha \to 1^+} e(Q_{n,s}, H_{\text{wal},\alpha,s,\gamma}) = \infty.$$ 

Proof. From (2.4.20) we have

$$e(Q_{n,s}, H_{\text{wal},\alpha,s,\gamma}) \geq -1 + \frac{1}{n} \prod_{j=1}^{s} (1 + \min(\gamma_j, 1)\mu(\alpha)).$$ 

As $\lim_{\alpha \to 1^+} \mu(\alpha) = \infty$ the result follows. \qed

2.5 Tractability error bounds of Monte Carlo

A simple Monte Carlo algorithm is of the form

$$\text{MC}_{n,s}(x_1, \ldots, x_n)(f) = \frac{1}{n} \sum_{h=1}^{n} f(x_h),$$ 

where the sample points $x_1, \ldots, x_n$ are independent, uniformly on $[0, 1]^s$ distributed random variables. (For simplicity we will write Monte Carlo algorithm or MC algorithm in the following instead of simple Monte Carlo algorithm.)

Here we are interested in the randomized error of a MC algorithm in the unit ball of $H_{\text{wal},s,\gamma}$. The randomized error of a MC algorithm is given by

$$e_{n,s}^{\text{mc}} := \sup_{f \in H_{\text{wal},s,\gamma}, \|f\|_{H_{\text{wal},s,\gamma}} \leq 1} \mathbb{E}(\|I_s(f) - \text{MC}_{n,s}(x_1, \ldots, x_n)(f)\|_2^2)^{1/2},$$
where the expectation $E$ is taken with respect to independent, uniformly on $[0,1]^s$ distributed sample points $x_1, \ldots, x_n$. It is well known (see for example \cite[Theorem 1.1]{42}) that
\[
\varepsilon_{n,s}^{mc} = \frac{1}{n^{1/2}} \sup_{f \in H_{wal,s,\gamma}, \|f\|_{wal,s,\gamma} \leq 1} \left( I_s(|f|^2) - |I_s(f)|^2 \right)^{1/2}.
\]

By Parseval’s identity we have
\[
I_s(|f|^2) = \sum_{k \in \mathbb{N}_0^s} |\hat{f}_{wal}(k)|^2.
\]

Further we have $I_s(f) = \hat{f}_{wal}(0)$, and hence we obtain
\[
I_s(|f|^2) - |I_s(f)|^2 = \sum_{k \in \mathbb{N}_0^s, k \neq 0} |\hat{f}_{wal}(k)|^2
\]
\[
= \sum_{k \in \mathbb{N}_0^s, k \neq 0} r(\alpha, \gamma, k)^{-1} |\hat{f}_{wal}(k)|^2 r(\alpha, \gamma, k)
\]
\[
\leq \|f\|_{wal,s,\gamma}^2 \max_{k \in \mathbb{N}_0^s, k \neq 0} r(\alpha, \gamma, k).
\]
\[(2.5.21)\]

Let $k^* \in \mathbb{N}_0^s$, $k^* \neq 0$ be such that $\max_{k \in \mathbb{N}_0^s, k \neq 0} r(\alpha, \gamma, k) = r(\alpha, \gamma, k^*)$ and let $f(x) = wal_{k^*}(x)$. For this special choice we have
\[
\hat{f}_{wal}(k) = \int_{[0,1]^s} wal_k(x)wal_{k^*}(x) \, dx = \begin{cases} 1, & \text{if } k = k^*, \\ 0, & \text{otherwise}. \end{cases}
\]

Now it is easy to check that for our special choice for $f$ inequality (2.5.21) becomes an equality. This leads to
\[
\varepsilon_{n,s}^{mc} = \frac{1}{n^{1/2}} \left( \max_{k \in \mathbb{N}_0^s, k \neq 0} r(\alpha, \gamma, k) \right)^{1/2}.
\]

Now we have
\[
\max_{k \in \mathbb{N}_0^s, k \neq 0} r(\alpha, \gamma, k) = \max_{k=1,\ldots,s} \prod_{j=1}^k \gamma_j,
\]
that is, the right hand side is just the product of the $\gamma_j$ which are greater than 1, except $\gamma_1$ is always included. Therefore
\[
\varepsilon_{n,s}^{mc} = \frac{1}{n^{1/2}} \left( \max_{k=1,\ldots,s} \prod_{j=1}^k \gamma_j^{1/2} \right).
\]

So the randomized error of MC algorithms in the weighted Hilbert space $H_{wal,s,\gamma}$ is the same as in the weighted Korobov space, see \cite[Section 5]{64}.
Let \( n_{mc}^{mc}(s, \varepsilon) \) denote the minimal number of function evaluations needed to guarantee that \( e_{mc}^{mc} \leq \varepsilon \). As before, we say that integration in the space \( H_{wal,s,\gamma} \) is strongly MC-tractable if \( n_{mc}^{mc}(s, \varepsilon) \) is bounded by a polynomial in \( \varepsilon^{-1} \) for all \( s \geq 1 \) and \( \varepsilon \in (0, 1) \), and MC-tractable if \( n_{mc}^{mc}(s, \varepsilon) \) is bounded by a polynomial in \( s \) and \( \varepsilon^{-1} \) for all \( s \geq 1 \) and \( \varepsilon \in (0, 1) \).

Now we can proceed as in [64, Section 5] to obtain the following theorem.

**Theorem 2.5.1**  
1. Multivariate integration in the weighted Hilbert space \( H_{wal,s,\gamma} \) is strongly MC-tractable iff
   \[
   \sum_{j=1}^{\infty} \max(\log \gamma_j, 0) < \infty.
   \]
   If so, then the minimal number \( n_{mc}^{mc}(s, \varepsilon) \) of function evaluations needed to guarantee the randomized error of MC to be at most \( \varepsilon \) is given by
   \[
   \sup_{s=1, 2, \ldots} n_{mc}^{mc}(s, \varepsilon) = C \varepsilon^{-2},
   \]
   where \( C = \sup_{s=1, 2, \ldots} \prod_{j=1}^{s} \gamma_j < \infty. \)

2. Integration in the weighted Hilbert space \( H_{wal,s,\gamma} \) is MC-tractable iff
   \[
   q := \limsup_{s \to \infty} \sum_{j=1}^{s} \frac{\max(\log \gamma_j, 0)}{\log s} < \infty.
   \]
   If so, then
   \[
   n_{mc}^{mc}(s, \varepsilon) \leq s^{q+o(1)} \varepsilon^{-2} \quad \text{as } s \to \infty.
   \]

### 2.6 Multivariate integration in weighted Sobolev spaces

In this section we consider the integration problem in weighted Sobolev spaces. We will establish a connection between the worst-case error of integration in weighted Hilbert spaces \( H_{wal,s,\gamma} \) and in weighted Sobolev spaces. This will allow us to use results from Section 2.4.

#### 2.6.1 Digital shift invariant kernels

In this section we introduce ‘digital \( b \)-adic shift invariant kernels’. Consider the reproducing kernel of the weighted Hilbert space \( H_{wal,s,\gamma} \). Let \( \sigma = (\sigma_1, \ldots, \sigma_s) \in [0, 1)^s \) and let \( \sigma_j = \sum_{i=1}^{\infty} \frac{\sigma_j}{b^i} \). As before \( x \oplus \sigma = (x_1 \oplus \sigma_1, \ldots, x_s \oplus \sigma_s) \). Then

\[
K_{wal,s,\gamma}(x \oplus \sigma, y \oplus \sigma) = \sum_{k \in \mathbb{N}_0} r(\alpha, \gamma, k) \, \text{walk}_b(x \oplus \sigma) \, \text{walk}_b(y \oplus \sigma)
\]

\[
= \sum_{k \in \mathbb{N}_0} r(\alpha, \gamma, k) \, \text{walk}_b(x) \, \text{walk}_b(y)
\]

\[
= K_{wal,s,\gamma}(x, y),
\]
as

\[ b_{\text{wal}}(x \oplus \sigma) b_{\text{wal}}(y \oplus \sigma) = b_{\text{wal}}(x) b_{\text{wal}}(y) b_{\text{wal}}(\sigma) b_{\text{wal}}(\sigma) = b_{\text{wal}}(x) b_{\text{wal}}(y). \]

This means that the reproducing kernel of the weighted Hilbert space \( H_{\text{wal},s,\gamma} \) stays unchanged under applying the same digital \( b \)-adic shift to both coordinates. We call a reproducing kernel with this property ‘digital \( b \)-adic shift invariant’. Again, if it is clear with respect to which base \( b \) the kernel is digital \( b \)-adic shift invariant we just write ‘digital shift invariant’ kernel.

For an arbitrary reproducing kernel we associate a digital shift invariant kernel. This association is given in the following definition.

**Definition 2.6.1** For an arbitrary reproducing kernel \( K \) we define the associated digital shift invariant kernel by

\[ K_{\text{ds}}(x, y) := \int_{[0,1]^s} K(x \oplus \sigma, y \oplus \sigma) d\sigma. \]

The kernel \( K_{\text{ds}} \) is indeed digital shift invariant as

\[ K_{\text{ds}}(x \oplus \sigma, y \oplus \sigma) = \int_{[0,1]^s} K(x \oplus \sigma \oplus \Delta, y \oplus \sigma \oplus \Delta) d\Delta \]
\[ = \int_{[0,1]^s} K(x \oplus \Delta, y \oplus \Delta) d\Delta \]
\[ = K_{\text{ds}}(x, y). \]

Recall that for a reproducing kernel \( K(x, y) : [0,1]^s \times [0,1]^s \rightarrow \mathbb{C} \) we have \( K(x, y) = \bar{K}(y, x) \). Therefore, in order to obtain a simple notation, we define, for a reproducing kernel \( K \in L_2([0,1)^{2s}) \) and for \( k, k' \in \mathbb{N}_0^s \),

\[ \hat{K}(k, k') = \int_{[0,1]^s} \int_{[0,1]^s} K(x, y) b_{\text{wal}}(x) b_{\text{wal}}(y) \, dx \, dy, \]

and so the reproducing kernel can be written as

\[ K(x, y) = \sum_{k \in \mathbb{N}_0^s} \sum_{k' \in \mathbb{N}_0^s} \hat{K}(k, k') b_{\text{wal}}(x) b_{\text{wal}}(y). \quad (2.6.22) \]

From \( K(x, y) = \bar{K}(y, x) \) then follows that \( \hat{K}(k, k') = \bar{\hat{K}}(k', k) \) and therefore \( \hat{K}(k, k') \in \mathbb{R} \). This is also consistent with the definition of \( K_{\text{wal},s,\gamma} \). In the following lemma we show how the digital shift invariant kernel can easily be found from (2.6.22).

**Lemma 2.6.2** Let \( K \in L_2([0,1)^{2s}) \) be a reproducing kernel. Then the digital shift invariant kernel \( K_{\text{ds}} \) is given by

\[ K_{\text{ds}}(x, y) = \sum_{k \in \mathbb{N}_0^s} \hat{K}(k, k) b_{\text{wal}}(x) b_{\text{wal}}(y). \]
where

\[ \hat{K}(k, k') = \int_{[0,1]^2} \int_{[0,1]^2} K(x, y) b_{\text{wal}}(x) b_{\text{wal}}(y) \, dx \, dy. \]

**Proof.** Using the definition of the digital shift invariant kernel we have

\[
K_{ds}(x, y) = \int_{[0,1]^2} K(x \oplus \sigma, y \oplus \sigma) \, d\sigma
= \int_{[0,1]^2} \sum_{k' \in \mathbb{N}_0} \sum_{k' \in \mathbb{N}_0} \hat{K}(k, k') b_{\text{wal}}(x \oplus \sigma) b_{\text{wal}}(y \oplus \sigma) \, d\sigma
= \sum_{k \in \mathbb{N}_0} \sum_{k' \in \mathbb{N}_0} \hat{K}(k, k') \int_{[0,1]^2} b_{\text{wal}}(x \oplus \sigma) b_{\text{wal}}(y \oplus \sigma) \, d\sigma.
\]

Due to the orthogonality properties of the Walsh functions we have

\[
\int_{[0,1]^2} b_{\text{wal}}(x \oplus \sigma) b_{\text{wal}}(y \oplus \sigma) \, d\sigma = \begin{cases} 0 & \text{if } k \neq k', \\ b_{\text{wal}}(x) b_{\text{wal}}(y) & \text{if } k = k'. \end{cases}
\]

Therefore

\[ K_{ds}(x, y) = \sum_{k \in \mathbb{N}_0} \hat{K}(k, k) b_{\text{wal}}(x) b_{\text{wal}}(y), \]

as claimed. \( \square \)

For a point set \( P_n = \{x_1, \ldots, x_n\} \) and a \( \sigma \in [0,1]^s \), let \( P_{n,\sigma} = \{x_1 \oplus \sigma, \ldots, x_n \oplus \sigma\} \) be the digitally shifted point set. Let, for a reproducing kernel Hilbert space \( H \) with reproducing kernel \( K \) and a point set \( P_n \), the worst-case error \( e(P_n, K) \) be defined as

\[ e(P_n, K) := \sup_{f \in H, \|f\| \leq 1} |I(f) - Q_{n,d}(P_n, f)|. \]

Further let the mean square worst-case error \( \hat{e}^2(P_n, K) \) be given by

\[ \hat{e}^2(P_n, K) := \mathbb{E}[e^2(P_n, \sigma, K)] = \int_{[0,1]^s} e^2(P_n, \sigma, K) \, d\sigma. \] (2.6.23)

We have the following theorem.

**Theorem 2.6.3** For any real valued reproducing kernel \( K \in L_2([0,1)^2s) \) and point set \( P_n \) we have

\[ \mathbb{E}[e^2(P_n, \sigma, K)] = e^2(P_n, K_{ds}). \]
Proof. From (2.1.6) we have

\[
e^{2(P_n, K)} = \int_{[0,1]^{2^n}} K(x, y) \, dx \, dy - \frac{2}{n} \sum_{h=1}^{n} \int_{[0,1]^n} K(x_h, y) \, dy + \frac{1}{n^2} \sum_{h,i=1}^{n} K(x_h, x_i).
\]

Therefore we have

\[
\int_{[0,1]^n} e^{2(P_{n,\sigma}, K)} \, d\sigma \n\]

\[
= \int_{[0,1]^{2^n}} K(x, y) \, dx \, dy - \frac{2}{n} \sum_{h=1}^{n} \int_{[0,1]^n} K(x_h \oplus \sigma, y) \, dy \, d\sigma + \frac{1}{n^2} \sum_{h,i=1}^{n} \int_{[0,1]^n} K(x_h \oplus \sigma, x_i \oplus \sigma) \, d\sigma.
\]

Using item 4 of Proposition 2.2.3 we obtain

\[
\int_{[0,1]^{2^n}} K(x, y) \, dx \, dy = \int_{[0,1]^n} K(x \oplus \sigma, y \oplus \sigma) \, dx \, dy \, d\sigma = \int_{[0,1]^n} K(x \oplus \sigma, y \oplus \sigma) \, d\sigma \, dx \, dy = \int_{[0,1]^{2^n}} K_{ds}(x, y) \, dx \, dy.
\]

Further we have

\[
\int_{[0,1]^n} \int_{[0,1]^n} K(x_h \oplus \sigma, y) \, dy \, d\sigma = \int_{[0,1]^n} K(x_h \oplus \sigma, y \oplus \sigma) \, dy \, d\sigma = \int_{[0,1]^n} K(x_h \oplus \sigma, y \oplus \sigma) \, d\sigma \, dy = \int_{[0,1]^n} K_{ds}(x_h, y) \, dy
\]

and

\[
\int_{[0,1]^n} K(x_h \oplus \sigma, x_i \oplus \sigma) \, d\sigma = K_{ds}(x_h, x_i).
\]

Thus we obtain

\[
\int_{[0,1]^n} e^{2(P_{n,\sigma}, K)} \, d\sigma = e^{2(P_n, K_{ds})},
\]

and the result follows.
2.6.2 Weighted Sobolev spaces

In this section we introduce the weighted Sobolev space $H_{\text{sob},s,\gamma}$ with the reproducing kernel given by (see [11, 65])

$$K(x,y) = \prod_{j=1}^{s} K_{\gamma_j}(x_j,y_j) = \prod_{j=1}^{s} (1 + \gamma_j(\frac{1}{2}B_2(\{x_j - y_j\}) + (x_j - \frac{1}{2})(y_j - \frac{1}{2})), $$

where $B_2(x) = x^2 - x + \frac{1}{6}$ is the second Bernoulli polynomial and $\{x\} = x - \lfloor x \rfloor$. The inner product in $H_{\text{sob},s,\gamma}$ is given by

$$\langle f,g \rangle_{H_{\text{sob},s,\gamma}} := \sum_{u \subseteq \{1,\ldots,s\}} \prod_{j \in u} \gamma_j^{-1} \int_{[0,1]^{|u|}} \int_{[0,1]^{s-|u|}} \frac{\partial^{[u]} f(x)}{\partial x_u}(x) dx_{-u} \int_{[0,1]^{s-|u|}} \frac{\partial^{[u]} g(x)}{\partial x_u}(x) dx_{-u} dx_u,$$

where for $x = (x_1,\ldots,x_s)$ we use the notation $x_u = (x_j)_{j \in u}$ and $x_{-u} = (x_j)_{j \in \{1,\ldots,s\}\setminus u}$.

It is shown in Appendix A that the digital shift invariant kernel in base $b \geq 2$ of $H_{\text{sob},s,\gamma}$ is given by

$$K_{\text{ds},b,\gamma}(x,y) = \prod_{j=1}^{\infty} \left( \sum_{k=1}^{\kappa_b - 1} \hat{r}_{b}(\gamma_j,k) b^{\text{walk}_k(x_j)} b^{\text{walk}_k(y_j)} \right)$$

$$= \sum_{k \in \mathbb{N}_0} \hat{r}_{b}(\gamma,k) b^{\text{walk}_k(x)} b^{\text{walk}_k(y)}, \quad (2.6.25)$$

with $\hat{r}_{b}(\gamma,k) := \prod_{j=1}^{s} \hat{r}_{b}(\gamma_j,k_j)$, where, for $k = \kappa_{a-1}b^{a-1} + \ldots + \kappa_1b + \kappa_0$ with $\kappa_{a-1} \neq 0$, we define

$$\hat{r}_{b}(\gamma,k) := \begin{cases} 
1, & \text{if } k = 0, \\
\frac{\gamma}{2\pi b^{a}} \left( \frac{1}{\sin^2(\kappa_{a-1}\pi/b)} - \frac{1}{3} \right), & \text{if } k > 0.
\end{cases}$$

For $x = \frac{x_1}{b} + \frac{x_2}{b^2} + \ldots$ and $y = \frac{y_1}{b} + \frac{y_2}{b^2} + \ldots$ we define

$$\phi_{\text{ds},b}(x,y) := \begin{cases} 
\frac{1}{b}, & \text{if } x = y, \\
\frac{1}{b^i} - \frac{|x_{0} - y_{0}|(b^{-i}|x_{0} - y_{0}|)}{b^{i+1}}, & \text{if } x_{i_0} \neq y_{i_0} \text{ and } x_i = y_i \text{ for } i = 1,\ldots,i_0 - 1.
\end{cases} \quad (2.6.26)$$

Again, we consider a fixed base $b \geq 2$. Therefore we will write $K_{\text{ds},\gamma}$, $\hat{r}$ and $\phi_{\text{ds}}$ instead of $K_{\text{ds},b,\gamma}$, $\hat{r}_{b}$ and $\phi_{\text{ds},b}$. It is shown in Appendix B that we can write the digital shift invariant kernel as

$$K_{\text{ds},\gamma}(x,y) = \prod_{j=1}^{s} (1 + \gamma_j \phi_{\text{ds}}(x_j,y_j)). \quad (2.6.27)$$

The following results on the mean square worst-case error given by (2.6.23) translate from our analysis of the weighted Hilbert space $H_{\text{wal},s,\gamma}$ by using (2.6.25). The next theorem is a consequence of (2.6.24), Theorem 2.6.3, (2.6.25) and (2.6.27).
Theorem 2.6.4 The mean square worst-case error \( \hat{e}^2(P_n, H_{\text{sob},s,\gamma}) \) for multivariate integration in the weighted Sobolev space \( H_{\text{sob},s,\gamma} \) by using a random digital shift in base \( b \geq 2 \) on the point set \( P_n = \{x_1, \ldots, x_n\} \), with \( x_h = (x_{h,1}, \ldots, x_{h,s}) \), is given by

\[
\hat{e}^2(P_n, H_{\text{sob},s,\gamma}) = -1 + \frac{1}{n^2} \sum_{h,i=1}^{n} \sum_{k \in \mathbb{N}_0^s} \hat{r}_b(\gamma, k) b \text{wal}_k(x_h) b \text{wal}_k(x_i)
\]

where the function \( \phi_{ds,b} \) is given by \((2.6.26)\).

As shown in Theorem 2.4.3 for \( H_{\text{wal},s,\gamma} \), we also obtain an error bound for \( \hat{e}^2(P, H_{\text{sob},s,\gamma}) \), where \( P \) is a randomly digitally shifted digital \((t,m,s)\)-net over \( \mathbb{Z}_b \). In order to stress the dependence of the worst-case error on the generating matrices \( C_1, \ldots, C_s \) we write \( \hat{e}^2(P, H_{\text{sob},s,\gamma}) \) instead of \( \hat{e}^2(P, H_{\text{sob},s,\gamma}) \).

Theorem 2.6.5 Let \( \{x_1, \ldots, x_{bm}\} \) be a digital \((t,m,s)\)-net over \( \mathbb{Z}_b \) generated by the matrices \( C_1, \ldots, C_s \).

1. The mean square worst-case error for integration in the weighted Sobolev space \( H_{\text{sob},s,\gamma} \) using the randomly digitally shifted point set \( \{x_1, \ldots, x_{bm}\} \), where the digital shift is in base \( b \), is given by

\[
\hat{e}^2_{bm}(C_1, \ldots, C_s) = \sum_{k \in D_{\text{net}}} \hat{r}_b(\gamma, k),
\]

where

\[
D_{\text{net}} = \{k \in \mathbb{N}_0^s \setminus \{0\} : C_1^T \text{tr}_m(\vec{k}_1) + \ldots + C_s^T \text{tr}_m(\vec{k}_s) = 0\}.
\]

2. Let \( x_h = (x_{h,1}, \ldots, x_{h,s}) \) for \( h = 1, \ldots, bm \), then we have

\[
\hat{e}^2(P_{bm}, H_{\text{sob},s,\gamma}) = -1 + \frac{1}{bm} \sum_{h=1}^{bm} \prod_{j=1}^{s} (1 + \gamma_j \phi_{ds,b}(x_{h,j}, 0)),
\]

where \( \phi_{ds,b} \) given by \((2.6.26)\).

Before we state bounds on the worst-case error using digitally shifted digital nets, we calculate the sum of \( \hat{r}_b(\gamma, k) \) and calculate (for \( b = 2,3 \)) and give a bound (for \( b > 3 \)) on the sum of \((\hat{r}_b(\gamma, k))^\lambda \) over all \( k \geq 1 \). This is done in the following.

For any \( b \geq 2 \) we have

\[
\sum_{k=1}^{\infty} \hat{r}_b(\gamma, k) = \sum_{\alpha=1}^{\infty} \sum_{k=b^{\alpha-1}}^{b^{\alpha}-1} \hat{r}_b(\gamma, k) = \sum_{\alpha=1}^{\infty} \gamma b^{\alpha-1} \sum_{k=1}^{b-1} \left( \frac{1}{2b^{2\alpha}} \sum_{n=1}^{b-1} \left( \frac{1}{\sin^2((\pi n/b) - \frac{1}{3})} \right) \right).
\]
We have
\[ \sum_{a=1}^{\infty} b^{-a} = \frac{1}{b-1} \quad \text{and} \quad \sum_{\kappa_{a-1}=1}^{b-1} \frac{1}{\sin^2(\kappa_{a-1} \pi/b)} = \frac{b^2 - 1}{3}, \]
where the second equality is shown in Appendix C. Thus we obtain
\[ \sum_{k=1}^{\infty} \hat{r}_b(\gamma, k) = \frac{\gamma}{2b(b-1)} \left( \frac{b^2 - 1}{3} - \frac{b-1}{3} \right) = \frac{\gamma}{6}. \]
Further we have \( \hat{r}_2(\gamma, k) = \frac{\gamma}{3 \cdot 2^{\lambda}} \) for \( k > 0 \) and therefore
\[ \sum_{k=1}^{\infty} \hat{r}_2(\gamma, k)^{\lambda} = \sum_{a=1}^{\infty} \sum_{k=2a-1}^{2a-1} \hat{r}_2(\gamma, k)^{\lambda} = \sum_{a=1}^{\infty} \frac{\gamma^{a\lambda} 2^{a-1}}{3 \cdot 2^{2a\lambda}} \]
for any \( 1/2 < \lambda \leq 1 \). For \( b > 2 \) we estimate \( \sin(\kappa_{a-1} \pi/b) \geq \sin(\pi/b) \geq \frac{3\sqrt{3}}{2b} \)
and therefore
\[ \frac{1}{\sin^2(\kappa_{a-1} \pi/b)} - \frac{1}{3} \leq \frac{4b^2 - 9}{27}. \]
Using this estimation we get
\[ \sum_{k=1}^{\infty} \hat{r}_b(\gamma, k)^{\lambda} \leq \sum_{a=1}^{\infty} \frac{b^{a\lambda}(b-1)(4b^2 - 9)^{\lambda}}{2^{\lambda} b^{2a\lambda} 2^{7\lambda}} = \frac{\gamma \lambda (b-1)(4b^2 - 9)^{\lambda}}{54 \lambda (2^{\lambda} - b)}, \]
for any \( 1/2 < \lambda \leq 1 \). We note that the inequality becomes an equality for \( b = 3 \). We are ready to give upper bounds on the worst-case error.

We define the average of the mean square worst-case error over all generating matrices for digital nets. Let \( M_{b,m} \) be the set of all \( m \times m \) matrices with entries in \( \{0, \ldots, b-1\} \) and let \( C_b := \{(C_1, \ldots, C_s) : C_j \in M_{b,m} \text{ for } j = 1, \ldots, s\} \).
Then we define
\[ \hat{A}_{b,m,s} := \frac{1}{b^{m^2s}} \sum_{(C_1, \ldots, C_s) \in C_b} \hat{e}_{b,m,s}^2(C_1, \ldots, C_s). \quad (2.6.28) \]
In the following lemma we give a formula and an upper bound for this average (compare Lemma 2.4.5).

**Lemma 2.6.6** 1. For \( \hat{A}_{b,m,s} \) defined by (2.6.28) we have
\[ \hat{A}_{b,m,s} = -1 + \frac{1}{b^m} \prod_{j=1}^{s} \left( 1 + \frac{\gamma_j}{6} \right) + \left( 1 - \frac{1}{b^m} \right) \prod_{j=1}^{s} \left( 1 + \frac{\gamma_j}{6b^{2m}} \right). \]
2. Further we have

\[ \hat{A}_{b^m, s} \leq \frac{2}{b^m} \prod_{j=1}^{s} \left( 1 + \frac{\gamma_j}{6} \right). \]

As above (see Theorem 2.4.6) we obtain the following theorem.

**Theorem 2.6.7**

1. There exists a digital \((t, m, s)\)-net over \(\mathbb{Z}_b\) such that the mean square worst-case error for multivariate integration of functions in the weighted Sobolev space \(H_{\text{sob}, s, \gamma}\) is bounded by

\[ \hat{e}_{b^m, s}^2 \leq \hat{c}_{s, b, \gamma, \lambda} b^{-m/\lambda}, \]

where

\[ \hat{c}_{s, 2, \gamma, \lambda} = 2^{1/\lambda} \prod_{j=1}^{s} \left( 1 + \frac{\gamma_j^\lambda}{3^\lambda(2^{2\lambda} - 2)} \right)^{1/\lambda} \]

and

\[ \hat{c}_{s, b, \gamma, \lambda} = 2^{1/\lambda} \prod_{j=1}^{s} \left( 1 + \frac{\gamma_j^\lambda(b - 1)(4b^2 - 9)^\lambda}{54^\lambda(b^2 - b)} \right)^{1/\lambda}, \]

for \(b > 2\),

for any \(1/2 < \lambda \leq 1\).

2. For some \(\lambda \in (1/2, 1]\) assume

\[ \sum_{j=1}^{\infty} \gamma_j^\lambda < \infty. \]  \(\text{(2.6.29)}\)

Then \(\hat{c}_{s, b, \gamma, \lambda} \leq \hat{c}_{\infty, b, \gamma, \lambda} < \infty\) and we have

\[ \hat{e}_{b^m, s}^2 \leq \hat{c}_{\infty, b, \gamma, \lambda} b^{-m/\lambda} \quad \text{for all } s \geq 1. \]

Thus, assuming (2.6.29), there exists a digital \((t, m, s)\)-net over \(\mathbb{Z}_b\) such that the mean square worst-case error is bounded independently of the dimension. Let \(\lambda_0\) be the minimum over all \(\lambda\) which satisfy (2.6.29). Then the \(\epsilon\)-exponent lies in the interval \([1/2, \lambda_0]\).

3. Under the assumption

\[ A := \limsup_{s \to \infty} \frac{\sum_{j=1}^{s} \gamma_j}{\log s} < \infty \]  \(\text{(2.6.30)}\)

we obtain \(\hat{c}_{s, b, \gamma, 1} \leq \tilde{c}_3 s^{C(b)(A+\delta)}\) and therefore

\[ \hat{e}_{b^m, s}^2 \leq \tilde{c}_3 s^{C(b)(A+\delta)} b^{-m} \]

for any \(\delta > 0\), where the constant \(\tilde{c}_3\) depends only on \(\delta\) and \(C(2) = 1/12\) and \(C(b) = (4b^2 - 9)/54b\) for \(b > 2\). Thus, assuming (2.6.30), there exists a digital \((t, m, s)\)-net over \(\mathbb{Z}_b\) such that the worst-case error satisfies bound which depends only polynomially on the dimension. Hence the \(\epsilon\)-exponent is at most 2 and \(s\)-exponent at most \(C(b)A\).
4. Let $\nu$ be the equiprobable measure on the set $C_b$, i.e., $\mu(C_1, \ldots, C_s) = b^{-m/s}$ for all $(C_1, \ldots, C_s) \in C_b$. For $c > 1$ and $1/2 < \lambda \leq 1$, define the set

$$C_b(c, \lambda) = \left\{(C_1, \ldots, C_s) \in C_b : \hat{e}_{b^m,s}(C_1, \ldots, C_s) \leq c^{1/\lambda} \sqrt{\hat{e}_{s,b,\lambda} b^{-m/\lambda}}\right\}. $$

Then

$$\nu(C_b(c, \lambda)) > 1 - c^{-2}. $$

In item 1 of the theorem above we showed that the average over all possible shifts obtains a certain error bound. From this result we can also follow that there exists a shift $\sigma$ such that this error bound is satisfied. We have the following corollary.

**Corollary 2.6.8** There exists a digital shift $\sigma \in [0,1)^s$ and a digital $(t,m,s)$-net over $\mathbb{Z}_b$, with generating matrices $C_1, \ldots, C_s$, such that the worst-case error $e_{b^m,s}((C_1, \ldots, C_s), \sigma)$ for multivariate integration of functions in the weighted Sobolev space $H_{\text{sob},s,\gamma}$ is bounded by

$$e_{b^m,s}((C_1, \ldots, C_s), \sigma) \leq \hat{e}_{s,b,\gamma,\lambda} b^{-m/\lambda},$$

where

$$\hat{e}_{s,2,\gamma,\lambda} = 2^{1/\lambda} \prod_{j=1}^s \left(1 + \frac{\gamma_j^2 \lambda^2}{3^\lambda (2^{2\lambda} - 2)}\right)^{1/\lambda}$$

and

$$\hat{e}_{s,b,\gamma,\lambda} = 2^{1/\lambda} \prod_{j=1}^s \left(1 + \frac{\gamma_j^2 (b-1)(4b^2 - 9)^{\lambda}}{54^\lambda (b^2 - b)}\right)^{1/\lambda}, \text{ for } b > 2,$$

for any $1/2 < \lambda \leq 1$.

### 2.6.3 Concluding remarks

The conditions for tractability and strong tractability in item 2 and 3 in Theorem 2.6.7 are the same as for the worst-case error of integration in weighted Sobolev spaces using shifted lattice rules (see [64]). In [64] it was also shown that there exists a lattice rule which achieves a strong tractability worst-case error bound (and in [7, 26] it was shown that such lattice rules can be constructed with the component-by-component algorithm). Here we showed that similar results can also be achieved using digitally shifted digital $(t,m,s)$-nets.

Indeed, the upper bound in [64] and the upper bound presented in item 1 of Theorem 2.6.7 and in Corollary 2.6.8 are almost the same. The constant for the upper bounds on the worst-case error in the weighted Sobolev space $H_{\text{sob},s,\gamma}$ using shifted lattice rules is of the form

$$\prod_{j=1}^s \left(1 + \frac{2\zeta(2\lambda) \gamma_j^2}{(2\pi^2)^{\lambda}}\right)^{1/\lambda}, \quad (2.6.31)$$
for $1/2 < \lambda \leq 1$. The rest of the error bound is the same. (We note that for lattice rules with $2^m$ points the constant $(2.6.31)$ has to be multiplied with $2^{1/\lambda}$, see [7].) For example, if one takes a lattice rule with $2^m$ points and chooses $\lambda = 1$, then the constant shown here for $b = 2$ and the constant in the error bound for lattice rules is exactly the same.

Note that in item 1 of Theorem 2.6.7 and in Corollary 2.6.8 upper bounds of $O((\log n)^{s-1} n^{-1})$ can be obtained by using ideas of [7]. The upper bound of the worst-case error for shifted lattice rules is of the same convergence order (see [7]) and it is optimal up to a power of $\log n$ as the best possible rate of convergence is $O(n^{-1})$ even for $s = 1$ (see [64]).

For unweighted Sobolev spaces (that is, take all $\gamma_j = 1$) the rate of convergence of the worst-case error for lattice rules is shown to be $O((\log n)^{s-1} n^{-1})$, see [18, 42]. Further, scrambled $(t, m, s)$-nets achieve a convergence rate of $O((\log n)^{(s-1)/2} n^{-1})$ (see for example [16, 19, 21, 52]), which yields a slight improvement in the $\log n$ factor. On the other hand, the importance of suitably chosen weights was pointed out, see for example [71, 72].

In [69, 70], Wang showed that Halton-, Sobol- and Niederreiter sequences achieve strong tractability error bounds under a stronger condition on the weights than presented here. The convergence order for those sequences is shown to be $n^{-1+\delta}$, for any $\delta > 0$. Thus, those results give a constructive approach to strong tractability with the optimal rate of convergence. On the other hand, Corollary 2.6.8 shows the existence of a $(t, m, s)$-net which achieves a strong tractability worst-case error bound with the optimal rate of convergence under a far weaker condition on the weights. Unfortunately, it is not known how to find such a $(t, m, s)$-net. It remains an open question whether known nets and sequences satisfy this existence result.

## 2.7 Appendices

### 2.7.1 Appendix A: Calculation of the digital shift invariant kernel

It can easily be seen that the reproducing kernel

$$K(x, y) = \prod_{j=1}^{s} K_{\gamma_j}(x_j, y_j) = \prod_{j=1}^{s} \left(1 + \gamma_j \left(\frac{1}{2} B_2(\{x_j - y_j\}) + (x_j - \frac{1}{2})(y_j - \frac{1}{2})\right)\right)$$

is in $L_2([0, 1)^{2s})$. Therefore we can use Lemma 2.6.2 to calculate the digital shift invariant kernel. Further, as our Sobolev space is a tensor product of one-dimensional Hilbert spaces and the reproducing kernel is the product of the kernels $K_{\gamma_j}$, we only need to find the digital shift invariant kernels $K_{\text{ds}, \gamma_j}$.
associated to $K_{\gamma}$, that is,

\[
K_{bs, b, \gamma}(x, y) = \int_0^1 K_\gamma(x \oplus \sigma, y \oplus \sigma) \, d\sigma
\]

where we used the facts that $\int_0^{1/2} B_2 \{ (x \oplus \sigma) - (y \oplus \sigma) \} + ((x \oplus \sigma) - \frac{1}{2})((y \oplus \sigma) - \frac{1}{2}) \, d\sigma$

where the digital shift is in base $b$. The digital shift invariant kernel of the $s$-dimensional space will then again just be the product of $K_{bs, b, \gamma_j}$. It is convenient for our analysis to represent this kernel in terms of Walsh functions in base $b$.

We need to find the Walsh coefficients $\hat{K}_b(k, k)$ of the kernel $K_\gamma$. It proves to be convenient to find the Walsh representation of the function $x - \frac{1}{2}$. Let $x = \frac{a_0}{b} \frac{a_1 + \ldots + a_{b-1}}{b} + \ldots + \kappa_1 b + \kappa_0$, where $\kappa_{a-1} \neq 0$. Then we have

\[
\int_0^1 \left( x - \frac{1}{2} \right) \overline{\text{wal}_k(x)} \, dx
\]  

\[
= \frac{1}{b^a} \sum_{x_1=0}^{b-1} \ldots \sum_{x_a=0}^{b-1} e^{-2\pi i (x_1 \kappa_0 + \ldots + x_a \kappa_{a-1})/b} \int_{\frac{x_1}{b} + \ldots + \frac{x_a}{b}}^{\frac{x_1}{b} + \ldots + \frac{x_a}{b}} \left( x - \frac{1}{2} \right) \, dx
\]

where we used the facts that

\[
\int_{\frac{x_1}{b} + \ldots + \frac{x_a}{b}}^{\frac{x_1}{b} + \ldots + \frac{x_a}{b}} \left( x - \frac{1}{2} \right) \, dx = \frac{1}{b^a} \left( \frac{x_1}{b} + \ldots + \frac{x_a}{b} \right) + \frac{1}{2} \cdot \frac{1}{b^a} \left( \frac{1}{b^a} - 1 \right)
\]

and

\[
\sum_{x_a=0}^{b-1} e^{-2\pi i x_a \kappa_{a-1}/b} = 0 \quad \text{for} \quad \kappa_{a-1} \neq 0.
\]

For any $x_1, \ldots, x_{a-1} \in \{0, \ldots, b-1\}$ we have

\[
\sum_{x_a=0}^{b-1} \left( \frac{x_1}{b} + \ldots + \frac{x_{a-1}}{b} + \frac{x_a}{b} \right) e^{-2\pi i x_a \kappa_{a-1}/b} = \sum_{x_a=0}^{b-1} \frac{x_a}{b^a} e^{-2\pi i x_a \kappa_{a-1}/b}
\]

\[
= \frac{b}{b^a \left( e^{-2\pi i \kappa_{a-1}/b} - 1 \right)}
\]

as for $\kappa_{a-1} \neq 0$ we have

\[
\sum_{x_a=0}^{b-1} e^{-2\pi i x_a \kappa_{a-1}/b} = 0 \quad \text{and} \quad \sum_{x_a=0}^{b-1} \frac{b}{e^{-2\pi i \kappa_{a-1}/b} - 1} = \frac{b}{e^{-2\pi i \kappa_{a-1}/b} - 1}.
\]

Therefore we obtain from (2.7.33)

\[
\int_0^1 \left( x - \frac{1}{2} \right) \overline{\text{wal}_k(x)} \, dx
\]

\[
= \frac{1}{b^{2a-1} \left( e^{-2\pi i \kappa_{a-1}/b} - 1 \right)} \sum_{x_1=0}^{b-1} e^{-2\pi i x_1 \kappa_0/b} \ldots \sum_{x_{a-1}=0}^{b-1} e^{-2\pi i x_{a-1} \kappa_{a-2}/b}.
\]
For $0 \leq \kappa < b$ we use
\[
\sum_{x=0}^{b-1} e^{-2\pi i x \kappa / b} = \begin{cases} 
 b, & \text{if } \kappa = 0, \\
 0, & \text{if } \kappa \neq 0,
\end{cases}
\]
and obtain
\[
\int_0^1 \left( x - \frac{1}{2} \right) \tilde{b}\text{wal}_k(x) \, dx = \begin{cases} 
 \frac{1}{b^a(e^{-2\pi i \kappa_{a-1} / b} - 1)}, & \text{if } \kappa_0 = \ldots = \kappa_{a-2} = 0, \\
 0, & \text{otherwise}.
\end{cases}
\]
Thus, for $x \in [0, 1)$, we have
\[
x - \frac{1}{2} = \sum_{a=1}^{\infty} \sum_{\kappa=1}^{b-1} \frac{1}{b^a(e^{-2\pi i \kappa / b} - 1)} \tilde{b}\text{wal}_{\kappa b^a - 1}(x)
\]
\[
= \sum_{a=1}^{\infty} \sum_{\kappa=1}^{b-1} \frac{1}{b^a(e^{2\pi i \kappa / b} - 1)} \tilde{b}\text{wal}_{\kappa b^a - 1}(x),
\]
as $x - \frac{1}{2} = \frac{x - 1}{2}$.

Using these equalities we get for all $x, y \in [0, 1)$ that
\[
(x - \frac{1}{2})(y - \frac{1}{2})
\]
\[
= \sum_{a=1}^{\infty} \sum_{\kappa=1}^{b-1} \sum_{a'=1}^{b-1} \sum_{\kappa'=1}^{b-1} \frac{1}{b^a(e^{-2\pi i \kappa / b} - 1)(e^{2\pi i \kappa' / b} - 1)} \tilde{b}\text{wal}_{\kappa b^a - 1}(x) \tilde{b}\text{wal}_{\kappa' b^{a'} - 1}(y).
\]
Note that Lemma 2.6.2 is true for any function in $L_2([0, 1)^2)$, so that
\[
\int_0^1 ((x \oplus \sigma) - \frac{1}{2})((y \oplus \sigma) - \frac{1}{2}) \, d\sigma
\]
\[
= \sum_{a=1}^{\infty} \sum_{\kappa=1}^{b-1} \frac{1}{b^a(e^{2\pi i \kappa / b} - 1)^2} \tilde{b}\text{wal}_{\kappa b^a - 1}(x) \tilde{b}\text{wal}_{\kappa b^{a'} - 1}(y).
\]
We have $B_2(x) = x^2 - x + \frac{1}{6}$ and it can be shown that $B_2(\{x\}) = B_2(|x|)$ for all $x \in (-1, 1)$. For our investigations here we use $B_2(|x - y|)$ instead of $B_2(\{x - y\})$. By using (2.7.34) we obtain
\[
|x - y|^2
\]
\[
= \left( \sum_{a=1}^{\infty} \sum_{\kappa=1}^{b-1} \frac{1}{b^a(e^{-2\pi i \kappa / b} - 1)} \tilde{b}\text{wal}_{\kappa b^a - 1}(x) - \sum_{a=1}^{b-1} \sum_{\kappa=1}^{b-1} \frac{1}{b^a(e^{-2\pi i \kappa / b} - 1)} \tilde{b}\text{wal}_{\kappa b^a - 1}(y) \right)^2
\]
\[
= \sum_{a=1}^{\infty} \sum_{\kappa=1}^{b-1} \sum_{a'=1}^{b-1} \sum_{\kappa'=1}^{b-1} \frac{1}{b^a(e^{-2\pi i \kappa / b} - 1)(e^{2\pi i \kappa' / b} - 1)} \tilde{b}\text{wal}_{\kappa b^a - 1}(x) \tilde{b}\text{wal}_{\kappa' b^{a'} - 1}(x)
\]
\[
+ \sum_{a=1}^{\infty} \sum_{\kappa=1}^{b-1} \sum_{a'=1}^{b-1} \sum_{\kappa'=1}^{b-1} \frac{1}{b^a(e^{-2\pi i \kappa / b} - 1)} \tilde{b}\text{wal}_{\kappa b^a - 1}(y) \tilde{b}\text{wal}_{\kappa' b^{a'} - 1}(y)
\]
\[
- 2 \sum_{a=1}^{\infty} \sum_{\kappa=1}^{b-1} \sum_{a'=1}^{b-1} \sum_{\kappa'=1}^{b-1} \frac{1}{b^a(e^{-2\pi i \kappa / b} - 1)(e^{2\pi i \kappa' / b} - 1)} \tilde{b}\text{wal}_{\kappa b^a - 1}(x) \tilde{b}\text{wal}_{\kappa' b^{a'} - 1}(y).
\]
By using Lemma 2.6.2 again we obtain
\[
\int_0^1 |(x \oplus \sigma) - (y \oplus \sigma)|^2 \, d\sigma \\
= 2 \sum_{a=1}^{\infty} \sum_{\kappa=1}^{b-1} \frac{1}{b^2a|e^{2\pi i\kappa/b} - 1|^2} - 2 \sum_{a=1}^{\infty} \sum_{\kappa=1}^{b-1} \frac{1}{b^2a|e^{2\pi i\kappa/b} - 1|^2} \, \bar{\text{wal}}_{k\kappa a^{-1}}(x) \, \text{wal}_{k\kappa a^{-1}}(y) \\
= \frac{1}{6} - 2 \sum_{a=1}^{\infty} \sum_{\kappa=1}^{b-1} \frac{1}{b^2a|e^{2\pi i\kappa/b} - 1|^2} \, \bar{\text{wal}}_{k\kappa a^{-1}}(x) \, \text{wal}_{k\kappa a^{-1}}(y),
\] (2.7.36)
where the last equality follows from
\[
2 \sum_{a=1}^{\infty} \sum_{\kappa=1}^{b-1} \frac{1}{b^2a|e^{2\pi i\kappa/b} - 1|^2} = \int_0^1 \int_0^1 |x - y|^2 \, dx \, dy = \frac{1}{6},
\] (2.7.37)

which in turn follows from Lemma 2.6.2.

For the last part, namely \( \int_0^1 |(x \oplus \sigma) - (y \oplus \sigma)| \, d\sigma \), we cannot use the argument above. Instead, by Lemma 2.6.2, there are \( \tau_b(k) \) such that
\[
\int_0^1 |(x \oplus \sigma) - (y \oplus \sigma)| \, d\sigma = \sum_{k=0}^{\infty} \tau_b(k) \, \text{wal}_k(x) \, \text{wal}_k(y). \tag{2.7.38}
\]

For \( k \geq 0 \) we have
\[
\tau_b(k) = \int_0^1 \int_0^1 \int_0^1 |(x \oplus \sigma) - (y \oplus \sigma)| \, \bar{\text{wal}}_k(x) \, \text{wal}_k(y) \, d\sigma \, dx \, dy \\
= \int_0^1 \int_0^1 \int_0^1 |x - y| \, \bar{\text{wal}}_k(x \oplus \sigma) \, \text{wal}_k(y \oplus \sigma) \, d\sigma \, dx \, dy \\
= \int_0^1 \int_0^1 |x - y| \, \bar{\text{wal}}_k(x) \, \text{wal}_k(y) \, dx \, dy.
\]

First, one can show that \( \int_0^1 \int_0^1 |x - y| \, dx \, dy = \frac{1}{3} \) and therefore \( \tau_b(0) = \frac{1}{3} \). For \( k > 0 \) let \( k = \kappa_{a-1}b^{a-1} + \ldots + \kappa_1b + \kappa_0 \), where \( a \) is such that \( \kappa_{a-1} \neq 0 \), \( u = u_{a-1}b^{a-1} + \ldots + u_1b + u_0 \) and \( v = v_{a-1}b^{a-1} + \ldots + v_1b + v_0 \). Then
\[
\tau_b(k) = \int_0^1 \int_0^1 |x - y| \, \bar{\text{wal}}_k(x) \, \text{wal}_k(y) \, dx \, dy \\
= \sum_{u=0}^{b^{a-1}-1} \sum_{u=0}^{b^{a-1}-1} e^{2\pi i(k_0(u_{a-1}-v_{a-1}) + \ldots + \kappa_{a-1}(u_0-v_0))} / b \int_{u/b^a}^{(u+1)/b^a} \int_{v/b^a}^{(v+1)/b^a} |x - y| \, dx \, dy.
\]

We have the following equalities: let \( 0 \leq u < b^a \), then
\[
\int_{u/b^a}^{(u+1)/b^a} \int_{u/b^a}^{(u+1)/b^a} |x - y| \, dx \, dy = \frac{1}{3b^{3a}}
\]
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and for $0 \leq u, v < b^a$, $u \neq v$, we have
\[
\int_{(u+1)/b^a}^{(u+1)/b^a} \int_{v/b^a}^{(v+1)/b^a} |x - y| \, dx \, dy = \frac{|u - v|}{b^{3a}}.
\]
Thus
\[
\tau_b(k) = \sum_{u=0}^{b^a-1} \sum_{v=0}^{b^a-1} e^{2\pi i \kappa_0(u-a-1)+\kappa_1(u_0-v_0))/b} \frac{|u - v|}{b^{3a}}
\]
\[
= \frac{1}{3b^{3a}} + \frac{2}{b^{3a}} \sum_{u=0}^{b^a-1} \sum_{v=0}^{b^a-1} (v - u) e^{2\pi i (\kappa_0(u-a-1)+\kappa_1(u_0-v_0))/b},
\]
(2.7.39)
In the following we will determine the values of $\tau_b(k)$ for any $k > 0$. Let
\[
\theta(u, v) = (v - u) e^{2\pi i (\kappa_0(u-a-1)+\kappa_1(u_0-v_0))/b}.
\]
In order to find the value of the double sum in the expression for $\tau_b(k)$ let $u = u_{a-1}b^{a-1} + \ldots + u_1 b$ and let $v = v_{a-1}b^{a-1} + \ldots + v_1 b$, where $v > u$. Observe that $u$ and $v$ are divisible by $b$, that is $u_0 = v_0 = 0$, and that $k = \kappa_{a-1}b^{a-1} + \ldots + \kappa_1 b + \kappa_0$, where $a$ is such that $\kappa_{a-1} \neq 0$. We have
\[
\left| \sum_{u_0=0}^{b-1} \sum_{v_0=0}^{b-1} \theta(u + u_0, v + v_0) \right| = \left| \sum_{u_0=0}^{b-1} \sum_{v_0=0}^{b-1} (v_0 - u_0) e^{2\pi i \kappa_{a-1}(u_0-v_0)/b} \right|,
\]
(2.7.40)
which follows from $|e^{2\pi i (\kappa_0(u-a-1)+\kappa_1(u_1-v_1))/b}| = 1$ and $\sum_{u_0=0}^{b-1} \sum_{v_0=0}^{b-1} (v_0 - u_0) e^{2\pi i \kappa_{a-1}(u_0-v_0)/b} = 0$. We will show that the sum (2.7.40) is indeed 0. This can be seen by the following:
\[
\sum_{u_0=0}^{b-1} \sum_{v_0=0}^{b-1} (v_0 - u_0) e^{2\pi i \kappa_{a-1}(u_0-v_0)/b}
\]
\[
= \sum_{u_0=0}^{b-1} \sum_{v_0=0}^{b-1} (v_0 - u_0) e^{2\pi i \kappa_{a-1}(u_0-v_0)/b}
\]
\[
= \sum_{u_0=0}^{b-1} \sum_{v_0=u_0+1}^{b-1} (v_0 - u_0) e^{2\pi i \kappa_{a-1}(u_0-v_0)/b} + \sum_{u_0=0}^{b-1} \sum_{v_0=u_0+1}^{b-1} (v_0 - u_0) e^{2\pi i \kappa_{a-1}(u_0-v_0)/b}
\]
\[
= \sum_{u_0=0}^{b-1} \sum_{v_0=u_0+1}^{b-1} (v_0 - u_0) (e^{2\pi i \kappa_{a-1}(u_0-v_0)/b} - e^{-2\pi i \kappa_{a-1}(u_0-v_0)/b})
\]
\[
= 21 \sum_{u_0=0}^{b-1} \sum_{v_0=u_0+1}^{b-1} (v_0 - u_0) \sin(2\pi \kappa_{a-1}(u_0 - v_0)/b).
\]
Let $M = \{(u_0, v_0) : 0 \leq u_0 < v_0 \leq b - 1\}$. For $c \in \{1, \ldots, b - 1\}$ consider the sets $J_c = \{(u_0, v_0) \in M : v_0 - u_0 = c\}$. Let $|J_c|$ be the number of
elements in the set \( J_c \), then \( |J_c| = b - c \). Further we have \( \sin(2\pi \kappa_{a-1} c/b) = -\sin(-2\pi \kappa_{a-1} c/b) = -\sin(2\pi \kappa_{a-1} (b - c)/b) \). If \( c = b - c \), that is \( b = 2c \) we have \( \sin(2\pi \kappa_{a-1} c/b) = \sin(\pi \kappa_{a-1}) = 0 \) and for \( c \in \{1, \ldots, b - 1\} \) with \( c \neq b - c \) we have

\[
\sum_{(u_0, v_0) \in J_c} (v_0 - u_0) - \sum_{(u_0, v_0) \in J_{b-c}} (v_0 - u_0) = |J_c| c - |J_{b-c}| (b - c) = (b - c) c - c(b - c) = 0.
\]

Thus it follows that

\[
\sum_{u_0=0}^{b-1} \sum_{v_0=0}^{b-1} \theta(u + u_0, v + v_0) = 0 \tag{2.7.41}
\]

for any \( 0 \leq u < v \leq b^a - 1 \) which are divisible by \( b \).

Therefore most terms in the double sum in (2.7.39) cancel out. We are left with the following terms: \( \theta(u + u_0, u + v_0) \) for \( u = 0, \ldots, b^a - b \), where \( b | u \), and \( 0 \leq u_0 < v_0 \leq b - 1 \). We have

\[
\theta(u + u_0, u + v_0) = \left( u + v_0 - u - u_0 \right) e^{2\pi i (\kappa_0 (u_0 - u_0) + \ldots + \kappa_{a-2} (u_1 - u_1) + \kappa_{a-1} (u_0 - v_0)) / b}
= \left( v_0 - u_0 \right) e^{2\pi i \kappa_{a-1} (u_0 - v_0) / b}
= \theta(u_0, v_0). \tag{2.7.42}
\]

The sum over all remaining \( \theta(u_0, v_0) \) can be calculated using geometric series. By doing that we obtain

\[
\sum_{u_0=0}^{b-2} \sum_{v_0=u_0+1}^{b-1} \theta(u_0, v_0) = \frac{2b e^{2\pi i \kappa_{a-1} / b}}{(e^{2\pi i \kappa_{a-1} / b} - 1)^2} = -\frac{b}{2 \sin^2(\kappa_{a-1} \pi / b)} \tag{2.7.43}
\]

Combining (2.7.39), (2.7.41), (2.7.42) and (2.7.43) we obtain for \( k > 0 \) that

\[
\tau_b(k) = \frac{1}{3b^{2a}} - \frac{2}{b^{2a}} \frac{b^a}{2 \sin^2(\kappa_{a-1} \pi / b)} = \frac{1}{b^{2a}} \left( \frac{1}{3} - \frac{1}{\sin^2(\kappa_{a-1} \pi / b)} \right), \tag{2.7.44}
\]

where \( k = \kappa_{a-1} b^{a-1} + \ldots + \kappa_1 b + \kappa_0 \), with \( \kappa_{a-1} \neq 0 \). Further we repeat that \( \tau_b(0) = \frac{1}{3} \).

Therefore we obtain from (2.7.32), (2.7.35), (2.7.36), (2.7.38) and (2.7.44) that the digital shift invariant kernel \( K_{ds,b,\gamma} \) associated to \( K_\gamma \) is given by

\[
K_{ds,b,\gamma}(x, y) = 1 + \gamma \left( \sum_{k=1}^{\infty} \frac{-\tau_b(k)}{2} \overline{b_{\text{wal}}(x)} \overline{b_{\text{wal}}(y)} \right).
\]

### 2.7.2 Appendix B: Simplification of the digital shift invariant kernel

Here we will show that the digital shift invariant kernel can be simplified further. For \( a \geq 1 \) and \( 1 \leq \kappa \leq b - 1 \) we define

\[
D_{a,\kappa}(x, y) := \sum_{k=\kappa b^{a-1}}^{(\kappa+1)b^{a-1}-1} \overline{b_{\text{wal}}(x)} \overline{b_{\text{wal}}(y)}.
\]

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Let now \( x = \frac{x_1}{b} + \frac{x_2}{b^2} + \ldots, \ y = \frac{y_1}{b} + \frac{y_2}{b^2} + \ldots \) and \( k = \kappa_{a-1}b^{a-1} \ldots + \kappa_1b + \kappa_0 \) with \( \kappa_{a-1} \neq 0 \). Then for \( 1 \leq \kappa_{a-1} \leq b-1 \) we have

\[
D_{a,\kappa_{a-1}}(x, y) = \sum_{k=\kappa_{a-1}}^{(\kappa_{a-1}+1)b^{a-1}-1} b\text{wal}_k(x) b\text{wal}_k(y)
\]

\[
= \sum_{k=\kappa_{a-1}}^{(\kappa_{a-1}+1)b^{a-1}-1} e^{2\pi i(\kappa_0(x_1-y_1)+\ldots+\kappa_{a-1}(x_a-y_a))/b}
\]

\[
= e^{2\pi i\kappa_{a-1}(x_a-y_a)/b} \sum_{\kappa_0=0}^{b-1} e^{2\pi i\kappa_0(x_1-y_1)/b} \ldots \sum_{\kappa_{a-2}=0}^{b-1} e^{2\pi i\kappa_{a-2}(x_{a-1}-y_{a-1})/b}
\]

\[
= \begin{cases} 
  b^{a-1}e^{2\pi i\kappa_{a-1}(x_a-y_a)/b}, & \text{if } x_i = y_i \text{ for } i \in \{1, \ldots, a-1\}, \\
  0, & \text{otherwise.}
\end{cases}
\]

For \( x = y \) we obtain \( D_{a,\kappa_{a-1}}(x, x) = b^{a-1} \) for all \( a \) and \( \kappa_{a-1} \). Therefore, by using

\[
\sum_{\kappa=1}^{b-1} \frac{1}{\sin^2(\kappa\pi/b)} = \frac{b^2-1}{3},
\]

which is shown in Appendix C, we obtain

\[
\sum_{k=1}^{\infty} \frac{-\tau_b(k)}{2} b\text{wal}_k(x) b\text{wal}_k(x) = \sum_{a=1}^{\infty} \sum_{\kappa=1}^{b^{a-1}} \left( \frac{1}{2 \sin^2(\kappa\pi/b)} - \frac{1}{6} \right) D_{a,\kappa}(x, x)
\]

\[
= \sum_{a=1}^{\infty} \frac{b^2-b}{6b^{2a}} b^{a-1}
\]

\[
= \frac{1}{6}.
\]
Now let \( x \neq y \), more precisely, let \( x_i = y_i \) for \( i = 1, \ldots, i_0 - 1 \) and \( x_{i_0} \neq y_{i_0} \). Then we have

\[
\sum_{k=1}^{\infty} \frac{-\tau_b(k)}{2} \text{walk}_k(x) \text{walk}_k(y)
\]

\[
= \sum_{a=1}^{b-1} \sum_{\kappa=1}^{i_0-1} \frac{1}{b^{2a}} \left( \frac{1}{2 \sin^2(\kappa \pi/b)} - \frac{1}{6} \right) D_{a,\kappa}(x, y)
\]

\[
= \sum_{a=1}^{i_0-1} \sum_{\kappa=1}^{i_0-1} \frac{1}{b^{2a}} \left( \frac{1}{2 \sin^2(\kappa \pi/b)} - \frac{1}{6} \right) D_{i_0,\kappa}(x, y)
\]

\[
+ \sum_{\kappa=1}^{i_0-1} \frac{\frac{1}{b^{2i_0}}}{b^{2a+1}} \sum_{\kappa=1}^{b-1} \left( \frac{1}{2 \sin^2(\kappa \pi/b)} - \frac{1}{6} \right)
\]

\[
+ 1 \frac{\frac{1}{b^{i_0+1}}}{b^{2a+1}} \sum_{\kappa=1}^{b-1} \left( e^{2\pi i \kappa(x_{i_0} - y_{i_0})/b} - \frac{e^{2\pi i \kappa(x_{i_0} - y_{i_0})/b}}{6} \right).
\]

It is shown in Appendix C that

\[
\sum_{\kappa=1}^{b-1} \frac{e^{2\pi i \kappa(x_{i_0} - y_{i_0})/b}}{\sin^2(\kappa \pi/b)} = 2|x_{i_0} - y_{i_0}|(|x_{i_0} - y_{i_0}| - b) + \frac{b^2 - 1}{3}
\]

and

\[
\sum_{\kappa=1}^{b-1} \frac{1}{\sin^2(\kappa \pi/b)} = \frac{b^2 - 1}{3}.
\]

Further, as \( x_{i_0} \neq y_{i_0} \), we have

\[
\sum_{\kappa=1}^{b-1} e^{2\pi i \kappa(x_{i_0} - y_{i_0})/b} = -1.
\]

Therefore we obtain

\[
\sum_{k=1}^{\infty} \frac{-\tau_b(k)}{2} \text{walk}_k(x) \text{walk}_k(y)
\]

\[
= \sum_{a=1}^{i_0-1} \frac{b^2 - b}{6b^{2a+1}} + \frac{1}{b^{i_0+1}} \left( |x_{i_0} - y_{i_0}|(|x_{i_0} - y_{i_0}| - b) + \frac{b^2}{6} \right)
\]

\[
= \frac{1}{6} - \frac{|x_{i_0} - y_{i_0}|(b - |x_{i_0} - y_{i_0}|)}{b^{i_0+1}}.
\]
Thus
\[
\sum_{k=1}^{\infty} \frac{-\tau_b(k)}{2} b \text{wal}_k(x) b \text{wal}_k(y)
\]

\[
= \begin{cases} 
\frac{1}{6}, & \text{if } x = y, \\
\frac{1}{6} - \frac{|x_{i_0} - y_{i_0}|(b - |x_{i_0} - y_{i_0}|)}{b^{i_0+1}}, & \text{if } x_{i_0} \neq y_{i_0} \text{ and } x_i = y_i \text{ for } i = 1, \ldots, i_0 - 1.
\end{cases}
\]

### 2.7.3 Appendix C: A sum

First we show that for \(l \in \{- (b - 1), \ldots, b - 1\}\) we have

\[
\sum_{\kappa=1}^{b-1} \frac{e^{2\pi i \kappa l / b}}{|e^{2\pi i \kappa / b} - 1|^2} = \frac{|l||l| - b}{2} + \frac{b^2 - 1}{12}.
\]  

(2.7.45)

From (2.7.37) we obtain

\[
\frac{1}{12} = \sum_{a=1}^{\infty} \frac{1}{b^{2a}} \sum_{\kappa=1}^{b-1} \frac{1}{|e^{2\pi i \kappa / b} - 1|^2} = \frac{1}{b^2 - 1} \sum_{\kappa=1}^{b-1} \frac{1}{|e^{2\pi i \kappa / b} - 1|^2}.
\]

Hence (2.7.45) holds for \(l = 0\). For \(l \in \{1, \ldots, b - 1\}\) we use (2.7.36), which states that for any \(x, y \in [0, 1)\) we have

\[
\int_0^1 |(x \oplus \sigma) - (y \oplus \sigma)|^2 \, d\sigma = \frac{1}{6} - 2 \sum_{a=1}^{\infty} \sum_{\kappa=1}^{b-1} \frac{1}{b^{2a} |e^{2\pi i \kappa / b} - 1|^2} b \text{wal}_{b\kappa-1}(x) b \text{wal}_{b\kappa-1}(y).
\]

Take \(x = \frac{l}{b}\) and \(y = 0\), then the left hand side of the equation yields

\[
\int_0^1 |(x \oplus \sigma) - (y \oplus \sigma)|^2 \, d\sigma = \int_0^1 \left| \left(\frac{l}{b} \oplus \sigma\right) - \sigma \right|^2 \, d\sigma
\]

\[
= \int_0^{b - l} \left| \left(\frac{l}{b} + \sigma\right) - \sigma \right|^2 \, d\sigma + \int_{b - l}^{1} \left| \left(\frac{l - b}{b} + \sigma\right) - \sigma \right|^2 \, d\sigma
\]

\[
= \int_0^{b - l} \frac{l^2}{b^2} \, d\sigma + \int_{b - l}^{1} \frac{(l - b)^2}{b^2} \, d\sigma
\]

\[
= \frac{b - l}{b} \frac{l^2}{b^2} + \left( 1 - \frac{b - l}{b} \right) \frac{(l - b)^2}{b^2}
\]

\[
= \frac{l(b - l)}{b^2},
\]

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and for the right hand side we obtain

\[
\frac{1}{6} - 2 \sum_{a=1}^{\infty} \sum_{\kappa=1}^{b-1} \frac{1}{b^{2a} |e^{2\pi i \kappa/b} - 1|^2} \operatorname{wal}_{\kappa b^a-1}(x) \operatorname{wal}_{\kappa b^a-1}(y)
\]

\[
= \frac{1}{6} - 2 \sum_{\kappa=1}^{b-1} \frac{e^{2\pi i l \kappa/b}}{b^{2} |e^{2\pi i \kappa/b} - 1|^2} - 2 \sum_{a=2}^{\infty} \frac{1}{b^{2a}} \sum_{\kappa=1}^{b-1} \frac{1}{|e^{2\pi i \kappa/b} - 1|^2}
\]

\[
= \frac{b^2 - 1}{6b^2} - 2 \sum_{\kappa=1}^{b-1} \frac{e^{2\pi i l \kappa/b}}{b^{2} |e^{2\pi i \kappa/b} - 1|^2}.
\]

Thus (2.7.45) follows for \( l \in \{0, \ldots, b - 1\} \). To show that (2.7.45) holds for \( l \in \{-(b - 1), \ldots, -1\} \), use \( x = 0 \) and \( y = -\frac{l}{b} \) in the argument above. The details are omitted.

Further observe that \( |e^{2\pi i \kappa/b} - 1|^2 = |e^{\pi i \kappa/b}|^2 |e^{\pi i \kappa/b} - e^{-\pi i \kappa/b}|^2 = 4 \sin^2(\kappa \pi/b) \) and therefore we have for \( l \in \{-(b - 1), \ldots, b - 1\} \) that

\[
\sum_{\kappa=1}^{b-1} \frac{e^{2\pi i l \kappa/b}}{\sin^2(\kappa \pi/b)} = 2(|l|(|l| - b) + \frac{b^2 - 1}{3}).
\]
Chapter 3

Construction algorithms for polynomial lattice rules for multivariate integration

3.1 Polynomial lattice rules

In [41] (see also [42, Section 4.4]) Niederreiter introduced a special family of digital \((t, m, s)\)-nets over a finite field \(F_b\). Those nets are obtained from rational functions over finite fields. For a prime-power \(b\) let \(F_b((x^{-1}))\) be the field of formal Laurent series over \(F_b\). Elements of \(F_b((x^{-1}))\) are formal Laurent series,

\[
L = \sum_{l=w}^{\infty} t_l x^{-l},
\]

where \(w\) is an arbitrary integer and all \(t_l \in F_b\). Note that \(F_b((x^{-1}))\) contains the field of rational functions over \(F_b\) as a subfield. Further let \(F_b[x]\) be the set of all polynomials over \(F_b\).

**Definition 3.1.1** For a given dimension \(s \geq 1\), choose \(p \in F_b[x]\) with \(\deg(p) = m \geq 1\) and let \(q_1, \ldots, q_s \in F_b[x]\). For \(1 \leq j \leq s\), consider the expansions

\[
\frac{q_j(x)}{p(x)} = \sum_{l=w_j}^{\infty} u_{l}^{(j)} x^{-l} \in F_b((x^{-1}))
\]

where \(w_j \leq 1\). Consider the \(m \times m\) matrices \(C_1, \ldots, C_s\) over \(F_b\) where the elements \(c_{i,r}^{(j)}\) of the matrix \(C_j\) are given by

\[
c_{i,r}^{(j)} = u_{r+i}^{(j)} \in F_b, \tag{3.1.1}
\]

for \(1 \leq j \leq s\), \(1 \leq i \leq m\), \(0 \leq r \leq m - 1\) and construct a digital \((t, m, s)\)-net over \(F_b\) with these \(s\) matrices. The digital net given by the polynomials \(p\) and \(q := (q_1, \ldots, q_s) \in F_b^s\) is denoted by \(S_p(q)\). A quasi-Monte Carlo rule using the point set \(S_p(q)\) is called a polynomial lattice rule.
For the determination of the quality parameter $t$ of $S_p(q)$ we refer to [42, Theorem 4.42]. We continue by stating some properties of the construction of the point set $S_p(q)$.

**Remark 3.1.2** Note that the matrices defined by (3.1.1) are of the form

$$C_j = C_{q_j,p} = \begin{pmatrix} u_1^{(j)} & u_2^{(j)} & \cdots & u_m^{(j)} \\ u_2^{(j)} & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ u_m^{(j)} & \cdots & u_{2m-1}^{(j)} \end{pmatrix},$$

i.e., the matrix $C_j$ is a so-called Hankel matrix associated with the linear recurring sequence $(u_1^{(j)}, u_2^{(j)}, \ldots)$. If $\gcd(q_j, p) = 1$, then $p$ is called the minimal polynomial of the linear recurring sequence (see for example [42, Appendix A]) and $C_j$ is non-singular (see [36, Theorem 6.75]).

**Remark 3.1.3** From Remark 3.1.2 it follows that if $\gcd(q_j, p) = 1$ for all $1 \leq j \leq s$, then each one-dimensional projection of the point set $S_p(q)$ to the $j$-th coordinate is a $(0, m, 1)$-net over $\mathbb{F}_b$.

The following result concerning the determination of the Laurent series coefficients of rational functions was already stated in [34, 54, 59] for the case $b = 2$.

**Proposition 3.1.4** For $p \in \mathbb{F}_b[x]$, $p = x^m + p_1 x^{m-1} + \cdots + p_{m-1} x + p_m$ and $q \in \mathbb{F}_b[x]$, $q(x) = c_1 x^{m-1} + \cdots + c_{m-1} x + c_m$, the coefficients $u_l$, $l \geq 1$, in the Laurent series expansion of

$$\frac{q(x)}{p(x)} = \sum_{l=1}^{\infty} u_l x^{-l}$$

can be computed as follows: the first $m$ coefficients $u_1, \ldots, u_m$ are obtained by solving the linear system

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ p_1 & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ p_{m-1} & \cdots & p_1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix},$$

and for $l > m$, $u_l$ is obtained from the linear recursion in $\mathbb{F}_b$,

$$u_l + u_{l-1} p_1 + u_{l-2} p_2 + \cdots + u_{l-m} p_m = 0.$$

**Proof.** Consider $q(x) = p(x) \sum_{l=1}^{\infty} u_l x^{-l}$ and compare the coefficients of $x^l$, $l \in \mathbb{Z}$, on both sides of the equation. \qed
Remark 3.1.5 If $b$ is a prime, then there is an equivalent but simpler form of the construction of the point set $S_p(q)$, see [43]. For an integer $m \geq 1$ let $v_m$ be the map from $\mathbb{F}_b((x^{-1}))$ to the interval $[0, 1)$ defined by

$$v_m \left( \sum_{l=w}^{\infty} t_l x^{-l} \right) = \sum_{l=\max(1, w)}^{m} t_l b^{-l}.$$ 

Choose $p$ and $q$ as above. Since $b$ is prime, \{0, \ldots, b - 1\} and $\mathbb{F}_b$ can be identified and so the bijection $\varphi$ from Definition 2.3.2 can be taken as identity map.

For $0 \leq h < b^m$ let $h = h_0 + h_1 b + \cdots + h_{m-1} b^{m-1}$ be the $b$-adic expansion of $h$. With each such $h$ we associate the polynomial

$$h(x) = \sum_{r=0}^{m-1} h_r x^r \in \mathbb{F}_b[x].$$

Then $S_p(q)$ is the point set consisting of the $b^m$ points

$$x_h = \left( v_m \left( \frac{h(x)q_1(x)}{p(x)} \right), \ldots, v_m \left( \frac{h(x)q_s(x)}{p(x)} \right) \right) \in [0, 1)^s,$$

for $0 \leq h < b^m$.

Finally we introduce some notation: for arbitrary $k = (k_1, \ldots, k_s) \in \mathbb{F}_b[x]^s$ and $q = (q_1, \ldots, q_s) \in \mathbb{F}_b[x]^s$, we define the ‘inner product’

$$k \cdot q = \sum_{j=1}^{s} k_j q_j \in \mathbb{F}_b[x]$$

and we write $q \equiv 0 \pmod{p}$ if $p$ divides $q$ in $\mathbb{F}_b[x]$. Further, as in Remark 3.1.5, for $b$ prime we associate a non-negative integer $k = \kappa_0 + \kappa_1 b + \cdots + \kappa_a b^a$ with the polynomial $k(x) = \kappa_0 + \kappa_1 x + \cdots + \kappa_a x^a \in \mathbb{F}_b[x]$.

3.2 Multivariate integration in weighted Hilbert spaces based on Walsh functions

In this section we introduce and analyze two construction algorithms for polynomial lattice rules based on the worst-case error, see Subsections 3.2.1 and 3.2.2. A formula for the worst-case error is presented in the following subsection.

Henceforth, in order to have a simple notation, we restrict $b$ to prime numbers. In this case the finite field $\mathbb{F}_b$ is just $\mathbb{Z}_b$, the least residue ring modulo $b$, and the bijections in Definition 2.3.2 can be chosen as identities. We remark that the subsequent results can also be obtained for arbitrary finite fields $\mathbb{F}_b$.  

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3.2.1 The component-by-component construction of polynomial lattice rules for integration in $H_{\text{wal},b,s,\alpha,\gamma}$

For a non-negative integer $k$ with $b$-adic expansion $k = \kappa_0 + \kappa_1 b + \cdots$, we write
$$\text{tr}_m(k) = \kappa_0 + \kappa_1 b + \cdots + \kappa_m b^m,$$
thus the associated polynomial $\text{tr}_m(k)(x) \in \mathbb{Z}_b[x]$ has degree $< m$. For a vector $k \in \mathbb{N}_0^s$, $\text{tr}_m(k)$ is defined component wise. In the following lemma $\text{tr}_m(k)$ is considered as a vector of polynomials in $\mathbb{Z}_b[x]$, see Section 3.1.

**Lemma 3.2.1** Let $p \in \mathbb{Z}_b[x]$ with $\deg(p) = m \geq 1$ and let $q = (q_1, \ldots, q_s) \in \mathbb{Z}_b[x]^s$. Let $\{x_0, \ldots, x_{b^m-1}\}$ be the point set $S_p(q)$. Then the square worst-case error for integration in the weighted Hilbert space $H_{\text{wal},b,s,\alpha,\gamma}$ is given by
$$e_{b,m,s}^2(q, p) = \sum_{k \in D_{pl}} r_b(\alpha, \gamma, k),$$
where
$$D_{pl} = \{k \in \mathbb{N}_0^s \setminus \{0\} : \text{tr}_m(k) \cdot q \equiv 0 \pmod{p}\}.$$

**Remark 3.2.2** Compare the above lemma with [64, formula (15)], which states that the squared worst-case error $\bar{e}_{n,s}^2$ for integration in the weighted Korobov space using an $n$ point lattice rule with generating vector $z$ is given by
$$\bar{e}_{n,s}^2(z) = \sum_{k \in \mathcal{L}} \bar{r}(\alpha, \gamma, k),$$
where $\mathcal{L} = \{k \in \mathbb{Z}^s \setminus \{0\} : k \cdot z \equiv 0 \pmod{n}\}$ and $\bar{r}(\alpha, \gamma, k) = \prod_{j=1}^s \bar{r}(\alpha, \gamma_j, k_j)$ with $\bar{r}(\alpha, \gamma, 0) = 1$ and $\bar{r}(\alpha, \gamma, k) = \gamma|k|^{-\alpha}$ for $k \neq 0$. The set $\mathcal{L}$ is called the dual lattice (see [60]), accordingly we call $D_{pl}$ the dual polynomial lattice.

**Proof.** The result follows from the fact that if $C_1, \ldots, C_s$ are generating matrices for the point set $S_p(q)$, then for any $k = (k_1, \ldots, k_s) \in \mathbb{N}_0^s \setminus \{0\}$ we have
$$C_1^T \text{tr}_m(k_1) + \cdots + C_s^T \text{tr}_m(k_s) = \vec{0} \quad \text{iff} \quad \text{tr}_m(k) \cdot q \equiv 0 \pmod{p}.$$ (3.2.2)
This was first proved in [42, Lemma 4.40].

Let $R_{b,m}$ be the set of all non-zero polynomials in $\mathbb{Z}_b[x]$ with degree at most $m - 1$, that is,
$$R_{b,m} := \{q \in \mathbb{Z}_b[x] : \deg(q) < m \text{ and } q \neq 0\}.$$ Obviously we have $|R_{b,m}| = b^m - 1$.

It follows from the construction principle presented in Section 3.1 that the polynomials $q_j$ can be restricted to the set $R_{b,m}$. Using 2 of Theorem 2.4.3 we can therefore use computers to search for good polynomials $q_j$. In the following we present an efficient algorithm for such a computer search.
Algorithm 1 Given a prime number $b$, a dimension $s$, an integer $m \geq 1$ and weights $\gamma = (\gamma_j)_{j \geq 1}$.

1. Choose an irreducible polynomial $p \in \mathbb{Z}_b[x]$ with $\deg(p) = m$.

2. Set $q_1 = 1$.

3. For $d = 2, 3, \ldots, s$, find $q_d \in \mathbb{R}_b,m$ by minimizing $e_{b,m}^2((q_1, \ldots, q_d), p)$.

Theorem 3.2.3 Let $b$ be prime and $p \in \mathbb{Z}_b[x]$ be irreducible, with $\deg(p) = m \geq 1$. Suppose $(q_1^*, \ldots, q_s^*) \in \mathbb{R}_b,m$ is constructed by Algorithm 1. Then for all $d = 1, 2, \ldots, s$ we have

$$e_{b,m}^2((q_1^*, \ldots, q_d^*), p) \leq (b^m - 1)^{-\frac{1}{2}} \prod_{j=1}^{d} (1 + \mu(\alpha \lambda) \gamma_j^\lambda)^\frac{1}{2}$$

for all $\frac{1}{\alpha} < \lambda \leq 1$, where $\mu$ is given by (2.2.8).

Remark 3.2.4 We have $(b^m - 1)^{-1} \leq 2b^{-m}$ for all $m \geq 2$ and therefore it follows that digital nets satisfying the existence result of Theorem 2.4.6 can be constructed by Algorithm 1.

Proof. It is not hard to show that the result is true for $d = 1$. Suppose for some $1 \leq d < s$ we have $q^*_d \in \mathbb{R}_b,m$ and

$$e_{b,m}^2(q^*, p) \leq (b^m - 1)^{-\frac{1}{2}} \prod_{j=1}^{d} (1 + \mu(\alpha \lambda) \gamma_j^\lambda)^\frac{1}{2}, \quad (3.2.3)$$

for all $\frac{1}{\alpha} < \lambda \leq 1$. Now we consider $e_{b,m,d+1}^2((q^*, q_{d+1}), p)$. It follows from Lemma 3.2.1 that

$$e_{b,m,d+1}^2((q^*, q_{d+1}), p) = \sum_{(k,k_{d+1}) \in \mathbb{N}_0 \setminus \{0\}} r_b(\alpha, \gamma, k) \prod_{j=1}^{d+1} (1 + \mu(\alpha \lambda) \gamma_j^\lambda)^\frac{1}{2}$$

where we have separated out the $k_{d+1} = 0$ terms and

$$e_{b,m,d+1}^2((q^*, q_{d+1}), p) = e_{b,m,d}^2(q^*, p) + \theta(q_{d+1}), \quad (3.2.4)$$

where

$$\theta(q_{d+1}) = \sum_{k_{d+1}=1}^{\infty} \left( r_b(\alpha, \gamma, k_{d+1}) \sum_{k \in \mathbb{N}_0} r_b(\alpha, \gamma, k) \right)$$

We see from the algorithm that $q^*_{d+1}$ is chosen to minimize $e_{b,m,d+1}^2((q^*, q_{d+1}), p)$. Since the only dependency on $q_{d+1}$ is in $\theta(q_{d+1})$, we have $\theta(q^*_{d+1}) \leq \theta(q_{d+1})$.
for all \( q_{d+1} \in R_{b,m} \), which implies that for any \( \lambda \leq 1 \) we have \([\theta(q_{d+1}^*)]^\lambda \leq [\theta(q_{d+1})]^\lambda\) for all \( q_{d+1} \in R_{b,m} \). This leads to

\[
\theta(q_{d+1}^*) \leq \left( \frac{1}{b^m - 1} \sum_{q_{d+1} \in \mathbb{Z}} [\theta(q_{d+1})]^\lambda \right)^{\frac{1}{\lambda}}. \tag{3.2.5}
\]

We will obtain a bound on \( \theta(q_{d+1}^*) \) through this last inequality.

Let \( \lambda \) satisfy \( \frac{1}{\alpha} < \lambda \leq 1 \). It follows from Jensen’s inequality and the property \([r_b(\alpha, \gamma, k)]^\lambda = r_b(\alpha \lambda, \gamma^\lambda, k)\) that

\[
[\theta(q_{d+1}^*)]^\lambda \leq \sum_{k_{d+1} = 1}^{\infty} \left( r_b(\alpha \lambda, \gamma_{d+1}^\lambda, k_{d+1}) \sum_{k \in \mathbb{N}_0^d} r_b(\alpha \lambda, \gamma^\lambda, k) \right)_{tr_m(k) \equiv -tr_m(k_{d+1}) \cdot q_{d+1} \pmod{p}},
\]

where \( \gamma^\lambda \) denotes the sequence \( (\gamma_j^\lambda)_{j \geq 1} \). If \( k_{d+1} \) is a multiple of \( b^m \), then \( tr_m(k_{d+1}) = 0 \) and the corresponding term in the sum is independent of \( q_{d+1} \). If \( k_{d+1} \) is not a multiple of \( b^m \), then \( tr_m(k_{d+1}) \) can have any value between 1 and \( b^m - 1 \). Moreover, since \( q_{d+1} \neq 0 \) and \( p \) is irreducible, \( tr_m(k_{d+1}) \cdot q_{d+1} \) is never a multiple of \( p \).

By averaging over all \( q_{d+1} \in R_{b,m} \), with the above discussion in mind, we obtain

\[
\frac{1}{b^m - 1} \sum_{q_{d+1} \in \mathbb{Z}} [\theta(q_{d+1})]^\lambda \leq \sum_{k_{d+1} = 1}^{\infty} \left( r_b(\alpha \lambda, \gamma_{d+1}^\lambda, k_{d+1}) \sum_{k \in \mathbb{N}_0^d} r_b(\alpha \lambda, \gamma^\lambda, k) \right)_{tr_m(k) \equiv 0 \pmod{p}} + \frac{1}{b^m - 1} \sum_{k_{d+1} = 1}^{\infty} \left( r_b(\alpha \lambda, \gamma_{d+1}^\lambda, k_{d+1}) \sum_{k \in \mathbb{N}_0^d} r_b(\alpha \lambda, \gamma^\lambda, k) \right)_{tr_m(k) \equiv 0 \pmod{p}} \leq \frac{\mu(\alpha \lambda) \gamma_{d+1}^\lambda}{b^m - 1} \prod_{j=1}^{d} (1 + \mu(\alpha \lambda) \gamma_j^\lambda), \tag{3.2.6}
\]

where the first inequality follows from the fact that if \( k_{d+1} \) is not a multiple of \( b^m \) then

\[
\sum_{q_{d+1} \in \mathbb{Z}} \sum_{k \in \mathbb{N}_0^d} \sum_{tr_m(k) \equiv -tr_m(k_{d+1}) \cdot q_{d+1} \pmod{p}} r_b(\alpha \lambda, \gamma^\lambda, k) = \sum_{k \in \mathbb{N}_0^d} r_b(\alpha \lambda, \gamma^\lambda, k), \quad \sum_{tr_m(k) \equiv 0 \pmod{p}} r_b(\alpha \lambda, \gamma^\lambda, k).
\]
and the second inequality is obtained using the following:

\[
\sum_{k_{d+1}=1}^{\infty} r_b(\alpha \lambda, \gamma_{d+1}, k_{d+1}) = \frac{\mu(\alpha \lambda) \gamma_{d+1}}{b^{\lambda m}},
\]

\[
\sum_{k_{d+1}=1}^{\infty} r_b(\alpha \lambda, \gamma_{d+1}, k_{d+1}) = \sum_{k_{d+1}=1}^{\infty} r_b(\alpha \lambda, \gamma_{d+1}, k_{d+1}) = \mu(\alpha \lambda) \gamma_{d+1},
\]

\[
\sum_{\mathbf{tr}_m(k) \not\equiv 0 (\text{mod } p)} r_b(\alpha \lambda, \gamma_{d+1}, k_{d+1}) = \sum_{\mathbf{tr}_m(k) \not\equiv 0 (\text{mod } p)} r_b(\alpha \lambda, \gamma_{d+1}, k_{d+1}).
\]

Thus we have from (3.2.5) and (3.2.6) that

\[
\theta(q_{d+1}^*) \leq \mu(\alpha \lambda) \frac{1}{2} \gamma_{d+1} (b^m - 1)^{-\frac{1}{2}} \prod_{j=1}^{d} (1 + \mu(\alpha \lambda) \gamma_{j})^{\frac{1}{2}},
\]

which, together with (3.2.3) and (3.2.4), yields

\[
e_{b^m,d+1}(q^*, q_{d+1}^*) = e_{b^m,d}(q^*, p) + \theta(q_{d+1}^*)
\]

\[
\leq \left(1 + \mu(\alpha \lambda) \frac{1}{2} \gamma_{d+1}\right) (b^m - 1)^{-\frac{1}{2}} \prod_{j=1}^{d} (1 + \mu(\alpha \lambda) \gamma_{j})^{\frac{1}{2}}
\]

\[
\leq (b^m - 1)^{-\frac{1}{2}} \prod_{j=1}^{d+1} (1 + \mu(\alpha \lambda) \gamma_{j})^{\frac{1}{2}}.
\]

Hence by induction the result follows for all \(d = 1, 2, \ldots, s\). \(\square\)

**Corollary 3.2.5** Let \(b\) be prime, \(p \in \mathbb{Z}_b[x]\) be irreducible with \(\deg(p) = m \geq 1\) and \(n = b^m\). Suppose \(q^* \in R_{b,m}^s\) is constructed by Algorithm 1.

1. We have

\[
e_{n,s}(q^*, p) \leq c_{s,\alpha,\gamma,\delta} n^{-\frac{\alpha - 1}{2}} + \delta \quad \text{for all } 0 < \delta \leq \frac{\alpha - 1}{2},
\]

where

\[
c_{s,\alpha,\gamma,\delta} := 2^{\frac{\alpha - \delta}{2}} \prod_{j=1}^{s} \left(1 + \mu \left(\frac{\alpha}{\alpha - 2\delta}\right) \gamma_{j}^{-\frac{1}{2} - \delta}\right)^{\frac{\alpha}{2} - \delta}.
\]

2. Suppose

\[
\sum_{j=1}^{\infty} \gamma_{j}^{-\frac{1}{2} - \delta} < \infty.
\]

Then \(c_{s,\alpha,\gamma,\delta} \leq c_{\in\infty,\alpha,\gamma,\delta} < \infty\) and we have

\[
e_{n,s}(q^*, p) \leq c_{\in\in\in,\alpha,\gamma,\delta} n^{-\frac{\alpha - 1}{2} + \delta} \quad \text{for all } 0 < \delta \leq \frac{\alpha - 1}{2}.
\]

Thus the worst-case error is bounded independently of the dimension.
3. Under the assumption

$$A := \limsup_{s \to \infty} \frac{\sum_{j=1}^{s} \gamma_j}{\log s} < \infty$$

we obtain $c_{s,\alpha,\gamma,(\alpha-1)/2} \leq \tilde{c}_\eta s^{\mu(\alpha)(A+\eta)}$ and therefore

$$e_{n,s}(q^*,p) \leq \tilde{c}_\eta s^{\mu(\alpha)(A+\eta)/2} n^{-\frac{1}{2}}$$

for all $\eta > 0$,

where the constant $\tilde{c}_\eta$ depends only on $\eta$. Thus the worst-case error satisfies a bound which depends only polynomially on the dimension.

**Proof.** The first part of the theorem follows from Theorem 3.2.3 by setting $\frac{1}{\chi} = \alpha - 2\delta$ and noting that $b^n - 1 = n - 1 \geq \frac{n}{2}$ for all $n \geq 2$.

Further we have

$$\prod_{j=1}^{s} \left(1 + \mu \left(\frac{\alpha}{\alpha - 2\delta}\right) \gamma_j^{\frac{1}{\alpha - 2\delta}}\right) = \exp \left(\sum_{j=1}^{s} \log \left(1 + \mu \left(\frac{\alpha}{\alpha - 2\delta}\right) \gamma_j^{\frac{1}{\alpha - 2\delta}}\right)\right) \leq \exp \left(\mu \left(\frac{\alpha}{\alpha - 2\delta}\right) \sum_{j=1}^{s} \gamma_j^{\frac{1}{\alpha - 2\delta}}\right),$$

where we used the fact that $\log(1+x) \leq x$ for all $x \geq 0$. Therefore $c_{\infty,\alpha,\gamma,\delta} < \infty$ provided that (3.2.7) is satisfied. Obviously $c_{s,\alpha,\gamma,\delta} \leq c_{\infty,\alpha,\gamma,\delta}$ and so the second part follows.

For the third part of the theorem observe that $A < \infty$ and therefore for any positive $\eta$ there exists a positive $s_\eta$ such that

$$\sum_{j=1}^{s} \gamma_j \leq (A + \eta) \log s \quad \text{for all } s \geq s_\eta.$$}

Hence

$$c_{s,\alpha,\gamma,(\alpha-1)/2}^2 = 2 \prod_{j=1}^{s} \left(1 + \gamma_j \mu(\alpha)\right) = 2 s^{\sum_{j=1}^{s} \log(1 + \gamma_j \mu(\alpha))/\log s} \leq 2 s^{\mu(\alpha) \sum_{j=1}^{s} \gamma_j/\log s} \leq 2 s^{\mu(\alpha)(A+\eta)},$$

for any $\eta > 0$ and all $s \geq s_\eta$. Thus there is a constant $\tilde{c}_\eta$ such that

$$c_{s,\alpha,\gamma,(\alpha-1)/2} \leq \tilde{c}_\eta s^{\mu(\alpha)(A+\eta)/2}$$

and the result follows. \qed

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3.2.2 A Korobov type construction of polynomial lattice rules for integration in $H_{\text{wal}, b, s, \alpha, \gamma}$

In the method of good lattice points one often restricts the attention to lattice points whose coordinates are successive powers of a single integer. Such a choice was first proposed by Korobov, see [24], and therefore such lattice points are often called \textit{Korobov lattice points}. Here we consider now $s$-tuples $\mathbf{q} = (q_1, \ldots, q_s)$ of polynomials that are obtained by taking a polynomial $q \in R_{b,m}$ and putting $q_j \equiv q^j - 1 \pmod{p}$ with $\deg q_j < m$, for $1 \leq j \leq s$. For such $s$-tuples we use the notation $v_s(q) \equiv (1, q, q^2, \ldots, q^{s-1}) \pmod{p}$.

\textbf{Algorithm 2} Given a prime number $b$, a dimension $s \geq 2$, an integer $m \geq 1$ and weights $\gamma = (\gamma_j)_{j \geq 1}$.

1. Choose an irreducible polynomial $p \in \mathbb{Z}_b[x]$ with $\deg(p) = m$.
2. Find $\tilde{q} \in R_{b,m}$ by minimizing $e_{b,m}^2(v_s(q), p)$.

We have the following error bound.

\textbf{Theorem 3.2.6} Let $b$ be prime, $s \geq 2$ and let $p \in \mathbb{Z}_b[x]$ be irreducible with $\deg(p) = m \geq 1$. A minimizer $\tilde{q}$ obtained from Algorithm 2 satisfies

$$e_{b,m}^2(v_s(\tilde{q}), p) \leq \frac{s - 1}{b^m - 1} \prod_{j=1}^{s} (1 + \gamma_j \mu(\alpha)).$$

\textbf{Proof.} Let $\tilde{q}$ be a minimizer of $e_{b,m}^2(v_s(q), p)$. We are interested in how small $e_{b,m}^2(v_s(\tilde{q}), p)$ is. To this end we define

$$M_s(p) := \frac{1}{b^m - 1} \sum_{q \in R_{b,m}} e_{b,m}^2(v_s(q), p).$$

From Lemma 3.2.1 we obtain

$$M_s(p) = \frac{1}{b^m - 1} \sum_{q \in R_{b,m}} \sum_{k \in D_{pl}} r_b(\alpha, \gamma, k)$$
$$= \frac{1}{b^m - 1} \sum_{k \in \mathbb{N}_0^s \setminus \{0\}} r_b(\alpha, \gamma, k) \sum_{q \in R_{b,m}, \text{tr}_m(k) \cdot v_s(q) \equiv 0 \pmod{p}} 1.$$

Now we recall that for an irreducible polynomial $p \in \mathbb{Z}_b[x]$ with $\deg(p) = m \geq 1$, and a nonzero $(k_1, \ldots, k_s) \in \mathbb{Z}_b[x]^s$ with $\deg(k_j) < m$, $j = 1, \ldots, s$, the congruence

$$k_1 + k_2q + \cdots + k_sq^{s-1} \equiv 0 \pmod{p}$$

has no solution if $k_2 = \cdots = k_s = 0$, and it has at most $s - 1$ solutions $q \in R_{b,m}$ otherwise.

For $k = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$, $k \neq 0$, we consider two cases:
1. For \( j = 2, \ldots, s \) let \( k_j = b^m l_j, \ l_j \geq 0 \). In this case we have \( \text{tr}_m(k_j) = 0 \) for \( 2 \leq j \leq s \) and therefore
\[
\sum_{q \in R_{b,m} \atop \text{tr}_m(k) \cdot w_s(q) \equiv 0 \pmod{p}} 1 = 0.
\]

2. For \( j = 2, \ldots, s \) let \( k_j = k_j^* + b^m l_j, \ l_j \geq 0, \ 0 \leq k_j^* \leq b^m - 1 \) and \((k_2^*, \ldots, k_s^*) \neq (0, \ldots, 0)\). Then we obtain
\[
\sum_{q \in R_{b,m} \atop \text{tr}_m(k) \cdot w_s(q) \equiv 0 \pmod{p}} 1 \leq s - 1.
\]

Now we have

\[
M_s(p) \leq \frac{s - 1}{b^m - 1} \left[ \sum_{k_1=0}^{\infty} \sum_{k_2^* \ldots k_s^* = 0}^{b^m - 1} \sum_{(k_2^*, \ldots, k_s^*) \neq (0, \ldots, 0)} r_b(\alpha, \gamma_1, k_1) \prod_{j=2}^{s} r_b(\alpha, \gamma_j, k_j^* + l_j b^m) \right]
\]

\[
= \frac{s - 1}{b^m - 1} (1 + \gamma_1 \mu(\alpha)) \left[ \sum_{k_2^*, \ldots, k_s^* = 0}^{\infty} \prod_{j=2}^{s} r_b(\alpha, \gamma_j, k_j) - \sum_{l_2^*, \ldots, l_s^* = 0}^{\infty} \prod_{j=2}^{s} r_b(\alpha, \gamma_j, l_j b^m) \right]
\]

\[
= \frac{s - 1}{b^m - 1} (1 + \gamma_1 \mu(\alpha)) \left[ \prod_{j=2}^{s} (1 + \gamma_j \mu(\alpha)) - \prod_{j=2}^{s} \left(1 + \gamma_j b^m \mu(\alpha) \right) \right]
\]

and therefore

\[
M_s(p) \leq \frac{s - 1}{b^m - 1} \prod_{j=1}^{s} (1 + \gamma_j b^m \mu(\alpha))
\]

But since \( e_{b,m,s}(v_s(\tilde{q}), p) \) is no larger than the average \( M_s(p) \) the result follows. \( \square \)

**Corollary 3.2.7** Let \( b \) be prime, \( s \geq 2, \ p \in \mathbb{Z}_b[x] \) be irreducible with \( \deg(p) = m \geq 1 \) and \( n = b^m \). Suppose \( \tilde{q} \in R_{b,m} \) is constructed by Algorithm 2.

1. We have
\[
e_{n,s}(v_s(\tilde{q}), p) \leq c_{s,\alpha, \gamma, \delta} s^{\frac{\alpha}{2} - \delta} n^{\frac{\alpha}{2} + \delta} \text{ for all } 0 < \delta \leq \frac{\alpha - 1}{2},
\]

where
\[
c_{s,\alpha, \gamma, \delta} := 2^{\frac{\alpha}{2} - \delta} \prod_{j=1}^{s} \left(1 + \gamma_j b^m \mu(\frac{\alpha}{\alpha - 25}) \right)^{\frac{\alpha}{2} - \delta}.
\]
2. Under the assumption

\[ A := \limsup_{s \to \infty} \frac{\sum_{j=1}^{s} \gamma_j}{\log s} < \infty \]

we obtain \( c_{s,\alpha,\gamma,\gamma}^{(\alpha-1)/2} \leq \tilde{c}_\eta s^{(1+\mu(\alpha)(A+\eta))/2} \) and therefore

\[ e_{n,s}(\nu_s(\hat{q}), p) \leq \tilde{c}_\eta s^{(1+\mu(\alpha)(A+\eta))/2} n^{-\frac{1}{2}} \] for all \( \eta > 0 \),

where the constant \( \tilde{c}_\eta \) depends only on \( \eta \). Thus the worst-case error satisfies a bound which depends only polynomially on the dimension.

**Proof.** In this proof we denote the square worst-case error by \( e_{n,s}^2(\alpha, \gamma, \nu_s(q)) \) to stress the dependence on \( \alpha \) and \( \gamma \).

By Jensen's inequality we obtain

\[ e_{n,s}^2(\alpha, \gamma, \nu_s(q)) \leq \left[ e_{n,s}^2(\alpha \lambda, \gamma \lambda, \nu_s(q)) \right]^{1/\lambda} \] for all \( 1/\alpha < \lambda \leq 1 \).

From Theorem 3.2.6 we find that there exists a polynomial \( q_s \in R_{b,m} \) such that

\[ e_{n,s}^2(\alpha \lambda, \gamma \lambda, \nu_s(q_s)) \leq \frac{s-1}{n-1} \prod_{j=1}^{s} \left( 1 + \gamma_j \mu(\alpha \lambda) \right) \leq \frac{2s}{n} \prod_{j=1}^{s} \left( 1 + \gamma_j \mu(\alpha \lambda) \right). \]

Therefore for the minimizer \( \hat{q} \) we obtain

\[ e_{n,s}(\alpha, \gamma, \nu_s(\hat{q})) \leq \frac{(2s)^{\frac{1}{\lambda}}}{n^{\frac{1}{\lambda}}} \prod_{j=1}^{s} \left( 1 + \gamma_j \mu(\alpha \lambda) \right)^{\frac{1}{\lambda}}. \]

Now set \( \lambda = 1/(\alpha - 2\delta) \) with \( 0 < \delta \leq (\alpha - 1)/2 \) and the first result follows. The second part can be shown as in the proof of Corollary 3.2.5.

\[ \square \]

### 3.3 Multivariate integration in weighted Sobolev spaces

In this section we develop the theory of the previous chapter also for the integration problem in weighted Sobolev spaces. We use randomly digitally shifted polynomial lattice rules.

#### 3.3.1 Weighted Sobolev spaces

In this section we introduce the weighted Sobolev space \( H_{sob,s,w,\gamma} \) with reproducing kernel given by (see also [7, 26, 61, 62])

\[ K_{sob,s,w,\gamma}(x, y) = \prod_{j=1}^{s} (1 + \gamma_j \theta_{w_j}(x_j, y_j)), \]
where $\mathbf{w} = (w_1, \ldots, w_s) \in [0, 1]^s$ and
\[
g_\mathbf{w}(x, y) = \frac{|x - w| + |y - w| - |x - y|}{2} = \begin{cases} \min(|x - w|, |y - w|), & \text{if } (x - w)(y - w) \geq 0, \\ 0, & \text{otherwise}. \end{cases}
\]

The inner product in $H_{\text{sob}, s, w, \gamma}$ is given by
\[
\langle f, g \rangle_{H_{\text{sob}, s, w, \gamma}} := f(\mathbf{w})g(\mathbf{w}) + \sum_{u \subseteq \{1, \ldots, s\}, u \neq \emptyset} \prod_{j \in u} \gamma_j^{-1} \int_{[0,1]^{|u|}} \frac{\partial^{\left|u\right|} f}{\partial x_u}(x_u, \mathbf{w}) \frac{\partial^{\left|u\right|} g}{\partial x_u}(x_u, \mathbf{w}) \, dx_u,
\]
where for $\mathbf{x} = (x_1, \ldots, x_s)$ and $u \subseteq \{1, \ldots, s\}, u \neq \emptyset$, we use the notation $x_u = (x_j)_{j \in u}$ and $(x_u, \mathbf{w})$ denotes the $s$-dimensional vector whose $j$-th component is $x_j$ if $j \in u$ and $w_j$ if $j \notin u$.

Note that the reproducing kernel and the inner product are slightly different from the Sobolev space $H_{\text{sob}, s, w, \gamma}$ introduced in Subsection 2.6.2. As in [11] we call the space $H_{\text{sob}, s, w, \gamma}$ the ‘anchored Sobolev space’ with anchor $\mathbf{w}$ and the space $H_{\text{sob}, s, w, \gamma}$, introduced in Subsection 2.6.2, is called the ‘unanchored Sobolev space’. Note that the theory developed subsequently for the anchored Sobolev space can also be obtained for the unanchored Sobolev space. The construction algorithms for lattice rules focused on the anchored Sobolev space and as our aim here is to compare the performances of lattice rules with polynomial lattice rules we also use the anchored Sobolev space. (We might note that construction algorithms for lattice rules can also be obtained for integration in unanchored Sobolev spaces.)

The digital shift invariant kernel associated to the reproducing kernel of the anchored Sobolev space $H_{\text{sob}, s, w, \gamma}$ is very similar to the digital shift invariant kernel associated to the unanchored Sobolev space. In fact, using the calculations from Section 2.7, Appendix A, it can easily be verified that the associated digital shift invariant kernel $K_{\text{ds}, b, w, \gamma}(\mathbf{x}, \mathbf{y})$ in base $b$ is given by
\[
K_{\text{ds}, b, w, \gamma}(\mathbf{x}, \mathbf{y}) = \prod_{j=1}^s \left( \sum_{k=1}^{\infty} \hat{r}_b(w_j, \gamma_j, k) \overline{\text{wal}_b}(x_j) \overline{\text{wal}_b}(y_j) \right) = \sum_{k \in \mathbb{N}_0} \hat{r}_b(\mathbf{w}, \gamma, k) \overline{\text{wal}_b}(\mathbf{x}) \overline{\text{wal}_b}(\mathbf{y}), \quad (3.3.8)
\]
with $\mathbf{w} = (w_1, \ldots, w_s) \in [0, 1]^s$ and $\hat{r}_b(\mathbf{w}, \gamma, k) = \prod_{j=1}^s \hat{r}_b(w_j, \gamma_j, k_j)$, where now, for $k = \kappa_{a-1}b^{a-1} + \ldots + \kappa_1 b + \kappa_0$ with $\kappa_{a-1} \neq 0$, we define
\[
\hat{r}_b(w, \gamma, k) = \begin{cases} 1 + \gamma(w^2 - w + \frac{1}{2}), & \text{if } k = 0, \\ \frac{\gamma}{2b^a} \left( \frac{1}{\sin^2(\kappa_{a-1} \pi/b)} - \frac{1}{3} \right), & \text{if } k > 0. \end{cases}
\]
Further, for $x = \frac{y_1}{b} + \frac{y_2}{b^2} + \ldots$ and $y = \frac{w_1}{b} + \frac{w_2}{b^2} + \ldots$, we define

$$
\phi_{\text{ds},b,w}(x, y) = \begin{cases} 
  w^2 - w + \frac{1}{4}, & \text{if } x = y, \\
  w^2 - w + \frac{1}{2} - \frac{|x_{i_0} - y_{i_0}(b^{-1})|}{b^{N_0+1}}, & \text{if } x_{i_0} \neq y_{i_0} \text{ and } x_i = y_i \text{ for } i = 1, \ldots, i_0 - 1.
\end{cases}
$$

Using the ideas of Section 2.7, Appendix B, or otherwise, it can be checked that

$$
K_{\text{ds},b,w,\gamma}(x, y) = \prod_{j=1}^{s} (1 + \gamma_j \phi_{\text{ds},b,w}(x_j, y_j)).
$$

(3.3.10)

Now with the same arguments used to prove Theorem 2.6.5, one can show that the mean square worst-case error \( \bar{e}^2(P_n, K_{\text{sob},s,w,\gamma}) \) for multivariate integration in the weighted Sobolev space \( H_{\text{sob},s,w,\gamma} \) by using a random digital shift in base \( b \) on the point set \( P_n = \{ x_0, \ldots, x_{n-1} \} \), with \( x_h = (x_{h,1}, \ldots, x_{h,s}) \), is given by

$$
\bar{e}^2(P_n, K_{\text{sob},s,w,\gamma}) = \prod_{j=1}^{s} (1 + \gamma_j \phi_{\text{ds},b,w}(x_j, y_j))
$$

(3.3.11)

where the function \( \phi_{\text{ds},b,w} \) is given by (3.3.9). For the special case where the point set \( P \) is a digital \( (t, m, s) \)-net over \( \mathbb{Z}_b \) with generating matrices \( C_1, \ldots, C_s \), the mean-square worst case error \( \bar{e}^2_{\text{sob},s}(C_1, \ldots, C_s) := \bar{e}^2(P, K_{\text{sob},s,w,\gamma}) \) can be written as

$$
\bar{e}^2_{\text{sob},s}(C_1, \ldots, C_s) = \sum_{k \in \mathcal{D}_p} \hat{r}_b(w, \gamma, k),
$$

(3.3.12)

where

$$
\mathcal{D}_p = \{ k \in \mathbb{N}_0^s \setminus \{0\} : C_1^T \text{tr}_m(k_1) + \cdots + C_s^T \text{tr}_m(k_s) = 0 \}.
$$

Further we have

$$
\bar{e}^2(P, K_{\text{sob},s,w,\gamma}) = \prod_{j=1}^{s} (1 + \gamma_j \phi_{\text{ds},b,w}(x_{h,j}, y_{i,j}))
$$

(3.3.13)

where \( \phi_{\text{ds},b,w} \) is given by (3.3.9).

As can be seen from (3.3.9), the function values of \( \phi_{\text{ds},b,w} \) can be computed easily for any \( x \) and \( y \) and therefore \( \bar{e}^2(P, K_{\text{sob},s,w,\gamma}) \) can be computed in \( O(b^{m s}) \)
operations for a given digital net $P$. We will use this fact in the following subsection to search for good digital nets.

Note that, as opposed to the previous section, the base $b$ is not part of the definition of the space, but part of the randomization method. Therefore the base $b$ can be chosen arbitrarily, whereas in the previous section the $b$ was determined by the space $H_{\text{wal},b,s,\alpha,\gamma}$.

### 3.3.2 The component-by-component construction of polynomial lattice rules for integration in weighted Sobolev spaces

In this section we consider digital nets based on Niederreiter’s construction using polynomials. We have the following lemma.

**Lemma 3.3.1** Let $p \in \mathbb{Z}_b[x]$ with $\deg(p) = m \geq 1$ and let $q = (q_1, \ldots, q_s) \in \mathbb{Z}_b[x]^s$. The mean square worst-case error $\hat{e}_{b,m}^2(q, p) := \hat{e}_{b,m}^2(S_p(q), H_{\text{so},b,m,s,\alpha,\gamma})$ for integration in the weighted Sobolev space $H_{\text{so},b,m,s,\alpha,\gamma}$ is given by

$$\hat{e}_{b,m}^2(q, p) = \sum_{k \in D_{pl}} \hat{r}_b(w, \gamma, k),$$

where

$$D_{pl} = \{ k \in \mathbb{N}_0^s \setminus \{0\} : \text{tr}_m(k) \cdot q \equiv 0 \pmod{p} \}.$$ 

**Proof.** The result follows from (3.3.11) together with (3.2.2). \qed

Using the above lemma suitable polynomials can be found using the following algorithm.

**Algorithm 3** Given a dimension $s$, an integer $m \geq 1$ and weights $\gamma = (\gamma_j)_{j \geq 1}$.

1. Choose a prime number $b$ and an irreducible polynomial $p \in \mathbb{Z}_b[x]$ with $\deg(p) = m$.
2. Set $q_1 = 1$.
3. For $d = 2, 3, \ldots, s$, find $q_d \in R_{b,m}$ by minimizing $\hat{e}_{b,m}^2((q_1, \ldots, q_d), p)$.

As for the space $H_{\text{wal},b,s,\alpha,\gamma}$ we also obtain the following bound.

**Theorem 3.3.2** Let $b$ be prime and $p \in \mathbb{Z}_b[x]$ be irreducible, with $\deg(p) = m \geq 1$. Suppose $(q_1^*, \ldots, q_s^*) \in R_{b,m}^*$ is constructed by Algorithm 3. Then for all $d = 1, 2, \ldots, s$ we have

$$\hat{e}_{b,m}^2((q_1^*, \ldots, q_d^*), p) \leq (b^m - 1)^{\frac{1}{2}} \prod_{j=1}^d \left( (1 + \gamma_j \left[ w_j^2 - w_j + \frac{1}{3} \right])^\lambda + \tau_b(\lambda) \gamma_j^\lambda \right)^{\frac{1}{2}},$$

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for all $\frac{1}{2} < \lambda \leq 1$. Here, for $\lambda = 1$, $\tau_2(1) := \frac{1}{6}$ and for $\frac{1}{2} < \lambda < 1$ we define

$$\tau_2(\lambda) := \frac{1}{3\lambda(2^{2\lambda} - 2)} \quad \text{and} \quad \tau_b(\lambda) := \frac{(4b^2 - 9)^{\lambda}}{54^{\lambda}(b^{2\lambda} - b)} \quad \text{for } b > 2.$$ 

**Proof.** The proof follows exactly the lines of the proof of Theorem 3.2.3. We just note the following equalities: for any $b > 2$ we have

$$\sum_{k=1}^{\infty} \hat{r}_b(w, \gamma, k) = \sum_{a=1}^{\infty} \sum_{k=b^{a-1}}^{b^{a-1}} \hat{r}_b(w, \gamma, k) = \sum_{a=1}^{\infty} \frac{\gamma b^{a-1}}{2b^{2a}} \sum_{\kappa_{a-1}=1}^{b^{a-1}} \left( \frac{1}{\sin^{2}(\kappa_{a-1}\pi/b)} - \frac{1}{3} \right).$$

We have

$$\sum_{a=1}^{\infty} b^{-a} = \frac{1}{b-1} \quad \text{and} \quad \sum_{\kappa_{a-1}=1}^{b^{a-1}} \frac{1}{\sin^{2}(\kappa_{a-1}\pi/b)} = \frac{b^2 - 1}{3},$$

where the second equality is shown in Section 2.7, Appendix C. Thus we obtain

$$\sum_{k=1}^{\infty} \hat{r}_b(w, \gamma, k) = \frac{\gamma}{2b(b-1)} \left( \frac{b^2 - 1}{3} - \frac{b-1}{3} \right) = \frac{\gamma}{6} = \gamma \tau_b(1).$$

Further we have $\hat{r}_2(w, \gamma, k) = \frac{\gamma}{32\pi}$ for $k > 0$ and therefore

$$\sum_{k=1}^{\infty} \hat{r}_2(w, \gamma, k)^{\lambda} = \sum_{a=1}^{\infty} \sum_{k=2^{a-1}}^{2^{a-1}} \hat{r}_2(w, \gamma, k)^{\lambda} = \sum_{a=1}^{\infty} \frac{\gamma^{\lambda}2^{a-1}}{3^{2\lambda}} = \frac{\gamma^{\lambda}}{2 \cdot 3^{\lambda}} \sum_{a=1}^{\infty} \frac{1}{2^{a(2\lambda-1)}}$$

$$= \frac{\gamma^{\lambda}}{3\lambda(2^{2\lambda} - 2)} = \gamma^{\lambda} \tau_2(\lambda),$$

for any $1/2 < \lambda \leq 1$. For $b > 2$ we estimate $\sin(\kappa_{a-1}\pi/b) \geq \sin(\pi/b) \geq \frac{3\sqrt{3}}{2b}$ and therefore

$$\frac{1}{\sin^{2}(\kappa_{a-1}\pi/b)} - \frac{1}{3} \leq \frac{4b^2 - 9}{27}.$$

Using this estimation we get

$$\sum_{k=1}^{\infty} \hat{r}_b(w, \gamma, k)^{\lambda} \leq \sum_{a=1}^{\infty} \sum_{k=b^{a-1}}^{b^{a-1}} \hat{r}_b(w, \gamma, k)^{\lambda}$$

$$\leq \sum_{a=1}^{\infty} \frac{\gamma^{\lambda}b^{a-1}(b-1)(4b^2 - 9)^{\lambda}}{2b^{2\lambda}27^{\lambda}} = \frac{\gamma^{\lambda}(b-1)(4b^2 - 9)^{\lambda}}{54^{\lambda}(b^{2\lambda} - b)} = \gamma^{\lambda} \tau_b(\lambda),$$

for any $1/2 < \lambda \leq 1$. \hfill $\square$

As above (see Corollary 3.2.5) we obtain the following corollary.

**Corollary 3.3.3** Let $b$ be prime, $p \in \mathbb{Z}_b[x]$ be irreducible with $\deg(p) = m \geq 1$ and $n = b^m$. Suppose $q^* \in R_{b,m}^*$ is constructed by Algorithm 3.
1. We have
\[ \hat{e}_{n,s}(\mathbf{q}^*, p) \leq c_{s,w,\gamma,\delta} n^{-1+\delta} \quad \text{for all } 0 < \delta \leq \frac{1}{2}, \]
where
\[ c_{s,w,\gamma,\delta} := 2^{1-\delta} \prod_{j=1}^{s} \left( 1 + \gamma_j \left[ \left( \frac{1}{s+1} \right)^{\frac{1}{1-s}} \right] \right)^{1-\delta}. \]

2. Suppose
\[ \sum_{j=1}^{\infty} \gamma_j^{\frac{1}{1-s+j}} < \infty. \]
Then \( c_{s,w,\gamma,\delta} \leq c_{\infty,w,\gamma,\delta} < \infty \) and we have
\[ \hat{e}_{n,s}(\mathbf{q}^*, p) \leq c_{\infty,w,\gamma,\delta} n^{-1+\delta} \quad \text{for all } 0 < \delta \leq \frac{1}{2}. \]
Thus the root mean square worst-case error of the point set \( S_p(\mathbf{q}) \) is bounded independently of the dimension.

3. Under the assumption
\[ A := \limsup_{s \to \infty} \frac{\sum_{j=1}^{s} \gamma_j}{\log s} < \infty \]
we obtain \( c_{s,w,\gamma,1/2} \leq \tilde{c}_\eta s^{(A+\eta)/2} \) and therefore
\[ \hat{e}_{n,s}(\mathbf{q}^*, p) \leq \tilde{c}_\eta s^{(A+\eta)/2} n^{-\frac{1}{2}} \quad \text{for all } \eta > 0, \]
where the constant \( \tilde{c}_\eta \) depends only on \( \eta \). Thus the root mean square worst-case error of the point set \( S_p(\mathbf{q}) \) satisfies a bound which depends only polynomially on the dimension.

### 3.3.3 A Korobov type construction of polynomial lattice rules for integration in weighted Sobolev spaces

We can also obtain results for Korobov type rules. As before we use the notation \( \mathbf{v}_s(\mathbf{q}) \equiv (1, q, q^2, \ldots, q^{s-1}) \) (mod \( p \)). We have the following algorithm.

**Algorithm 4** Given a dimension \( s \geq 2 \), an integer \( m \geq 1 \) and weights \( \gamma = (\gamma_j)_{j \geq 1} \).

1. Choose a prime number \( b \) and an irreducible polynomial \( p \in \mathbb{Z}_b[x] \) with \( \deg(p) = m \).
2. Find \( \tilde{q} \in R_{b,m} \) by minimizing \( \hat{e}_{b,m,s}^2(\mathbf{v}_s(\mathbf{q}), p) \).
The following results can be shown as in Subsection 3.2.2.

**Theorem 3.3.4** Let \( b \) be prime, \( s \geq 2 \) and \( p \in \mathbb{Z}_b[x] \) be irreducible, with \( \deg(p) = m \geq 1 \). Let \( \tau_b \) be defined as in Theorem 3.3.2. Suppose \( \tilde{q} \in R_{b,m} \) is constructed by Algorithm 4. Then we have

\[
\hat{e}_{b,m,s}(v_s(\tilde{q}), p) \leq \left( \frac{d-1}{q^m - 1} \right)^{\frac{1}{m}} \prod_{j=1}^{d} \left( (1 + \gamma_j \left[ w_j^2 - w_j + \frac{1}{3} \right])^{\lambda} + \tau_b(\lambda)^{\lambda} \right)^{\frac{1}{\lambda}}
\]

for all \( \frac{1}{2} < \lambda \leq 1 \).

**Corollary 3.3.5** Let \( b \) be prime, \( s \geq 2 \), \( p \in \mathbb{Z}_b[x] \) be irreducible with \( \deg(p) = m \geq 1 \) and \( n = b^m \). Suppose \( \tilde{q} \in R_{b,m} \) is constructed by Algorithm 4.

1. We have

\[
\hat{e}_{n,s}(v_s(\tilde{q}), p) \leq c_{s,w,\gamma,\delta} s^{1-\delta} n^{-1+\delta} \quad \text{for all } 0 < \delta \leq \frac{1}{2},
\]

where

\[
c_{s,w,\gamma,\delta} := 2^{1-\delta} \prod_{j=1}^{s} \left( 1 + \gamma_j \left[ w_j^2 - w_j + \frac{1}{3} \right]^{\frac{1}{2(1-\delta)}} + \tau_b \left( \frac{1}{2(1-\delta)} \right) \right)^{1-\delta}.
\]

2. Under the assumption

\[
A := \limsup_{s \to \infty} \frac{\sum_{j=1}^{s} \gamma_j}{\log s} < \infty
\]

we obtain \( \hat{e}_{s,w,\gamma,1/2} \leq \tilde{c}_n(A+\eta)/2 \) and therefore

\[
\hat{e}_{n,s}(v_s(\tilde{q}), p) \leq \tilde{c}_n s^{(1+(A+\eta))/2} n^{-\frac{1}{2}} \quad \text{for all } \eta > 0,
\]

where the constant \( \tilde{c}_n \) depends only on \( \eta \). Thus the root mean square worst-case error of the point set \( S_p(v_s(\tilde{q})) \) satisfies a bound which depends only polynomially on the dimension.

### 3.4 Numerical results

In this section we present numerical results for the worst-case error of integration in the anchored Sobolev spaces. The aim is to compare the performance of polynomial lattice rules with those of lattice rules. As previously done for lattice rules we choose \( w_j = 1 \) for \( j = 1, \ldots, s \), the dimension \( s = 100 \) and we consider the weights

\[
\gamma_j = 1, \quad \gamma_j = 0.5^j, \quad \gamma_j = \frac{1}{j^2}, \quad \gamma_{j,s} = \frac{10}{s} = \frac{1}{10}.
\]
Note that we can also allow the weights $\gamma_j$ to depend on the dimension $s$, that is $\gamma_j = \gamma_{j,s}$, see [11].

The simplest and also most efficient way for our construction algorithms is obtained by choosing $b = 2$. The mean-square worst-case error, see (3.3.12), is then given by

$$\hat{e}^2(S_p(q), K_{\text{sob}, s, 1, \gamma}) = -\prod_{j=1}^{s} \left( 1 + \frac{\gamma_j^3}{3} \right) + \frac{1}{n} \sum_{h=0}^{2m-1} \prod_{j=1}^{s} \left( 1 + \gamma_j \phi_{d, 2, 1}(x_{h,j}, 0) \right),$$

where (see (3.3.9))

$$\phi_{d, 2, 1}(x, 0) = \begin{cases} \frac{1}{2}, & \text{if } x = 0, \\ \frac{1}{2} - \frac{1}{2^{i_0+1}}, & \text{otherwise}, \end{cases}$$

with $i_0$ the index of the first non-zero digit in the base 2 representation of $x$.

Tables of the worst-case error of polynomial lattice rules (plr) and lattice rules (lr), which are constructed by a component-by-component algorithm (CBC) or by Korobov’s construction method (Korobov), are presented in Subsection 3.4.2. (The irreducible polynomials $p \in \mathbb{Z}_2[x]$ used in these computations are given in Table 3.1.)

Further we compare the worst-case errors obtained from different choices of irreducible polynomials $p \in \mathbb{Z}_2[x]$. These results are shown in Tables 3.6 and 3.7. Again we choose $s = 100$ and we consider $n = 1024$ and 2048 and $\gamma_j = j^{-2}$ and $\gamma_j = \frac{1}{10}$.

### 3.4.1 Concluding Remarks

The upper bounds on the mean square worst-case error for integration in weighted Sobolev spaces using randomly digitally shifted polynomial lattice rules are almost the same as for randomly shifted lattice rules. Surprisingly enough we obtain exactly the same constant in Theorem 3.3.2 and Theorem 3.3.4 by taking $\lambda = 1$ as for the corresponding bounds for lattice rules, see [7, 73].

It appears that polynomial lattice rules constructed by a component-by-component algorithm are slightly better (see Tables 3.2 to 3.5) than lattice rules constructed by a component-by-component algorithm. For the Korobov construction sometimes polynomial lattice rules, sometimes lattice rules are better. In any case, the difference is rather small.

Tables 3.6 and 3.7 suggest that the choice of the polynomial $p$ has a small influence on the outcome. This influence seems to be slightly stronger for the Korobov construction. Still it seems that the particular choice of the irreducible polynomial $p$ is of no significance. This is also validated by our theory, which only depends on the degree of the polynomial $p$, but not on the particular choice.

The construction algorithms presented in this chapter show how polynomial lattice rules can be extended in practice in the dimension. In [44] on the other
hand it was shown that it is possible to extend such point sets in both, the number of points and the dimension. How this can be done in practice is an interesting, but seemingly difficult, problem which is left for future work.

3.4.2 Appendix: Tables of numerical results

Table 3.1: Irreducible polynomial $p \in \mathbb{Z}_2[x]$ 

<table>
<thead>
<tr>
<th>$m$</th>
<th>$p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>$x^8 + x^5 + x^4 + x^3 + 1$</td>
</tr>
<tr>
<td>9</td>
<td>$x^9 + x^8 + x^7 + x^5 + x^4 + x^2 + 1$</td>
</tr>
<tr>
<td>10</td>
<td>$x^{10} + x^7 + x^3 + x + 1$</td>
</tr>
<tr>
<td>11</td>
<td>$x^{11} + x^{10} + x^8 + x^6 + x^4 + x^2 + 1$</td>
</tr>
<tr>
<td>12</td>
<td>$x^{12} + x^9 + x^8 + x^7 + x^6 + x^4 + x^2 + x + 1$</td>
</tr>
</tbody>
</table>

Table 3.2: Worst-case error, $\gamma_j = 1$

<table>
<thead>
<tr>
<th>$n$</th>
<th>CBC plr</th>
<th>CBC lr</th>
<th>Korobov plr</th>
<th>Korobov lr</th>
</tr>
</thead>
<tbody>
<tr>
<td>256</td>
<td>3.98437e+07</td>
<td>3.98456e+07</td>
<td>3.98443e+07</td>
<td>3.98456e+07</td>
</tr>
<tr>
<td>512</td>
<td>2.81719e+07</td>
<td>2.81721e+07</td>
<td>2.81721e+07</td>
<td>2.81721e+07</td>
</tr>
<tr>
<td>1024</td>
<td>1.99186e+07</td>
<td>1.99194e+07</td>
<td>1.99187e+07</td>
<td>1.99193e+07</td>
</tr>
<tr>
<td>2048</td>
<td>1.40828e+07</td>
<td>1.40840e+07</td>
<td>1.40828e+07</td>
<td>1.40841e+07</td>
</tr>
<tr>
<td>4096</td>
<td>9.95656e+06</td>
<td>9.95785e+06</td>
<td>9.95642e+06</td>
<td>9.95826e+06</td>
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</tbody>
</table>

Table 3.3: Worst-case error, $\gamma_j = 0.5^j$

<table>
<thead>
<tr>
<th>$n$</th>
<th>CBC plr</th>
<th>CBC lr</th>
<th>Korobov plr</th>
<th>Korobov lr</th>
</tr>
</thead>
<tbody>
<tr>
<td>256</td>
<td>2.51805e−03</td>
<td>2.59907e−03</td>
<td>2.75401e−03</td>
<td>2.86635e−03</td>
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<tr>
<td>512</td>
<td>1.33062e−03</td>
<td>1.36727e−03</td>
<td>1.49945e−03</td>
<td>1.46307e−03</td>
</tr>
<tr>
<td>1024</td>
<td>6.95360e−04</td>
<td>7.14036e−04</td>
<td>7.84960e−04</td>
<td>7.75159e−04</td>
</tr>
<tr>
<td>2048</td>
<td>3.61270e−04</td>
<td>3.80296e−04</td>
<td>4.14176e−04</td>
<td>4.20300e−04</td>
</tr>
<tr>
<td>4096</td>
<td>1.90239e−04</td>
<td>1.94805e−04</td>
<td>2.23073e−04</td>
<td>2.15116e−04</td>
</tr>
</tbody>
</table>
Table 3.4: Worst-case error, $\gamma_j = j^{-2}$

<table>
<thead>
<tr>
<th>$n$</th>
<th>CBC plr</th>
<th>CBC lr</th>
<th>Korobov plr</th>
<th>Korobov lr</th>
</tr>
</thead>
<tbody>
<tr>
<td>256</td>
<td>4.23326e-03</td>
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<td>5.75309e-03</td>
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<td>512</td>
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<td>1024</td>
<td>1.23355e-03</td>
<td>1.28702e-03</td>
<td>1.75583e-03</td>
<td>1.75881e-03</td>
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<td>4096</td>
<td>3.62609e-04</td>
<td>3.74410e-04</td>
<td>5.48164e-04</td>
<td>5.47949e-04</td>
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</tbody>
</table>

Table 3.5: Worst-case error, $\gamma_j = \frac{1}{10}$

<table>
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<th>$n$</th>
<th>CBC plr</th>
<th>CBC lr</th>
<th>Korobov plr</th>
<th>Korobov lr</th>
</tr>
</thead>
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<tr>
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<td>2.86692e-01</td>
<td>2.77131e-01</td>
<td>2.86441e-01</td>
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<tr>
<td>1024</td>
<td>1.84695e-01</td>
<td>1.84526e-01</td>
<td>1.81462e-01</td>
<td>1.80904e-01</td>
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<tr>
<td>2048</td>
<td>1.21283e-01</td>
<td>1.21268e-01</td>
<td>1.18402e-01</td>
<td>1.20839e-01</td>
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<tr>
<td>4096</td>
<td>8.00544e-02</td>
<td>8.15918e-02</td>
<td>7.97847e-02</td>
<td>8.18734e-02</td>
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</tbody>
</table>

Table 3.6: Worst-case error for various choices of the irreducible polynomial $p$
with $\gamma_j = j^{-2}$

<table>
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<tr>
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<th>$p$</th>
<th>CBC plr</th>
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</thead>
<tbody>
<tr>
<td>1024</td>
<td>$x^{10} + x^7 + x^3 + x + 1$</td>
<td>1.23355e-03</td>
<td>1.75583e-03</td>
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<tr>
<td></td>
<td>$x^{10} + x^9 + x^9 + x^4 + x^3 + x^2 + x + 1$</td>
<td>1.23383e-03</td>
<td>1.72422e-03</td>
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<tr>
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<td>$x^{10} + x^9 + x^7 + x^6 + x^4 + x^3 + x + 1$</td>
<td>1.22844e-03</td>
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<tr>
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<td>$x^{10} + x^8 + x^4 + x^3 + 1$</td>
<td>1.22893e-03</td>
<td>1.71933e-03</td>
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<tr>
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<td>$x^{10} + x^8 + x^7 + x^6 + 1$</td>
<td>1.23561e-03</td>
<td>1.68367e-03</td>
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<tr>
<td>2048</td>
<td>$x^{11} + x^{10} + x^8 + x^6 + x^4 + x^2 + 1$</td>
<td>6.68382e-04</td>
<td>9.31863e-04</td>
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<td>$x^{11} + x^{10} + x^9 + x^5 + x^2 + x + 1$</td>
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<td>9.43137e-04</td>
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<td>9.40263e-04</td>
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<td>$x^{11} + x^{10} + x^8 + x^6 + x^5 + x^4 + 1$</td>
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<td>1.00080e-03</td>
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<td>$x^{11} + x^{10} + x^8 + x^5 + x^4 + 1$</td>
<td>6.63566e-04</td>
<td>9.41526e-04</td>
</tr>
</tbody>
</table>

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Table 3.7: Worst-case error for various choices of the irreducible polynomial $p$ with $\gamma_j = \frac{1}{10}$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$p$</th>
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<th>Korobov plr</th>
</tr>
</thead>
<tbody>
<tr>
<td>1024</td>
<td>$x^{10} + x^9 + x^7 + x^3 + x + 1$</td>
<td>1.84695e - 01</td>
<td>1.81462e - 01</td>
</tr>
<tr>
<td></td>
<td>$x^{10} + x^9 + x^8 + x^7 + x^6 + x^4 + x^3 + x + 1$</td>
<td>1.83927e - 01</td>
<td>1.77861e - 01</td>
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<tr>
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<td>$x^{10} + x^8 + x^4 + x^3 + 1$</td>
<td>1.83857e - 01</td>
<td>1.84145e - 01</td>
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<tr>
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<td>$x^{10} + x^8 + x^7 + x^6 + 1$</td>
<td>1.84438e - 01</td>
<td>1.84699e - 01</td>
</tr>
<tr>
<td></td>
<td>$x^{10} + x^8 + x^7 + x^6 + x + 1$</td>
<td>1.84385e - 01</td>
<td>1.82713e - 01</td>
</tr>
<tr>
<td>2048</td>
<td>$x^{11} + x^{10} + x^9 + x^6 + x^4 + x^3 + 1$</td>
<td>1.21283e - 01</td>
<td>1.18402e - 01</td>
</tr>
<tr>
<td></td>
<td>$x^{11} + x^{10} + x^9 + x^8 + x^6 + x^4 + x + 1$</td>
<td>1.21869e - 01</td>
<td>1.20605e - 01</td>
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<td>1.19697e - 01</td>
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<tr>
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<td>$x^{11} + x^{10} + x^8 + x^6 + x^3 + 1$</td>
<td>1.21290e - 01</td>
<td>1.19272e - 01</td>
</tr>
<tr>
<td></td>
<td>$x^{11} + x^{10} + x^8 + x^6 + x^5 + x^4 + 1$</td>
<td>1.21721e - 01</td>
<td>1.20039e - 01</td>
</tr>
</tbody>
</table>
Chapter 4

On the mean square weighted $L_2$ discrepancy of randomized digital $(t, m, s)$-nets over $\mathbb{Z}_2$

4.1 The $L_2$ discrepancy

In this chapter we study distribution properties of point sets in the $s$-dimensional unit-cube $[0,1)^s$. There are various measures for the equidistribution of such point sets (see for example [13, 18, 25, 38, 41]). The one we consider here is based on the following function. For a point set $P_N = \{x_0, \ldots, x_{N-1}\}$ of points in the $s$-dimensional unit-cube $[0,1)^s$ the discrepancy function is defined as

$$\Delta(t_1, \ldots, t_s) = \frac{A_N([0,t_1) \times \cdots \times [0,t_s))}{N} - t_1 \cdots t_s,$$

where $0 \leq t_j \leq 1$ and $A_N([0,t_1) \times \cdots \times [0,t_s))$ denotes the number of indices $n$ with $x_n \in [0,t_1) \times \cdots \times [0,t_s)$.

The discrepancy function measures the difference of the portion of points in an axis parallel box containing the origin and the volume of this box. Hence it is a measure of the irregularity of distribution of a point set in $[0,1)^s$. There are of course other functions which serve a comparable purpose, though this function has drawn a great deal of attention as various connections with applications have been pointed out, notably numerical integration of functions (see for example [41, 63]). Further, we can use different norms of the discrepancy function, again yielding different quality measures. Amongst those norms especially the $L_2$ norm and the $L_\infty$ norm have been of considerable interest and have been studied extensively (see for example [41, 63]). In the following we introduce some notation and subsequently we define the weighted $L_2$ discrepancy of a point set, which will be the focus of this chapter.

Let $D$ denote the index set $D = \{1, \ldots, s\}$. For $u \subseteq D$ let $\gamma_u$ be a non-negative real number, $|u|$ the cardinality of $u$ and for a vector $x \in [0,1)^s$ let $x_u$ denote the vector from $[0,1)^{|u|}$ containing all components of $x$ whose indices are
in $u$. Further let $d x_u = \prod_{j \in u} d x_j$ and let $(x_u, 1)$ be the vector from $[0, 1)^s$ with all components whose indices are not in $u$ replaced by 1. Then the weighted $L_2$ discrepancy of $P_N$ is defined as (see [63])

$$L_{2,N,\gamma}(P_N) = \left( \sum_{u \subseteq D} \gamma_u \int_{[0,1]^{|u|}} \Delta((x_u, 1))^2 d x_u \right)^{1/2}. \tag{4.1.1}$$

In [63, Theorem 1] (see also [17]) it was shown that the weighted $L_2$ discrepancy coincides with the worst-case error of integration in the anchored Sobolev space $H_{sob,s,1,\gamma}$ with anchor $w = 1 = (1, \ldots, 1)$, see Subsection 3.3.1.

The weighted $L_2$ discrepancy is a generalization of the classical $L_2$ discrepancy. By choosing $\gamma_D = 1$ and $\gamma_u = 0$ for all $u \subset D$ we obtain the classical $L_2$ discrepancy and if we choose $\gamma_u = 1$ for all $u \subseteq D$ we obtain the unweighted $L_2$ discrepancy. Note that in this definition we also include the lower dimensional projections (see [16]). In [37] it has been pointed out that the classical $L_2$ discrepancy of $N$ copies of the point $(1, \ldots, 1)$ can almost yield the best value if the dimension is high compared to $N$. (Note that the $L_2$ discrepancy does not change by considering point sets in $[0, 1]^s$ rather than $[0, 1)^s$.) Such a point set is obviously not well distributed in an intuitive sense. Including the lower dimensional projections much reduces this effect. The weights $\gamma_u$ are then introduced to modify the importance of the discrepancy of the projections, with the intention to adjust the measure to the usage of the point set (see [11, 63]). For example it has been observed that in many applications the importance of higher dimensional projections is considerably lower than the importance of lower dimensional projections.

There is a well known formula for the classical $L_2$ discrepancy of a point set by Warnock (see for example [38]), which can easily be generalized to obtain a formula for the weighted $L_2$ discrepancy. This formula is given in the following proposition (for a hint on how to prove this formula see for example [35] or [38]).

**Proposition 4.1.1** Let $P_N = \{x_0, \ldots, x_{N-1}\}$ be a point set in $[0, 1)^s$. Then we have

$$L_{2,N,\gamma}(P_N) = \sum_{u \subseteq D} \gamma_u \left[ \frac{1}{3^{|u|}} - \frac{2}{N} \sum_{n=0}^{N-1} \prod_{j \in u} \frac{1-x_{n,j}^2}{2} + \frac{1}{N^2} \sum_{n,m=0}^{N-1} \prod_{j \in u} \min(1-x_{n,j}, 1-x_{m,j}) \right],$$

where $x_{n,j}$ is the $j$-th component of the point $x_n$.

As the choice of weights is determined by the task (for example, approximating the integral of a function) and therefore not known a priori, we wish to find point sets which ‘work well’ for many (if not all) choices of weights. In this chapter we consider randomized digital $(t, m, s)$-nets over the finite field $\mathbb{Z}_2$.  

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The bijections in Definition 2.3.2 are chosen as identities. In the following section we introduce the randomization method considered in this chapter.

4.2 The digital shift of depth \( m \)

For practical applications it is often useful to have a random element in the point set used (see [38]). On the other hand we wish to preserve the structure which a point set already has. That is in our case, we wish to randomize a \((t, m, s)\)-net such that the resulting point set is again a \((t, m, s)\)-net with the same quality parameter \( t \). Several randomization methods for \((t, m, s)\)-nets have been introduced (see [38, 49, 76]). The randomization method considered in this chapter is a digital shift of depth \( m \) (see also [38]). The aim of this chapter is to analyze the expected value of the weighted \( \mathcal{L}_2 \) discrepancy of digitally shifted digital \((t, m, s)\)-nets.

In the following we introduce the digital shift of depth \( m \) for the one dimensional case. For higher dimensions each coordinate is randomized independently and therefore one just needs to apply the one dimensional randomization method to each coordinate independently.

Let the point set \( P_{2^m} = \{x_0, \ldots, x_{2^m-1}\} \) be a digital \((t, m, 1)\)-net over \( \mathbb{Z}_2 \) generated by the matrix \( C \). Let
\[
x_n = \frac{x_{n,1}}{2} + \frac{x_{n,2}}{2^2} + \cdots
\]
be the dyadic digit expansion of \( x_n \). In Chapter 2 and 3 we considered a randomization method which uses a digital shift \( \sigma = \sigma_1 + \sigma_2 2 + \cdots \), where the shift \( \sigma \in [0,1) \) was chosen randomly. Here we modify this randomization method in the following way: at first we choose the digits \( \sigma_1, \ldots, \sigma_m \in \{0,1\} \) i.i.d.. Then we define
\[
z_{n,i} \equiv x_{n,i} + \sigma_i \pmod{2}
\]
for \( i = 1, \ldots, m \) with \( z_{n,i} \in \{0,1\} \). Further, for \( n = 0, \ldots, 2^m - 1 \), we choose \( \delta_n \in [0, \frac{1}{2^m}) \) i.i.d.. Then the randomized point set \( \tilde{P}_{2^m} = \{z_0, \ldots, z_{2^m-1}\} \) is given by
\[
z_n = \frac{z_{n,1}}{2} + \cdots + \frac{z_{n,m}}{2^m} + \delta_n.
\]
This means we apply the same digital shift to the first \( m \) digits, whereas the following digits are shifted independently for each \( x_n \). Therefore we call it a digital shift of depth \( m \) (see again [38]).

In this chapter we will sometimes write digital shift or simply shift instead of digital shift of depth \( m \). When we use a digital shift of depth \( m' \) in conjunction with digital \((t, m, s)\)-nets we always assume that \( m' = m \).

For arbitrary \( s \geq 1 \) it can be shown that a \((t, m, s)\)-net in base 2 randomized by a digital shift of depth \( m \) independently in each coordinate is again a \((t, m, s)\)-net in base 2 with the same quality parameter \( t \). As this result is not essential for the following we omit the proof. Similar results have been shown before (see for example Lemma 2.3.5 or also [49]).
4.3 On the mean square weighted $L_2$ discrepancy of randomized nets

In the following subsection we prove a formula for the mean square weighted $L_2$ discrepancy of randomized digital nets. This formula depends on the generating matrices of the digital net. Subsequently we use this formula to derive the exact value of the mean square weighted $L_2$ discrepancy for digital $(0,m,s)$-nets over $\mathbb{Z}_2$ for $s = 2, 3$. Following this we obtain a bound for the general case, that is, for the mean square weighted $L_2$ discrepancy of digital $(t,m,s)$-nets over $\mathbb{Z}_2$.

4.3.1 A formula for the mean square weighted $L_2$ discrepancy of randomized nets

The aim of this subsection is to prove the following theorem.

**Theorem 4.3.1** Let $P_{2^m}$ be a digital $(t,m,s)$-net over $\mathbb{Z}_2$ with generating matrices $C_1, \ldots, C_s$. Let $\tilde{P}_{2^m}$ be the point set obtained after applying an i.i.d. random digital shift of depth $m$ independently to each coordinate of each point of $P_{2^m}$. Then the mean square weighted $L_2$ discrepancy of $\tilde{P}_{2^m}$ is given by

$$E[L^2_{2^m, \gamma}(\tilde{P}_{2^m})] = \sum_{u \subseteq D} \gamma_u \left[ \frac{1}{2^{m+|u|}} \left( 1 - \left( 1 - \frac{1}{3 \cdot 2^m} \right)^{|u|} \right) + \frac{1}{3^{|u|}} \sum_{v \subseteq u, v \neq \emptyset} \left( \frac{3}{2} \right)^{|v|} B(v) \right],$$

where for $v = \{v_1, \ldots, v_e\}$ we have

$$B(v) = \sum_{k_1, \ldots, k_e=1}^{2^m-1} \prod_{j=1}^e \psi(k_j),$$

with $\psi(k) = \frac{1}{6 \cdot 4^r(k)}$ and $r(k)$ such that $2^{r(k)} \leq k < 2^{r(k)+1}$.

The proof of this theorem is based on the Walsh series representation of the formula for the $L_2$ discrepancy given in Proposition 4.1.1. As we will see later, the function $\psi$ in the theorem above is related to Walsh coefficients of a certain function appearing in the formula for the $L_2$ discrepancy. We need several lemmas.

**Lemma 4.3.2** Let $x_1, x_2 \in [0,1)$ and let $z_1, z_2 \in [0,1)$ be the points obtained after applying an i.i.d. random digital shift of depth $m$ to $x_1$ and $x_2$. Then we have

$$E[\text{wal}_k(z_1)\text{wal}_l(z_2)] = \begin{cases} \text{wal}_k(x_1 \oplus x_2) & \text{if } 0 \leq k = l < 2^m, \\ 0 & \text{otherwise}. \end{cases}$$
Proof. Let \( x_n = \frac{x_{n,1}}{2} + \frac{x_{n,2}}{2^2} + \cdots \) for \( n = 1, 2 \). Further let \( \sigma_1, \ldots, \sigma_m \in \{0, 1\} \) i.i.d. and for \( n = 1, 2 \) let \( \delta_n = \frac{\delta_{n,m+1}}{2^{m+1}} + \frac{\delta_{n,m+2}}{2^{m+2}} + \cdots \in [0, \frac{1}{2^{m}}) \) be i.i.d.. Then define \( z_{n,i} \equiv x_{n,i} + \sigma_i \pmod{2} \) for \( i = 1, \ldots, m \) and \( z_n = \frac{z_{n,1}}{2} + \cdots + \frac{z_{n,m}}{2^m} + \delta_n \) for \( n = 1, 2 \).

First let \( k, l \in \mathbb{N} \), more precisely, let \( k = k_u 2^u + \cdots + k_1 2 + k_0 \) and \( l = l_v 2^v + \cdots + l_1 2 + l_0 \) be the dyadic expansion of \( k \) and \( l \) with \( k_u = l_v = 1 \). Further we set \( k_{u+1} = k_{u+2} = \cdots = 0 \) and also \( l_{u+1} = l_{u+2} = \cdots = 0 \). Then

\[
\mathbb{E}[\text{wal}_k(z_1)\text{wal}_l(z_2)] = (-1)^{k_0 x_{1,1} + \cdots + k_{m-1} x_{1,m}} (-1)^{l_0 x_{2,1} + \cdots + l_{m-1} x_{2,m}}
\]

\[
\frac{1}{2} \sum_{\sigma_1=0}^1 (-1)^{(k_0 + l_0)\sigma_1} \cdots \frac{1}{2} \sum_{\sigma_m=0}^1 (-1)^{(k_{m-1} + l_{m-1})\sigma_m}
\]

\[
\frac{1}{2} \sum_{\delta_{1,m+1}=0}^1 (-1)^{k_m \delta_{1,m+1}} \frac{1}{2} \sum_{\delta_{1,m+2}=0}^1 (-1)^{k_{m+1} \delta_{1,m+2}} \cdots
\]

\[
\frac{1}{2} \sum_{\delta_{2,m+1}=0}^1 (-1)^{l_m \delta_{2,m+1}} \frac{1}{2} \sum_{\delta_{2,m+2}=0}^1 (-1)^{l_{m+1} \delta_{2,m+2}} \cdots (4.3.2)
\]

(The product above consists only of finitely many factors as \( k_{u+1} = k_{u+2} = \cdots = 0 \) and for \( \kappa \geq \max(m, u + 1) \) we have \( \frac{1}{2} \sum_{\delta_{1,\kappa+1}=0}^1 (-1)^{k_0 \delta_{1,\kappa+1}} = 1 \). The same argument holds for the last line in the equation above.)

First we consider the case where \( u \geq m \). We have

\[
\frac{1}{2} \sum_{\delta_{1,u+1}=0}^1 (-1)^{k_0 \delta_{1,u+1}} = 0
\]

and therefore \( \mathbb{E}[\text{wal}_k(z_1)\text{wal}_l(z_2)] = 0 \). The same holds if \( v \geq m \). Now assume that there is an \( \omega \in \{0, \ldots, m-1\} \) such that \( k_\omega \neq l_\omega \). Then \( k_\omega + l_\omega \equiv 1 \pmod{2} \) and

\[
\frac{1}{2} \sum_{\sigma_{\omega+1}=0}^1 (-1)^{(k_\omega + l_\omega)\sigma_{\omega+1}} = 0.
\]

Therefore we obtain in this case \( \mathbb{E}[\text{wal}_k(z_1)\text{wal}_l(z_2)] = 0 \). Now let \( k = l \) and \( k \in \{0, \ldots, 2^m - 1\} \). It follows from (4.3.2) that

\[
\mathbb{E}[\text{wal}_k(z_1)\text{wal}_l(z_2)] = (-1)^{k_0 (x_{1,1} + x_{2,1}) + \cdots + k_{m-1} (x_{1,m} + x_{2,m})}
\]

and the result follows. \( \Box \)

In the following lemma we calculate Walsh coefficients of the function \( |z_1 - z_2| \). This function appears in the formula for the \( L_2 \) discrepancy through the equation \( \min(z_1, z_2) = \frac{1}{2}(z_1 + z_2 - |z_1 - z_2|) \).

Lemma 4.3.3 Let \( z_1, z_2 \in [0, 1) \). We have

\[
|z_1 - z_2| = \sum_{k,l=0}^{\infty} \tau(k,l)\text{wal}_k(z_1)\text{wal}_l(z_2),
\]

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where \( \tau(0) := \tau(0,0) = \frac{1}{3} \) and \( \tau(k) := \tau(k,k) = \frac{1}{64^{r+1}} \) for \( k > 0 \). For \( k > 0 \), \( r(k) \) denotes the unique integer \( r \) such that \( 2^r \leq k < 2^{r+1} \).

**Proof.** As \( |z_1 - z_2| \in \mathcal{L}_2([0,1)^2) \) it follows from Proposition 2.2.3 that the function \( |z_1 - z_2| \) can be represented by Walsh functions. We have

\[
\tau(k,l) = \int_0^1 \int_0^1 |z_1 - z_2| \text{wal}_k(z_1) \text{wal}_l(z_2) \, dz_1 \, dz_2.
\]

As \( \text{wal}_0(z) = 1 \) for all \( z \in [0,1) \) we have

\[
\tau(0,0) = \int_0^1 \int_0^1 |z_1 - z_2| \, dz_1 \, dz_2 = \frac{1}{3}.
\]

Let now \( k = l > 0 \) and \( k = k_r 2^r + \cdots + k_1 2 + k_0 \), where \( r \) is such that \( k_r = 1 \), \( u = u_r 2^r + \cdots + u_1 2 + u_0 \) and \( v = v_r 2^r + \cdots + v_1 2 + v_0 \). Then

\[
\tau(k,k) = \int_0^1 \int_0^1 |z_1 - z_2| \text{wal}_k(z_1 \oplus z_2) \, dz_1 \, dz_2
\]

\[
= \sum_{u=0}^{2^r+1} \sum_{v=0}^{2^r+1} (-1)^{k_0(u_r+v_r)+\cdots+k_r(u_0+v_0)} \times \int_{u/2^r+1}^{(u+1)/2^r+1} \int_{v/2^r+1}^{(v+1)/2^r+1} |z_1 - z_2| \, dz_1 \, dz_2.
\]

We have the following equalities: let \( 0 \leq u < 2^{r+1} \), then

\[
\int_{u/2^r+1}^{(u+1)/2^r+1} \int_{v/2^r+1}^{(v+1)/2^r+1} |z_1 - z_2| \, dz_1 \, dz_2 = \frac{1}{3} \cdot \frac{1}{2^{3(r+1)}}
\]

and for \( 0 \leq u, v < 2^{r+1}, u \neq v \), we have

\[
\int_{u/2^r+1}^{(u+1)/2^r+1} \int_{v/2^r+1}^{(v+1)/2^r+1} |z_1 - z_2| \, dz_1 \, dz_2 = \frac{|u - v|}{2^{3(r+1)}}.
\]

Thus

\[
\tau(k,k) = \sum_{u=0}^{2^r+1} \frac{1}{3} \cdot \frac{1}{2^{3(r+1)}} + \sum_{u=0}^{2^r+1} \sum_{v=0}^{2^r+1} (-1)^{k_0(u_r+v_r)+\cdots+k_r(u_0+v_0)} \frac{|u - v|}{2^{3(r+1)}}
\]

\[
= \frac{1}{3} \cdot \frac{1}{2^{2(r+1)}} + \frac{1}{2^{3(r+1)}} \sum_{u=0}^{2^r+1} \sum_{v=u+1}^{2^r+2} (-1)^{k_0(u_r+v_r)+\cdots+k_r(u_0+v_0)} (v - u).
\]

We define

\[
\theta(u,v) = (-1)^{k_0(u_r+v_r)+\cdots+k_r(u_0+v_0)} (v - u).
\]

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In order to find the value of the double sum in the expression for \( \tau(k, k) \) let \( u = u_r 2^r + \cdots + u_1 2 \) and let \( v = v_r 2^r + \cdots + v_1 2 \), where \( v > u \). We consider now the sum of the following four terms, namely \( \theta(u, v), \theta(u + 1, v), \theta(u, v + 1) \) and \( \theta(u + 1, v + 1) \). Observe that \( u \) and \( v \) are even, that is \( u_0 = v_0 = 0 \), and that \( k = k_r 2^r + \cdots + k_1 2 + k_0 \), where \( r \) is such that \( k_r = 1 \). We obtain now

\[
|\theta(u, v) + \theta(u + 1, v) + \theta(u, v + 1) + \theta(u + 1, v + 1)|
= (v - u) - ((v + 1) - u) - (v - (u + 1)) + ((v + 1) - (u + 1))
= 0.
\]

After applying this procedure we are left with the following terms:

\[ \theta(0, 1), \theta(2, 3), \ldots, \theta(2^r - 1, 2^r), \ldots, \theta(2^r, 2^r + 1) \].

Observe that in all cases we have \( v - u = 1 \), hence \( u_i = v_i \) for \( i = 1, \ldots, r \) and that \( u_0 = 0 \) and \( v_0 = 1 \). Therefore we have

\[
(-1)^{k_0 (u_r + v_r) + \cdots + k_{r-1} (u_1 + v_1) + k_r (u_0 + v_0)} = -1.
\]

Thus we obtain

\[
\tau(k, k) = \frac{1}{3} \cdot 2^{2r+1} + \frac{1}{3} \cdot 2^{2r+1} - \frac{2^r}{2^{2r+2}} = -\frac{1}{6} \cdot 2^{2r}.
\]

**Lemma 4.3.4** Let \( x_1, x_2 \in [0, 1) \) and let \( z_1, z_2 \in [0, 1) \) be the points obtained after applying an i.i.d. random digital shift of depth \( m \) to \( x_1 \) and \( x_2 \).

1. We have
   \[ \mathbb{E}[z_1] = \frac{1}{2} \quad \text{and} \quad \mathbb{E}[z_1^2] = \frac{1}{3}. \]

2. We have
   \[ \mathbb{E}[|z_1 - z_2|] = \sum_{k=0}^{2^m - 1} \tau(k) \text{wal}_k(x_1 \oplus x_2), \]
   where \( \tau(0) = \frac{1}{3} \) and \( \tau(k) = -\frac{1}{6^{2^r(r+1)}} \) for \( k > 0 \). For \( k > 0 \), \( r(k) \) denotes the unique integer \( r \) such that \( 2^r \leq k < 2^{r+1} \).

3. We have
   \[ \mathbb{E}[\min(1 - z_1, 1 - z_2)] = \frac{1}{2} \left( 1 - \sum_{k=0}^{2^m - 1} \tau(k) \text{wal}_k(x_1 \oplus x_2) \right). \]

**Proof.**

1. As \( z_1 \) is uniformly distributed in \([0, 1)\), we have
   \[ \mathbb{E}[z_1] = \int_0^1 z_1 \, dz_1 = \frac{1}{2} \quad \text{and} \quad \mathbb{E}[z_1^2] = \int_0^1 z_1^2 \, dz_1 = \frac{1}{3}. \]
2. In Lemma 4.3.3 it was shown that
\[ |z_1 - z_2| = \sum_{k,l=0}^{\infty} \tau(k,l) \text{wal}_k(z_1) \text{wal}_l(z_2), \]
where
\[ \tau(k) = \tau(k,k) = -\frac{1}{6 \cdot 4^r(k)}, \]
for \( k > 0 \) and \( \tau(0,0) = \frac{1}{3} \). (We do not need to know \( \tau(k,l) \) for \( k \neq l \) for our purposes here.) The result now follows from the linearity of the expectation value and Lemma 4.3.2.

3. This result follows from item 1. and 2. together with the formula
\[ \min(z_1, z_2) = \frac{1}{2} (z_1 + z_2 - |z_1 - z_2|). \]

We are now ready to prove Theorem 4.3.1.

**Proof of Theorem 4.3.1.** Let \( \tilde{P}_{2m} = \{z_0, \ldots, z_{2m-1}\} \) and \( z_n = (z_{n,1}, \ldots, z_{n,s}) \). From Proposition 4.1.1 and the linearity of expectation we get
\[
E[\mathcal{L}_{2,N,\gamma}(\tilde{P}_{2m})] = \sum_{u \subseteq D \atop u \neq \emptyset} \gamma_u \left[ \frac{1}{3|u|} \frac{2}{2m} \sum_{n=0}^{2m-1} \prod_{j \in u} \frac{1 - E[z_{n,j}^2]}{2} \right. \\
+ \left. \frac{1}{2^{2m}} \sum_{n,h=0}^{2m-1} \prod_{j \in u} E[\min(1 - z_{n,j}, 1 - z_{h,j})] \right]
\]
\[
= \sum_{u \subseteq D \atop u \neq \emptyset} \gamma_u \left[ \frac{1}{3|u|} + \frac{1}{2^{2m}} \sum_{n=0}^{2m-1} \prod_{j \in u} E[1 - z_{n,j}] \right. \\
+ \left. \frac{1}{2^{2m}} \sum_{n,h=0 \atop n \neq h}^{2m-1} \prod_{j \in u} E[\min(1 - z_{n,j}, 1 - z_{h,j})] \right].
\]

Now we use Lemma 4.3.4 to obtain
\[
E[\mathcal{L}_{2,N,\gamma}(\tilde{P}_{2m})] = \sum_{u \subseteq D \atop u \neq \emptyset} \gamma_u \left[ \frac{1}{3|u|} + \frac{1}{2^{2m}} \frac{1}{2|u|} \right. \\
+ \left. \frac{1}{2^{2m}} \sum_{n,h=0 \atop n \neq h}^{2m-1} \prod_{j \in u} \left( 1 - \sum_{k=0}^{2m-1} \tau(k) \text{wal}_k(x_{n,j} \oplus x_{h,j}) \right) \right].
\]

We have
\[
\prod_{j \in u} \left( 1 - \sum_{k=0}^{2m-1} \tau(k) \text{wal}_k(x_{n,j} \oplus x_{h,j}) \right) = 1 + \sum_{w \subseteq u \atop w \neq \emptyset} (-1)^{|w|} \times \\
\sum_{k_1=0}^{2m-1} \cdots \sum_{k_d=0}^{2m-1} \tau(k_1) \cdots \tau(k_d) \text{wal}_{k_1,\ldots,k_d}(x_{n,w_1} \oplus x_{h,w_1}, \ldots, x_{n,w_d} \oplus x_{h,w_d}).
\]
Thus

\[ \mathbb{E}[\mathcal{L}^2_{2,N,\gamma}(\tilde{P}_{2m})] = \sum_{u \subseteq D \atop u \neq \emptyset} \gamma_u \left[ -\frac{1}{3|u|} + \frac{1}{2^m} \frac{1}{2|u|} + \frac{1}{2^{2m}} \sum_{n,h=0 \atop n \neq h}^{2m-1} \frac{1}{2|u|} \right] + \frac{1}{2|u|} \frac{1}{2^{2m}} \sum_{n,h=0 \atop n \neq h}^{2m-1} \sum_{w \subseteq u \atop w \neq \emptyset} \frac{1}{2^{2m}} \sum_{w = \{w_1, \ldots, w_d\}} (-1)^d \times \sum_{k_1=0}^{2^m-1} \cdots \sum_{k_d=0}^{2^m-1} \prod_{i=1}^{d} \tau(k_i) wal_{k_i}(x_{n,w_i} \oplus x_{h,w_i}) \].

We have

\[ \sum_{k=0}^{2^m-1} \tau(k) = \frac{1}{3} - \sum_{r=0}^{m-2} \sum_{k=2^r}^{2^m-1} \frac{1}{6 \cdot 4^r} = \frac{1}{3 \cdot 2^m}, \]

and therefore

\[ \sum_{w \subseteq u \atop w \neq \emptyset} \sum_{k_1, \ldots, k_{|w|}=0}^{2^m-1} \prod_{i=1}^{|w|} \tau(k_i) = \sum_{|w| \geq 1} \left( \frac{|w|}{|u|} \right) \left( -\frac{1}{3 \cdot 2^m} \right)^{|w|} = \sum_{r=1}^{\infty} \left( \frac{|u|}{r} \right) \left( -\frac{1}{3 \cdot 2^m} \right)^r = \left( 1 - \frac{1}{3 \cdot 2^m} \right)^{|u|} - 1. \]

By adding and subtracting this in the above expression we obtain

\[
\mathbb{E}[\mathcal{L}^2_{2,N,\gamma}(\tilde{P}_{2m})] = \sum_{u \subseteq D \atop u \neq \emptyset} \gamma_u \left[ -\frac{1}{3|u|} + \left( 1 - \left( 1 - \frac{1}{3 \cdot 2^m} \right) \right) \frac{1}{2^m} \frac{1}{2|u|} \right] + \frac{1}{2^m} \frac{1}{2|u|} \sum_{n,h=0 \atop n \neq h}^{2m-1} \sum_{w \subseteq u \atop w \neq \emptyset} \frac{1}{2^{2m}} \sum_{w = \{w_1, \ldots, w_d\}} (-1)^d \times \sum_{k_1=0}^{2^m-1} \cdots \sum_{k_d=0}^{2^m-1} \prod_{i=1}^{d} \tau(k_i) wal_{k_i}(x_{n,w_i} \oplus x_{h,w_i}) \].

Since \( \tau(0) = \frac{1}{3} \) we have

\[ \frac{1}{2^{2m}} \frac{1}{2|u|} \sum_{n,h=0 \atop n \neq h}^{2m-1} \sum_{w \subseteq u \atop w \neq \emptyset} (-1)^{|w|} \tau(0)^{|w|} = \frac{1}{3|u|} - \frac{1}{2|u|}. \]
Hence

\[ \mathbb{E}[\mathcal{L}_{2,N,\gamma}^2(\widetilde{P}_{2m})] \]

\[ = \sum_{u \subseteq D \atop u \neq \emptyset} \gamma_u \left[ \frac{1}{2^{|u|}} - \frac{1}{3^{|u|}} + \left( 1 - \left( 1 - \frac{1}{3 \cdot 2^m} \right)^{|u|} \right) \right] \frac{1}{2^m} \frac{1}{2^{|u|}} + \frac{1}{3^{|u|}} - \frac{1}{2^{|u|}} \]

\[ + \frac{1}{2^{|u|}} \frac{1}{2^{2m}} \sum_{w \subseteq u \atop w \neq \emptyset} \left( -1 \right)^d \sum_{k_1, \ldots, k_d = 0 \atop (k_1, \ldots, k_d) \neq (0, \ldots, 0)} \sum_{n,h=0}^{2m-1} \prod_{i=1}^{d} \tau(k_i) \omega_{k_i}(x_{n,w_j} \oplus x_{h,w_j}) \].

From the group structure of digital nets (see Lemma 2.3.3) and Lemma 2.3.4 it follows that for any digital net \( \{x_0, \ldots, x_{2^{m-1}}\} \) generated by the \( m \times m \) matrices \( C_1, \ldots, C_s \), we have

\[ \frac{1}{2^{2m}} \sum_{n,h=0}^{2m-1} \omega_{k_1,\ldots,k_s}(x_n \oplus x_h) = \frac{1}{2^m} \sum_{n=0}^{2m-1} \omega_{k_1,\ldots,k_s}(x_n) \]

\[ = \begin{cases} 1 & \text{if } C_1^T \vec{k}_1 + \cdots + C_s^T \vec{k}_s = \vec{0}, \\ 0 & \text{otherwise.} \end{cases} \]

Since we have that the \( d \)-dimensional projection of a digital \( (t, m, d) \)-net is again a digital \( (t, m, d) \)-net (see Introduction) we get (with \( \mathbf{w} = \{w_1, \ldots, w_d\} \))

\[ = 2^{2m} \sum_{v \subseteq \mathbf{w} \atop v \neq \emptyset} \frac{1}{3^{|\mathbf{w}|} - |\mathbf{w}|} \sum_{k_1,\ldots,k_e = 1 \atop (k_1,\ldots,k_e) \neq (0,\ldots,0)} \prod_{j=1}^{e} \tau(k_j). \]

As \( \prod_{j=1}^{e} \tau(k_j) = (-1)^e \prod_{j=1}^{e} \psi(k_j) \) we have

\[ \sum_{k_1,\ldots,k_d = 0 \atop (k_1,\ldots,k_d) \neq (0,\ldots,0)} \prod_{j=1}^{d} \tau(k_j) \omega_{k_j}(x_{n,w_j} \oplus x_{h,w_j}) = \frac{2^{2m}}{3^{|\mathbf{w}|} \sum_{u \subseteq \mathbf{w} \atop u \neq \emptyset} (-3)^{|u|} B(\mathbf{v})}. \]

Thus we obtain

\[ \mathbb{E}[\mathcal{L}_{2,N,\gamma}^2(\widetilde{P}_{2m})] = \sum_{u \subseteq D \atop u \neq \emptyset} \gamma_u \left[ \frac{1}{2^{m+|u|}} \left( 1 - \left( 1 - \frac{1}{3 \cdot 2^m} \right)^{|u|} \right) \right] \]

\[ + \frac{1}{2^{|\mathbf{w}|}} \sum_{v \subseteq \mathbf{w} \atop v \neq \emptyset} \left( -\frac{1}{3} \right)^{|\mathbf{w}|} \sum_{u \subseteq \mathbf{w} \atop u \neq \emptyset} (-3)^{|u|} B(\mathbf{v}) \].

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Let now $u, v$, with $\emptyset \neq v \subset u \subset D$, be fixed. Then $v \subseteq w \subseteq u$ is equivalent to $(w \setminus v) \subseteq (u \setminus v)$, provided that $v \subseteq w$. Therefore, for $|v| \leq w \leq |u|$, there are $\binom{|u| - |v|}{|w| - |v|}$ sets $w$ such that $|w| = w$ and $v \subseteq w \subseteq u$. Hence

$$
\sum_{u \subseteq v} \left( -\frac{1}{3} \right)^{|v|} \sum_{w \subseteq |v|} (-3)^{|w|} B(v) = \sum_{u \subseteq v} \sum_{w \subseteq |v|} \left( \frac{|u| - |v|}{w - |v|} \right) \left( -\frac{1}{3} \right)^{w} (-3)^{|v|} B(v)
$$

$$
= \sum_{u \subseteq v} \sum_{w = 0}^{w - |v|} \left( \frac{|u| - |v|}{w} \right) \left( -\frac{1}{3} \right)^{w} B(v)
$$

$$
= \sum_{u \subseteq v} \left( \frac{2}{3} \right)^{|u| - |v|} B(v)
$$

and the result follows. \hfill \Box

### 4.3.2 The mean square weighted $L_2$ discrepancy of randomized digital $(0, m, s)$-nets over $\mathbb{Z}_2$ for $s = 2, 3$

In this subsection we calculate the exact value of the mean square weighted $L_2$ discrepancy of randomized digital $(0, m, s)$-nets for $s = 2, 3$. We have the following theorem.

**Theorem 4.3.5** For $s = 2, 3$ let $P_{s,2^m}$ be a digital $(0, m, s)$-net over $\mathbb{Z}_2$. Let $\tilde{P}_{s,2^m}$ be the point set obtained after applying an i.i.d. random digital shift of depth $m$ independently to each coordinate of each point of $P_{s,2^m}$. Then the mean square weighted $L_2$ discrepancy of $\tilde{P}_{s,2^m}$ for $s = 2$ is given by

$$
\mathbb{E}[L_{2,2^m}^2, \gamma(\tilde{P}_{2,2^m})] = \frac{\gamma(1,2)}{24} \frac{m}{2^m} + \frac{1}{2^m} \left( \frac{\gamma(1)}{6} + \frac{\gamma(2)}{6} + \frac{5\gamma(1,2)}{36} \right)
$$

and for $s = 3$ the mean square weighted $L_2$ discrepancy is given by

$$
\mathbb{E}[L_{2,2^m}^2, \gamma(\tilde{P}_{3,2^m})] = \gamma(1,2,3) \left( \frac{1}{192} \frac{m^2}{2^m} + \frac{23}{576} \frac{m}{2^m} + \frac{19}{216} \frac{1}{2^m} \right)
$$

$$
+ (\gamma(1,2) + \gamma(1,3) + \gamma(2,3)) \left( \frac{1}{24} \frac{m}{2^m} + \frac{5}{36} \right)
$$

$$
+ (\gamma(1) + \gamma(2) + \gamma(3)) \frac{1}{6} \frac{1}{2^m}.
$$

**Proof.** Let $C_1$ and $C_2$ denote the generating matrices of the digital $(0, m, 2)$-net over $\mathbb{Z}_2$. For $s = 2$ we obtain from Theorem 4.3.1

$$
\mathbb{E}[L_{2,2^m}^2, \gamma(\tilde{P}_{2,2^m})] = \sum_{u \subseteq (1,2)} \gamma_u \left[ \frac{1}{2m + |u|} \left( 1 - \left( 1 - \frac{1}{3 \cdot 2^m} \right)^{|u|} \right) + \frac{1}{3 |u|} \sum_{v \subseteq u \atop v \neq \emptyset} \left( \frac{3}{2} \right)^{|v|} B(v) \right],
$$

(4.3.3)
where for \( v = \{v_1, \ldots, v_e\} \) we have

\[
B(v) = \sum_{k_1, \ldots, k_e = 1}^{2m-1} \prod_{j=1}^{e} \psi(k_j),
\]

with \( \psi(k) = \frac{1}{6 \cdot 4^m} \) and \( r(k) \) such that \( 2^r(k) \leq k < 2^{r(k)+1} \).

First we note, as \( C_1 \) and \( C_2 \) generate a \((0, m, 2)\)-net, that \( C_1 \) and \( C_2 \) are regular. Therefore \( B(v) = 0 \) for \( |v| = 1 \). Thus for \( |u| = 1 \) we obtain \( \frac{1}{2^m} \gamma_u \).

Now let \( u = \{1, 2\} \). We have

\[
\frac{\gamma\{1,2\}}{2^m+|u|} \left( 1 - \left( 1 - \frac{1}{3 \cdot 2^m} \right) \right) = \frac{\gamma\{1,2\}}{6 \cdot 2^{2m}} - \frac{\gamma\{1,2\}}{36 \cdot 2^{3m}}.
\]

In the following we calculate \( B(\{1,2\}) \). Since the generating matrices \( C_1 \) and \( C_2 \) of a digital \((0, m, 2)\)-net over \( \mathbb{Z}_2 \) must be regular, and since multiplying \( C_1 \) and \( C_2 \) by a regular matrix \( A \) does not change the point set (only its order) we may assume in the following that \( C_1 \) is the \( m \times m \) identity matrix. Hence we have

\[
C_1^T \vec{k}_1 + C_2^T \vec{k}_2 = \vec{0}
\]

iff

\[
\vec{k}_1 = C_2^T \vec{k}_2 =: \vec{k}_1(k_2).
\]

Now we use the definition of \( \psi \) and we get

\[
\sum_{k_1, k_2 = 1}^{2m-1} \psi(k_1) \psi(k_2) = \frac{1}{36} \sum_{k_2 = 1}^{2^{m-1}} \frac{1}{4^r(k_1(k_2))} \frac{1}{4^{k_2}} = \frac{1}{36} \sum_{u = 0}^{m-1} \sum_{k_2 = 2^u}^{2^{n+1}-1} \frac{1}{4^u} \sum_{k_2 = 2^u}^{2^{n+1}-1} \frac{1}{4^r(k_1(k_2))}.
\]

Consider the innermost sum in the above expression. We have

\[
\Sigma(u) := \sum_{k_2 = 2^u}^{2^{n+1}-1} \frac{1}{4^r(k_1(k_2))} = \sum_{w = 0}^{m-1} \frac{1}{4^w} \sum_{k_2 = 2^u}^{2^{n+1}-1} \frac{1}{4^r(k_1(k_2))}.
\]

From [30, Proof of Theorem 1] we find that

\[
\sum_{k_2 = 2^u}^{2^{n+1}-1} 1 = \begin{cases} 
0 & \text{if } u + w \leq m - 2, \\
1 & \text{if } u + w = m - 1, \\
2^{w+m} & \text{if } u + w \geq m.
\end{cases}
\]

Thus

\[
\Sigma(u) = \frac{1}{4^{m-1-u}} + \sum_{w = m-u}^{m-1} \frac{2^{u+w-m}}{4^w} = \frac{6 \cdot 4^u}{4^m} - \frac{2 \cdot 2^u}{4^m}.
\]

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and therefore

\[
\sum_{c_1^T k_1 + c_2^T k_2 = 0}^{2^m - 1} \psi(k_1) \psi(k_2) = \frac{1}{36} \sum_{u=0}^{m-1} \frac{1}{4^u} \left( 6 \cdot 4^u - 2 \cdot 2^u \right)
\]

\[
= \frac{m}{6 \cdot 4^m} - \frac{1}{36 \cdot 4^m} \frac{2}{2^m} \left( 1 - \frac{1}{2^m} \right)
\]

\[
= \frac{m}{6 \cdot 4^m} - \frac{1}{9 \cdot 4^m} + \frac{4}{36 \cdot 2^m}.
\]

Now we insert this result in equation (4.3.3) and get

\[
\mathbb{E}[L_{2,2m}^2, \gamma(\tilde{P}_{2,2m})] = \frac{1}{2^{2m}} \left( \frac{\gamma(1)}{6} + \frac{\gamma(2)}{6} + \frac{\gamma(1,2)}{6} \right) - \frac{\gamma(1,2)}{36 \cdot 2^m}
\]

\[
+ \frac{\gamma(1,2)}{4} \left( \frac{m}{6 \cdot 4^m} - \frac{1}{9 \cdot 4^m} + \frac{4}{36 \cdot 2^m} \right)
\]

\[
= \frac{\gamma(1,2)}{24} \frac{m}{2^{2m}} + \frac{1}{2^{2m}} \left( \frac{\gamma(1)}{6} + \frac{\gamma(2)}{6} + \frac{5\gamma(1,2)}{36} \right),
\]

which is the desired result for \( s = 2 \).

We turn to the case where \( s = 3 \). Let \( C_1, C_2 \) and \( C_3 \) denote the generating matrices of the digital \((0, m, 3)\)-net over \( \mathbb{Z}_2 \). As the quality parameter \( t \) is zero it is clear that \( C_1, C_2 \) and \( C_3 \) are regular. Hence \( B(v) = 0 \) for \( |v| = 1 \). Further, for \( v \subseteq \{1, 2, 3\} \) with \( |v| = 2 \) we obtain from the first part of the proof

\[
B(v) = \frac{m}{6 \cdot 4^m} - \frac{1}{9 \cdot 4^m} + \frac{1}{9 \cdot 2^m}.
\]

So it remains to calculate \( B(\{1, 2, 3\}) \). As above we may assume in the following that \( C_1 \) is the \( m \times m \) identity matrix. Hence we have

\[
C_1^T \tilde{k}_1 + C_2^T \tilde{k}_2 + C_3^T \tilde{k}_3 = 0
\]

iff

\[
\tilde{k}_1 = C_2^T \tilde{k}_2 + C_3^T \tilde{k}_3 =: \tilde{k}_1(k_2, k_3).
\]

Now we get

\[
B(\{1, 2, 3\}) = \sum_{c_1^T k_1 + c_2^T k_2 + c_3^T k_3 = 0}^{2^m - 1} \psi(k_1) \psi(k_2) \psi(k_3)
\]

\[
= \frac{1}{216} \sum_{k_1=0}^{2^m - 1} \frac{1}{4^r(k_1(k_2, k_3))} \frac{1}{4^r(k_2) + 4^r(k_3)}
\]

\[
= \frac{1}{216} \sum_{u=0}^{m-1} \frac{1}{4^{u+v}} \sum_{k_2=2^u}^{2^{u+1}-1} \sum_{k_3=2^v}^{2^{v+1}-1} \frac{1}{4^r(k_1(k_2, k_3))},
\]

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The innermost double-sum in the above expression equals

$$\Sigma(u, v) := \sum_{k_2=2^u}^{2^{u+1}-1} \sum_{k_3=2^v}^{2^{v+1}-1} \frac{1}{4^r(k_1(k_2,k_3))} = \sum_{w=0}^{m-1} \frac{1}{4^w} \sum_{k_2=2^u}^{2^{u+1}-1} \sum_{k_3=2^v}^{2^{v+1}-1} 1,$$

From [51, Proof of Theorem 1] we find that

$$\sum_{k_2=2^u}^{2^{u+1}-1} \sum_{k_3=2^v}^{2^{v+1}-1} 1 = \begin{cases} 0 & \text{if } u + v + w \leq m - 3, \\ 1 & \text{if } u + v + w = m - 2, \\ 0 & \text{if } u + v + w = m - 1, \\ 2^{u+v+w-m} & \text{if } u + v + w \geq m. \end{cases}$$

Therefore we get

$$B(\{1, 2, 3\}) = \frac{1}{216} \sum_{u,v,w=0}^{m-1} \frac{1}{4^{u+v+w}} + \frac{1}{216} \sum_{u,v,w=0}^{m-1} \frac{1}{2^{u+v+w}}.$$

For the first sum we have

$$\sum_{u,v,w=0}^{m-1} \frac{1}{4^{u+v+w}} = \frac{1}{4^{m-2}} \binom{m}{2}.$$

The second sum can be written as

$$\sum_{u,v,w=0}^{m-1} \frac{1}{2^{u+v+w}} = \sum_{l=m}^{3m-3} \frac{1}{2^l} \sum_{u,v,w=0}^{m-1} 1.$$

Define

$$f(k) := \sum_{u,v=0}^{m-1} 1,$$

then we have

$$f(k) = \begin{cases} k + 1 & \text{if } 0 \leq k \leq m - 1, \\ 2m - k - 1 & \text{if } m \leq k \leq 2m - 2, \\ 0 & \text{if } k \geq 2m - 1. \end{cases}$$

Now we obtain

$$\sum_{u,v,w=0}^{m-1} \frac{1}{2^{u+v+w}} = \sum_{k=0}^{2m-2} f(k) \sum_{w=0}^{m-1} 1 = \sum_{k=0}^{2m-2} f(k) = \sum_{k=0}^{\min(l, 2m-2)} f(k).$$
Therefore we obtain
\[
\sum_{l=m}^{3m-3} \frac{1}{2^l} \sum_{u,v,w=0}^{m-1} 1 = \sum_{l=m}^{3m-3} \frac{1}{2^l} \sum_{k=\max(0,l-m+1)}^{\min(l,2m-2)} f(k)
\]
\[
= \sum_{l=m}^{3m-3} \frac{1}{2^l} \sum_{k=l-m+1}^{\min(l,2m-2)} f(k)
\]
\[
= \sum_{l=m}^{2m-2} \frac{1}{2^l} \sum_{k=l-m+1}^{l} f(k) + \sum_{l=2m-1}^{3m-3} \frac{1}{2^l} \sum_{k=l-m+1}^{2m-2} f(k).
\]

After some straightforward but tedious calculations we obtain the formula for \(B(\{1, 2, 3\})\). The result then follows by inserting the above results in the formula from Theorem 4.3.1.

Note that the generating matrices \(C_1, \ldots, C_s\) do not appear in our formula and therefore the mean square weighted \(L_2\) discrepancy is the same for any digital \((0, m, s)\)-net over \(\mathbb{Z}_2\), for \(s = 2, 3\). This is also true for scrambled \((0, m, s)\)-nets in a base \(b\) (see [16, 38]). Furthermore, the expected value of the \(L_2\) discrepancy of scrambled \((0, m, 2)\)-nets is the same as for \((0, m, 2)\)-nets which are randomized using a digital shift of depth \(m\), see [38].

In the following we consider the classical \(L_2\) discrepancy, that is, we choose \(\gamma_D = 1\) and \(\gamma_u = 0\) for \(u \subset D\). We denote this choice of weights by \(\gamma_e\). Roth [56] proved that for any dimension \(s \geq 2\) there exists a constant \(c(s) > 0\) such that for any point set \(P_N\) consisting of \(N\) points in the \(s\)-dimensional unit-cube we have
\[
\int_{[0,1]^s} \Delta(x)^2 \, dx \geq c(s) \frac{(\log N)^{s-1}}{N^2}.
\]

Therefore we obtain
\[
L_{2,N,\gamma_e}(P_N) \geq c(s)^{\frac{1}{2}} \frac{(\log N)^{s-1}}{N}.
\]
(See Section 4.4 for more details.) Thus Theorem 4.3.5 shows that the mean square \(L_2\) discrepancy of randomized digital \((0, m, s)\)-nets over \(\mathbb{Z}_2\) achieves the best possible rate of convergence for \(s = 2\) and 3.

In the following we compare the constants. For \(N = 2^m\) the constant of [56] can be improved to
\[
c(2)^{\frac{1}{2}} = \frac{3}{2^8 \sqrt{\log 2}} = 0.01407 \ldots \quad \text{and} \quad c(3)^{\frac{1}{2}} = \frac{3}{2^{10} \sqrt{2} \log 2} = 0.00298 \ldots.
\]

In the following let for \(s = 2\) and 3 and each \(m \in \mathbb{N}\) the set \(P_{s,2m,\sigma_m,s}\) be a digital \((0, m, s)\)-net over \(\mathbb{Z}_2\) shifted by the digital shift \(\sigma_{m,s}\) of depth \(m\). We obtain the following corollary.
Corollary 4.3.6 For \( s = 2, 3 \) there exist sequences of digital shifts \((\sigma_{m,s})_{m \geq 1}\) of depth \( m \) and digital \((0, m, s)\)-nets over \( \mathbb{Z}_2 \), \((P_{s,2^m})_{m \geq 1}\) such that the sequences of shifted nets \((P_{s,2^m,\sigma_{m,s}})_{m \geq 1}\) satisfy

\[
\limsup_{m \to \infty} \frac{2^m \mathcal{L}_{2,2^m}(P_{2^m,\sigma_{m,2}})}{\sqrt{\log 2^m}} \leq \frac{1}{\sqrt{24 \log 2}} = 0.24518 \ldots
\]

and

\[
\limsup_{m \to \infty} \frac{2^m \mathcal{L}_{2,2^m}(P_{3^m,\sigma_{m,3}})}{\log 2^m} \leq \frac{1}{\sqrt{192 \log 2}} = 0.10411 \ldots
\]

We remark that it might be possible to improve the constant in Corollary 4.3.6 by finding the best shift for each digital \((0, m, s)\)-net over \( \mathbb{Z}_2 \).

Note that one can also obtain the constants for the weighted \( L_2 \) discrepancy: the constant of a weighted lower bound can be obtained from the definition of weighted \( L_2 \) discrepancy (4.1.1) and (4.3.4) and the constant for the upper bound can be obtained from Theorem 4.3.5.

4.3.3 An upper bound on the mean square weighted \( L_2 \) discrepancy of randomized digital \((t, m, s)\)-nets over \( \mathbb{Z}_2 \)

In this subsection we derive an upper bound on the formula shown in Theorem 4.3.1. We have the following theorem.

Theorem 4.3.7 Let \( P_{2^m} \) be a digital \((t, m, s)\)-net over \( \mathbb{Z}_2 \) with \( t < m \). Let \( \widetilde{P}_{2^m} \) be the point set obtained after applying an i.i.d. random digital shift of depth \( m \) independently to each coordinate of each point of \( P_{2^m} \). Then the mean square weighted \( L_2 \) discrepancy of \( P_{2^m} \) is bounded by

\[
E[\mathcal{L}_{2,2^m}(\gamma(\widetilde{P}_{2^m}))] \leq \frac{1}{2^{2(m-t)}} \sum_{u \subseteq D \setminus \emptyset} \gamma_u(m-t)^{|u|-1}.
\]

As for the exact value of the mean square weighted \( L_2 \) discrepancy for \((0, m, s)\)-nets with \( s = 2, 3 \), the generating matrices \( C_1, \ldots, C_s \) do not appear in the upper bound, which depends now only on the quality parameter \( t \). For \( t > 0 \) the exact value of \( B(v) \) (see Theorem 4.3.1) depends on the generating matrices and therefore we prove a bound.

For fixed \( s \geq 1 \) there is a \( t \geq 0 \) such that for every \( m > t \) there is a digital \((t, m, s)\)-net over \( \mathbb{Z}_2 \) (see for example [41]). Thus Theorem 4.3.7 shows that the convergence order of the mean square weighted \( L_2 \) discrepancy is best possible by the lower bound by Roth [56], see (4.3.5).

We need two lemmas for the proof of the above theorem.
Lemma 4.3.8 For \( b > 1 \) and integers \( k, t_0 > 0 \), we have
\[
\sum_{t=t_0}^{\infty} \binom{t+k-1}{k-1} b^{-t} \leq b^{-t_0} \binom{t_0 + k-1}{k-1} \left( 1 - \frac{1}{b} \right)^{-k}.
\]

Proof. For completeness we give a short proof of the lemma which is taken from Matoušek [38]. By the binomial theorem we have
\[
\sum_{t=t_0}^{\infty} \binom{t+k-1}{k-1} b^{-t} = b^{-t_0} \binom{t_0 + k-1}{k-1} \left( 1 - \frac{1}{b} \right)^{-k}.
\]
Now use the inequality \( \binom{t+k-1}{k-1} / \binom{t_0+k-1}{k-1} = \frac{(t+k-1)(t+k-2)\cdots(t-t_0+k)}{(t_0-1)\cdots(t_0-t+1)} \leq \frac{t_0+k-1}{k-1} \) and we are done. \( \square \)

Lemma 4.3.9 Let \( C_1, \ldots, C_s \) be the generating matrices of a digital \((t, m, s)\)-net over \( \mathbb{Z}_2 \). Further define \( B \) as in Theorem 4.3.1. Then for any \( v \subseteq D \) we have
\[
B(v) \leq \frac{2^{2t}}{2^{2m}} \left( \frac{8}{9} \right)^{|v|} \left( m - t + \frac{1}{8} \right)^{|v|-1}.
\]

Proof. To simplify the notation we show the result only for \( v = \{1, \ldots, s\} \). The other cases follow by the same arguments. We have
\[
B(\{1, \ldots, s\}) = \frac{1}{6^s} \sum_{v_1, \ldots, v_s=0}^{m-1} \sum_{k_1=2^{v_1}}^{2^{v_1+1}-1} \cdots \sum_{k_s=2^{v_s}}^{2^{v_s+1}-1} \sum_{C_1^T k_1 + \cdots + C_s^T k_s = 0} 1.
\]
Now we write
\[
\Sigma(v_1, \ldots, v_s) := \sum_{k_1=2^{v_1}}^{2^{v_1+1}-1} \cdots \sum_{k_s=2^{v_s}}^{2^{v_s+1}-1} \sum_{C_1^T k_1 + \cdots + C_s^T k_s = 0} 1. \quad (4.3.6)
\]
For \( 1 \leq j \leq s \) and \( 1 \leq i \leq m \) let \( \bar{c}_j^T \) denote the \( i \)-th row vector of the matrix \( C_j \).

For \( 2^v \leq k_j \leq 2^{v+1} - 1 \), the binary digit expansion of \( k_j \) is of the form
\[
k_j = k_{j,0} + k_{j,1} 2 + \cdots + k_{j,v-1} 2^{v-1} + 2^v.
\]
Hence the condition in our sum (4.3.6) can be written as
\[
\bar{c}_{1,1} k_{1,0} + \cdots + \bar{c}_{1,v_1} k_{1,v_1-1} + \bar{c}_{1,v_1+1} + \bar{c}_{2,1} k_{2,0} + \cdots + \bar{c}_{2,v_2} k_{2,v_2-1} + \bar{c}_{2,v_2+1} + \cdots + \bar{c}_{s,1} k_{s,0} + \cdots + \bar{c}_{s,v_s} k_{s,v_s-1} + \bar{c}_{s,v_s+1} = 0. \quad (4.3.7)
\]

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Since by the digital \((t, m, s)\)-net property (see Definition 2.3.2) the vectors
\[
\vec{c}_{1,1}, \ldots, \vec{c}_{1,v_1+1}, \ldots, \vec{c}_{s,1}, \ldots, \vec{c}_{s,v_s+1}
\]
are linearly independent as long as \((v_1 + 1) + \cdots + (v_s + 1) \leq m - t\), we must have
\[
v_1 + \cdots + v_s \geq m - t - s + 1.
\] (4.3.8)

Let now \(A\) denote the \(m \times (v_1 + \cdots + v_s)\) matrix with column vectors
\[
\vec{c}_{1,1}, \ldots, \vec{c}_{1,v_1}, \ldots, \vec{c}_{s,1}, \ldots, \vec{c}_{s,v_s},
\]
i.e.,
\[
\begin{array}{c}
\vec{c}_{1,1}, \\
\vec{c}_{1,v_1}, \\
\vdots \\
\vec{c}_{s,1}, \\
\vec{c}_{s,v_s},
\end{array}
\]
Further let
\[
\vec{f} := \vec{c}_{1,v_1+1} \oplus \cdots \oplus \vec{c}_{s,v_s+1} \in \mathbb{Z}_2^m
\]
and
\[
\vec{k} := (k_{1,0}, \ldots, k_{1,v_1-1}, \ldots, k_{s,0}, \ldots, k_{s,v_s-1})^T \in \mathbb{Z}_2^{v_1 + \cdots + v_s}.
\]
Then the linear equation system (4.3.7) can be written as
\[
A\vec{k} = \vec{f}
\] (4.3.9)
and hence
\[
\Sigma(v_1, \ldots, v_s) = \sum_{\vec{k} \in \mathbb{Z}_2^{v_1 + \cdots + v_s}} 1 = \#\{\vec{k} \in \mathbb{Z}_2^{v_1 + \cdots + v_s} : A\vec{k} = \vec{f}\}.
\]
By the definition of the matrix \(A\) and since \(C_1, \ldots, C_s\) are the generating matrices of a digital \((t, m, s)\)-net over \(\mathbb{Z}_2\) we have
\[
\text{rank}(A) = \begin{cases} 
  v_1 + \cdots + v_s & \text{if } v_1 + \cdots + v_s \leq m - t, \\
  \geq m - t & \text{else.}
\end{cases}
\]
Let \(L\) denote the linear space of solutions of the homogen system \(A\vec{k} = \vec{0}\) and let \(\text{dim}(L)\) denote the dimension of \(L\). Then it follows that
\[
\text{dim}(L) = \begin{cases} 
  0 & \text{if } v_1 + \cdots + v_s \leq m - t, \\
  \leq v_1 + \cdots + v_s - m + t & \text{else.}
\end{cases}
\]
Hence if \(v_1 + \cdots + v_s \leq m - t\) we find that the system (4.3.9) has at most 1 solution and if \(v_1 + \cdots + v_s > m - t\) the system (4.3.9) has at most \(2^{v_1 + \cdots + v_s - m + t}\) solutions, i.e.,
\[
\Sigma(v_1, \ldots, v_s) \leq \begin{cases} 
  1 & \text{if } v_1 + \cdots + v_s \leq m - t, \\
  2^{v_1 + \cdots + v_s - m + t} & \text{if } v_1 + \cdots + v_s > m - t.
\end{cases}
\]
Together with condition (4.3.8) we obtain
\[
B(\{1, \ldots, s\}) \leq \frac{1}{6^s} \sum_{v_1, \ldots, v_s = 0}^{m-1} \frac{1}{4^{v_1+\cdots+v_s}} + \frac{1}{6^s} \sum_{v_1, \ldots, v_s = 0}^{m-1} \frac{1}{4^{v_1+\cdots+v_s}} 2^{v_1+\cdots+v_s-m+t} =: \Sigma_1 + \Sigma_2. \tag{4.3.10}
\]

Now we have to estimate the sums \(\Sigma_1\) and \(\Sigma_2\). First we have
\[
\Sigma_2 = \frac{1}{6^s} \frac{2^l}{2^m} \sum_{l=m-t+1}^{s(m-1)} \frac{1}{2^l} \sum_{v_1, \ldots, v_s = 0}^{m-1} 1 \leq \frac{1}{6^s} \frac{2^l}{2^m} \sum_{l=m-t+1}^{\infty} \binom{l+s-1}{s-1} \frac{1}{2^l},
\]
where we used the fact that for fixed \(l\) the number of non-negative integer solutions of \(v_1 + \cdots + v_s = l\) is given by \(\binom{l+s-1}{s-1}\). Now we apply Lemma 4.3.8 and obtain
\[
\Sigma_2 \leq \frac{1}{6^s} \frac{2^l}{2^m} \frac{1}{2^{m-t+1}} \binom{m-t+s}{s-1} 2^s = \frac{1}{3^s} \frac{4^t}{4^m} \frac{1}{2^t} \binom{m-t+s}{s-1}. \tag{4.3.11}
\]

Finally, since
\[
\binom{m-t+s}{s-1} = \frac{(m-t+2)(m-t+3)\cdots(m-t+s)}{1 \cdot 2 \cdots (s-1)} \leq (m-t+2)^{s-1},
\]
we obtain
\[
\Sigma_2 \leq \frac{1}{3^s} \frac{4^t}{4^m} \frac{1}{2^t} (m-t+2)^{s-1}.
\]

Now we estimate \(\Sigma_1\). If \(m-t \geq s-1\) we proceed similar to above and obtain
\[
\Sigma_1 = \frac{1}{6^s} \sum_{l=m-t-s+1}^{m-t} \binom{l+s-1}{s-1} \frac{1}{4^l} \leq \frac{1}{6^s} \frac{1}{4^m} \frac{1}{2^{m-t-s+1}} \binom{m-t}{s-1} \left(\frac{3}{4}\right)^{-s} \tag{4.3.12}
\]
\[
\leq \frac{8^s \frac{4^t}{4^m} \frac{1}{2^t}}{9^s \frac{4^m}{4}(s-1)!}.
\]

For this case we obtain
\[
B(\{1, \ldots, s\}) \leq \frac{8^s \frac{4^t}{4^m} \frac{1}{2^t}}{9^s \frac{4^m}{4}(s-1)!} \leq \frac{8^s \frac{4^t}{4^m} \frac{1}{2^t}}{9^s \frac{4^m}{4}(s-1)!} + \frac{3}{8} \frac{3}{8} \frac{1}{2^t} (m-t+2)^{s-1}
\]
\[
\leq \frac{8^s \frac{4^t}{4^m} \frac{1}{2^t}}{9^s \frac{4^m}{4}(s-1)!} + \frac{3}{8} \frac{3}{8} \frac{1}{2^t} (m-t+2)^{s-1}.
\]
As \( \frac{3}{8}(m-t) + \frac{6}{8} \leq m - t + \frac{1}{8} \) provided that \( m - t > 0 \) we have
\[
B(\{1, \ldots, s\}) \leq \frac{4^t}{4^m} \left( \frac{8}{9} \right)^s \left( m - t + \frac{1}{8} \right)^{s-1},
\]
which is the desired bound.

Now we consider the case where \( m - t < s - 1 \). We have
\[
\Sigma_1 = \frac{1}{6^s} \sum_{l=0}^{m-t} \binom{l + s - 1}{s - 1} \frac{1}{4^l}
\leq \frac{1}{6^s} \sum_{l=0}^{\infty} \binom{l + s - 1}{s - 1} \frac{1}{4^l} = \left( \frac{2}{9} \right)^s \leq \frac{1}{16} 9^s 4^m.
\]
Thus we obtain
\[
B(\{1, \ldots, s\}) \leq \frac{1}{16} 9^s 4^m + \frac{1}{3^s 4^m} \frac{1}{2} (m - t + 2)^{s-1}
\leq \frac{8^s 4^t}{9^s 4^m} \left( \frac{1}{16} + \frac{31}{82} \left( \frac{3}{8} (m - t) + \frac{6}{8} \right)^{s-1} \right).
\]
The result now follows using the same arguments as above. \( \Box \)

We are now ready to prove Theorem 4.3.7.

**Proof of Theorem 4.3.7.** We use the formula of Theorem 4.3.1 together with Lemma 4.3.9 to obtain
\[
\mathbb{E}[L_{2,2m}^4(\tilde{P}_{2^m})] \leq \sum_{u \subseteq D, u \neq \emptyset} \gamma_u \left[ \frac{1}{2^{m+|u|}} \left( 1 - \left( 1 - \frac{1}{3 \cdot 2^m} \right)^{|u|} \right) \right]
+ \frac{1}{3^{|u|} 2^{2m}} \sum_{v \subseteq u, v \neq \emptyset} \binom{4}{3}^{|v|} \left( m - t + \frac{1}{8} \right)^{|v| - 1}.
\]
We have
\[
\frac{1}{3^{|u|}} \sum_{v \subseteq u, v \neq \emptyset} \binom{4}{3}^{|v|} \left( m - t + \frac{1}{8} \right)^{|v| - 1} \leq (m - t)^{-1} \left( \frac{1}{3} + \frac{4}{9} \left( m - t + \frac{1}{8} \right) \right)^{|u|}
\leq \left( \frac{5}{6} \right)^{|u|} (m - t)^{|u| - 1},
\]
provided that \( m - t > 0 \). Since for \( x < y \) we have \( y^s - x^s = \zeta^{s-1} (y - x) \) for a \( x < \zeta < y \) we have
\[
1 - \left( 1 - \frac{1}{3 \cdot 2^m} \right)^{|u|} \leq \frac{|u|}{3 \cdot 2^m}.
\]
As $\frac{|u|}{2^m} \leq \frac{1}{2}$ for $|u| \geq 1$ we obtain

$$\frac{1}{2^{m+|u|}} \left(1 - \left(1 - \frac{1}{3 \cdot 2^m}\right)^{|u|}\right) + \frac{1}{3^{2|u|}} \sum_{u \in D \atop u \neq \emptyset} \left(\frac{4}{3}\right)^{|u|} \left(m - t + \frac{1}{8}\right)^{|u| - 1}$$

$$\leq \frac{1}{2^{2m}} \left[\frac{1}{6} + 2^{2t} \left(\frac{5}{6}\right)(m - t)|u| - 1\right]$$

$$\leq \frac{1}{2^{2(m-t)}} (m - t)|u| - 1$$

(4.3.14)

and the result follows.

In the following corollary we refine the bound in Theorem 4.3.7 by including the $t$-values of the lower dimensional projections. Observe that it follows easily from Definition 2.3.2 that any projection of a digital $(t, m, s)$-net on the coordinates of $\emptyset \neq u \subseteq D$ is again a digital $(t_u, m, |u|)$-net with some $t_u \leq t$. In the following we write digital $(t_u, m, |u|)$-net to denote a digital $(t, m, s)$-net where the projections on $\emptyset \neq u \subseteq D$ have quality parameter $t_u$. The subsequent corollary can be obtained by using (4.3.14).

**Corollary 4.3.10** Let $P_{2^m}$ denote a digital $(t_u, m, s)$-net over $\mathbb{Z}_2$ such that $\max_{\emptyset \neq u \subseteq D} t_u < m$. Let $\tilde{P}_{2^m}$ be the point set obtained after applying an i.i.d. random digital shift of depth $m$ independently to each coordinate of each point of $P_{2^m}$. Then the mean square weighted $L_2$ discrepancy of $\tilde{P}_{2^m}$ is bounded by

$$\mathbb{E}[L_{2, 2^m, \gamma} (\tilde{P}_{2^m})] \leq \frac{1}{2^{2m}} \sum_{u \subseteq D \atop u \neq \emptyset} \gamma_u 2^{2t_u} (m - t_u)|u| - 1.$$  

**4.4 Asymptotics**

In this section we investigate the asymptotic behaviour of the $L_2$ discrepancy. We consider the classical $L_2$ discrepancy, that is, $\gamma_D = 1$ and $\gamma_u = 0$ for $u \subset D$. As before we denote these weights with $\gamma_c$. (We remark that the results in this section, except Subsection 4.4.2, can be generalized to arbitrary weights.)

Let $\log_2$ denote the logarithm in base 2. By an extension of the result of Roth [56] to dimension $s$ we obtain that for any point set $P_N$, consisting of $N$ points, in the $s$-dimensional unit-cube

$$L_{2, N, \gamma_c} (P_N) \geq \frac{1}{N} \sqrt{\left(\frac{\lceil \log_2 N \rceil + s + 1}{s - 1}\right)} \frac{1}{2^{2s+4}}.$$

For point sets consisting of $N = 2^m$ points the result can be slightly improved, we obtain

$$L_{2, 2^m, \gamma_c} (P_{2^m}) \geq \frac{1}{2^m} \sqrt{\left(\frac{m + s + 1}{s - 1}\right)} \frac{3}{2^{2s+4}}.$$  

(4.4.15)
From
\[
\binom{m + s + 1}{s - 1} \geq \frac{m^{s-1}}{(s-1)!}
\]
and \( m = \frac{\log N}{\log 2} \) it follows that
\[
L_{2,2^m,\gamma_c}(P_{2^m}) \geq \frac{(\log N)^{(s-1)/2}}{N} \frac{3}{2^{2s+4}(\log 2)^{(s-1)/2}} \frac{1}{(s-1)!}.
\] (4.4.16)

In the following subsection we consider the asymptotic behaviour of the classical \( L_2 \) discrepancy of certain shifted \((t, m, s)\)-nets. In Subsection 4.4.2 we consider shifted Niederreiter-Xing nets and show that the lower bound by Roth is essentially best possible in \( N \) and \( s \).

### 4.4.1 Asymptotics of the classical \( L_2 \) discrepancy of shifted digital \((t, m, s)\)-nets over \( \mathbb{Z}_2 \)

In the following let for an \( s \in \mathbb{N} \) and each \( m \in \mathbb{N} \) the set \( P_{t,2^m,\sigma_{m,s}} \) be a digital \((t, m, s)\)-net over \( \mathbb{Z}_2 \) shifted by the digital shift \( \sigma_{m,s} \) of depth \( m \). We obtain the following theorem.

**Theorem 4.4.1** Let \( s > 3, 0 \leq t < m \) and \( m - t \geq s \) be such that a digital \((t, m, s)\)-net over \( \mathbb{Z}_2 \) exists. Then there exists a digital shift \( \sigma_{m,s} \) of depth \( m \) such that for the shifted net \( P_{t,2^m,\sigma_{m,s}} \) we have
\[
L_{2,2^m,\gamma_c}(P_{t,2^m,\sigma_{m,s}}) \leq \frac{2^t}{2^m} \sqrt{\binom{m - t + s}{s - 1}} \left( \frac{2}{3} \right)^s + O\left( \frac{m^{(s-2)/2}}{2^m} \right).
\]

**Proof.** We obtain from Theorem 4.3.1
\[
E[L_{2,2^m,\gamma_c}(\widetilde{P}_{2^m})] = \frac{1}{2^{m+s}} \left( 1 - \left( 1 - \frac{1}{3 \cdot 2^m} \right) ight) + \frac{3}{3^s} \sum_{v \in D, v \neq 0} \left( \frac{3}{2} \right)^{|v|} B(v). \quad (4.4.17)
\]

Lemma 4.3.9 shows that, in order to find the constant of the leading term, we only need to consider \( B(\{1, \ldots, s\}) \). From (4.3.10), (4.3.11) and (4.3.12) we obtain
\[
B(\{1, \ldots, s\}) \leq \frac{2^t}{2^m} \left( \frac{1}{2} \right)^{1/3} \left( \frac{1}{3} \right)^s \binom{m - t + s}{s - 1} + \frac{1}{4s} \binom{m - t}{s - 1}.
\]

As the bound in Theorem 4.3.1 was obtained by averaging over all shifts it follows that there exists a shift which yields an \( L_2 \) discrepancy smaller than or equal to this bound. The result follows. \( \square \)

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Observe that the bound in Theorem 4.4.1 is for large \( m \), apart from the \( t \), similar to (4.4.15). We consider now \((t,s)\)-sequences. A \((t,s)\)-sequence in base 2 is a sequence of points \((x_n)_{n \geq 0}\) such that for all \( m > t \) and \( l \geq 0 \) we have that \( \{x_n : 2^m \leq n < (l+1)2^m\} \) is a \((t,m,s)\)-net in base 2. A digital \((t,s)\)-sequence over \( \mathbb{Z}_2 \) is obtained by using \( \infty \times \infty \) generating matrices \( C_1, \ldots, C_s \) over \( \mathbb{Z}_2 \).

From [46] it follows that for every dimension \( s \) there exists a digital \((t,s)\)-sequence over \( \mathbb{Z}_2 \) such that \( t \leq 5s \). Thus it follows that for all \( s \geq 1 \) and \( m > 5s \) there is a digital \((5s,m,s)\)-net over \( \mathbb{Z}_2 \). (Note that if there is a digital \((t,m,s)\)-net then it follows that also a digital \((t+1,m,s)\)-net exists.) Let \( P_{5s,2^m,\sigma_{m,s}} \) denote a digital \((5s,m,s)\)-net over \( \mathbb{Z}_2 \) shifted by the digital shift \( \sigma_{m,s} \) of depth \( m \). We are interested in the asymptotic behaviour of the \( L_2 \) discrepancy. Therefore, for \( m \) much larger than \( s \) and \( t = 5s \), we have

\[
\binom{m-t+s}{s-1} \leq \binom{m}{s-1} \leq \frac{m^{s-1}}{(s-1)!}.
\]

Further let \( N = 2^m \), then \( m = \frac{\log N}{\log 2} \). The following corollary follows now from Theorem 4.4.1.

**Corollary 4.4.2** For every \( s \geq 1 \) and \( m \geq 5s \) there exists a shifted digital \((5s,m,s)\)-net \( P_{5s,2^m,\sigma_{m,s}} \) over \( \mathbb{Z}_2 \) shifted by the digital shift \( \sigma_{m,s} \) of depth \( m \) such that

\[
L_2(2^m,\gamma_c(P_{5s,2^m,\sigma_{m,s}})) \leq \frac{(\log N)^{(s-1)/2}}{N} \frac{22^s}{(\log 2)^{(s-1)/2} \sqrt{(s-1)!}} + O\left(\frac{(\log N)^{(s-2)/2}}{N}\right),
\]

where \( N = 2^m \).

We note that the convergence of \( O((\log N)^{(s-1)/2}N^{-1}) \) is best possible, see (4.4.16).

In the remaining part of this subsection we discuss the constant depending on \( s \). Note that

\[
C(s) := \frac{22^s}{(\log 2)^{(s-1)/2} \sqrt{(s-1)!}}
\]
tends faster than exponentially to zero. The best constant of the leading term of an upper bound known to the authors was derived by Hickernell [16]. He showed that for scrambled \((0,m,s)\)-nets in base \( b \geq s - 1 \), where \( b \) is a prime power, the constant of the leading term is

\[
A(s) = \frac{(s-s^{-1})(s-1)/2}{6^{s/2} \sqrt{(s-1)!}((\log s)^{(s-1)/2} \approx \left(\frac{e^s}{\sqrt{6\pi s} s^{s/2}((\log s)^{(s-1)/2}}}\right)^{1/2} \quad \text{as } s \to \infty.
\]

The right hand side is obtained by Stirling’s formula. It can easily be checked that \( C(s) \) tends to zero much faster than \( A(s) \). Thus our result improves the result by Hickernell considerably.
Compared to (4.4.16) our constant $C(s)$ is not quite as good. It is known that for $(t,s)$-sequences we always have

$$t > s \log_2 \frac{3}{2} - 4 \log_2(s - 2) - 23$$

for all $s \geq 3$

by a result of Schmid [58]. Hence for digital $(t,m,s)$-nets obtained from digital $(t,s)$-sequences Theorem 4.4.1 can not yield a constant of the form $a^{-s/2}((s - 1)!)^{-1/2}$ for some $a > 1$. On the other hand it is possible that for special choices of $m$ and $s$ the $t$-value of a digital $(t,m,s)$-net can be considerably lower than the $t$-value of the best $(t,s)$-sequence. This will be investigated in the next subsection.

### 4.4.2 On the $L_2$ discrepancy of shifted Niederreiter-Xing nets

In this subsection we derive an upper bound on the classical $L_2$ discrepancy of shifted Niederreiter-Xing nets, see [47] (see also [75] for a recent survey article). This enables us to show that (4.4.15) is essentially best possible.

Niederreiter and Xing [47, Corollary 3] showed that for every integer $d \geq 2$ there exists a sequence of digital $(t_k,t_k+d,s_k)$-nets over $\mathbb{Z}_2$ with $s_k \to \infty$ as $k \to \infty$ such that

$$\lim_{k \to \infty} \frac{t_k}{\log_2 s_k} = \left\lfloor \frac{d}{2} \right\rfloor,$$

and that this is best possible. (We remark that the sequence of digital nets from the result of Niederreiter and Xing can be constructed explicitly.) Therefore for any $d \geq 1$ there exists a sequence of digital $(t_k,t_k+d,s_k)$-nets over $\mathbb{Z}_2$ and a $k_d$ such that

$$\left\lfloor \frac{t_k}{\log_2 s_k} \right\rfloor = d \quad \text{and} \quad s_k \geq 2d + 2 \quad \text{for all } k \geq k_d. \quad (4.4.19)$$

(Note that if a digital $(t_k,t_k+d,s_k)$-net exists, then there exists also a digital $(t_k+1,t_k+d+1,s_k)$-net. Further, for $d = 1$ there exists a digital $(t,t+1,s)$-net for all $t, s \geq 1$.) For a point set $P$ in $[0,1)^s$ with $2^m$ points let

$$D_{m,s}(P) := \frac{2^m L_{t_k+d,s_k}(P)}{\sqrt{\binom{m+s+1}{s-1}}}. \quad (4.4.20)$$

The bound in Theorem 4.3.7 was obtained by averaging over all shifts. Hence for any digital $(t,m,s)$-net there is always a shift $\sigma^*$ which yields an $L_2$ discrepancy smaller or equal to this bound. Let $P_k(d)$ denote a shifted digital $(t_k,t_k+d,s_k)$-net over $\mathbb{Z}_2$ satisfying (4.4.19), which is shifted by such a shift $\sigma^*$. We prove an upper bound on $D_{t_k+d,s_k}(P_k(d))$ for fixed $d$. 

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In the following let $k \geq k_d$. Let $v \subseteq \{1, \ldots, s_k\}$ and $l := |v|$. First we consider the case where $l \geq d + 2$. Note that $m - t = d$ for the nets considered here. Then (4.3.11) and (4.3.13) yield

$$B(v) \leq \frac{1}{3^l} \frac{1}{4^d} \frac{2^l}{d+1} + \frac{2^l}{g^l}.$$  

For $0 < l \leq d + 1$ we obtain from (4.3.11) and (4.3.12) that

$$B(v) \leq \frac{1}{3^l} \frac{1}{4^d} \frac{2^l}{d+1} + \frac{8^l}{g^l} \frac{1}{4^d} \frac{d}{l-1}.$$  

Therefore we obtain

$$\frac{1}{3^v} \sum_{v \subseteq D, v \neq \emptyset} \left( \frac{3}{2} \right)^{|v|} B(v) \leq \frac{1}{3^v} \sum_{l=1}^{d+1} \left( \frac{3}{2} \right)^l \left( \frac{s_k}{l} \right) \left( \frac{1}{3^l} \frac{1}{4^d} \frac{2^l}{d+1} + \frac{8^l}{g^l} \frac{1}{4^d} \frac{d}{l-1} \right) + \frac{1}{3^v} \sum_{l=d+2}^{s_k} \left( \frac{3}{2} \right)^l \left( \frac{s_k}{l} \right) \left( \frac{1}{3^l} \frac{2^l}{d+1} + \frac{2^l}{g^l} \right).$$  

Now we have

$$\frac{1}{3^v} \sum_{l=1}^{s_k} \left( \frac{3}{2} \right)^l \left( \frac{s_k}{l} \right) \frac{1}{3^l} \frac{1}{4^d} \frac{2^l}{d+1} \leq \frac{1}{2} \frac{1}{3^v} \frac{1}{4^d} \left( \frac{d}{l} \right) \frac{1}{3^l} \frac{1}{4^d} \frac{2^l}{d+1} \leq \frac{1}{2} \frac{1}{3^v} \frac{1}{4^d} \left( \frac{d}{l} \right) \frac{1}{3^l} \frac{1}{4^d} \frac{2^l}{d+1}$$  

and

$$\frac{1}{3^v} \sum_{l=1}^{d+1} \left( \frac{3}{2} \right)^l \left( \frac{s_k}{l} \right) \frac{8^l}{g^l} \frac{1}{4^d} \frac{d}{l-1} + \frac{1}{3^v} \sum_{l=d+2}^{s_k} \left( \frac{3}{2} \right)^l \left( \frac{s_k}{l} \right) \frac{2^l}{g^l} \leq \frac{1}{3} \frac{7}{12} \frac{d}{s_k} \frac{1}{3^v} \left( \frac{s_k}{d+1} \right) + \left( \frac{4}{9} \right)^{s_k},$$

as $\max_{l=1, \ldots, d+1} \left( \frac{s_k}{l} \right) = \left( \frac{s_k}{d+1} \right)$ for $s_k \geq 2d + 2$. Thus we obtain

$$\frac{1}{3^v} \sum_{v \subseteq D, v \neq \emptyset} \left( \frac{3}{2} \right)^{|v|} B(v) \leq \frac{1}{2} \frac{1}{3^v} \frac{1}{4^d} \left( \frac{d+s_k}{s_k-1} \right) + \frac{1}{3} \frac{7}{12} \frac{d}{3^v} \frac{1}{4^d} \left( \frac{s_k}{d+1} \right) + \left( \frac{4}{9} \right)^{s_k}.$$  

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Further we have
\[ 1 - \left( 1 - \frac{1}{3 \cdot 2^m} \right)^s \leq \frac{s}{3 \cdot 2^m}. \]

Hence it follows from the definition of \( P_k(d) \) and (4.4.17) that
\[ L^2_{2^k+4, \gamma_c} (P_k(d)) \]
\[ \leq \frac{1}{2^{2^k}} \frac{1}{4} \left( \frac{d + s_k}{s_k - 1} \right) + \frac{1}{3} \left( \frac{7}{12} \right)^d \frac{1}{3^s} \left( \frac{s_k}{d + 1} \right) + \frac{4}{9} \frac{s_k}{s_k + 1} + \frac{s_k}{3 \cdot 2^{(t_k + d)s_k + s_k}}. \]

In order to get a bound on \( D^2_{t_k + d, s_k} (P_k(d)) \) we need to multiply the inequality above with \( 4^{t_k + d} \left( \frac{t_k + d + s_k}{s_k - 1} \right)^{-1} \). For the first term in the bound of (4.4.21) we get
\[ \frac{1}{2^{2^k}} \frac{1}{4} \left( \frac{d + s_k}{s_k - 1} \right) 4^t \left[ \left( \frac{t_k + d + s_k + 1}{s_k - 1} \right) \right]^{-1} \]
\[ = \frac{1}{2^{2^k}} \frac{4^t}{(t_k + d + s_k) \cdots (d + 3)} \cdot \frac{(d + s_k) \cdots (d + 2)}{(s_k - 1)} \cdot \frac{(t_k + d + s_k + 1) \cdots (t_k + d + 3)}{\left( t_k + d + s_k + 1 \right) \cdots (t_k + d + 3)} \cdot \frac{s_k}{s_k + 1}. \]

Let \( r \geq 1 \) be an integer which will be chosen later. From (4.4.18) follows that for large enough \( k \) we have \( rt_k < s_k \). Further we have \( t_k > 0 \). We get
\[ \frac{(t_k + d + s_k + 1) \cdots (t_k + d + 3)}{(d + s_k) \cdots (d + 3)} = \frac{1 + \frac{t_k + 1}{d + s_k}}{\left( 1 + \frac{t_k + 1}{d + s_k} \right) \cdots \left( 1 + \frac{t_k + 1}{d + 2} \right)} \]
\[ \geq \prod_{j=1}^{r} \left( 1 + \frac{t_k + 1}{j t_k + d + 1} \right)^{t_k} \cdot \frac{r}{j}. \]

Now we have
\[ \prod_{j=1}^{r} \left( 1 + \frac{t_k + 1}{j t_k + d + 1} \right) \rightarrow \prod_{j=1}^{r} \left( 1 + \frac{1}{j} \right) = (r + 1) \quad \text{as} \quad t_k \rightarrow \infty. \]

Therefore, for large enough \( k \), we obtain
\[ r^t \leq \frac{(t_k + d + s_k + 1) \cdots (t_k + d + 3)}{(d + s_k) \cdots (d + 2)} \]
and
\[ \frac{1}{2^{2^k}} \frac{1}{4^t} \left( \frac{d + s_k}{s_k - 1} \right) 4^{t_k + d} \left[ \left( \frac{t_k + d + s_k + 1}{s_k - 1} \right) \right]^{-1} \leq \frac{1}{2^{2^k}} \frac{4}{r} \cdot \frac{t_k}{s_k - 1} \]
for all \( k \geq K_1(r, d) \), for some well chosen \( K_1(r, d) \). Further one can show that the other terms on the right hand side of (4.4.21) decay faster than \( 2^{-s_k} (4/r)^{t_k} \). From (4.4.19) it follows that \( t_k \geq (d - 1) \log_2 s_k \) for all \( k \geq K_2(d) \). Let \( r = 8 \), then we have \( (4/r)^{t_k} \leq s_k^{-d} \). Therefore there exists a \( K_d \) such that for all \( k \geq K_d \) we have
\[ D^2_{t_k + d, s_k} (P_k(d)) \leq \frac{1}{2^{2^k}} \frac{1}{s^{d-1}}. \]

We summarize the result in the following theorem.
Theorem 4.4.3 For any $d \geq 1$ there exists an integer $K_d > 0$ and a sequence of shifted digital $(t_k, t_k + d, s_k)$-nets over $\mathbb{Z}_2$, $(P_k(d))_{k \geq 1}$, with $s_k \to \infty$ as $k \to \infty$ and
\[
\left\lfloor \frac{t_k}{\log_2 s_k} \right\rfloor = d \quad \text{for all } k \geq K_d,
\]
such that for all $k \geq K_d$ we have
\[
\mathcal{L}_{2,2^{t_k+d},\gamma_c}(P_k(d)) \leq \frac{1}{2^{t_k+d}} \frac{1}{2^{s_k/2}} \frac{1}{s_k^{(d-1)/2}} \sqrt{\frac{(t_k + d + s_k + 1)}{s_k - 1}}.
\]

We use (4.4.20) again. Then by using (4.4.15) and the result above we obtain that for any $d > 0$ and for all $k \geq K_d$ we have
\[
\frac{3}{16} \frac{1}{2^{2s_k}} \leq D_{t_k+d,s_k}(P_k(d)) \leq \frac{1}{2^{s_k/2}} \frac{1}{s_k^{(d-1)/2}}.
\]
(4.4.22)

This shows that the lower bound of Roth is also in $s$ of the best possible form. The small remaining gap in the constant is not surprising as the result in Theorem 4.4.3 was obtained by averaging over well distributed point sets. Some attempts have been made in improving the lower bound of Roth, but no considerable progress has been made (see [37]). For small point sets there exist other lower bounds which yield numerically better results than the bound of Roth, but do not show the higher convergence rate (see [37]).

We note that the results in this section are, apart from the digital shift, constructive as they are based on Niederreiter-Xing constructions of digital nets and sequences. It would also be desirable to have fully deterministic point sets with a small $L_2$ discrepancy (like the constructions in [3]). In our context this amounts to finding an appropriate digital shift for a given digital $(t, m, s)$-net. This work remains to be done.
Chapter 5

Conclusion

In this thesis we have shown that digital lattice rules can be used in very much the same way as lattice rules. A result of the theory established here are the construction algorithms for polynomial lattice rules which were previously unknown. The results follow along the same lines as those for lattice rules, though it was not obvious that this can actually be done, at least not for the case of the weighted Sobolev space. For the Hilbert space based on Walsh functions the results follow in a rather straightforward manner by analogy. The roots of this similarity of course go back to the similarities between natural numbers and polynomials over finite fields. Natural numbers and polynomials over finite fields behave in many ways in very much the same way, see [36]. The analogies are carried further with the invention of polynomial lattices rules by Niederreiter [42] and the introduction of the Hilbert space based on Walsh functions. In this sense, lattice rules and polynomial lattice rules are examples of a general strategy of establishing a theory for numerical integration, with the basic ingredients being a point set with a group structure, functions which form a character over this group and a function space which is based on these characters. Hence many of the arguments are actually the same, but the details are different, as lattice rules are based on natural numbers, exponential functions, Korobov spaces and shifts, whereas polynomial lattice rules are based on polynomials over finite fields, Walsh functions, reproducing kernel Hilbert spaces based on Walsh functions and digital shifts. Other examples which use the same ingredients (that is a point set with a group structure, functions which form a character over this group and a function space which is based on these characters) might be possible, but are not (yet) known.

On the other hand, comparing lattice rules with polynomial lattice rules we observe several differences. Generally speaking, there is more freedom in the construction of polynomial lattice rules. For example, there are many more irreducible polynomials $f \in \mathbb{Z}_b[x]$ with degree smaller or equal to some integer $m$, than there are prime numbers smaller than $b^m$, for a given prime base $b$. Another example is the variety of randomization methods, ranging from Owen’s scrambling and subsequent developments to the digital shift and digital shift of depth $m$ established here. This gives rise to further research.
as there might be some possibilities to improve the performance of polynomial lattice rules.

The randomization methods considered here could be considered as special cases of Owen’s scrambling method. Much effort has been put into simplifying this randomization method as it is computationally too costly to carry out. The optimum from this point of view is of course what has been coined a derandomized randomized QMC rule, that is, finding the best QMC rule among those rules in the sample space. It is hoped that future research will prove that the results established here are fruitful in obtaining a complete derandomization, as the digital shift used here is amongst the simplest randomization methods for nets.

In this way, the thesis not only establishes new results on digital nets and sequences, but, maybe even more importantly, gives raise to future developments of QMC methods.
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Chapter 6

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