Higher order nets and sequences

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Abstract

Quasi-Monte Carlo rules are equal weight integration formulas used to approximate integrals over the unit cube, which are typically of high dimension. Quasi-Monte Carlo point sets can roughly be divided into integration lattices and nets. This thesis focuses on nets, as introduced by Niederreiter. Nets are defined geometrically using elementary intervals, which are subintervals of the unit cube. An important special case of nets are digital nets, which are constructed using matrix-vector multiplications over finite fields. A further interesting special case of digital nets are polynomial lattice point sets, whose construction is based on polynomials over finite fields.

Recently, Dick introduced higher order digital nets, which generalize digital nets and have the desirable property that quasi-Monte Carlo rules based on higher order digital nets achieve almost optimal convergence rates when used in a quasi-Monte Carlo rule to approximate integrals over the unit cube. Subsequently, Dick and Pillichshammer introduced higher order polynomial lattice point sets, which are special cases of higher order digital nets and generalize polynomial lattice point sets. They established the existence of higher order polynomial lattice point sets, which achieve almost optimal convergence rates when used in a quasi-Monte Carlo rule to approximate integrals over the unit cube, but did not show how to construct them.

The contributions of this thesis are as follows:

i) higher order nets are introduced, which generalize both, the higher order digital nets introduced by Dick and the nets introduced by Niederreiter. Subsequently, duality theory and propagation rules for higher order nets are presented, and higher order nets are used in quasi-Monte Carlo rules for numerical integration,

ii) we provide explicit constructions of higher order polynomial lattice point sets achieving almost optimal convergence rates when used in a quasi-Monte Carlo rule to approximate integrals over the unit cube,

iii) applying Owen's scrambling algorithm to polynomial lattice point sets, we show that the resulting quasi-Monte Carlo rules are almost optimal for functions of bounded variation.

The thesis concludes by applying quasi-Monte Carlo rules to a finance problem, showing that quasi-Monte Carlo rules can be used to solve problems of practical interest.

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Chapter One

Introduction

Quasi-Monte Carlo rules are equal weight integration formulas used to approximate integrals over the unit cube, which are typically of high dimension; such problems can arise in mathematical finance, see e.g. [85], physics, see e.g. [62], or statistics, see e.g. [53], to name but a few areas. Quasi-Monte Carlo point sets can roughly be divided into integration lattices, see e.g. [66; 97] and nets, see e.g. [35; 66]. The focal point of this thesis will be on nets, which were formally introduced by Niederreiter [64], but Faure [37] and Sobol' [104], provided explicit constructions before, which were encapsulated by the framework provided in [64].

Niederreiter observed in [69] that nets were developed with a view to providing deterministic sample points for quasi-Monte Carlo rules, see e.g. [35; 66; 68]. However, some aspects of nets also have the flavor of discrete mathematics. This thesis aims to explore both the connections with discrete mathematics and also the application to quasi-Monte Carlo rules.

Regarding the history of nets, in [64], Niederreiter introduced so-called (t, m, s)-nets in base b: such point sets consist of b^m points in $[0, 1)^s$, whose quality is governed by t, in particular, lower values of t correspond to (t, m, s)-nets of higher quality. Niederreiter defined (t, m, s)-nets geometrically in terms of the number of points lying in so-called elementary intervals, which are subsets of $[0, 1)^s$. Such a definition does not suggest how to construct (t, m, s)-nets; this was also addressed in [64], where so-called digital (t, m, s)-nets were introduced, a special case of (t, m, s)-nets. Digital (t, m, s)-nets are (t, m, s)-nets constructed using matrix-vector multiplications over finite fields and many explicit constructions are known, see e.g. [37; 66; 69; 70; 75; 104]. A further interesting special case of digital (t, m, s)-nets are polynomial lattice point sets, which were introduced in [65]. Though a special case of digital (t, m, s)-nets, similarities between polynomial lattice point sets and integration lattices in the sense of [97] exist and have been successfully exploited for the purpose of constructing polynomial lattice point sets, see [29].

As mentioned before (t, m, s)-nets were introduced with a view to numerical integration. An important result in this direction is the following: if a quasi-Monte Carlo (qMC) rule based on a (t, m, s)-net in base b is employed to numerically integrate a function enjoying finite variation in the sense of Hardy and Krause, the integration error decreases at a rate of $\mathcal{O}(b^{-(m-t)}m^{s-1})$, see e.g. [66, Theorem 2.11, Theorem 4.10]. However, for a function known to enjoy more smoothness, say square-integrable mixed partial derivatives of order α, it was not clear how to make use of this smoothness to achieve faster convergence rates of the integration error. On the other hand, for integration lattices, for particular function classes, smoothness can be exploited, see e.g. [52; 66; 97], likewise for example for Smolyak cubature formulas, which are based on one-dimensional integration formulas known to integrate polynomials up to a certain degree exactly, see e.g. [14; 103], and see also [39].

We now recall, in chronological order, results showing how (t, m, s)-nets in base b can exploit the smoothness of the function under consideration, where we set $N = b^m$:

- in [82] a randomization method referred to as scrambling, [80], was applied to (t, m, s)-nets and qMC rules based on the resulting point sets were shown to achieve a convergence rate of the integration error of $N^{-3/2}(\log N)^{(s-1)/2}$,
- in [18] it was shown that applying the tent transform to randomly digitally shifted (t, m, s)-nets produces qMC rules achieving a convergence rate of the integration error of $N^{-2+\delta}$, for all $\delta > 0$.

However, the above results only show how to exploit the smoothness for specific examples of function spaces: in [82] it was assumed that the mixed partial derivatives of the function under consideration satisfy a Lipschitz condition, in [18] the authors considered functions having square integrable mixed partial derivatives of order two.

In [20] it was shown how (t, m, s)-nets are to be modified so that the resulting qMC rules can numerically integrate smooth periodic functions having square integrable mixed partial derivatives of order α at a rate of $N^{-\alpha}$ multiplied by a power of a $\log N$ factor. Furthermore, in [21] it was shown how to modify (t, m, s)-nets so that functions which are not necessarily periodic, but have square integrable mixed partial derivatives of order α, can be numerically integrated at a rate of $N^{-\alpha}$ multiplied by a power of a $\log N$ factor using qMC rules corresponding to the modified (t, m, s)-nets. The point sets introduced in [21] are from now on referred to as higher order digital nets. Regarding nomenclature, the qualifier "higher order" is addedgr to emphasize that quasi-Monte Carlo rules based on these point sets can achieve convergence rates faster than N^{-1}.

Higher order digital nets are constructed using matrix-vector multiplications over finite fields, analogous to the way in which digital (t, m, s)-nets are constructed, and explicit constructions of higher order digital nets based on digital (t, m, s)-nets were presented in [21].

Obviously the contributions in [20; 21] initiated a lot of research, to name only but a few contributions: a link between Smolyak cubature formulas and higher order digital nets was established in [30], duality theory and propagation rules for higher order digital nets were discussed in [27] and higher order polynomial lattice point sets, an analogue of polynomial lattice point sets in the sense of [65], were introduced in [34]; as with digital (t, m, s)-nets and polynomial lattice point sets, higher order polynomial lattice point sets are special cases of higher order digital nets. We remark that in [34] the existence of higher order polynomial lattice point sets achieving integration errors of order $N^{-\alpha}$ multiplied by a power of a $\log N$ factor when used in a quasi-Monte Carlo rule was established, but explicit constructions were not presented. Furthermore, it was shown in [28] that an exhaustive computer search for higher order polynomial lattice point sets can result in point sets superior to those constructed using the original method for constructing higher order digital nets from [21].

We can now explain the contributions of this thesis:

- firstly, we introduce a generalization of higher order digital nets, referred to as higher order nets. Higher order nets generalize higher order digital nets in the same way in which (t, m, s)-nets generalize digital (t, m, s)-nets. Furthermore, higher order nets include (t, m, s)-nets as special cases. Consequently, higher order nets are studied in detail: we introduce duality theory and propagation rules for higher order nets and apply higher order nets to numerical integration,
- secondly, we provide explicit constructions of higher order polynomial lattice point sets achieving convergence rates of order $N^{-\alpha}$ multiplied by a power of a $\log N$ factor when used in a qMC rule, mainly relying on ideas from [29],
- thirdly, we apply the scrambling algorithm introduced in [80] to polynomial lattice point sets. For functions of bounded variation of order α, $0 < \alpha \leq 1$, we construct polynomial lattice point sets achieving convergence rates of order $N^{-(1+2\alpha)+\epsilon}$, for all $\epsilon > 0$, when used in a qMC rule. We establish the optimality of qMC rules based on scrambled polynomial lattice point sets for a large class of randomized algorithms including adaptive ones and finally show how to implement this algorithm in a way that is computationally efficient.

Regarding the observation by Niederreiter from [69], we point out that the first contribution explores the link between nets and discrete mathematics, the second and the third contributions are concerned with quasi-Monte Carlo point sets employed for numerical integration.

The introduction is concluded by a brief overview of the thesis detailing my contributions to the material presented. This overview is intended to be non-technical, however, each chapter commences with a motivation assuming that the reader is familiar with the material covered in the preceding chapters. The remaining structure of the thesis is as follows: in Chapter 2 we introduce concepts needed to make this thesis self-contained: (t,m,s)-nets and (t,s)-sequences, digital (t,m,s)-nets and digital (t,s)-sequences and polynomial lattice point sets are introduced. Subsequently, higher order digital nets and higher order digital sequences are introduced and some illustrations are provided. A special case of higher order digital nets, namely higher order polynomial lattice point sets, is introduced. In Section 2.4 we introduce Walsh functions and show how they can be used to characterize (t,m,s)-nets. Next, we introduce the Walsh function spaces $W_{\alpha,s,\gamma}$ and $V_{\alpha,s,\gamma}$. The function space $W_{\alpha,s,\gamma}$ is the setting of Chapters 5 and 6, the function space $V_{\alpha,s,\gamma}$ is used in Chapter 7. Consequently, we review different randomization algorithms, in particular the digital shift, the digital shift of depth m and Owen's scrambling algorithm. The digital shift and the digital shift of depth m are used in Section 5.3, the scrambling algorithm in Chapter 7. Finally, in Section 2.7 we recall results from the literature dealing with numerical integration of functions in $W_{\alpha,s,\gamma}$ using higher order digital nets and of functions in $V_{\alpha,s,\gamma}$ using scrambled digital (t,m,s)-nets.

In Chapter 3 higher order nets and higher order sequences are introduced. This chapter is based on work which appeared in [10; 25]. We firstly generalize the concept of an elementary interval, [64], which then allows us to define higher order nets and higher order sequences. Having defined higher order nets and higher order sequences, we show some simple propagation rules and show that higher order digital nets and higher order digital sequences are special cases of higher order nets and higher order sequences. Furthermore, the rate at which the quality parameter of the higher order sequences worsens is studied. In Section 3.4 we characterize higher order nets using Walsh functions, which is analogous to Subsection 2.4.2. This characterization can be used to study the randomization of higher order nets, see Section 3.5, but also to discuss numerical integration using higher order nets, see Chapter 5. The definition of higher order nets and higher order sequences resulted from a discussion with Josef Dick, the proof of Theorem 3.19 showing that higher order digital nets and higher order digital sequences are special cases of higher order nets and higher order sequences was my contribution. Josef Dick and Friedrich Pillichshammer suggested that the characterization of higher order nets in terms of Walsh functions could be useful, but the proofs of the results in Section 3.4 are my own work. It was my idea to study randomizations of higher order nets and Section 3.5 is my own work.

In Chapter 4, which is based on work that appeared in [11], duality theory for not necessarily digital nets is introduced. This new duality theory is then used to establish propagation rules for higher order nets and higher order sequences. In a nutshell propagation rules for nets show how to obtain new nets from existing ones. Josef Dick and Friedrich Pillichshammer came up with the ideas underpinning the section on duality theory, Section 4.2, and I proved Theorem 4.5, which connects duality theory to the characterization of higher order nets from Section 3.4. Section 4.3 is largely my own work, Lemmas 4.9 and 4.14 were joint work with Josef Dick and Friedrich Pillichshammer, Proposition 4.31 was proven by Josef Dick. It was my idea to study propagation rules for higher order sequences and Section 4.4 is my own work.

In Chapter 5, which is based on work that appeared in [10], we study numerical integration based on qMC rules employing higher order nets. In Section 5.2, we show that integration errors associated with functions in $W_{\alpha,s,\gamma}$ converge at a rate of $N^{-(\alpha-1)}$ multiplied by a power of a $\log N$ factor. This is of course not optimal, but does show that higher order nets can exploit the smoothness of the integrand under consideration. Consequently, in Section 5.3, we show that root mean-square integration errors based on qMC rules using the randomized higher order nets from Section 3.5 converge at a rate of $N^{-(\alpha-1/2)}$ multiplied by a power of a $\log N$ factor, which is again not optimal but improves on the result from Section 5.2. Section 5.2 is joint work with Josef Dick and Friedrich Pillichshammer, I came up with the idea to study integration errors associated with randomized higher order nets, so Section 5.3 is my own work.

In Chapter 6 we study higher order polynomial lattice point sets, in particular the construction thereof. In Section 6.3, we show how a component-by-component approach can be used to construct higher order polynomial lattice point sets. The resulting qMC rules produce integration errors converging at a rate of $N^{-\alpha}$ multiplied by a power of a $\log N$ factor for functions in $W_{\alpha,s,\gamma}$. In Section 6.4 we consider the construction of point sets which achieve a convergence rate of $N^{-\alpha}$ multiplied by a power of a $\log N$ factor for functions in $W_{\alpha,s,\gamma}$, where α can assume values in a predetermined range, when used in a qMC rule; hence the point sets adjust themselves to the almost optimal convergence rate, which is $N^{-\alpha}$ multiplied by a $\log N$ factor. In Section 6.5 we consider a different type of higher order polynomial lattice point set: in particular we are interested in higher order polynomial lattice point sets exhibiting a particular structure, namely we study higher order polynomial lattice point sets which are of Korobov type. This structure means that the construction is very simple and we manage to prove results analogous to those presented in Section 6.4 for higher order polynomial lattice point sets of Korobov type. Section 6.3 is joint work with Josef Dick and Friedrich Pillichshammer, Section 6.4 is my own work. Regarding Section 6.5 I realized that the techniques from Section 6.4 can also be used to construct higher order polynomial lattice point sets of Korobov type.

In Chapter 7, which is based on work that appeared in [6], we study the construction of scrambled polynomial lattice point sets. In Section 7.3 we find that the variance of a qMC rule based on a scrambled polynomial lattice point set constructed using a component-by-component approach decays at a rate of $N^{-(1+2\alpha)+\epsilon}$, for all $\epsilon > 0$, assuming that the function under consideration is of bounded variation of order α. The same result is obtained for Korobov polynomial lattice point sets, see Section 7.4. It follows from the results presented in Subsection 2.7.2 that these point sets are almost optimal for the function space considered in this chapter. The implementation of the component-by-component approach is discussed in Section 7.5, in particular we show how to reduce the computational cost associated with it. Finally, in Section 7.6 numerical results comparing scrambled polynomial lattice point sets and scrambled digital nets are presented. Josef Dick suggested that we study integration errors associated with scrambled polynomial lattice point sets for functions in $V_{\alpha,s,\gamma}$. I proved Theorem 2.54 and the results presented in Sections 7.2, 7.3 and 7.4. The numerical experiments presented in Section 7.6 are my own work.

Applying classical and higher order nets to numerical integration, the convergence rate of the integration error is a natural object to study, [35; 66; 97]. This approach assumes that the integration problem has been formulated; however, for some applications of practical interest, it is difficult to formulate the integration problem in the first place; having formulated the integration problem, one might find that it is possible to reformulate the problem in such a way that the performance of qMC rules is enhanced. This topic is discussed in Chapter 8, which is based on work that appeared in [4], where we study a particular problem from financial mathematics. To be more precise, we study how to price a particular financial derivative in the Kou model [50; 51]. Firstly, we formulate the integration problem, which is non-trivial and secondly, we show how to reformulate the problem resulting in an enhancement of the performance of qMC rules. We remark that Chapter 8 is of a different flavor than the preceding chapters; it clearly demonstrates how qMC rules can be applied to problems of practical interest. Studying problems of practical interest is of course important in its own right, however, the study of such problems can also motivate new theoretical developments. Lastly, Chapter 8 also shows that the research area concerned with qMC rules is indeed a very rich one dealing with problems of both theoretical and practical interest which can also go hand-in-hand.

Finally, we conclude the introduction by remarking that this thesis is based on the following contributions:

- [4], see Chapter 8,
- [6], see Chapter 7,
- [7], see Chapter 6,
- [10], see Chapters 3 and 5,
- [11], see Chapter 4,
- [25], see Chapter 3.

Chapter Two

Nets, sequences, and integration

We begin with the motivation for this chapter.

2.1 Motivation

The purpose of this chapter is to make this thesis self-contained by introducing all the concepts needed later on.

Firstly, we need to fix some nomenclature. From now on, (t, m, s)-nets ((t, s)-sequences, respectively) will be referred to as classical nets (classical sequences, respectively), digital (t, m, s)-nets (digital (t, s)-sequences, respectively) will be referred to as classical digital nets (classical digital sequences, respectively) and finally polynomial lattice point sets in the sense of [65] will be referred to as classical polynomial lattice point sets.

We begin by introducing classical nets and classical sequences, classical digital nets and classical digital sequences and classical polynomial lattice point sets. Recalling these concepts we rely on [29; 35; 66]. Consequently, we introduce higher order digital nets and higher order digital sequences following [21], however, we also illustrate these novel concepts using some examples taken from [25]. In Subsection 2.3.2 we introduce higher order polynomial lattice point sets relying on [34]. Walsh functions are an important tool when analyzing classical nets. This was first realized in [58] and subsequently it was shown how to characterize classical nets in terms of Walsh functions, [45]. For this reason, we introduce Walsh functions and also recall the characterization of classical nets in terms of Walsh functions; an analogous result, but for a more general class of point sets, namely higher order nets, is presented in Section 3.4. In Section 2.5 we introduce the function spaces considered for numerical integration in this thesis, the Walsh spaces $W_{\alpha,s,\gamma}$ and $V_{\alpha,s,\gamma}$. These are weighted function spaces in the sense of [101] meaning that a non-increasing sequence of weights $\{\gamma_j\}_{j=1}^{\infty}$ is used to model the importance of the different abscissas. Membership of these function spaces is restricted to functions whose Walsh coefficients exhibit a particular decay property, see Equations (2.11) and (2.13).

We remark that the function space $W_{\alpha,s,\gamma}$ first appeared in [21], but was subsequently the setting for a number of papers concerned with numerical integration using qMC rules based on higher order digital nets. The reason is that qMC rules based on higher order digital nets turn out to be exactly the right tool to exploit this decay property for purposes of numerical integration, see Subsection 2.7.1, which follows [21]. The function space $V_{\alpha,s,\gamma}$ first appeared in [6], but related ideas were presented in [35, Section 13.5]. Next, we discuss randomizations of classical nets: we introduce the digital shift, the digital shift of depth m and Owen's scrambling algorithm. The latter is used in Subsection 2.7.2, where we establish the optimality of qMC rules based on scrambled digital nets for numerical integration in $V_{\alpha,s,\gamma}$.

Regarding notation, unless stated otherwise, we fix the dimension $s \geq 1$ and an integer $b \geq 2$. We use $\mathbb{N}$ to denote the natural numbers, $\mathbb{N}_0$ to denote non-negative integers and λ_s to denote the s-dimensional Lebesgue measure. Also, we set $[s] = \{1, 2, \dots, s\}$. The finite field of order b is denoted by $\mathbb{Z}_b$, vectors in finite fields are denoted by $\vec{h}$ and $\boldsymbol{h}$ is used for vectors of integers and real numbers. Also, $\mathbb{Z}_b[x]$ denotes the set of all polynomials over $\mathbb{Z}_b$, polynomials over $\mathbb{Z}_b$ are denoted by $h(x)$ and vectors of polynomials over $\mathbb{Z}_b[x]$ by $\boldsymbol{h}(x)$.

We now formally introduce qMC rules.

Definition 2.1. *For a point set $\mathcal{P} = \{\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_{N-1}\} \subseteq [0,1)^s$, $N \in \mathbb{N}$, let*

$$Q_N(f, \mathcal{P}) = \frac{1}{N} \sum_{h=0}^{N-1} f(\boldsymbol{x}_h) \,. \tag{2.1}$$

Then $Q_N(f, \mathcal{P})$ is the quasi-Monte Carlo rule for f based on the N-element point set $\mathcal{P}$.

The effectiveness of a quasi-Monte Carlo rule of course depends on the underlying point set $\mathcal{P}$. Consequently in Sections 2.2 and 2.3 we discuss several point sets which are used in quasi-Monte Carlo rules. As usual in the area, for our purposes a point set is a "multiset" in the sense of combinatorics, i.e. a set in which multiplicities of elements are allowed and taken into account.

2.2 Introducing classical nets and classical sequences

We now introduce classical nets and classical sequences following [66].

2.2.1 Classical nets and classical sequences

A concept fundamental to the study of classical nets and classical sequences is the concept of an elementary interval, which we now recall. We remark that the notation used in this thesis to define elementary intervals is slightly different from the traditional one, see [66]; the reason is that we aim to facilitate the comparison with generalized elementary intervals introduced in Subsection 3.2, which in turn form the basis of the definition of higher order nets.

We need the following notation: let $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_s)$, $\nu_j \in \mathbb{N}_0$, $j = 1, \ldots, s$, let $|\boldsymbol{\nu}|_1 = \sum_{j=1}^{s} \nu_j$ and let $\boldsymbol{a}_{\boldsymbol{\nu}} \in \{0, \ldots, b-1\}^{|\boldsymbol{\nu}|_1}$, and let $\boldsymbol{a}_{\boldsymbol{\nu}} = (a_{1,1}, \ldots, a_{1,\nu_1}, \ldots, a_{s,1}, \ldots, a_{s,\nu_s})$, where the components $a_{j,l}$, $l = 1, \ldots, \nu_j$ do not appear in the vector $\boldsymbol{a}_{\boldsymbol{\nu}}$ in case $\nu_j = 0$. An elementary interval in base b is now defined as follows, see also Figure 2.1 for an illustration:

Definition 2.2. *Let $\boldsymbol{\nu} \in \{0, \ldots, m\}^s$, $m \in \mathbb{N}_0$, and $\boldsymbol{a}_{\boldsymbol{\nu}}$ be defined as above. Then a subset $E(\boldsymbol{\nu}, \boldsymbol{a}_{\boldsymbol{\nu}})$ of $[0,1)^s$ of the form*

$$E(\boldsymbol{\nu}, \boldsymbol{a}_{\boldsymbol{\nu}}) = \prod_{j=1}^{s} \left[\frac{a_{j,1}}{b} + \cdots + \frac{a_{j,\nu_j}}{b^{\nu_j}}, \frac{a_{j,1}}{b} + \cdots + \frac{a_{j,\nu_j}}{b^{\nu_j}} + \frac{1}{b^{\nu_j}} \right)$$

is called an elementary interval in base b, where $\frac{a_{j,1}}{b} + \cdots + \frac{a_{j,\nu_j}}{b^{\nu_j}} = 0$ for $\nu_j = 0$.

Next, we state the following two easy lemmas, which are to be compared with Lemmas 3.2 and 3.3.

Lemma 2.3. *Let $\boldsymbol{\nu} \in \{0, \ldots, m\}^s$ and $\boldsymbol{a}_{\boldsymbol{\nu}}$ be defined as above and fixed. Then the elementary intervals $E(\boldsymbol{\nu}, \boldsymbol{a}_{\boldsymbol{\nu}})$ for $\boldsymbol{a}_{\boldsymbol{\nu}} \in \{0, \ldots, b-1\}^{|\boldsymbol{\nu}|_1}$ form a partition of $[0,1)^s$, i.e. $\bigcup_{\boldsymbol{a}_{\boldsymbol{\nu}} \in \{0,\ldots,b-1\}^{|\boldsymbol{\nu}|_1}} E(\boldsymbol{\nu}, \boldsymbol{a}_{\boldsymbol{\nu}}) = [0,1)^s$ and $E(\boldsymbol{\nu}, \boldsymbol{a}_{\boldsymbol{\nu}}) \bigcap E(\boldsymbol{\nu}, \boldsymbol{a}'_{\boldsymbol{\nu}}) = \emptyset$, $\boldsymbol{a}_{\boldsymbol{\nu}} \neq \boldsymbol{a}'_{\boldsymbol{\nu}} \in \{0, \ldots, b-1\}^{|\boldsymbol{\nu}|_1}$.*

Lemma 2.4. *Let $\boldsymbol{\nu}$ and $\boldsymbol{a}_{\boldsymbol{\nu}}$ be as in Lemma 2.3. Then $\lambda_s(E(\boldsymbol{\nu}, \boldsymbol{a}_{\boldsymbol{\nu}})) = b^{-|\boldsymbol{\nu}|_1}$.*

We are now in a position to introduce classical nets.

Definition 2.5. *Let $0 \leq t \leq m$ be integers and $\mathcal{P} = \{\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{b^m-1}\} \subseteq [0,1)^s$, where $s \geq 1$. We say that $\mathcal{P}$ is a (t,m,s)-net in base b, if for all $\boldsymbol{\nu} \in \{0, \ldots, m\}^s$, for which*

$$\sum_{j=1}^{s} \nu_j = m - t\,,$$

the elementary intervals $\mathsf{E}(\boldsymbol{\nu}, \boldsymbol{a}_{\boldsymbol{\nu}})$ contain exactly $b^{m-|\boldsymbol{\nu}|_1}$ points of $\mathcal{P}$ for all $\boldsymbol{a}_{\boldsymbol{\nu}} \in \{0, \ldots, b-1\}^{|\boldsymbol{\nu}|_1}$. A (t,m,s)-net in base b is called a strict (t,m,s)-net in base b, if it is not a (u,m,s)-net in base b with $u < t$.

The next remark gives a geometric interpretation of (t,m,s)-nets, see [66, Remark 4.3].

Remark 2.6. *Let $\mathcal{P}$ be a (t,m,s)-net in base b and $E(\boldsymbol{\nu},\boldsymbol{a}_{\boldsymbol{\nu}})$ an elementary interval in base b for which $\lambda_s(E(\boldsymbol{\nu},\boldsymbol{a}_{\boldsymbol{\nu}})) \geq b^{t-m}$. Then Definition 2.5 says that the proportion of points of $\mathcal{P}$ in $E(\boldsymbol{\nu},\boldsymbol{a}_{\boldsymbol{\nu}})$, which is given by $|\mathcal{P}(E(\boldsymbol{\nu},\boldsymbol{a}_{\boldsymbol{\nu}}))|/|\mathcal{P}([0,1)^s)|$, equals the volume of $E(\boldsymbol{\nu},\boldsymbol{a}_{\boldsymbol{\nu}})$, where $|\mathcal{P}(E)|$ denotes the number of points of $\mathcal{P}$ in E.*

The next remark shows how to interpret the quality parameter t of (t,m,s)-nets, see again [66, Remark 4.3].

Remark 2.7. *If we deal with an elementary interval $E(\boldsymbol{\nu},\boldsymbol{a}_{\boldsymbol{\nu}})$, for which $\lambda_s(E(\boldsymbol{\nu},\boldsymbol{a}_{\boldsymbol{\nu}})) \geq b^{t-m}$, or a disjoint union of such elementary intervals, then the number of points in $E(\boldsymbol{\nu},\boldsymbol{a}_{\boldsymbol{\nu}})$ equals $b^m\lambda_s(E(\boldsymbol{\nu},\boldsymbol{a}_{\boldsymbol{\nu}}))$; this is due to the fact that $E(\boldsymbol{\nu},\boldsymbol{a}_{\boldsymbol{\nu}})$ can be expressed as a disjoint union of elementary intervals in base b of volume b^{t-m}. This implies that a (t,m,s)-net in base b is also a (u,m,s)-net in base b for integers $t \leq u \leq m$, hence smaller values of t mean stronger regularity properties.*

Next, we introduce the concept of a (t,s)-sequence.

Definition 2.8. *Let $t \geq 0$ be an integer and $\mathcal{S} = \{\boldsymbol{x}_0, \boldsymbol{x}_1, \dots\}$ be a sequence of points in $[0,1)^s$. Then $\mathcal{S}$ is a (t,s)-sequence in base b, if for all integers $k \geq 0$ and $m \geq t$, the point set consisting of the $\boldsymbol{x}_h$ with $kb^m \leq h < (k+1)b^m$ is a (t,m,s)-net in base b. A (t,s)-sequence in base b is called a strict (t,s)-sequence in base b, if it is not a (u,s)-sequence in base b with $u < t$.*

We have the following remark regarding the quality parameter t of a (t,s)-sequence.

Remark 2.9. *Using the same argument as in Remark 2.7, it can be shown that any (t,s)-sequence in base b is also a (u,s)-sequence in base b for integers $u \geq t$.*

We point out that the definitions of (t,m,s)-nets in base b and (t,s)-sequences in base b do not suggest how to practically construct such point sets; this is addressed in the next subsection.

2.2.2 Classical digital nets and classical digital sequences

We first introduce the classical digital construction scheme, which classical digital nets and classical digital sequences are based upon. We avoid too many technical notions by restricting b to prime numbers; for a more general definition (over arbitrary finite commutative rings) see for example [54; 56; 66]. Following [29] we introduce the classical digital construction scheme.

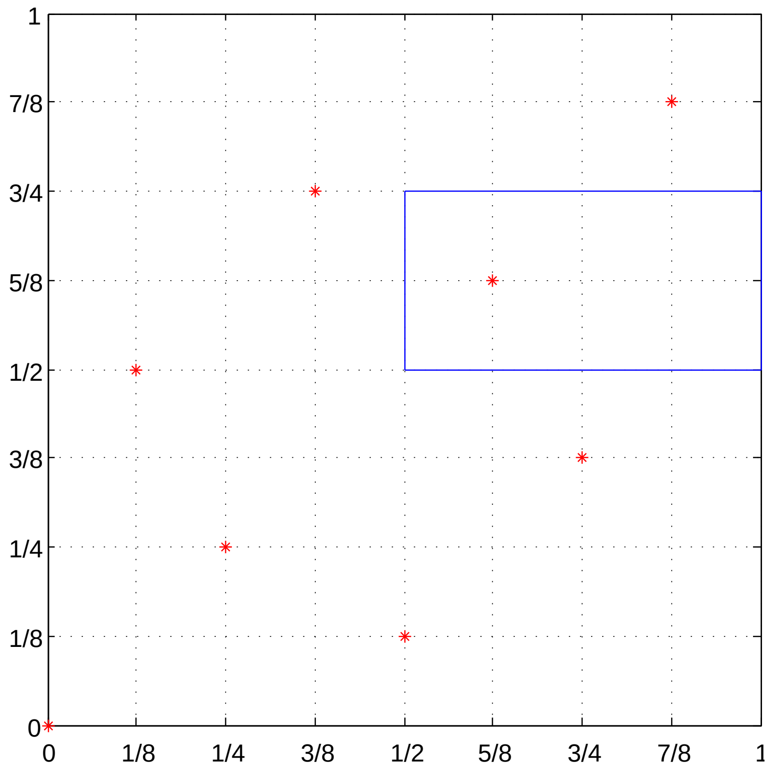

Figure 2.1. *The picture shows a* $(0,3,2)$*-net in base* 2 *and an elementary interval* $E(\boldsymbol{\nu},\boldsymbol{a}_{\boldsymbol{\nu}})$*, where* $\nu_1 = 1$ *and* $\nu_2 = 2$ *and* $a_{1,1} = 1$*,* $a_{2,1} = 1$ *and* $a_{2,2} = 0$*.*

Definition 2.10. *Let* b *be a prime and* $m \geq 1$ *be an integer. Let* $C_1, \ldots, C_s$ *be* $m \times m$ *matrices over the finite field* $\mathbb{Z}_b$*. We construct* b^m *points in* $[0,1)^s$ *in the following way: for* $0 \leq h < b^m$ *let* $h = h_0 + h_1 b + \ldots h_{m-1} b^{m-1}$ *be the* b*-adic expansion of* h*. Identify* h *with the vector* $\vec{h} = (h_0, \ldots, h_{m-1})^\top \in \mathbb{Z}_b^m$*, where* $\top$ *denotes the transpose of the vector. For* $1 \leq j \leq s$ *we multiply the matrix* C_j *by* $\vec{h}$*, i.e.*

$$C_j \vec{h} =: (y_{j,1}(h), \ldots, y_{j,m}(h))^\top \in \mathbb{Z}_b^m$$

and set

$$x_{h,j} := \frac{y_{j,1}(h)}{b} + \cdots + \frac{y_{j,m}(h)}{b^m}.$$

The point set $\{\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{b^m-1}\}$ *is called a classical digital net (over* $\mathbb{Z}_b$*) (with generating matrices* $C_1, \ldots, C_s$*).*

For $m = \infty$*, we obtain a sequence* $\{\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots\}$*, which is called a classical digital sequence (over* $\mathbb{Z}_b$*) (with generating matrices* $C_1, \ldots, C_s$*).*

Having introduced the classical digital construction scheme, we are now in a position to define classical digital nets and classical digital sequences; we remark that the manner in which the definitions are presented makes for easy comparison with the definitions of higher order digital nets and higher order digital sequences given in Section 2.3.

Definition 2.11. *Let b be a prime, $t \in \mathbb{N}_0$, and $m \geq 1$ a natural number, $\mathbb{Z}_b$ the finite field of order b and $C_1, \dots, C_s \in \mathbb{Z}_b^{m \times m}$ with $C_j = (c_{j,1}, \dots, c_{j,m})^\top$. If $\forall d_j, j = 1, \dots, s$, $0 \leq d_j \leq m$, such that $\sum_{j=1}^s d_j = m - t$, the vectors $\{c_{j,i}, i = 1, \dots, d_j, j = 1, \dots, s\}$ are linearly independent, then the matrices $C_1, \dots, C_s$ generate a digital (t, m, s)-net over $\mathbb{Z}_b$. A digital (t, m, s)-net over $\mathbb{Z}_b$ generated by the matrices $C_1, \dots, C_s$ is called a strict digital (t, m, s)-net over $\mathbb{Z}_b$, if the matrices $C_1, \dots, C_s$ do not generate a digital (u, m, s)-net over $\mathbb{Z}_b$ with $u < t$.*

Of course, see e.g. [66, Theorem 4.28], a digital (t, m, s)-net over $\mathbb{Z}_b$ is a (t, m, s)-net in base b. Next, we introduce digital (t, s)-sequences.

Definition 2.12. *Let b be a prime, $\mathbb{Z}_b$ the finite field of order b and $C_1, \dots, C_s \in \mathbb{Z}_b^{\infty \times \infty}$ with $C_j = (c_{j,1}, c_{j,2}, \dots)^\top$. Further, let $C_{j,m \times m}$ denote the left upper $m \times m$ submatrix of C_j. If for all $m \geq t$, $t \in \mathbb{N}_0$, the matrices $C_{1,m \times m}, \dots, C_{s,m \times m}$ generate a digital (t, m, s)-net over $\mathbb{Z}_b$, then the classical digital sequence with generating matrices $C_1, \dots, C_s$ is called a digital (t, s)-sequence over $\mathbb{Z}_b$. A digital (t, s)-sequences over $\mathbb{Z}_b$ generated by the matrices $C_1, \dots, C_s$ is called a strict digital (t, s)-sequence over $\mathbb{Z}_b$, if the matrices $C_1, \dots, C_s$ do not generate a digital (u, s)-sequence over $\mathbb{Z}_b$ with $u < t$.*

As with nets a digital (t, s)-sequence over $\mathbb{Z}_b$ is a (t, s)-sequence in base b, see e.g. [66, Theorem 4.36].

We now illustrate the above definitions using some examples, which will be used again in Subsection 2.3.1.

Example 2.13. The Hammersley net, [41], is an example of a digital $(0, m, 2)$-net over $\mathbb{Z}_2$. Its generating matrices are given by

$$C_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \text{ and } C_2 = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \iddots & \iddots & 0 \\ 0 & \iddots & \iddots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}.$$

◁

Example 2.14. The following matrices generate a strict digital $(1, 3, 4)$-net over $\mathbb{Z}_2$ and stem from a Niederreiter-Xing sequence as implemented by Pirsic [88]:

$$C_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, C_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, C_3 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, C_4 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

◁

Example 2.15. The following matrices generate a strict digital $(2, 3, 4)$-net over $\mathbb{Z}_2$:

$$K_1 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, K_2 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, K_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, K_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

◁

A concept crucial to the study of classical digital nets is the concept of a dual net, which was first introduced in [72] and which we now recall.

Definition 2.16. *Let $C_1, \ldots, C_s \in \mathbb{Z}_b^{m\times m}$ be the generating matrices of a classical digital net over $\mathbb{Z}_b$, where b is prime. Consequently, we define the dual net $\mathcal{D} = \mathcal{D}(C_1, \ldots, C_s)$ associated with the matrices $C_1, \ldots, C_s$ by*

$$\mathcal{D} = \left\{ \boldsymbol{k} \in \mathbb{N}_0^s : C_1^\top \vec{k}_1 + \cdots + C_s^\top \vec{k}_s = \vec{0} \right\}$$

where for $\boldsymbol{k} = (k_1, \ldots, k_s)$ with $k_j = \kappa_{j,0} + \kappa_{j,1} b + \ldots$ and $\kappa_{j,i} \in \{0, \ldots, b-1\}$ we let $\vec{k}_j = (\kappa_{j,0}, \ldots, \kappa_{j,m-1})^\top$.

Furthermore, we set $\mathcal{D}' = \mathcal{D} \setminus \{\mathbf{0}\}$. We remark that the analogous concept for classical digital sequences was introduced in [31]. Dual nets are useful for constructing classical digital nets, [72], but also appear in the context of numerical integration, see [32].

We conclude this subsection by noting that so far, generating matrices have been the starting point for the construction of classical digital nets. However, for a particular class of classical digital nets, an alternative construction is possible, namely one which closely resembles the construction of integration lattices as developed by Sloan and collaborators, [19; 52; 98; 99]. This class, classical polynomial lattice point sets, is discussed in the next subsection.

2.2.3 Classical polynomial lattice point sets

In this subsection we introduce classical polynomial lattice point sets as a special case of classical digital nets mainly relying on [29; 35]. Classical polynomial lattice point sets were first introduced in [65], see also [35; 66]. We restrict ourselves to the case where b is a prime and let $\mathbb{Z}_b((x^{-1}))$ be the field of formal Laurent series over $\mathbb{Z}_b$. Elements of $\mathbb{Z}_b((x^{-1}))$ are formal Laurent series,

$$L = \sum_{l=w}^{\infty} t_l x^{-l},$$

where w is an arbitrary integer and all $t_l \in \mathbb{Z}_b$. We now define a classical polynomial lattice point set.

Definition 2.17. *Let b be a prime, $m \geq 1$ and v_m the map from $\mathbb{Z}_b((x^{-1}))$ to the interval $[0,1)$ defined by*

$$v_m(\sum_{l=w}^{\infty} t_l x^{-l}) = \sum_{l=\max(1,w)}^{m} t_l b^{-l}\,. \tag{2.2}$$

We choose $p(x) \in \mathbb{Z}_b[x]$ of $deg(p(x)) = m$ and let $q_1(x),\dots,q_s(x) \in \mathbb{Z}_b[x]$. For $0 \leq h < b^m$, we let $h = h_0 + h_1 b + \dots + h_{m-1}b^{m-1}$ be the b-adic expansion of h. With such h, we associate the polynomial

$$h(x) = \sum_{r=0}^{m-1} h_r x^r \in \mathbb{Z}_b[x]\,.$$

Then the point set consisting of the b^m points

$$\boldsymbol{x}_h = \left(v_m(\frac{h(x)q_1(x)}{p(x)}), v_m(\frac{h(x)q_2(x)}{p(x)}),\dots, v_m(\frac{h(x)q_s(x)}{p(x)})\right) \in [0,1)^s$$

for $0 \leq h < b^m$ is called a classical polynomial lattice point set and denoted by $S_{p,m}(\boldsymbol{q})$, where $\boldsymbol{q}(x) = (q_1(x),\dots,q_s(x)) \in \mathbb{Z}_b^s[x]$. A quasi-Monte Carlo rule using the point set $S_{p,m}(\boldsymbol{q})$ is called a classical polynomial lattice rule.

It can be shown that the point set $\mathcal{P} = \{\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_{b^m-1}\}$ introduced in Definition 2.17 is a classical digital net in the sense of Definition 2.11. The generating matrices of this classical digital net can be obtained in the following way: for $1 \leq j \leq s$, consider the expansion

$$\frac{q_j(x)}{p(x)} = \sum_{l=w_j}^{\infty} u_l^{(j)} x^{-l} \in \mathbb{Z}_b((x^{-1}))\,,$$

where $w_j \leq 1$. Then the elements $c_{i,r}^{(j)}$ of the $m \times m$ matrix C_j over $\mathbb{Z}_b$ are given by

$$c_{i,r}^{(j)} = u_{r+i}^{(j)} \in \mathbb{Z}_b\,,$$

for $1 \leq j \leq s$, $1 \leq i \leq m$, $0 \leq r \leq m-1$.

Analogously to the dual net associated with a classical digital net, see Definition 2.16, we now define the concept of a dual polynomial lattice, which first appeared in [29].

For a non-negative integer k with b-adic expansion $k = \kappa_0 + \kappa_1 b + \dots$, we write

$$tr_m(k) = \kappa_0 + \kappa_1 b + \dots + \kappa_{m-1} b^{m-1} \tag{2.3}$$

and thus the associated polynomial

$$tr_m(k)(x) = \kappa_0 + \kappa_1 x + \dots + \kappa_{m-1} x^{m-1} \in \mathbb{Z}_b[x] \tag{2.4}$$

has degree $< m$. For a vector $\boldsymbol{k} \in \mathbb{N}_0^s$, we define $tr_m(\boldsymbol{k})$ componentwise. We can now introduce the concept of a dual polynomial lattice.

Definition 2.18. *Let b be prime and $\boldsymbol{q}(x) = (q_1(x), \ldots, q_s(x)) \in \mathbb{Z}_b^s[x]$, then the dual polynomial lattice of $S_{p,m}(\boldsymbol{q})$ is given by*

$$\mathcal{D} = \mathcal{D}_p(\boldsymbol{q}) = \{\boldsymbol{k} \in \mathbb{N}_0^s : tr_m(k_1)(x)q_1(x) + tr_m(k_2)(x)q_2(x) + \cdots + tr_m(k_s)(x)q_s(x) \equiv 0 \pmod{p(x)}\} .$$

Following Subsection 2.2.2, we set $\mathcal{D}' = \mathcal{D} \setminus \{\boldsymbol{0}\}$. Recalling that classical polynomial lattice point sets are special cases of digital (t, m, s)-nets, it would be interesting to determine the quality parameter t of a classical polynomial lattice point set interpreted as a digital (t, m, s)-net. In order to do so, we recall the concept of a figure of merit following [35].

Definition 2.19. *Let b be a prime and let $s, m \in \mathbb{N}$. For $p(x) \in \mathbb{Z}_b[x]$ of $deg(p(x)) = m$ and $\boldsymbol{q}(x) \in \mathbb{Z}_b^s[x]$, the figure of merit of the classical polynomial lattice point set $S_{p,m}(\boldsymbol{q})$ is given by*

$$\rho(S_{p,m}(\boldsymbol{q})) = s - 1 + \min_{\boldsymbol{h} \in \mathcal{D}'_p(\boldsymbol{q})} \sum_{j=1}^{s} deg(tr_m(h_j)(x)) ,$$

where $\boldsymbol{h} = (h_1, h_2, \ldots, h_s)$.

The figure of merit $\rho(S_{p,m}(\boldsymbol{q}))$ is closely related to the quality parameter t of the classical polynomial lattice point set $S_{p,m}(\boldsymbol{q})$ interpreted as a digital (t, m, s)-net.

Theorem 2.20. *Let b be a prime and let $m, s \in \mathbb{N}$. Let $p(x) \in \mathbb{Z}_b[x]$ of $deg(p(x)) = m$ and $\boldsymbol{q}(x) \in \mathbb{Z}_b^s[x]$. Then $S_{p,m}(\boldsymbol{q})$ is a strict digital (t, m, s)-net over $\mathbb{Z}_b$ with $t = m - \rho(S_{p,m}(\boldsymbol{q}))$.*

However, instead of trying to find suitable generating matrices to construct classical digital nets, Definition 2.17 suggests that we could alternatively try to find suitable polynomials $q_1(x), \ldots, q_s(x) \in \mathbb{Z}_b[x]$. Initially, see [92], this was done using computer searches, however in [29], it was shown how the ideas underlying the component-by-component algorithm, [100], the workhorse when it comes to constructing integration lattices, could be used to construct classical polynomial lattice point sets: in particular, such an algorithm chooses the polynomials $q_1(x), \ldots, q_s(x)$ one at a time in a greedy way; this algorithm will be explained in detail and greater generality in Chapter 6, see also Chapter 7.

2.3 Introducing higher order digital nets and higher order digital sequences

In this section we introduce higher order digital nets and higher order digital sequences and higher order polynomial lattice point sets, following [21; 34]. In particular, higher order digital nets and higher order digital sequences generalize classical digital nets and classical digital sequences, and higher order polynomial lattice point sets generalize classical polynomial lattice point sets. Furthermore, higher order polynomial lattice point sets are special cases of higher order digital nets.

2.3.1 Higher order digital nets and higher order digital sequences

In this subsection we introduce higher order digital nets and higher order digital sequences. In particular, we illustrate how they differ from the classical digital nets and classical digital sequences introduced in Subsection 2.2.2. Firstly, we generalize the classical digital construction scheme introduced in Definition 2.10.

Definition 2.21. *Let b be a prime and $n, m \geq 1$ be integers. Let $C_1, \ldots, C_s$ be $n \times m$ matrices over the finite field $\mathbb{Z}_b$ of order b. Now we construct b^m points in $[0,1)^s$: for $0 \leq h < b^m - 1$, let $h = h_0 + h_1 b + \cdots + h_{m-1} b^{m-1}$ be the b-adic expansion of h. Identify h with the vector $\vec{h} = (h_0, \ldots, h_{m-1})^\top \in \mathbb{Z}_b^m$, where $\top$ denotes the transpose of the vector. For $1 \leq j \leq s$ we multiply the matrix C_j by $\vec{h}$, i.e.,*

$$C_j \vec{h} =: (y_{j,1}(h), \ldots, y_{j,n}(h))^\top \in \mathbb{Z}_b^n ,$$

and set

$$x_{h,j} := \frac{y_{j,1}(h)}{b} + \cdots + \frac{y_{j,n}(h)}{b^n} .$$

The point set $\{\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{b^m-1}\}$ is called a higher order digital net (over $\mathbb{Z}_b$) (with generating matrices $C_1, \ldots, C_s$).

For $n, m = \infty$ we obtain a sequence $\{\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots\}$, which is called a higher order digital sequence (over $\mathbb{Z}_b$) (with generating matrices $C_1, \ldots, C_s$).

We point out that the generalized digital construction scheme introduced in Definition 2.21 differs from the classical one presented in Definition 2.10 by allowing n and m to differ. We are now in a position to define higher order digital nets and higher order digital sequences, see also [27, Definition 2].

Definition 2.22. *Let $n, m, \alpha \geq 1$ be natural numbers, let $0 < \beta \leq \min(1, \alpha m/n)$ be a real number, and let $0 \leq t \leq \beta n$ be a natural number. Let $\mathbb{Z}_b$ be the finite field of prime order b and let $C_1, \ldots, C_s \in \mathbb{Z}_b^{n \times m}$ with $C_j = (c_{j,1}, \ldots, c_{j,n})^\top$. If for all $1 \leq i_{j,\nu_j} < \cdots < i_{j,1} \leq n$, where $0 \leq \nu_j \leq m$ for all $j = 1, \ldots, s$ with*

$$\sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j,\alpha)} i_{j,l} \leq \beta n - t ,$$

the vectors

$$c_{1,i_{1,\nu_1}}, \ldots, c_{1,i_{1,1}}, \ldots, c_{s,i_{s,\nu_s}}, \ldots, c_{s,i_{s,1}}$$

are linearly independent over $\mathbb{Z}_b$, then the higher order digital net with generating matrices $C_1, \ldots, C_s$ is called a digital $(t, \alpha, \beta, n \times m, s)$-net over $\mathbb{Z}_b$. If t is the smallest nonnegative integer such that the higher order digital net generated by $C_1, \ldots, C_s$ is a digital $(t, \alpha, \beta, n \times m, s)$-net, then we call the higher order digital net a strict digital $(t, \alpha, \beta, n \times m, s)$-net.

Comparing Definitions 2.11 and 2.22, we note that the linear independence requirements on the generating matrices have changed. Furthermore, we note that a lot of new parameters have appeared in Definition 2.22, which are explained in the forthcoming Remark 2.23.

Remark 2.23. *In the following we explain the meaning of the parameters t, α, β, n, m and s introduced in Definition 2.22, see also [25, Remark 1].*

- *s denotes the dimensionality of the point set,*
- *the logarithm in base b of the number of points is m, i.e. a digital $(t, \alpha, \beta, n \times m, s)$-net has b^m points,*
- *n denotes the number of rows of the generating matrices and therefore corresponds to the maximum number of non-zero digits in the base b expansion of each point; hence n determines how precise each point is placed in the unit cube, which has a direct influence on the convergence of the integration error as can be seen from the next point and Section 2.7.1,*
- *$\beta n - t$ denotes the quality of the point set, which can be referred to as the strength of the net, in particular, the integration error is $\mathcal{O}(b^{-\beta n + t}(\beta n - t)^{\alpha s})$, see Section 2.7.1,*
- *digital $(t, \alpha, \beta, n \times m, s)$-nets were introduced in the context of numerical integration, where α is a variable parameter, which denotes the smoothness of the integrand. Unless stated otherwise, see e.g. Chapters 5 and 6, we assume that the smoothness α is not known explicitly. Furthermore, we point out that setting $\beta n = \alpha m$, integration errors of order $N^{-\alpha}$ multiplied by a power of a $\log N$ factor can be obtained, see the forthcoming Theorem 2.49.*

We now define digital $(t, \alpha, \beta, \sigma, s)$-sequences over $\mathbb{Z}_b$.

Definition 2.24. *Let $\alpha, \sigma \geq 1$ and $t \geq 0$ be integers and let $0 < \beta \leq \alpha/\sigma$ be a real number. Let $\mathbb{Z}_b$ be the finite field of prime order b and let $C_1, \ldots, C_s \in \mathbb{Z}_b^{\infty \times \infty}$ with $C_j = (c_{j,1}, c_{j,2}, \ldots)^\top$. Further, let $C_{j,\sigma m \times m}$ denote the left upper $\sigma m \times m$ submatrix of C_j. If for all $m > t/(\beta\sigma)$ the matrices $C_{1,\sigma m \times m}, \ldots, C_{s,\sigma m \times m}$ generate a digital $(t, \alpha, \beta, \sigma m \times m, s)$-net over $\mathbb{Z}_b$, then the* *higher order digital sequence with generating matrices $C_1, \ldots, C_s$ is called a digital $(t, \alpha, \beta, \sigma, s)$-sequence over $\mathbb{Z}_b$. If t is the smallest nonnegative integer such that the higher order digital sequence generated by $C_1, \ldots, C_s$ is a digital $(t, \alpha, \beta, \sigma, s)$-sequence, then we call the higher order digital sequence a strict digital $(t, \alpha, \beta, \sigma, s)$-sequence.*

The following remark describes the relationship between digital (t, m, s)-nets and digital $(t, \alpha, \beta, n \times m, s)$-nets and digital (t, s)-sequences and digital $(t, \alpha, \beta, \sigma, s)$-sequences.

Remark 2.25. *We remark that the definition of a digital $(t, 1, 1, m \times m, s)$-net coincides with the definition of a digital (t, m, s)-net and that the definition of a digital $(t, 1, 1, 1, s)$-sequence coincides with the definition of a digital (t, s)-sequence; this should be compared with Remarks 3.10 and 3.11.*

Finally, following [21] we now recall a method of explicitly constructing digital $(t, \alpha, \beta, n \times m, s)$-nets, which was first presented in [21, Section 4.4]; for other, more advanced methods of constructing digital $(t, \alpha, \beta, n \times m, s)$-nets, we refer to the propagation rules presented in [27]. The method presented here produces a digital $(t, \alpha, \min(1, \alpha/d), dm \times m, s)$-net over $\mathbb{Z}_b$, for all $\alpha \geq 1$, where $d \in \mathbb{N}$ is a parameter which can be chosen freely.

Let $d \geq 1$ and let $C_1, \ldots, C_{sd}$ be the generating matrices of a digital (t', m, sd)-net over $\mathbb{Z}_b$; we recall that many explicit examples of such generating matrices are known, see e.g. [37; 66; 69; 70; 75; 104] and the references therein. As we will see later, the choice of the underlying digital (t', m, sd)-net has a direct impact on the bound on the t-value of the digital $(t, \alpha, \min(1, \alpha/d), dm \times m, s)$-net over $\mathbb{Z}_b$, which was proven in [21]. Let $C_j = (c_{j,1}, \ldots, c_{j,m})^\top$ for $j = 1, \ldots, sd$, i.e. $c_{j,l}$ are the row vectors of C_j. Now let the matrix $C_j^{(d)}$ be comprised of the first rows of the matrices $C_{(j-1)d+1}, \ldots, C_{jd}$, then the second rows of $C_{(j-1)d+1}, \ldots, C_{jd}$ and so on, in the order described in the following: the matrix $C_j^{(d)}$ is a $dm \times m$ matrix, i.e. $C_j^{(d)} = (c_{j,1}^{(d)}, \ldots, c_{j,dm}^{(d)})^\top$, where $c_{j,l}^{(d)} = c_{u,v}$ with $l = (v - j)d + u$, $1 \leq v \leq m$ and $(j-1)d < u \leq jd$ for $l = 1, \ldots, dm$ and $j = 1, \ldots, s$. We remark that this construction can be extended to digital $(t, \alpha, \beta, \sigma, s)$-sequences by letting $\widetilde{C}_j = (\widetilde{c}_{j,1}, \widetilde{c}_{j,2}, \ldots)^\top$, for $j = 1, \ldots, sd$, denote the generating matrices of a digital (t', sd)-sequence; the resulting matrices are now $\infty \times \infty$ matrices $\widetilde{C}_j^{(d)}$, $j = 1, \ldots, s$, where again we have $\widetilde{C}_j^{(d)} = (\widetilde{c}_{j,1}^{(d)}, \widetilde{c}_{j,2}^{(d)}, \ldots)^\top$, where $\widetilde{c}_{j,l}^{(d)} = \widetilde{c}_{u,v}$ with $l = (v - j)d + u$, $v \geq 1$ and $(j-1)d < u \leq jd$ for $l = 1, 2, \ldots$ and $j = 1, \ldots, s$. The following result improves on [21, Theorem 4.11] for some cases, a proof is presented in [27].

Theorem 2.26. *Let $d \geq 1$ be a natural number and let $C_1, \ldots, C_{sd}$ be the generating matrices of a digital (t', m, sd)-net over the finite field $\mathbb{Z}_b$ of prime order b. Let $C_1^{(d)}, \ldots, C_s^{(d)}$ be defined as above. Then for any $\alpha \in \mathbb{N}$, the matrices $C_1^{(d)}, \ldots, C_s^{(d)}$ are the generating matrices of a digital $(t, \alpha, \min(1, \alpha/d), dm \times m, s)$-net over $\mathbb{Z}_b$ with*

$$t = \min(\alpha, d) \min\left(m, t' + \left\lfloor \frac{s(d-1)}{2} \right\rfloor\right). \tag{2.5}$$

Furthermore, the matrices $\widetilde{C}_1^{(d)}, \ldots, \widetilde{C}_s^{(d)}$ obtained from the generating matrices $\widetilde{C}_1, \ldots, \widetilde{C}_{sd}$ of a digital (t', sd)-sequence over $\mathbb{Z}_b$ are the generating matrices of a digital $(t, \alpha, \min(1, \alpha/d), d, s)$-sequence over $\mathbb{Z}_b$ with

$$t = \min(\alpha, d)\left(t' + \left\lfloor \frac{s(d-1)}{2} \right\rfloor\right).$$

The following example shows that Theorem 2.26 cannot be improved on in general.

Example 2.27. Let $d = 2$ and $s = 1$ and consider a digital $(t, \alpha, \min(1, \alpha/2), 2m \times m, 1)$-net over $\mathbb{Z}_b$ generated from a digital $(0, m, 2)$-net over $\mathbb{Z}_b$ (such nets exist, for example one can take the Hammersley net from Example 2.13). Then Theorem 2.26 implies that we can choose $t = \min(\alpha, 2)(0 + \lfloor 1/2 \rfloor) = 0$, which is already best possible. ◁

On the other hand it can be checked that the bound on the t-value in Theorem 2.26 for particular generalized digital nets is not necessarily best possible. That is, if we use a strict digital (t', m, sd)-net over $\mathbb{Z}_b$ for the construction of the generating matrices $C_1^{(d)}, \ldots, C_s^{(d)}$, then these generating matrices do not necessarily generate a strict digital $(t, \alpha, \beta, n \times m, s)$-net over $\mathbb{Z}_b$, where t is given by Equation (2.5). This is illustrated in the next example.

Example 2.28. The following matrices were presented in Example 2.14 and produce a strict digital $(1, 3, 4)$-net over $\mathbb{Z}_2$:

$$C_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, C_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, C_3 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, C_4 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

Using the method described earlier in this subsection with $d = 2$, we construct the generating matrices $C_1^{(2)}$ and $C_2^{(2)}$, which are given by:

$$C_1^{(2)} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, C_2^{(2)} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

For any $\alpha \geq 2$, Theorem 2.26 yields a digital $(4, \alpha, 1, 6 \times 3, 2)$-net and for $\alpha = 1$ a digital $(2, 1, 1/2, 6 \times 3, 2)$-net.

We now show that the exact t-value of this higher order digital net is smaller than the one obtained from Theorem 2.26. It can be confirmed by inspection that the matrices $C_1^{(2)}$ and $C_2^{(2)}$ generate a digital $(2, \alpha, 1, 6 \times 3, 2)$-net for all $\alpha \geq 2$, by checking that for all $1 \leq i_{j,\nu_j} < \cdots < i_{j,1}$, where $0 \leq \nu_j$, $j = 1, 2$, with

$$\sum_{j=1}^{2} \sum_{l=1}^{\min(\nu_j, \alpha)} i_{j,l} \leq 6 - 2 = 4$$

the vectors $c_{1,i_{1,\nu_1}}^{(2)}, \ldots, c_{1,i_{1,1}}^{(2)}, c_{2,i_{2,\nu_2}}^{(2)}, \ldots, c_{2,i_{2,1}}^{(2)}$ are linearly independent over $\mathbb{Z}_2$. Furthermore, it can be confirmed that the two matrices $C_1^{(2)}$ and $C_2^{(2)}$ do not generate a digital $(1, \alpha, 1, 6 \times 3, 2)$-net for any $\alpha \geq 2$, as for $\nu_1 = 0$, $\nu_2 = 2$, $i_{2,2} = 1$, $i_{2,1} = 4$, $c_{2,i_{2,1}}^{(2)}$ and $c_{2,i_{2,2}}^{(2)}$ are linearly dependent. Hence, for any $\alpha \geq 2$ the matrices $C_1^{(2)}$ and $C_2^{(2)}$ generate a strict digital $(2, \alpha, 1, 6 \times 3, 2)$-net.

For $\alpha = 1$ on the other hand, it can be checked that the matrices $C_1^{(2)}$ and $C_2^{(2)}$ generate a strict digital $(0, 1, 1/2, 6 \times 3, 2)$-net.

Thus, for this example Theorem 2.26 does not yield the best possible result for any $\alpha \geq 1$. $\triangleleft$

Next we present an example which might be counterintuitive at first: we present a strict digital $(2, 3, 4)$-net, which generates a strict digital $(1, \alpha, 1, 6 \times 3, 2)$-net for any $\alpha \geq 2$, and for $\alpha = 1$, a strict digital $(0, 1, 1/2, 6 \times 3, 2)$-net.

Example 2.29. The following matrices were presented in Example 2.15 and generate a strict digital $(2, 3, 4)$-net over $\mathbb{Z}_2$:

$$K_1 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, K_2 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, K_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, K_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Using the method described earlier in this subsection with $d = 2$, we construct the generating matrices $K_1^{(2)}$ and $K_2^{(2)}$, which are given by:

$$K_1^{(2)} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, K_2^{(2)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

For any $\alpha \geq 2$, Theorem 2.26 yields a digital $(6, \alpha, 1, 6 \times 3, 2)$-net, and for $\alpha = 1$ a digital $(3, 1, 1/2, 6 \times 3, 2)$-net.

As in Example 2.28 it can be confirmed by inspection that the matrices $K_1^{(2)}$ and $K_2^{(2)}$ generate a digital $(1, \alpha, 1, 6 \times 3, 2)$-net for all $\alpha \geq 2$. Furthermore, it can be confirmed that the two matrices $K_1^{(2)}$ and $K_2^{(2)}$ do not generate a digital $(0, \alpha, 1, 6 \times 3, 2)$-net for $\alpha \geq 2$, as for $\nu_1 = 2$, $\nu_2 = 2$, $i_{1,2} = 1$, $i_{1,1} = 2$, $i_{2,2} = 1$ and $i_{2,1} = 2$, $k_{1,i_{1,2}}^{(2)}$, $k_{1,i_{1,1}}^{(2)}$, $k_{2,i_{2,2}}^{(2)}$ and $k_{2,i_{2,1}}^{(2)}$ are linearly dependent, where $k_{j,i}^{(2)}$ denotes the ith row of the matrix $K_j^{(2)}$.

For $\alpha = 1$ on the other hand, it can be checked that the matrices $K_1^{(2)}$ and $K_2^{(2)}$ generate a strict digital $(0, 1, 1/2, 6 \times 3, 2)$-net. ◁

The last two examples show that Theorem 2.26 does not always yield the best possible bounds on the t-value for digital $(t, \alpha, \beta, n \times m, s)$-nets constructed from particular classical digital nets. (This could mean that it might be possible to improve the bound on the t-value for higher order digital nets constructed from particular classical nets (or sequences).) On the other hand, at least for digital $(t, \alpha, \beta, \sigma, s)$-sequences, we will see in the forthcoming Theorem 3.21 that Theorem 2.26 does yield the asymptotically optimal dependence of the t-value on α and s.

Remark 2.30. *Note that even though the strict digital $(1, 3, 4)$-net used in Example 2.28 has a better t-value (in the classical sense) than the strict digital $(2, 3, 4)$-net in Example 2.29, the latter generates the better digital $(t, \alpha, 1, 6 \times 3, 2)$-net for any $\alpha \geq 2$, as measured by the generalized t-value. However, it is possible to find a strict digital $(1, 3, 4)$-net which generates a strict digital $(1, \alpha, 1, 6 \times 3, 2)$-net for any $\alpha \geq 2$. Consider for example*

$$\tilde{K}_1 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \tilde{K}_2 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \tilde{K}_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tilde{K}_4 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Remark 2.31. *It can be checked that the matrices $C_1^{(2)}$ and $C_2^{(2)}$ from Example 2.28 can also be interpreted as generating matrices of a digital $(0, 3, 2)$-net over $\mathbb{Z}_2$. However, if we set $\tilde{C}_2 = C_2^{(2)}$, but*

$$\tilde{C}_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

we have an example of a strict digital $(2, \alpha, 1, 6 \times 3, 2)$*-net over* $\mathbb{Z}_2$*,* $\alpha \geq 2$*, which is a strict digital* $(1, 3, 2)$*-net.*

Finally, following Subsection 2.2.2 we now recall the concept of a dual net for higher order digital nets. We remark that dual nets for higher order digital nets were first introduced in [21] and studied in detail in [27]. As with dual nets for classical digital nets, they have been employed to construct new higher order digital nets, see [27], but also occur in the context of numerical integration, see [21] and Subsection 2.7.1. In Chapter 4 the analogous concept for not necessarily digital nets is introduced. For a higher order digital net with generating matrices $C_1, \ldots, C_s$ let $\mathcal{D} = \mathcal{D}(C_1, \ldots, C_s)$ be the dual net given by

$$\mathcal{D} = \left\{ \boldsymbol{k} \in \mathbb{N}_0^s : C_1^\top \vec{k}_1 + \cdots + C_s^\top \vec{k}_s = \vec{0} \right\} , \tag{2.6}$$

where for $\boldsymbol{k} = (k_1, \ldots, k_s)$ with $k_j = \kappa_{j,0} + \kappa_{j,1} b + \ldots$ and $\kappa_{j,i} \in \{0, \ldots, b-1\}$ we let $\vec{k}_j = (\kappa_{j,0}, \ldots, \kappa_{j,n-1})^\top$. Furthermore, we set $\mathcal{D}' = \mathcal{D} \setminus \{\boldsymbol{0}\}$ and for $\emptyset \neq \mathfrak{u} \subseteq [s]$ let $\mathcal{D}_\mathfrak{u} = \mathcal{D}((C_j)_{j \in \mathfrak{u}})$ and $\mathcal{D}_\mathfrak{u}^* = \mathcal{D}_\mathfrak{u} \cap \mathbb{N}^{|\mathfrak{u}|}$.

As in Section 2.2 we now introduce a special case of higher order digital nets, namely higher order polynomial lattice point sets.

2.3.2 Higher order polynomial lattice point sets

We now recall the concept of a higher order polynomial lattice point set. As in Subsection 2.2.3 we assume that b is a prime number and that $\mathbb{Z}_b((x^{-1}))$ is the field of formal Laurent series over $\mathbb{Z}_b$. Elements of $\mathbb{Z}_b((x^{-1}))$ are formal Laurent series,

$$L = \sum_{l=w}^{\infty} t_l x^{-l} ,$$

where w is an arbitrary integer and all $t_l \in \mathbb{Z}_b$. We now define a higher order polynomial lattice point set in a way which resembles Definition 2.17.

Definition 2.32. *Let* b *be a prime and* $1 \leq m \leq n$*. Let* v_n *be the map from* $\mathbb{Z}_b((x^{-1}))$ *to the interval* $[0, 1)$ *defined by*

$$v_n\Big(\sum_{l=w}^{\infty} t_l x^{-l}\Big) = \sum_{l=\max(1,w)}^{n} t_l b^{-l} . \tag{2.7}$$

For a given dimension $s \geq 1$*, choose* $p(x) \in \mathbb{Z}_b[x]$ *of* $deg(p(x)) = n \geq 1$ *and let* $q_1(x), q_2(x) \ldots, q_s(x) \in \mathbb{Z}_b[x]$*. For* $0 \leq h < b^m$ *let* $h = h_0 + h_1 b + \cdots + h_{m-1} b^{m-1}$ *be the b-adic expansion of h. With each such h we associate the polynomial*

$$h(x) = \sum_{r=0}^{m-1} h_r x^r \in \mathbb{Z}_b[x] .$$

Then the point set consisting of the b^m *points*

$$\boldsymbol{x}_h = \left(v_n(\frac{h(x)q_1(x)}{p(x)}), v_n(\frac{h(x)q_2(x)}{p(x)}), \ldots, v_n(\frac{h(x)q_s(x)}{p(x)})\right)$$

for $0 \le h < b^m$*, is called a higher order polynomial lattice point set and denoted by* $S_{p,m,n}(\boldsymbol{q})$*. A quasi-Monte Carlo rule using the point set* $S_{p,m,n}(\boldsymbol{q})$ *is called a higher order polynomial lattice rule.*

We point out that the definition of a classical polynomial lattice point set, Definition 2.17, is recovered from Definition 2.32 by setting $n = m$, hence classical polynomial lattice point sets are special cases of higher order polynomial lattice point sets. Also for dimension $s = 1$, for $m < n$, the points of a higher order polynomial lattice point set are not equally spaced in general, contrary to the case $n = m$, see [34, Remark 2.3]. As for the classical case, we now show that higher order polynomial lattice point sets are in fact a special case of higher order digital nets: in particular, the generating matrices of the higher order digital net associated with the higher order polynomial lattice point set can be obtained as follows: for $1 \le j \le s$, consider the expansions,

$$\frac{q_j(x)}{p(x)} = \sum_{l=w_j}^{\infty} u_l^{(j)} x^{-l} \in \mathbb{Z}_b((x^{-1}))$$

where $w_j \le 1$. Then the elements $c_{i,r}^{(j)}$ of the $n \times m$ matrix C_j over $\mathbb{Z}_b$ are given by

$$c_{i,r}^{(j)} = u_{r+i}^{(j)} \in \mathbb{Z}_b$$

for $1 \le j \le s$, $1 \le i \le n$, $0 \le r \le m-1$.

Following Subsection 2.2.3 we now introduce the concept of a dual polynomial lattice for higher order polynomial lattice point sets.

Definition 2.33. *Let* $\boldsymbol{q}(x) = (q_1(x), \ldots, q_s(x)) \in \mathbb{Z}_b^s[x]$*, then the dual polynomial lattice associated with* $S_{p,m,n}(\boldsymbol{q})$ *is given by*

$$\mathcal{D} = \mathcal{D}_p(\boldsymbol{q}) = \{\boldsymbol{k} \in \mathbb{N}_0^s : tr_n(k_1)(x)q_1(x) + tr_n(k_2)(x)q_2(x) + \cdots + tr_n(k_s)(x)q_s(x) \equiv a(x) \pmod{p(x)} \text{ with } deg(a(x)) < n - m\},$$

where $\boldsymbol{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$*,* $tr_n(.)(x)$ *is defined in Equation* (2.4) *and we set* $deg(0) = -1$*.*

Following Subsection 2.3.1, we set $\mathcal{D}' = \mathcal{D} \setminus \{\boldsymbol{0}\}$. We point out that for $m = n$, we recover Definition 2.17 and for $m < n$ we obtain a superset.

As in Subsection 2.2.3 we now introduce the concept of a generalized figure of merit, which first appeared in [28]. We firstly define the α-degree of a polynomial $k(x) \in \mathbb{Z}_b[x]$,

$$deg_\alpha(k(x)) = \sum_{l=1}^{\min(v,\alpha)} a_l\,,$$

where $k(x) = \kappa_v x^{a_v-1} + \cdots + \kappa_1 x^{a_1-1}$, with $\kappa_1, \ldots, \kappa_v \in \{1, \ldots, b-1\}$ and $a_1 > \cdots > a_v \geq 1$; furthermore, we set $deg_\alpha(0) = -1$ and note that we have $deg_1(k) = deg(k) + 1$. We now define the generalized figure of merit, see [28, Definition 4].

Definition 2.34. *Let $p(x) \in \mathbb{Z}_b[x]$, $deg(p(x)) = n$, $\boldsymbol{q}(x) \in \mathbb{Z}_b^s[x]$. For $\alpha \geq 1$, the generalized figure of merit of a higher order polynomial lattice point set $S_{p,m,n}(\boldsymbol{q})$ is given by*

$$\rho_\alpha(S_{p,m,n}(\boldsymbol{q})) = -1 + \min_{\boldsymbol{h} \in \mathcal{D}'_p(\boldsymbol{q})} \sum_{j=1}^{s} deg_\alpha(tr_n(h_j)(x)),$$

where $\boldsymbol{h} = (h_1, \ldots, h_s)$.

We remark that setting $n = m$ and $\alpha = 1$ in Definition 2.34, we recover Definition 2.19. Finally, we present the analogue of Theorem 2.20, which shows how to determine the quality parameter t of a higher order polynomial lattice point set interpreted as a digital $(t, \alpha, \beta, n \times m, s)$-net over $\mathbb{Z}_b$; this result first appeared as [28, Theorem 2].

Theorem 2.35. *Let $p(x) \in \mathbb{Z}_b[x]$ of $deg(p(x)) = n$, $\alpha \geq 1$ and $\boldsymbol{q}(x) \in \mathbb{Z}_b^s[x]$ be the generating vector of a higher order polynomial lattice point set $S_{p,m,n}(\boldsymbol{q})$. Then $S_{p,m,n}(\boldsymbol{q})$ is a digital $(t, \alpha, \beta, n \times m, s)$-net over $\mathbb{Z}_b$ for any $0 < \beta \leq \alpha m/n$ and $0 \leq t \leq \beta n$ which satisfy*

$$t = \lfloor \beta n \rfloor - \rho_\alpha(S_{p,m,n}(\boldsymbol{q})).$$

Finally, we remark that higher order polynomial lattice point sets were also studied in [28]. From the point of view of this thesis, an important result in [28] was that using the generalized figure of merit, the existence of higher order polynomial lattice point sets achieving a better bound on the quality parameter t than the higher order digital nets constructed using the method from Subsection 2.3.1 was established. However, the paper did not provide an explicit construction of higher order polynomial lattice point sets; this issue is addressed in Chapter 6. In particular, it was mentioned in Subsection 2.2.3 that in [29], a construction method for classical polynomial lattice point sets resembling the component-by-component algorithm for integration lattices was introduced. In [7] a similar algorithm but for the more general class of higher order polynomial lattice point sets was studied and is discussed in Chapter 6.

2.4 Walsh functions, Weyl sums, and classical nets

In this section we introduce Walsh functions and consequently show that they can be used to characterize classical nets.

2.4.1 Walsh functions and Weyl sums

We firstly define Walsh functions.

Definition 2.36. *For an integer $b \geq 2$, represent $k \in \mathbb{N}_0$ in base b, $k = \kappa_{a-1}b^{a-1} + \cdots + \kappa_0$, with $\kappa_l \in \{0, \ldots, b-1\}$. Further, let $\omega_b = \exp(2\pi \mathtt{i}/b)$ be the bth root of unity. Then the kth b-adic Walsh function, ${}_b\mathrm{wal}_k(x) : [0,1) \to \left\{1, \omega_b, \ldots, \omega_b^{b-1}\right\}$ is given by*

$$ {}_b\mathrm{wal}_k(x) = \omega_b^{\xi_1\kappa_0 + \cdots + \xi_a\kappa_{a-1}}, $$

for $x \in [0,1)$ with base b representation $x = \xi_1 b^{-1} + \xi_2 b^{-2} + \ldots$ (unique in the sense that infinitely many of the ξ_l are different from $b-1$).

For dimension $s \geq 2$, $\boldsymbol{x} = (x_1, \ldots, x_s) \in [0,1)^s$ and $\boldsymbol{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$, we define ${}_b\mathrm{wal}_{\boldsymbol{k}} : [0,1)^s \to \left\{1, \omega_b, \ldots, \omega_b^{b-1}\right\}$ by

$$ {}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) = \prod_{j=1}^{s} {}_b\mathrm{wal}_{k_j}(x_j). $$

It follows from the above definition that Walsh functions are piecewise constant functions. For more information on Walsh functions, see [16; 105]. Dealing with Walsh functions in conjunction with a (t, m, s)-net in base b or a higher order digital net over $\mathbb{Z}_b$, we will assume that the b-adic Walsh functions are in the same base b. If we deal with an arbitrary point set, which is not necessarily a (t, m, s)-net in base b or a higher order digital net over $\mathbb{Z}_b$, we will deal with an arbitrary but fixed integer base $b \geq 2$. Consequently, we will in the following often write wal instead of ${}_b\mathrm{wal}$.

The following notation will be used throughout the thesis: by $\oplus$ we denote the digitwise addition modulo b, i.e. for $x, y \in [0,1)$ with base b expansions $x = \sum_{l=1}^{\infty} \xi_l b^{-l}$ and $y = \sum_{l=1}^{\infty} \eta_l b^{-l}$ we define

$$ x \oplus y = \sum_{l=1}^{\infty} \zeta_l b^{-l}, $$

where $\zeta_l \in \{0, 1, \ldots, b-1\}$ is given by $\zeta_l \equiv \xi_l + \eta_l \pmod{b}$, and let $\ominus$ denote the digitwise subtraction modulo b (for short we use $\ominus x := 0 \ominus x$). In the same fashion we also define the digitwise addition and digitwise subtraction for nonnegative integers based on the b-adic expansion. For vectors in $[0,1)^s$ or $\mathbb{N}_0^s$, the operations $\oplus$ and $\ominus$ are carried out componentwise. Throughout the thesis, we always use the same base b for the operations $\oplus$ and $\ominus$ as is used for the Walsh functions. Further, we call $x \in [0,1)$ a b-adic rational if it can be written in a finite base b expansion. The following simple properties of Walsh functions are often used in the sequel: for $k, l \in \mathbb{N}_0$ and all $x, y \in [0,1)$, with the restriction that if x, y are not b-adic rationals, then $x \oplus y$ is not allowed to be a

b-adic rational, we have $\mathrm{wal}_k(x)\mathrm{wal}_l(x) = \mathrm{wal}_{k\oplus l}(x)$ and $\mathrm{wal}_k(x)\mathrm{wal}_k(y) = \mathrm{wal}_k(x \oplus y)$. Furthermore,

$$\overline{\mathrm{wal}_k(x)} = \mathrm{wal}_{\ominus k}(x).$$

We recall again that Walsh functions were used for the first time in [58] to analyze (t, m, s)-nets. Next, we recall the concept of a Weyl sum.

Definition 2.37. *For a point set $\mathcal{P} = \{\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{N-1}\} \subseteq [0,1)^s$, $N \in \mathbb{N}$, and $\boldsymbol{k} \in \mathbb{N}_0^s$, $Q_N(\mathrm{wal}_{\boldsymbol{k}}, \mathcal{P})$ is called a Weyl sum (based on Walsh functions), where $Q_N(f, \mathcal{P})$ is given by Equation* (2.1).

2.4.2 Characterizing classical nets using Weyl sums

In this subsection we recall results from [45], which will be generalized later, to characterize (t, m, s)-nets. We now introduce the following function, which in its general form first appeared in [20]. For $k \in \mathbb{N}$, with base b expansion $k = \kappa_1 b^{a_1-1} + \kappa_2 b^{a_2-1} + \cdots + \kappa_\nu b^{a_\nu-1}$, $1 \le a_\nu < \cdots < a_1$, $\nu \ge 1$, we define

$$\mu_\alpha(k) := a_1 + \cdots + a_{\min(\nu,\alpha)}\,. \tag{2.8}$$

Furthermore, $\mu_\alpha(0) := 0$ and for $\boldsymbol{k} \in \mathbb{N}_0^s$, $\boldsymbol{k} = (k_1, \ldots, k_s)$, $\mu_\alpha(\boldsymbol{k}) = \sum_{j=1}^s \mu_\alpha(k_j)$. The case $\alpha \ge 2$ will be relevant in Chapters 3 - 6, but for now we are interested in the case $\alpha = 1$, in which case we recover so-called weight functions, which were introduced in [63; 72; 96]. We now state [45, Lemma 1], which was first proven in [58, Lemma 2a].

Lemma 2.38. *Let $\mathcal{P} = \{\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{b^m-1}\}$ be a (t, m, s)-net in base b, $b \ge 2$ an arbitrary integer. Then*

$$Q_{b^m}(\mathrm{wal}_{\boldsymbol{k}}, \mathcal{P}) = 0 \quad \forall \boldsymbol{k} : 0 < \mu_1(\boldsymbol{k}) \le m - t.$$

The next lemma, which appeared as [45, Lemma 2], gives the converse.

Lemma 2.39. *Let $\mathcal{P} = \{\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{b^m-1}\}$ be a point set consisting of b^m points in $[0,1)^s$ and suppose that*

$$Q_{b^m}(\mathrm{wal}_{\boldsymbol{k}}, \mathcal{P}) = 0 \quad \forall \boldsymbol{k} : 0 < \mu_1(\boldsymbol{k}) \le m - t.$$

Then $\mathcal{P}$ is a (t, m, s)-net in base b.

Hence we have obtained the following characterization of (t, m, s)-nets in terms of Weyl sums, which first appeared as [45, Corollary 3].

Corollary 2.40. *Let $\mathcal{P} = \{\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{b^m-1}\}$ be a point set consisting of b^m points in $[0,1)^s$. Then $\mathcal{P}$ is a (t, m, s)-net in base b if and only if*

$$Q_{b^m}(\mathrm{wal}_{\boldsymbol{k}}, \mathcal{P}) = 0 \quad \forall \boldsymbol{k} : 0 < \mu_1(\boldsymbol{k}) \le m - t.$$

2.5 Two function spaces based on Walsh functions

In this section we define the function spaces studied in this thesis and motivate them. We remark that both function spaces are based on Walsh functions, which is due to the following observation: when studying integration errors resulting from the approximation of an integral based on a qMC rule employing a classical or higher order digital net, it is often useful to consider the Walsh series of the integrand f. In particular, for $f \in L_2([0,1]^s)$, the Walsh series of f is given by

$$f(\boldsymbol{x}) \approx \sum_{\boldsymbol{k} \in \mathbb{N}_0^s} \hat{f}(\boldsymbol{k}) \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) \,, \tag{2.9}$$

where the Walsh coefficients $\hat{f}(\boldsymbol{k})$ are defined by $\hat{f}(\boldsymbol{k}) = \int_{[0,1)^s} f(\boldsymbol{x}) \overline{\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x})} d\boldsymbol{x}$. In general, the Walsh series given in Equation (2.9) need not converge to f, however, for the space of Walsh series $W_{\alpha,s,\gamma}$ introduced in Subsection 2.5.1 this is the case; we point out that for the space of functions $V_{\alpha,s,\gamma}$ introduced in Subsection 2.5.2 this need not be the case. Both spaces, $W_{\alpha,s,\gamma}$ and $V_{\alpha,s,\gamma}$, are weighted spaces in the sense of [101], i.e $\boldsymbol{\gamma} = (\gamma_j)_{j=1}^\infty$ is a sequence of positive non-increasing weights, which are introduced to model the importance of different abscissas. We remark that in [46], the concept of weighted coordinates was also used. We recall that for $s \in \mathbb{N}$, $[s] = \{1, 2, \dots, s\}$ and for $\mathfrak{u} \subseteq [s]$, let $\gamma_{\mathfrak{u}} = \prod_{j \in \mathfrak{u}} \gamma_j$ be the weight associated with the projection onto those components whose indices are contained in $\mathfrak{u}$. We now introduce the function spaces.

2.5.1 Introducing the Walsh space $W_{\alpha,s,\gamma}$

The function space introduced in this subsection is the space $W_{\alpha,s,\gamma} \subseteq L_2([0,1]^s)$, which will be the setting of Chapters 5 and 6. We remark that for purposes of this subsection, α denotes the smoothness of functions in $W_{\alpha,s,\gamma}$, $\alpha \in \mathbb{N}$ and $\alpha \geq 2$. Recalling the definition of μ, see Equation (2.8), we define a function

$$r_\alpha(\gamma, k) := \begin{cases} 1 & \text{if } k = 0 \\ \gamma b^{-\mu_\alpha(k)} & \text{otherwise.} \end{cases}$$

If we consider a vector $\boldsymbol{k} \in \mathbb{N}_0^s$ of the form $\boldsymbol{k} = (k_1, \dots, k_s)$, we set

$$r_\alpha(\boldsymbol{\gamma}, \boldsymbol{k}) := \prod_{j=1}^s r_\alpha(\gamma_j, k_j) \,. \tag{2.10}$$

The space $W_{\alpha,s,\gamma}$ consists of all Walsh series $f = \sum_{\boldsymbol{k} \in \mathbb{N}_0^s} \hat{f}(\boldsymbol{k}) \mathrm{wal}_{\boldsymbol{k}}$ for which the norm

$$\|f\|_{W_{\alpha,s,\gamma}} := \sup_{\boldsymbol{k} \in \mathbb{N}_0^s} \frac{|\hat{f}(\boldsymbol{k})|}{r_\alpha(\boldsymbol{\gamma}, \boldsymbol{k})} \tag{2.11}$$

is finite. The reason for which this Walsh space is considered is the following: clearly, if $f \in W_{\alpha,s,\gamma}$, its Walsh coefficients exhibit the following decay property:

$$|\hat{f}(\boldsymbol{k})| \leq \|f\|_{W_{\alpha,s,\gamma}} r_\alpha(\boldsymbol{\gamma}, \boldsymbol{k}), \forall \boldsymbol{k} \in \mathbb{N}_0^s.$$

In [21] it was shown that qMC rules based on higher order digital nets are precisely the right tool to exploit this decay property, see the forthcoming Lemma 2.48. Of course it is interesting to check how the function space $W_{\alpha,s,\gamma}$ relates to function spaces more commonly encountered in numerical analysis. In this regard, the following results were presented in [21]: for $\alpha \geq 2$, let $f : [0,1]^s \to \mathbb{R}$ be such that all mixed partial derivatives up to order α in each variable are square-integrable, then $f \in W_{\alpha,s,\gamma}$. Furthermore, an inequality using a Sobolev type norm and the norm in Equation (2.11) was shown in [21, Corollary 3.11], see also [20; 22]. Consequently, the results we are going to establish in Subsection 2.7.1 for functions in $W_{\alpha,s,\gamma}$ also apply automatically to smooth functions. The assumption $\alpha > 1$ is needed to ensure that the sum of the absolute values of the Walsh coefficients converges. The case $\alpha = 1$ requires a different analysis, see [32].

2.5.2 Introducing the Walsh space $V_{\alpha,s,\gamma}$

The function space introduced in this subsection is the space $V_{\alpha,s,\gamma} \subseteq L_2([0,1]^s)$, which will be the setting of Chapter 7. We remark that as in Subsection 2.5.1, α is used to denote the smoothness of functions in $V_{\alpha,s,\gamma}$, however, in this subsection $0 < \alpha \leq 1$. For the space of Walsh series $V_{\alpha,s,\gamma}$, we do not necessarily have equality in Equation (2.9), however, the completeness of the Walsh function system $\{\mathrm{wal}_{\boldsymbol{k}} : \boldsymbol{k} \in \mathbb{N}_0^s\}$ (see for instance [35, Appendix A]) implies that

$$\mathrm{Var}(f) = \sum_{\boldsymbol{k} \in \mathbb{N}_0^s \setminus \{\boldsymbol{0}\}} |\hat{f}(\boldsymbol{k})|^2,$$

where $\mathrm{Var}(f) = \int_{[0,1]^s} (f(\boldsymbol{x}) - \overline{f})^2 d\boldsymbol{x}$, and where $\overline{f} = \int_{[0,1]^s} f(\boldsymbol{x}) d\boldsymbol{x}$.

Let $\sigma^2_{(\boldsymbol{l}_{\mathfrak{u}},\boldsymbol{0})} = \sum_{\boldsymbol{k} \in L_{(\boldsymbol{l}_{\mathfrak{u}},\boldsymbol{0})}} |\hat{f}(\boldsymbol{k})|^2$, where

$$L_{(\boldsymbol{l}_{\mathfrak{u}},\boldsymbol{0})} = \left\{\boldsymbol{k} \in \mathbb{N}_0^s : b^{l_i - 1} \leq k_i < b^{l_i} \text{ for } i \in \mathfrak{u} \text{ and } k_i = 0 \text{ for } i \in [s] \setminus \mathfrak{u}\right\}. \tag{2.12}$$

Further let $|\boldsymbol{l}|_1 = \sum_{j=1}^s l_j$ for $\boldsymbol{l} = (l_1, \ldots, l_s)$. For $0 < \alpha \leq 1$ we define a weighted norm for functions $f \in L_2([0,1]^s)$ by

$$\|f\|_\alpha = \max_{\mathfrak{u} \subseteq [s]} \gamma_{\mathfrak{u}}^{-1/2} \sup_{\boldsymbol{l}_{\mathfrak{u}} \in \mathbb{N}^{|\mathfrak{u}|}} b^{\alpha |\boldsymbol{l}_{\mathfrak{u}}|_1} \sigma_{(\boldsymbol{l}_{\mathfrak{u}},\boldsymbol{0})}(f). \tag{2.13}$$

For $0 < \alpha \leq 1$ define a space $V_{\alpha,s,\gamma} \subseteq L_2([0,1]^s)$ consisting of all functions f for which $\|f\|_\alpha < \infty$. (One could of course use some ℓ_p norm instead of the supremum-norm to define $\|\cdot\|_\alpha$ and the function space, however, these do not yield a quality criterion as given in Equation (7.5), which we use for our constructions in Sections 7.3 and 7.4, see Equation (7.9) and Lemma 7.3.)

The following observation stems from [35, Section 13.5]: for a subinterval $J = \prod_{j=1}^{s}[x_j, y_j)$ with $0 \le x_j < y_j \le 1$ and a function $f : [0,1)^s \to \mathbb{R}$, let the function $\Delta(f, J)$ denote the alternating sum of f at the vertices of J where adjacent vertices have opposite signs. (Hence for $f = \prod_{j=1}^{s} f_j$ we have $\Delta(f, J) = \prod_{j=1}^{s}(f_j(x_j) - f_j(y_j))$.)

We define the generalized variation in the sense of Vitali of order $0 < \alpha \le 1$ by

$$V_\alpha^{(s)}(f) = \sup_{\mathcal{P}} \left(\sum_{J \in \mathcal{P}} \mathrm{Vol}(J) \left| \frac{\Delta(f,J)}{\mathrm{Vol}(J)^\alpha} \right|^2 \right)^{1/2},$$

where the supremum is extended over all partitions $\mathcal{P}$ of $[0,1]^s$ into subintervals and $\mathrm{Vol}(J)$ denotes the volume of the subinterval J.

For $\alpha = 1$ and if the partial derivatives of f are continuous on $[0,1]^s$ we also have the formula

$$V_1^{(s)}(f) = \left(\int_{[0,1]^s} \left| \frac{\partial^s f}{\partial x_1 \cdots \partial x_s} \right|^2 \mathrm{d}\boldsymbol{x} \right)^{1/2}.$$

Until now we did not take projections to lower-dimensional faces into account.

For $\emptyset \ne \mathfrak{u} \subseteq [s]$, let $V_\alpha^{(|\mathfrak{u}|)}(f_\mathfrak{u}; \mathfrak{u})$ be the generalized Vitali variation with coefficient $0 < \alpha \le 1$ of the $|\mathfrak{u}|$-dimensional function

$$f_\mathfrak{u}(\boldsymbol{x}_\mathfrak{u}) = \int_{[0,1]^{s-|\mathfrak{u}|}} f(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}_{[s]\setminus\mathfrak{u}}.$$

For $\mathfrak{u} = \emptyset$ we have $f_\emptyset = \int_{[0,1]^s} f(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}$ and we define $V_\alpha^{(|\emptyset|)}(f_\emptyset; \emptyset) = |f_\emptyset|$. Then

$$V_\alpha(f) = \left(\sum_{\mathfrak{u} \subseteq [s]} \left(V_\alpha^{(|\mathfrak{u}|)}(f_\mathfrak{u}; \mathfrak{u}) \right)^2 \right)^{1/2} \tag{2.14}$$

is called the generalized Hardy and Krause variation of f on $[0,1]^s$.

A function f for which $V_\alpha(f) < \infty$ is said to be of finite variation of order α.

The following result is from [26] and [35, Section 13.5].

Corollary 2.41. *Let $b \ge 2$ be a natural number and let $f \in L_2([0,1]^s)$ have bounded variation $V_\alpha(f) < \infty$ of order $0 < \alpha \le 1$. Then*

$$\|f\|_\alpha \le \max \left(\|f\|_{L_2} \gamma_\emptyset^{-1}, V_\alpha(f) \max_{\emptyset \ne \mathfrak{u} \subseteq [s]} \gamma_\mathfrak{u}^{-1/2} (b-1)^{(\alpha - 1/2)_+ |\mathfrak{u}|} \right).$$

Hence every function $f \in L_2([0,1]^s)$ which has bounded variation of order $0 < \alpha \le 1$ is in $V_{\alpha,s,\gamma}$. For the extreme case $\alpha = 0$ one obtains $V_{0,s,\gamma} = L_2([0,1]^s)$, but this case is not included in our investigations: in Chapter 7, we construct classical polynomial lattice point sets resulting in qMC rules which are almost optimal for $V_{\alpha,s,\gamma}$, however, the criterion used to construct these classical polynomial lattice point sets, see Equation (7.5), is infinite for $\alpha = 0$, see Equation (7.9) and Lemma 7.3.

2.6 Randomizing classical nets

In this section we discuss the randomization of classical nets, see also Section 3.5, Section 5.3 and Chapter 7. Randomizations of classical nets are carried out for two reasons:

- it allows us to obtain statistical information on integration errors,
- using sophisticated methods of randomization can result in improved convergence rates of integration errors.

In particular, we will always check if the following two properties are satisfied:

- applying a randomization to a (t, m, s)-net, the resulting points are uniformly distributed in $[0, 1)^s$,
- applying a randomization to a (t, m, s)-net, the resulting point set is still a (t, m, s)-net.

We now discuss three randomization methods used in this thesis.

2.6.1 Digital shift

In this subsection we discuss a randomization technique referred to as digital shift, see e.g. [32; 35]. To fix notation, we let $\mathcal{P} = \{\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_{b^m-1}\}$ be a (t, m, s)-net in base b, $\boldsymbol{x}_h = (x_{h,1}, \dots, x_{h,s})$, for $0 \le h < b^m$, and we assume that the b-adic expansion of $x_{h,j}$ is given by

$$x_{h,j} = \frac{\xi_{h,j,1}}{b} + \cdots + \frac{\xi_{h,j,m}}{b^m} + \frac{\xi_{h,j,m+1}}{b^{m+1}} + \dots,$$

for $0 \le h < b^m$ and $1 \le j \le s$. Also, let $\boldsymbol{\Delta} = (\Delta_1, \dots, \Delta_s)$, where Δ_j, $1 \le j \le s$ are uniformly distributed in $[0, 1)$ and mutually independent. We also consider the b-adic expansion of each coordinate of $\boldsymbol{\Delta}$, i.e.

$$\Delta_j = \frac{\Delta_{j,1}}{b} + \frac{\Delta_{j,2}}{b^2} + \dots,$$

for $1 \le j \le s$. Regarding the digital shift in base b, the randomly digitally shifted point set $\mathcal{P}_{\boldsymbol{\Delta}} = \{\boldsymbol{z}_0, \boldsymbol{z}_1, \dots, \boldsymbol{z}_{b^m-1}\}$ is given by

$$\boldsymbol{z}_h = \boldsymbol{x}_h \oplus \boldsymbol{\Delta} = (z_{h,1}, \dots, z_{h,s}),$$

for $0 \le h < b^m$, where $\oplus$ is defined component-wise, where for $0 \le h < b^m$ and $1 \le j \le s$

$$z_{h,j} := \frac{\xi_{h,j,1} \oplus \Delta_{j,1}}{b} + \cdots + \frac{\xi_{h,j,m} \oplus \Delta_{j,m}}{b^m} + \frac{\xi_{h,j,m+1} \oplus \Delta_{j,m+1}}{b^{m+1}} + \ldots,$$

and where $\xi_{h,j,l} \oplus \Delta_{j,l} := \xi_{h,j,l} + \Delta_{j,l} \pmod b$ for $l \ge 1$. The following proposition summarizes important properties of $\mathcal{P}_{\boldsymbol{\Delta}}$.

Proposition 2.42. *Let $\mathcal{P}$ be a (t,m,s)-net in base b and $\mathcal{P}_{\boldsymbol{\Delta}}$ be defined as above. Then*

i) each point in $\mathcal{P}_{\boldsymbol{\Delta}}$ is uniformly distributed in $[0,1)^s$,

ii) $\mathcal{P}_{\boldsymbol{\Delta}}$ is a (t,m,s)-net in base b with probability 1.

Proof. For i), see [83, Proposition 3.1], for ii), see [32, Lemma 3]. □

Regarding the application of digital shifts to numerical integration, we remark that in [32], digital shifts were used to establish the existence of qMC rules based on (t,m,s)-nets achieving optimal convergence rates of worst-case integration errors in weighted Sobolev spaces; similar results, but for classical polynomial lattice point sets, were presented in [29].

2.6.2 Digital shift of depth m

In this subsection we discuss a randomization technique referred to as a digital shift of depth m, see e.g. [33]. Again, we let $\mathcal{P} = \{\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{b^m-1}\}$ be a (t,m,s)-net in base b, $\boldsymbol{x}_h = (x_{h,1}, \ldots, x_{h,s})$, for $0 \le h < b^m$, and we recall that the b-adic expansion of $x_{h,j}$ is given by

$$x_{h,j} = \frac{\xi_{h,j,1}}{b} + \cdots + \frac{\xi_{h,j,m}}{b^m} + \frac{\xi_{h,j,m+1}}{b^{m+1}} + \ldots,$$

for $0 \le h < b^m$ and $1 \le j \le s$. We choose digits $\Delta_{j,l}$, for $1 \le j \le s$ and $1 \le l \le m$, uniformly distributed on $\{0, \ldots, b-1\}$ and mutually independent and also choose $\delta_{h,j}$, for $0 \le h < b^m$ and $1 \le j \le s$, uniformly distributed on $[0, b^{-m})$ and mutually independent. Consequently, recalling the digit expansion of $x_{h,j}$, we define

$$z_{h,j,l} = \xi_{h,j,l} + \Delta_{j,l} \pmod b$$

for $0 \le h < b^m$, $1 \le j \le s$ and $1 \le l \le m$ and finally set

$$z_{h,j} = \frac{z_{h,j,1}}{b} + \cdots + \frac{z_{h,j,m}}{b^m} + \delta_{h,j},$$

to obtain the point set $\mathcal{P}_{\boldsymbol{\Delta},\delta} = \{\boldsymbol{z}_0, \boldsymbol{z}_1, \ldots, \boldsymbol{z}_{b^m-1}\}$. The following proposition gives important properties of the point set $\mathcal{P}_{\boldsymbol{\Delta},\delta}$.

Proposition 2.43. *Let $\mathcal{P}$ be a (t, m, s)-net in base b and $\mathcal{P}_{\boldsymbol{\Delta},\delta}$ be defined as above. Then*

i) each point in $\mathcal{P}_{\boldsymbol{\Delta},\delta}$ is uniformly distributed in $[0, 1)^s$,

ii) $\mathcal{P}_{\boldsymbol{\Delta},\delta}$ is a (t, m, s)-net in base b.

Proof. For i), see [83, Proposition 3.1], for ii), see [33]. □

Regarding the application of digital shifts of depth m to numerical integration, we remark that in [33], a mean square weighted L_2 discrepancy and also the classical discrepancy of (t, m, s)-nets subjected to a digital shift of depth m are studied and shown to achieve optimal convergence rates. For connections between discrepancies and numerical integration, see [35; 66; 101].

2.6.3 Owen's scrambling algorithm

In this subsection we discuss Owen's scrambling algorithm, which was introduced in [80]. We now describe the scrambling algorithm using a generic point $\boldsymbol{x} \in [0, 1)^s$, where, as in the preceding two subsections, $\boldsymbol{x} = (x_1, \ldots, x_s)$ and

$$x_j = \frac{\xi_{j,1}}{b} + \frac{\xi_{j,2}}{b^2} + \ldots .$$

Then the scrambled point shall be denoted by $\boldsymbol{y} \in [0, 1)^s$, where $\boldsymbol{y} = (y_1, \ldots, y_s)$,

$$y_j = \frac{\eta_{j,1}}{b} + \frac{\eta_{j,2}}{b^2} + \ldots .$$

The permutation applied to $\xi_{j,l}$, $j = 1, \ldots, s$, depends on $\xi_{j,k}$, for $1 \le k < l$. In particular, $\eta_{j,1} = \pi_j(\xi_{j,1})$, $\eta_{j,2} = \pi_{j,\xi_{j,1}}(\xi_{j,2})$, $\eta_{j,3} = \pi_{j,\xi_{j,1},\xi_{j,2}}(\xi_{j,3})$ and in general

$$\eta_{j,k} = \pi_{j,\xi_{j,1},\ldots,\xi_{j,k-1}}(\xi_{j,k}) \,, \; k \ge 2 \,,$$

where π_j and $\pi_{j,\xi_{j,1},\ldots,\xi_{j,k-1}}$, $k \ge 2$, are random permutations of $\{0, \ldots, b-1\}$. We assume that permutations with different indices are mutually independent. As in the preceding two subsections we use $\mathcal{P}$ to denote a (t, m, s)-net in base b and $\mathcal{P}_\pi$ to denote the point set resulting from the application of Owen's scrambling algorithm to the points in $\mathcal{P}$. The following proposition was first established in [80].

Proposition 2.44. *Let $\mathcal{P}$ be a (t, m, s)-net in base b and $\mathcal{P}_\pi$ be defined as above. Then*

i) each point in $\mathcal{P}_\pi$ is uniformly distributed in $[0, 1)^s$,

ii) $\mathcal{P}_\pi$ is a (t, m, s)-net in base b with probability 1.

Proof. For part i), see [80, Proposition 2], for part ii), see [80, Proposition 1]. □

We remark that the application of Owen's scrambling algorithm to numerical integration has been studied in many papers, see e.g. [35; 43; 47; 61; 81; 82; 106; 107; 108; 109] to name but a few contributions. Concluding this subsection we recall that in [81], it was shown that for functions whose mixed partial derivatives satisfy a Lipschitz condition, the variance of the corresponding qMC rule based on a scrambled (t, m, s)-net converges at a rate of $N^{-3}(\log N)^{s-1}$.

2.7 Numerical integration in Walsh function spaces

In this section we recall results from the literature on numerical integration in $W_{\alpha,s,\gamma}$ and $V_{\alpha,s,\gamma}$. In particular, we present upper bounds on integration errors for particular quasi-Monte Carlo algorithms and subsequently establish the optimality of these algorithms, up to powers of a $\log N$ factor.

2.7.1 Numerical integration in $W_{\alpha,s,\gamma}$ using higher order digital nets

In this subsection we aim to briefly summarize how qMC rules based on higher order digital nets can achieve convergence rates of order $\mathcal{O}(N^{-\alpha}(\log N)^{\alpha s})$ for functions in $W_{\alpha,s,\gamma}$ and subsequently establish the optimality of qMC rules based on higher order digital nets up to powers of a $\log N$ factor; we remark that the application of qMC rules based on higher order digital nets to numerical integration of functions in a Sobolev space containing functions having square-integrable mixed partial derivatives of order α was studied in [5]. We alert the reader to the fact that for purposes of this subsection, α is used to denote the smoothness of functions in $W_{\alpha,s,\gamma}$, $\alpha \in \mathbb{N}$ and $\alpha \geq 2$, see also Subsection 2.5.1. Finally, the material this subsection aims to summarize of course first appeared in [21], however, for a more expository paper, the reader is referred to [23].

As in Subsection 2.3.1, we assume that b is a prime number. The following lemma is useful when discussing numerical integration using qMC rules based on higher order digital nets and first appeared as [21, Lemma 4.2], see also [32, Lemma 2.5] and compare it with Lemma 2.38.

Lemma 2.45. *Let $\{\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{b^m-1}\}$ be a higher order digital net over $\mathbb{Z}_b$, generated by the $n \times m$ matrices $C_1, \ldots, C_s$ over $\mathbb{Z}_b$, where b is prime, $n, m \geq 1$. Then for any vector $\boldsymbol{k} \in \mathbb{N}_0^s$ we have*

$$\sum_{h=0}^{b^m-1} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_h) = \begin{cases} b^m & \text{if } \boldsymbol{k} \in \mathcal{D}, \\ 0 & \text{otherwise.} \end{cases}$$

The next lemma shows the central role played by the dual net introduced in Equation (2.6) in the context of numerical integration.

Lemma 2.46. *Let $f \in W_{\alpha,s,\gamma}$ and let $\mathcal{P} = \{\boldsymbol{x}_0, \boldsymbol{x}_1 \ldots, \boldsymbol{x}_{b^m-1}\}$ be a higher order digital net over $\mathbb{Z}_b$, where b is prime. Then we have*

$$|\int_{[0,1]^s} f(\boldsymbol{x}) d\boldsymbol{x} - \frac{1}{b^m} \sum_{h=0}^{b^m-1} f(\boldsymbol{x}_h)| \le \|f\|_{W_{\alpha,s,\gamma}} \sum_{\emptyset \neq \mathrm{u} \subseteq [s]} \gamma_{\mathrm{u}} \sum_{\boldsymbol{k}_{\mathrm{u}} \in \mathcal{D}^*_{\mathrm{u}}} r_\alpha(\mathbf{1}_{\mathrm{u}}, \boldsymbol{k}_{\mathrm{u}}) . \tag{2.15}$$

Proof. We proceed as follows:

$$\begin{aligned}
|\int_{[0,1]^s} f(\boldsymbol{x}) d\boldsymbol{x} - \frac{1}{b^m} \sum_{h=0}^{b^m-1} f(\boldsymbol{x}_h)| &= |\hat{f}(\mathbf{0}) - \frac{1}{b^m} \sum_{h=0}^{b^m-1} \sum_{\boldsymbol{k} \in \mathbb{N}_0^s} \hat{f}(\boldsymbol{k}) \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_h)| \\
&= |\hat{f}(\mathbf{0}) - \sum_{\boldsymbol{k} \in \mathbb{N}_0^s} \hat{f}(\boldsymbol{k}) \frac{1}{b^m} \sum_{h=0}^{b^m-1} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_h)| \\
&= |\sum_{\boldsymbol{k} \in \mathbb{N}_0^s \setminus \{\mathbf{0}\}} \hat{f}(\boldsymbol{k}) \frac{1}{b^m} \sum_{h=0}^{b^m-1} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_h)| \\
&= |\sum_{\boldsymbol{k} \in \mathcal{D}'} \hat{f}(\boldsymbol{k})| \le \sum_{\boldsymbol{k} \in \mathcal{D}'} |\hat{f}(\boldsymbol{k})| \\
&= \sum_{\emptyset \neq \mathrm{u} \subseteq [s]} \sum_{\boldsymbol{k}_{\mathrm{u}} \in \mathcal{D}^*_{\mathrm{u}}} |\hat{f}(\boldsymbol{k}_{\mathrm{u}}, \mathbf{0}_{[s] \setminus \mathrm{u}})| \\
&\le \|f\|_{W_{\alpha,s,\gamma}} \sum_{\emptyset \neq \mathrm{u} \subseteq [s]} \sum_{\boldsymbol{k}_{\mathrm{u}} \in \mathcal{D}^*_{\mathrm{u}}} r_\alpha(\gamma_{\mathrm{u}}, \boldsymbol{k}_{\mathrm{u}}) \\
&= \|f\|_{W_{\alpha,s,\gamma}} \sum_{\emptyset \neq \mathrm{u} \subseteq [s]} \gamma_{\mathrm{u}} \sum_{\boldsymbol{k}_{\mathrm{u}} \in \mathcal{D}^*_{\mathrm{u}}} r_\alpha(\mathbf{1}_{\mathrm{u}}, \boldsymbol{k}_{\mathrm{u}}) .
\end{aligned}$$

□

We point out that instead of studying the integration error associated with a given function $f \in W_{\alpha,s,\gamma}$, we might be interested in the worst-case integration error associated with the quasi-Monte Carlo rule $Q_N(\cdot, \mathcal{P})$, where

$$Q_N(f, \mathcal{P}) = \frac{1}{N} \sum_{h=0}^{N-1} f(\boldsymbol{x}_h) ,$$

for a given point set $\mathcal{P} = \{\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{N-1}\} \subseteq [0,1)^s$. The worst-case integration error for the Walsh space $W_{\alpha,s,\gamma}$ using the quasi-Monte Carlo rule $Q_N(\cdot, \mathcal{P})$ is given by

$$e(Q_N(\cdot, \mathcal{P}), W_{\alpha,s,\gamma}) = \sup_{\substack{f \in W_{\alpha,s,\gamma} \\ \|f\|_{W_{\alpha,s,\gamma}} \le 1}} |I(f) - Q_N(f, \mathcal{P})| ,$$

where $I(f) = \int_{[0,1]^s} f(\boldsymbol{x}) d\boldsymbol{x}$.

The initial error is given by

$$e(Q_0(\cdot, \emptyset), W_{\alpha,s,\gamma}) = \sup_{\substack{f \in W_{\alpha,s,\gamma} \\ \|f\|_{W_{\alpha,s,\gamma}} \le 1}} |I(f)|$$

and the following lemma presents the analogue of Lemma 2.46, but for the worst-case integration error.

Lemma 2.47. *Let $\mathcal{P}$ be a digital $(t, \alpha, \beta, n \times m, s)$-net over $\mathbb{Z}_b$, where b is prime and $Q_N(\cdot, \mathcal{P})$ the quasi-Monte Carlo rule based on $\mathcal{P}$. Then we have*

$$e(Q_{b^m}(\cdot, \mathcal{P}), W_{\alpha,s,\gamma}) = \sum_{\emptyset \neq \mathfrak{u} \subseteq [s]} \gamma_{\mathfrak{u}} \sum_{\boldsymbol{k}_{\mathfrak{u}} \in \mathcal{D}^*_{\mathfrak{u}}} r_\alpha(\mathbf{1}_{\mathfrak{u}}, \boldsymbol{k}_{\mathfrak{u}}) \,. \tag{2.16}$$

Proof. Choosing $\hat{f}(\boldsymbol{k}_{\mathfrak{u}}, \mathbf{0}_{[s]\setminus\mathfrak{u}}) = \gamma_{\mathfrak{u}} r_\alpha(\mathbf{1}_{\mathfrak{u}}, \boldsymbol{k}_{\mathfrak{u}})$, we can obtain equality in Equation (2.15) and the result follows. □

We now recall the following lemma, which first appeared as [21, Lemma 5.2], see also [5, Remark 23].

Lemma 2.48. *Let $\mathcal{P}$ be a digital $(t, \alpha, \beta, n \times m, s)$-net over $\mathbb{Z}_b$, where b is prime, and let $\emptyset \neq \mathfrak{u} \subseteq [s]$. Then*

$$\sum_{\boldsymbol{k}_{\mathfrak{u}} \in \mathcal{D}^*_{\mathfrak{u}}} r_\alpha(\boldsymbol{k}_{\mathfrak{u}}, \mathbf{1}_{[s]\setminus\mathfrak{u}}) \leq b^{-\alpha m + t} C_{\mathfrak{u},\alpha} (\alpha m - t + \alpha + 2)^{|\mathfrak{u}|\alpha} \,,$$

where $C_{\mathfrak{u},\alpha} = b^{|\mathfrak{u}|\alpha}(b^{-1} + (1 - b^{1/\alpha - 1})^{-|\mathfrak{u}|\alpha})$.

We remark that Lemma 2.48 can be improved on, see [8, Lemma 9], however, for purposes of this thesis, this improvement is not needed. Combining Lemmas 2.46, 2.47 and 2.48, we can state the following result, see [21, Theorem 5.4].

Theorem 2.49. *Let $f \in W_{\alpha,s,\gamma}$ and $\mathcal{P} = \{\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{b^m-1}\}$ be a digital $(t, \alpha, \beta, n \times m, s)$-net over $\mathbb{Z}_b$, where b is prime. Then*

$$\left| \int_{[0,1]^s} f(\boldsymbol{x}) d\boldsymbol{x} - \frac{1}{b^m} \sum_{h=0}^{b^m-1} f(\boldsymbol{x}_h) \right| \leq \|f\|_{W_{\alpha,s,\gamma}} b^{-\alpha m + t} \sum_{\emptyset \neq \mathfrak{u} \subseteq [s]} \gamma_{\mathfrak{u}} C_{\mathfrak{u},\alpha} (\alpha m - t + \alpha + 2)^{|\mathfrak{u}|\alpha} \,,$$

where $C_{\mathfrak{u},\alpha}$ is defined as in Lemma 2.48. Also, if $Q_{b^m}(\cdot, \mathcal{P})$ is the quasi-Monte Carlo rule based on $\mathcal{P}$, then

$$e(Q_{b^m}(\cdot, \mathcal{P}), W_{\alpha,s,\gamma}) \leq b^{-\alpha m + t} \sum_{\emptyset \neq \mathfrak{u} \subseteq [s]} \gamma_{\mathfrak{u}} C_{\mathfrak{u},\alpha} (\alpha m - t + \alpha + 2)^{|\mathfrak{u}|\alpha} \,.$$

We conclude this subsection by recalling that the bounds on the integration error presented in Theorem 2.49 are optimal up to powers of a $\log N$ factor. This result follows for example from [95], where it was shown that the bounds are optimal up to powers of a $\log N$ factor for the space of smooth periodic functions, which is contained in $W_{\alpha,s,\gamma}$.

2.7.2 Scrambled net variance for integrals of functions in $V_{\alpha,s,\gamma}$

In this subsection, we aim to briefly summarize how qMC rules based on scrambled classical digital nets, see e.g. [80; 81; 82], can achieve convergence rates of order $N^{-(1+2\alpha)+\epsilon}$, for all $\epsilon > 0$, for functions in $V_{\alpha,s,\gamma}$ and subsequently establish the optimality of qMC rules based on scrambled classical digital nets up to powers of a $\log N$ factor. We remark that as in Subsection 2.5.2, α is used to denote the smoothness of functions in $V_{\alpha,s,\gamma}$ and satisfies $0 < \alpha \leq 1$. The application of qMC rules based on scrambled classical digital nets to numerical integration in $V_{\alpha,s,\gamma}$ was essentially first studied in [35, Section 13.5], the optimality of scrambled classical digital nets was first established in [6].

In this subsection we discuss the variance of the estimator

$$Q_{b^m}(f, \mathcal{P}_\pi) = \frac{1}{b^m} \sum_{h=0}^{b^m-1} f(\boldsymbol{y}_h) , \tag{2.17}$$

where the point set $\mathcal{P}_\pi = \{\boldsymbol{y}_0, \ldots, \boldsymbol{y}_{b^m-1}\}$ is obtained by applying the scrambling algorithm to $\mathcal{P}$, which is a digital (t, m, s)-net over $\mathbb{Z}_b$, see Subsection 2.6.3. As in Subsection 2.2.2, we assume that b is a prime number. The following result is taken from [35], where it appeared as Corollary 13.7, where we use the following notation: for a non-negative integer k with b-adic expansion

$$k = \kappa_0 + \kappa_1 b + \ldots ,$$

we write $\vec{k} = (\kappa_0, \kappa_1, \ldots)^\top$, which is an infinite-dimensional vector, and we use

$$tr_m(\vec{k}) = (\kappa_0, \kappa_1, \ldots, \kappa_{m-1})^\top .$$

Lemma 2.50. *Let $f \in L_2([0,1]^s)$, $Q_{b^m}(f, \mathcal{P}_\pi)$ be given by Equation* (2.17) *and $\mathcal{P}$ be a digital (t, m, s)-net over $\mathbb{Z}_b$ with generating matrices $C_1, \ldots, C_s$ over $\mathbb{Z}_b$. Then*

$$\mathrm{Var}(Q_{b^m}(f, \mathcal{P}_\pi)) = \sum_{\emptyset \neq \mathfrak{u} \subseteq [s]} \frac{b^{|\mathfrak{u}|}}{(b-1)^{|\mathfrak{u}|}} \sum_{\boldsymbol{l}_\mathfrak{u} \in \mathbb{N}^{|\mathfrak{u}|}} \frac{\sigma^2_{(\boldsymbol{l}_\mathfrak{u}, \boldsymbol{0})}(f)}{b^{|\boldsymbol{l}_\mathfrak{u}|_1}} |L_{(\boldsymbol{l}_\mathfrak{u}, \boldsymbol{0})} \cap \mathcal{D}(C_1, \ldots, C_s)| ,$$

where $\mathcal{D}(C_1, \ldots, C_s)$ is given by Definition 2.16 and $L_{(\boldsymbol{l}_\mathfrak{u}, \boldsymbol{0})}$ is given by Equation (2.12).

The next lemma, which appeared as Lemma 13.8 in [35], will allow us to derive a result on the variance of $Q_{b^m}(f, \mathcal{P}_\pi)$ given by Equation (2.17), for $f \in L_2([0,1]^s)$.

Lemma 2.51. *Let $\mathcal{D}(C_1, \ldots, C_s)$ be the dual net of a digital (t, m, s)-net over $\mathbb{Z}_b$ with generating matrices $C_1, \ldots, C_s$. Then*

$$|L_{(\boldsymbol{l}_\mathfrak{u}, \boldsymbol{0})} \cap \mathcal{D}(C_1, \ldots, C_s)| \leq \begin{cases} 0 & \text{if } |\boldsymbol{l}_\mathfrak{u}|_1 \leq m - t , \\ (b-1)^{|\mathfrak{u}|} & \text{if } m - t < |\boldsymbol{l}_\mathfrak{u}|_1 \leq m - t + |\mathfrak{u}| , \\ (b-1)^{|\mathfrak{u}|} b^{|\boldsymbol{l}_\mathfrak{u}|_1 - (m-t+|\mathfrak{u}|)} & \text{if } |\boldsymbol{l}_\mathfrak{u}|_1 > m - t + |\mathfrak{u}| . \end{cases}$$

Combining Lemmas 2.50 and 2.51, we obtain the following theorem dealing with qMC rules based on scrambled classical digital nets and L_2 functions, see [35, Theorem 13.9].

Theorem 2.52. *Let $f \in L_2([0,1]^s)$, $Q_{b^m}(f, \mathcal{P}_\pi)$ be given by Equation* (2.17) *and $\mathcal{P}$ be a digital (t,m,s)-net over $\mathbb{Z}_b$. Then*

$$\mathrm{Var}(Q_{b^m}(f, \mathcal{P}_\pi)) \le b^{-(m-t)} \sum_{\emptyset \ne \mathfrak{u} \subseteq [s]} \sum_{|\boldsymbol{l}_\mathfrak{u}|_1 > m-t} b^{|\mathfrak{u}|-1} \sigma^2_{(\boldsymbol{l}_\mathfrak{u}, \boldsymbol{0})}(f)\,.$$

Theorem 2.52 shows that asymptotically qMC rules based on scrambled digital nets outperform Monte Carlo algorithms in the following sense, which was first shown in [81]:

$$b^m \mathrm{Var}(Q_{b^m}(f, \mathcal{P}_\pi)) = b^t \sum_{\emptyset \ne \mathfrak{u} \subseteq [s]} \sum_{|\boldsymbol{l}_\mathfrak{u}|_1 > m-t} b^{|\mathfrak{u}|-1} \sigma^2_{(\boldsymbol{l}_\mathfrak{u}, \boldsymbol{0})}(f) \to 0 \text{ as } m \to \infty\,,$$

whereas for Monte Carlo algorithms, we have

$$N \mathrm{Var}\left(\frac{1}{N} \sum_{h=0}^{N-1} f(\boldsymbol{u}_h)\right) = \sum_{\boldsymbol{l} \in \mathbb{N}_0^s \setminus \{\boldsymbol{0}\}} \sigma^2_{\boldsymbol{l}}(f)\,,$$

for all $N \in \mathbb{N}$, where the $\{\boldsymbol{u}_h\}_{h=0}^{N-1}$ are uniformly distributed in $[0,1]^s$ and independent. We now focus on the worst-case variance of multivariate integration in $V_{\alpha,s,\gamma}$ using a scrambled quasi-Monte Carlo point set $\mathcal{P}$:

$$\mathrm{Var}(Q_{b^m}(\cdot, \mathcal{P}_\pi), V_{\alpha,s,\gamma}) = \sup_{\substack{f \in V_{\alpha,s,\gamma} \\ \|f\|_\alpha \le 1}} \mathrm{Var}(Q_{b^m}(f, \mathcal{P}_\pi))\,,$$

where $Q_{b^m}(\cdot, \mathcal{P}_\pi)$ denotes the quasi-Monte Carlo rule based on the point set obtained by applying the scrambling algorithm to $\mathcal{P}$. We denote a quasi-Monte Carlo point set which is a digital (t,m,s)-net generated by $\boldsymbol{C} = (C_1, \ldots, C_s)$ by $\mathcal{P}(\boldsymbol{C})$ and the associated worst-case variance by $\mathrm{Var}(Q_{b^m}(\cdot, \mathcal{P}_\pi(\boldsymbol{C})), V_{\alpha,s,\gamma})$. Using the definition of the function space $V_{\alpha,s,\gamma}$, see Equation (2.13), and Theorem 2.52 we obtain the following result, see also [35, Theorem 13.24].

Theorem 2.53. *Let $\mathcal{P}(\boldsymbol{C})$ be a digital (t,m,s)-net over $\mathbb{Z}_b$ and $\mathrm{Var}(Q_{b^m}(\cdot, \mathcal{P}_\pi(\boldsymbol{C})), V_{\alpha,s,\gamma})$ be defined as above. Then we have*

$$\mathrm{Var}(Q_{b^m}(\cdot, \mathcal{P}_\pi(\boldsymbol{C})), V_{\alpha,s,\gamma}) \le b^{-(m-t)(1+2\alpha)-(1+2\alpha)} \sum_{\emptyset \ne \mathfrak{u} \subseteq [s]} \gamma_\mathfrak{u} \left(\frac{b^{2\alpha+1}}{b^{2\alpha}-1}\right)^{|\mathfrak{u}|} (m-t+2)^{|\mathfrak{u}|-1}\,.$$

We now establish that the convergence rate presented in Theorem 2.53 is optimal up to powers of a $\log N$ factor in the following sense: we consider a large class of randomized algorithms, including adaptive ones, and establish that the variance of any algorithm from this class cannot decrease at a rate faster than $N^{-(1+2\alpha)}$ for the function space $V_{\alpha,s,\gamma}$.

As we rely on [77, Section 2.2.4, Proposition 1] to establish the result, the class of algorithms to which our result applies is the same as the class considered in [77, Section 2.2.4, Proposition 1]. We now recall the definition of this class: for $I(f) = \int_{[0,1]^s} f(\boldsymbol{x}) d\boldsymbol{x}$, where $f \in V_{\alpha,s\gamma}$, we consider approximating $I : V_{\alpha,s,\gamma} \to \mathbb{R}$ using a mapping $\tilde{S} : V_{\alpha,s,\gamma} \to \mathbb{R}$; we remark that $\tilde{S}$ is not restricted to the class of quasi-Monte Carlo rules. As in [77, Section 1.1], we assume that in general, the function $f \in V_{\alpha,s,\gamma}$ is not known, but we have some information on f available, which is denoted by M, where $M : V_{\alpha,s\gamma} \to \mathbb{R}^N$, see Equation (2.18). An approximation $\tilde{S} : V_{\alpha,s,\gamma} \to \mathbb{R}$ only uses the information M if it can be written as $\tilde{S} = \phi \circ M$, where $\phi : \mathbb{R}^N \to \mathbb{R}$ is an arbitrary mapping referred to as an (idealized) algorithm in [77]. In particular, we allow our approximation nodes to be chosen adaptively and define the following information operator:

$$I_N^{ad} = \Big\{ M : V_{\alpha,s,\gamma} \to \mathbb{R}^N | M(f) = (f(\boldsymbol{a}_1), f(\boldsymbol{a}_2(f(\boldsymbol{a}_1))), \dots, f(\boldsymbol{a}_N(f(\boldsymbol{a}_1), \dots, f(\boldsymbol{a}_{N-1})))) , \\ \text{where } \boldsymbol{a}_1 \in [0,1]^s \text{ and } \boldsymbol{a}_i : \mathbb{R}^{i-1} \to [0,1]^s \text{ for } i = 2, \dots, s \Big\} \tag{2.18}$$

and we can now introduce the class of all approximations considered in this subsection:

$$A_N^{ad} = \left\{ \tilde{S} : V_{\alpha,s,\gamma} \to \mathbb{R} | \tilde{S} = \phi \circ M \text{ with } \phi : \mathbb{R}^N \to \mathbb{R} \text{ and } M \in I_N^{ad} \right\} .$$

We remark that non-adaptive algorithms are also included in A_N^{ad}, consider $\tilde{S} = \phi \circ \overline{M}$, where $\overline{M}(f) = (f(\boldsymbol{a}_1), \dots f(\boldsymbol{a}_n))$. Now, following [77, Section 2.1], we can define the randomized algorithms considered in this subsection, referred to as generalized Monte Carlo methods in [77]: a random variable $Q = (Q(\omega))_{\omega \in \Omega}$ is called a randomized algorithm in A_N^{ad} if (Ω, B, μ) is a probability space and $Q(\omega) \in A_N^{ad}$ for all $\omega \in \Omega$. The set of all randomized algorithms is denoted by ${}^*C(A_N^{ad})$, hence qMC rules based on scrambled classical digital nets and scrambled classical polynomial lattice point sets are also included in this set. We now present the lower bound on the worst-case variance, which applies to all randomized algorithms in ${}^*C(A_N^{ad})$.

Theorem 2.54. *Let ${}^*C(A_N^{ad})$ and $V_{\alpha,s,\gamma}$ be defined as above. Then*

$$\inf_{Q \in {}^*C(A_N^{ad})} \sup_{\substack{f \in V_{\alpha,s,\gamma} \\ \|f\|_\alpha \le 1}} \operatorname{Var}(Q(f)) \ge \tilde{C} N^{-\alpha - \frac{1}{2}} ,$$

for some constant $\tilde{C}$ independent of N, where

$$\operatorname{Var}(Q(f)) = \int_\Omega \left[Q(\omega)(f) - \int_\Omega Q(\omega')(f) d\mu(\omega') \right]^2 d\mu(\omega) .$$

Proof. We remark that this proof follows along the lines of the proof of [43, Theorem 10]. We only consider $s = 1$, since integration in $V_{\alpha,1,\gamma_1}$ is no harder than integration in $V_{\alpha,s,\gamma}$ with $s > 1$, as the one-dimensional space $V_{\alpha,1,\gamma_1}$ can be identified with the subspace of $V_{\alpha,s,\gamma}$ consisting of functions depending only on the first variable. We let N be any given integer and choose an integer m such that

$$b^{m-1} < 2N \leq b^m .$$

We define basic intervals

$$B_{m,a} = \left[\frac{a}{b^m}, \frac{a+1}{b^m}\right) , \; a = 0, 1, \ldots, b^m - 1 ,$$

and let $g_a(x) = \mathbf{1}_{B_{m,a}}(x)$ be the characteristic function of $B_{m,a}$. Then

$$\int_{[0,1]} g_a(x) g_c(x) dx = \begin{cases} b^{-m} & \text{if } a = c, \\ 0 & \text{otherwise.} \end{cases}$$

We now define

$$g = \sum_{a=0}^{b^m-1} \xi_a g_a ,$$

where $\xi_a \in \{1, -1\}$ and bound $\sigma_l^2(g)$. Using Plancherel's identity we obtain that for any $l \geq 0$ we have

$$\sigma_l^2(g) \leq \sum_{l'=0}^{\infty} \sigma_{l'}^2(g) = \int_0^1 g^2(x) dx = \sum_{a,c=0}^{b^m-1} \xi_a \xi_c \int_0^1 g_a(x) g_c(x) dx = \frac{1}{b^m} \sum_{a=0}^{b^m-1} \xi_a^2 = 1.$$

Further, for $k \geq b^m$ we have

$$\widehat{g}(k) = \int_0^1 g(x) \overline{\mathrm{wal}_k(x)} dx = \sum_{a=0}^{b^m-1} \xi_a \int_0^1 g_a(x) \overline{\mathrm{wal}_k(x)} dx = \sum_{a=0}^{b^m-1} \xi_a \int_{a/b^m}^{(a+1)/b^m} \overline{\mathrm{wal}_k(x)} dx = 0,$$

since $\int_{a/b^m}^{(a+1)/b^m} \overline{\mathrm{wal}_k(x)} dx = 0$ for $k \geq b^m$ and hence for $l > m$ we have

$$\sigma_l^2(g) = \sum_{k=b^{l-1}}^{b^l-1} |\widehat{g}(k)|^2 = 0.$$

We set $f_a = \gamma_1^{1/2} b^{-\alpha m} g_a$ for $a = 0, 1, \ldots, b^m - 1$. These f_a have disjoint support and

$$\int_{[0,1]} f_a(x) dx = \gamma_1^{1/2} b^{-(\alpha+1)m}.$$

Set

$$f = \gamma_1^{1/2} b^{-\alpha m} g = \sum_{a=0}^{b^m-1} \xi_a f_a,$$

then we get $\sigma_l^2(f) \leq \gamma_1^1 b^{-2\alpha m}$ for $0 \leq l \leq m$ and $\sigma_l^2(f) = 0$ for $l > m$. Hence

$$\|f\|_\alpha = \gamma_1^{-1/2} \sup_{l \in \mathbb{N}} b^{\alpha l} \sigma_l(f) \leq \gamma_1^{-1/2} \sup_{1 \leq l \leq m} b^{\alpha l} \gamma_1^{1/2} b^{-\alpha m} \leq 1$$

and the result now follows from [77, Section 2.2.4, Proposition 1(ii)]. □

Remark 2.55. *For a large class of randomized algorithms, including adaptive ones, we have shown that the worst-case variance in the Walsh function space $V_{\alpha,s,\gamma}$ behaves like $N^{-(1+2\alpha)}$. Theorem 2.53 establishes that qMC rules based on scrambled digital (t, m, s)-nets can achieve worst-case variances of order $N^{-(1+2\alpha)+\epsilon}$, for all $\epsilon > 0$, and are hence optimal, up to powers of a $\log N$ factor, for the class of algorithms ${}^*C(A_N^{ad})$. Furthermore, in Sections 7.3 and 7.4, we present two algorithms which achieve worst-case variances of order $N^{-(1+2\alpha)+\epsilon}$, for all $\epsilon > 0$, and are hence optimal, up to powers of a $\log N$ factor, for the class of algorithms ${}^*C(A_N^{ad})$.*

Chapter Three

Introducing higher order nets and higher order sequences

Before introducing higher order nets and higher order sequences we firstly motivate them.

3.1 Motivation

Higher order nets and higher order sequences are best motivated using the following tables: for nets we have Table 3.1.

	digital	non-digital
classical	digital (t, m, s)-nets	(t, m, s)-nets
generalized	digital $(t, \alpha, \beta, n \times m, s)$-nets	???

Table 3.1. *Motivating higher order nets*

Similarly, for sequences we have Table 3.2; we remark that "non-digital" is meant to emphasize that nets and sequences are not necessarily constructed using the classical or generalized digital construction scheme, see Definitions 2.10 and 2.21.

	digital	non-digital
classical	digital (t, s)-sequences	(t, s)-sequences
generalized	digital $(t, \alpha, \beta, \sigma, s)$-sequences	???

Table 3.2. *Motivating higher order sequences*

Regarding Tables 3.1 and 3.2 we note that (t, m, s)-nets ((t, s)-sequences) generalize digital (t, m, s)-nets (digital (t, s)-sequences) and that digital $(t, \alpha, \beta, n \times m, s)$-nets (digital $(t, \alpha, \beta, \sigma, s)$-sequences) generalize digital (t, m, s)-nets (digital (t, s)-sequences). Consequently, the natural question which arises is whether it is possible to fill the cells containing the question marks, i.e. can we find nets (sequences) which generalize both, digital $(t,\alpha,\beta,n\times m,s)$-nets (digital $(t,\alpha,\beta,\sigma,s)$-sequences) and also (t,m,s)-nets ((t,s)-sequences). Furthermore, we would like the point sets to be non-digital, i.e. defined in terms of a concept analogous to that of an elementary interval, see Definition 2.2. We remark that finding such point sets might help to assess the validity of the conjecture that non-digital nets and non-digital sequences of better quality than their digital counterparts may exist, [70].

In Section 3.2 we introduce higher order nets and higher order sequences. Higher order nets and higher order sequences, as will become apparent in Section 3.2, are non-digital point sets, i.e. not necessarily constructed using the classical or generalized digital construction scheme. The structure of this section closely resembles Subsection 2.2.1, so that we can very clearly see how higher order nets and higher order sequences generalize their classical counterparts. In Section 3.3 we state some properties of higher order nets and higher order sequences and also show that higher order digital nets and higher order digital sequences can be regarded as special cases of higher order nets and higher order sequences; finally, we comment on the rate at which the quality parameter of the higher order sequences deteriorates. In Section 3.4 we generalize the characterization from Subsection 2.4.2 to higher order nets and apply it in Section 3.5 to study randomizations of higher order nets.

3.2 Defining higher order nets and higher order sequences

This section closely follows Subsection 2.2.1 illustrating that higher order nets and higher order sequences generalize their classical counterparts.

We recall that the definition of (t,m,s)-nets is based on the concept of an elementary interval, see Definition 2.5. In the following we introduce a concept analogous to that of an elementary interval namely that of a generalized elementary interval. Before we do so, we need some notation: let $\boldsymbol{\nu}=(\nu_1,\ldots,\nu_s)$, $\nu_j\in\mathbb{N}_0$, $j=1,\ldots,s$, let $|\boldsymbol{\nu}|_1=\sum_{j=1}^s\nu_j$, $\boldsymbol{i}_{\boldsymbol{\nu}}=(i_{1,1},\ldots,i_{1,\nu_1},\ldots,i_{s,1},\ldots,i_{s,\nu_s})$, $\boldsymbol{a}_{\boldsymbol{\nu}}\in\{0,\ldots,b-1\}^{|\boldsymbol{\nu}|_1}$, and $\boldsymbol{a}_{\boldsymbol{\nu}}=(a_{1,i_{1,1}},\ldots,a_{1,i_{1,\nu_1}},\ldots,a_{s,i_{s,1}},\ldots,a_{s,i_{s,\nu_s}})$, where the components $i_{j,l}$ and $a_{j,l}$, $l=1,\ldots,\nu_j$, do not appear in the vectors $\boldsymbol{i}_{\boldsymbol{\nu}}$ and $\boldsymbol{a}_{\boldsymbol{\nu}}$ in case $\nu_j=0$. We are now in a position to define a generalized elementary interval.

Definition 3.1 (c.f Definition 2.2). *Let $\boldsymbol{\nu} \in \{0, \ldots, n\}^s$, $\boldsymbol{i}_{\boldsymbol{\nu}}$ and $\boldsymbol{a}_{\boldsymbol{\nu}}$ be defined as above. Then a subset $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ of $[0,1)^s$ of the form*

$$\begin{aligned} & J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}}) \\ = & \prod_{j=1}^{s} \bigcup_{\substack{a_{j,l}=0 \\ l \in \{1,\ldots,n\} \setminus \left\{i_{j,1},\ldots,i_{j,\nu_j}\right\}}}^{b-1} \left[\frac{a_{j,1}}{b} + \cdots + \frac{a_{j,n}}{b^n}, \frac{a_{j,1}}{b} + \cdots + \frac{a_{j,n}}{b^n} + \frac{1}{b^n}\right), \end{aligned} \tag{3.1}$$

where $b \geq 2$ is an integer and where for $j = 1, \ldots, s$ we have $1 \leq i_{j,\nu_j} < \cdots < i_{j,1} \leq n$ in case $\nu_j > 0$ and $\left\{i_{j,1}, \ldots, i_{j,\nu_j}\right\} = \emptyset$ in case $\nu_j = 0$, is called a generalized elementary interval in base b.

We note that a generalized elementary interval is not always an elementary interval, but can be a union of several elementary intervals, see for example Figure 3.1. However, we remark that setting $i_{j,k} = \nu_j - k + 1$, $k = 1, \ldots, \nu_j$, in Definition 3.1, we recover the definition of an elementary interval making them special cases of generalized elementary intervals.

Generalized elementary intervals posses properties similar to those of classical elementary intervals, as we show in the following lemmas.

Lemma 3.2 (c.f. Lemma 2.3). *Let $\boldsymbol{\nu} \in \{0, \ldots, n\}^s$ and $\boldsymbol{i}_{\boldsymbol{\nu}}$ be defined as above and fixed. Then the generalized elementary intervals $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ for $\boldsymbol{a}_{\boldsymbol{\nu}} \in \{0, \ldots, b-1\}^{|\boldsymbol{\nu}|_1}$, form a partition of $[0,1)^s$, i.e. $\bigcup_{\boldsymbol{a}_{\boldsymbol{\nu}} \in \{0,\ldots,b-1\}^{|\boldsymbol{\nu}|_1}} J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}}) = [0,1)^s$ and $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}}) \cap J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}'_{\boldsymbol{\nu}}) = \emptyset$, for all $\boldsymbol{a}_{\boldsymbol{\nu}} \neq \boldsymbol{a}'_{\boldsymbol{\nu}} \in \{0, \ldots, b-1\}^{|\boldsymbol{\nu}|_1}$.*

Proof. First we have

$$\begin{aligned} & \bigcup_{\boldsymbol{a}_{\boldsymbol{\nu}} \in \{0,\ldots,b-1\}^{|\boldsymbol{\nu}|_1}} J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}}) \\ = & \prod_{j=1}^{s} \bigcup_{\substack{a_{j,l}=0 \\ l \in \{1,\ldots,n\}}}^{b-1} \left[\frac{a_{j,1}}{b} + \cdots + \frac{a_{j,n}}{b^n}, \frac{a_{j,1}}{b} + \cdots + \frac{a_{j,n}}{b^n} + \frac{1}{b^n}\right) \\ = & [0,1)^s. \end{aligned}$$

To show the second part we note that, for $\boldsymbol{i}_{\boldsymbol{\nu}}$ fixed and $\boldsymbol{a}_{\boldsymbol{\nu}} \neq \boldsymbol{a}'_{\boldsymbol{\nu}}$, there exists a $j \in \{1, \ldots, s\}$ and a $k \in \left\{i_{j,1}, \ldots, i_{j,\nu_j}\right\}$, such that $a_{j,k} \neq a'_{j,k}$. Let $\boldsymbol{x} = (x_1, \ldots, x_s)$ where each coordinate x_j, $j = 1, \ldots, s$, has base b expansion $x_j = \xi_{j,1} b^{-1} + \xi_{j,2} b^{-2} + \ldots$ (we assume that for each $j \in \{1, \ldots, s\}$ infinitely many $\xi_{j,k} \neq b - 1$). Then $\boldsymbol{x} \in J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ if and only if for all $j = 1, \ldots, s$ and all $k \in \{i_{j,1}, \ldots, i_{j,\nu_j}\}$ we have $\xi_{j,k} = a_{j,k}$. But as there exists a j and k such that $a_{j,k} \neq a'_{j,k}$, $\boldsymbol{x}$ cannot be in $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ and $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}'_{\boldsymbol{\nu}})$ simultaneously. Hence $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}}) \cap J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}'_{\boldsymbol{\nu}}) = \emptyset$ and the result follows. □

In the following lemma we compute the volume of a generalized elementary interval.

Lemma 3.3 (c.f. Lemma 2.4). *Let $\boldsymbol{\nu}$, $\boldsymbol{i}_\nu$ and $\boldsymbol{a}_\nu$ be as above. Then the volume of $J(\boldsymbol{i}_\nu, \boldsymbol{a}_\nu)$ is $b^{-|\nu|_1}$.*

Proof. Let $\boldsymbol{\nu}$ and $\boldsymbol{i}_\nu$ be fixed. Then we have seen in Lemma 3.2 that the $J(\boldsymbol{i}_\nu, \boldsymbol{a}_\nu)$, $\boldsymbol{a}_\nu \in \{0, \dots, b-1\}^{|\nu|_1}$, form a partition of $[0,1)^s$. From the definition of a generalized elementary interval one can see that $\lambda_s(J(\boldsymbol{i}_\nu, \boldsymbol{a}_\nu)) = \lambda_s(J(\boldsymbol{i}_\nu, \boldsymbol{a}'_\nu))$ for all $\boldsymbol{a}_\nu, \boldsymbol{a}'_\nu \in \{0, \dots, b-1\}^{|\nu|_1}$, as the intervals $J(\boldsymbol{i}_\nu, \boldsymbol{a}_\nu)$ and $J(\boldsymbol{i}_\nu, \boldsymbol{a}'_\nu)$ are only shifted versions of each other. Hence

$$\lambda_s(J(\boldsymbol{i}_\nu, \boldsymbol{a}_\nu)) = \frac{1}{|\{\boldsymbol{a}_\nu \in \{0, \dots, b-1\}^{|\nu|_1}\}|} = \frac{1}{b^{|\nu|_1}}.$$

□

We are now in a position to present the definition of higher order nets, i.e. $(t, \alpha, \beta, n, m, s)$-nets, which is based on the concept of a generalized elementary interval and Lemma 3.3.

Definition 3.4. *Let $n, m, \alpha \geq 1$ be natural numbers, let $0 < \beta \leq 1$ be a real number, and let $0 \leq t \leq \beta n$ be an integer. Let $b \geq 2$ be an integer and $\mathcal{P} = \{\boldsymbol{x}_0, \dots, \boldsymbol{x}_{b^m-1}\} \subseteq [0,1)^s$ be a point set in the s-dimensional unit cube, $s \geq 1$. We say that $\mathcal{P}$ is a $(t, \alpha, \beta, n, m, s)$-net (in base b), if for all integers $1 \leq i_{j,\nu_j} < \dots < i_{j,1}$, where $\nu_j \geq 0$, with*

$$\sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j, \alpha)} i_{j,l} \leq \beta n - t,$$

where for $\nu_j = 0$ we set the empty sum $\sum_{l=1}^{0} i_{j,l} = 0$, the generalized elementary interval $J(\boldsymbol{i}_\nu, \boldsymbol{a}_\nu)$ contains exactly $b^{m-|\nu|_1}$ points of $\mathcal{P}$ for each $\boldsymbol{a}_\nu \in \{0, \dots, b-1\}^{|\nu|_1}$.

For purposes of this chapter, the parameter α is used as a net parameter and satisfies $\alpha \in \mathbb{N}$, $\alpha \geq 1$. We point out that for digital $(t, \alpha, \beta, n \times m, s)$-nets, the notation "$n \times m$" clearly emphasizes that these point sets are obtained via the generalized digital construction scheme, see Definition 2.21. On the other hand, we want to emphasize that higher order nets are non-digital, hence we choose to replace the "$n \times m$" by "n, m". We now give a geometric interpretation of $(t, \alpha, \beta, n, m, s)$-nets.

Remark 3.5 (c.f. Remark 2.6). *Note that $b^{m-|\nu|_1} = b^m \lambda_s(J(\boldsymbol{i}_\nu, \boldsymbol{a}_\nu))$. For an interval $J \subseteq [0,1)^s$ and a point set $\mathcal{P} \subset [0,1)^s$, let $|\mathcal{P}(J)|$ denote the number of points of $\mathcal{P}$ in J. Then Definition 3.4 says that the proportion of points of $\mathcal{P}$ in $J(\boldsymbol{i}_\nu, \boldsymbol{a}_\nu)$, which is given by $|\mathcal{P}(J(\boldsymbol{i}_\nu, \boldsymbol{a}_\nu))| / |\mathcal{P}([0,1)^s)|$, equals the volume of $J(\boldsymbol{i}_\nu, \boldsymbol{a}_\nu)$.*

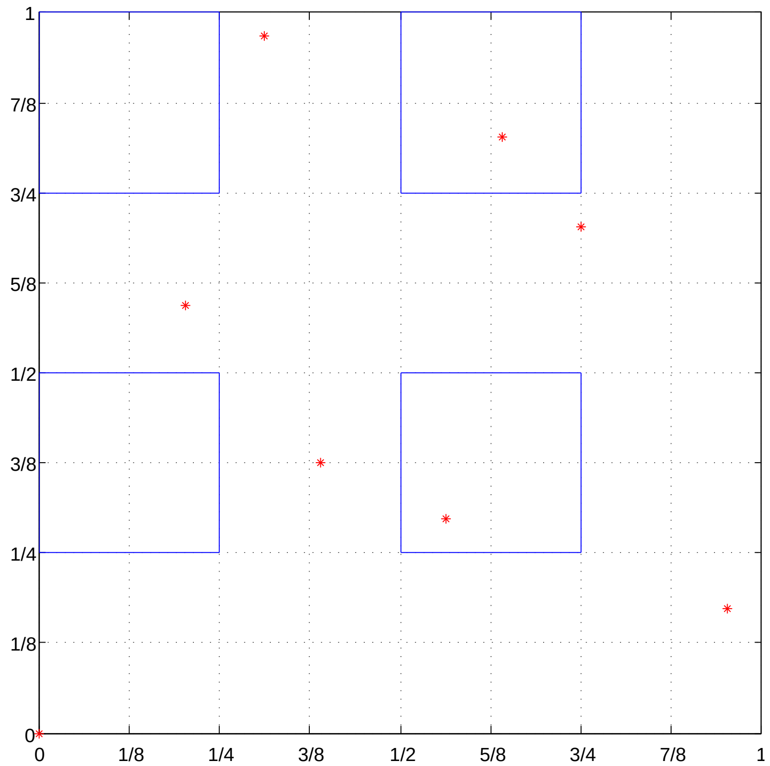

Figure 3.1. *The picture shows a* $(2, \alpha, 1, 6, 3, 2)$*-net in base 2 for any* $\alpha \geq 2$ *and a generalized elementary interval* $J(\boldsymbol{i}_\nu, \boldsymbol{a}_\nu)$*, where* $\nu_1 = \nu_2 = 1$*,* $i_{1,1} = i_{2,1} = 2$ *and* $a_{i_{1,1}} = 0$ *and* $a_{i_{2,1}} = 1$*.*

The next proposition shows how to interpret the quality parameter t of $(t, \alpha, \beta, n, m, s)$-nets.

Proposition 3.6 (c.f. Remark 2.7)**.** *Let* $\mathcal{P}$ *be a* $(t, \alpha, \beta, n, m, s)$*-net in base* b*. Then* $\mathcal{P}$ *is a* $(t', \alpha, \beta', n, m, s)$*-net for all* $0 < \beta' \leq \beta$ *and all* $t \leq t' \leq \beta' n$*.*

Proof. We note that $\beta' n - t' \leq \beta n - t$ and hence the condition on $\mathcal{P}$ in Definition 3.4 is either the same or weaker. □

We now introduce higher order sequences, i.e. $(t, \alpha, \beta, \sigma, s)$-sequences.

Definition 3.7 (c.f. Definition 2.8)**.** *Let* $\alpha, \sigma \geq 1$*,* $t \geq 0$ *be integers, and* $0 < \beta \leq 1$ *be a real number. Let* $\mathcal{S} = \{\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots\}$ *be a sequence of points in* $[0,1)^s$*. Then* $\mathcal{S}$ *is a* $(t, \alpha, \beta, \sigma, s)$*-sequence in base* b *if for all* $k \geq 0$ *and* $m > t/(\beta\sigma)$ *we have that* $\boldsymbol{x}_{kb^m}, \boldsymbol{x}_{kb^m+1}, \ldots, \boldsymbol{x}_{(k+1)b^m-1}$ *is a* $(t, \alpha, \beta, \sigma m, m, s)$*-net in base* b*.*

Following Subsection 2.2.1 we show how to interpret the quality parameter of a $(t, \alpha, \beta, \sigma, s)$-sequence.

Proposition 3.8 (c.f. Remark 2.9)**.** *Let* $\mathcal{S}$ *be a* $(t, \alpha, \beta, \sigma, s)$*-sequence in base* b*. Then* $\mathcal{S}$ *is a* $(t', \alpha, \beta', \sigma, s)$*-sequence for all* $0 < \beta' \leq \beta$ *and all* $t \leq t'$*.*

Proof. The proof follows from Proposition 3.6. □

The following remark is easy to prove and used in Section 4.4.

Remark 3.9. Let n, m, s, $\alpha \geq 1$ be natural numbers and let $0 < \beta \leq 1$ be a real number. It follows from Definition 3.4 that any multiset consisting of b^m points in $[0,1)^s$ is a $(\lfloor \beta n \rfloor, \alpha, \beta, n, m, s)$-net in base b.

We now comment on the relationship between (t,m,s)-nets and (t,α,β,n,m,s)-nets.

Remark 3.10. *We obtain the definition of a (t,m,s)-net from Definition 3.4 by setting $\alpha = \beta = 1$, $n = m$, and considering all $\nu_1, \ldots, \nu_s \geq 0$ so that $\sum_{j=1}^{s} \nu_j \leq m - t$, where we set $i_{j,k} = \nu_j - k + 1$, for $k = 1, \ldots, \nu_j$. Hence a $(t,1,1,m,m,s)$-net is a (t,m,s)-net.*

Analogously, we obtain the following relationship between (t,s)-sequences and $(t,\alpha,\beta,\sigma,s)$-sequences.

Remark 3.11. *We obtain the definition of a (t,s)-sequence from Definition 3.7 and Remark 3.10 by setting $\alpha = \beta = \sigma = 1$. Hence a $(t,1,1,1,s)$-sequence is a (t,s)-sequence.*

Finally, we point out that (t,α,β,n,m,s)-nets cannot exist for all choices of parameters t, α, β, n, m and s.

Remark 3.12. *Note that (t,α,β,n,m,s)-nets can only exist for parameters t, α, β, n, m, s where Definition 3.4 implies that $\nu_1 + \cdots + \nu_s \leq m$; of course, for (t,m,s)-nets Definition 2.5 does imply that $\nu_1 + \cdots + \nu_s \leq m$.*

Consider for example the choice of parameters $\beta = 1$, $t = \alpha = s = 2$, $m = 3$ and $n = 6$; such a $(2,2,1,6,3,2)$-net can exist, since if $\nu_1 + \nu_2 > 3$ we have for all choices of $1 \leq i_{j,\nu_j} < \cdots < i_{j,1} \leq 6$, for $j = 1,2$, that $\sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j,\alpha)} i_{j,l} > 4 = \beta n - t$. (On the other hand, that does not imply that such a net really does exist, it only allows for the possibility to exist.)

But a $(0,2,1,6,3,2)$-net, i.e. we set $t = 0$ and leave the remaining parameters unchanged, cannot exist, since we could choose $\nu_1 = \nu_2 = 2$, $i_{1,1} = i_{2,1} = 2$ and $i_{1,2} = i_{2,2} = 1$, in which case we have $i_{1,1} + i_{1,2} + i_{2,1} + i_{2,2} = 6 = \beta n - t$, and thereby obtain a generalized elementary interval which has to contain exactly $b^{m-\nu_1-\nu_2} = b^{-1}$ points, which is of course absurd. Hence $t = 0$ is not possible for this choice of parameters. (Regarding $t = 1$ we have explicitly constructed digital $(1,2,1,6 \times 3,2)$-nets in Example 2.29 and Remark 2.30, which by Theorem 3.19 below also form $(1,2,1,6,3,2)$-nets.)

3.3 Some properties of higher order nets and higher order sequences

In this section we aim to discuss further properties of (t,α,β,n,m,s)-nets and $(t,\alpha,\beta,\sigma,s)$-sequences. Firstly, we shall analyze the additional parameters α, β and n, which do not appear in the definition of (t,m,s)-nets. The case $\alpha = 1$ is closely related to (t,m,s)-nets. We can w.l.o.g. choose ν_j, $j = 1,\dots,s$, so that $\sum_{j=1}^{s} \nu_j = \lfloor \beta n \rfloor - t$ and set $i_{j,l} = \nu_j + 1 - l$, for $l = 1,\dots,\nu_j$, as in this case we obtain the most stringent condition on the points, i.e. all other conditions are automatically included in this choice of the $i_{j,l}$. Then a $(t,1,\beta,n,m,s)$-net is a classical (t',m,s)-net with $t' = m - \lfloor \beta n \rfloor + t$.

We have the following theorem.

Theorem 3.13. *Assume that $n,m,\alpha \in \mathbb{N}$, $0 < \beta \le 1$ a real number, and $0 \le t \le \beta n$ an integer, such that there exists a (t,α,β,n,m,s)-net in base b. For $1 \le j_0 \le s$, let $0 \le \ell_{j_0} < j_0$ be given by $\ell_{j_0} \equiv m \pmod{j_0}$. Then for $j_0 = 1,\dots,s$ we have*

$$\beta n - t < \alpha m - j_0 \frac{\alpha(\alpha-1)}{2} + \alpha, \qquad \text{for } m \ge \alpha j_0,$$

and

$$\beta n - t < \frac{1}{2}\alpha j_0 \left\lfloor \frac{m}{j_0} \right\rfloor + (\ell_{j_0}+1)\left(\left\lfloor \frac{m}{j_0} \right\rfloor + 1 \right), \qquad \text{for } m < \alpha j_0.$$

Proof. As elaborated in Remark 3.12 for every choice of $1 \le i_{j,\nu_j} < \cdots < i_{j,1}$, $\nu_j \ge 0$, for $j = 1,\dots,s$, with $\sum_{j=1}^{s}\sum_{l=1}^{\min(\nu_j,\alpha)} i_{j,l} \le \beta n - t$, we must have that $|\boldsymbol{\nu}|_1 \le m$.

Let $1 \le j_0 \le s$ and let

$$\nu_j = \begin{cases} \lfloor m/j_0 \rfloor + 1, & \text{for } 1 \le j \le \ell_{j_0} + 1, \\ \lfloor m/j_0 \rfloor, & \text{for } \ell_{j_0} + 2 \le j \le j_0, \\ 0, & \text{for } j_0 + 1 \le j \le s. \end{cases}$$

Further, set $i_{j,l} = \nu_j + 1 - l$, for $l = 1,\dots,\nu_j$, for $j = 1,\dots,j_0$. Note that for this choice of $\nu_1,\dots,\nu_s$ we have

$$|\boldsymbol{\nu}|_1 = j_0 \left\lfloor \frac{m}{j_0} \right\rfloor + \ell_{j_0} + 1 = j_0 \frac{m - \ell_{j_0}}{j_0} + \ell_{j_0} + 1 = m + 1.$$

Consider the case where $\alpha \le \lfloor m/j_0 \rfloor$. Then

$$\begin{aligned} \sum_{j=1}^{s}\sum_{l=1}^{\min(\nu_j,\alpha)} i_{j,l} &= j_0 \left(\left\lfloor \frac{m}{j_0} \right\rfloor + \left\lfloor \frac{m}{j_0} \right\rfloor - 1 + \cdots + \left\lfloor \frac{m}{j_0} \right\rfloor - (\alpha-1) \right) + \alpha(\ell_{j_0}+1) \\ &= \alpha j_0 \left\lfloor \frac{m}{j_0} \right\rfloor - j_0 \frac{\alpha(\alpha-1)}{2} + \alpha(\ell_{j_0}+1) \end{aligned}$$

$$
\begin{aligned}
&= \alpha j_0 \frac{m-\ell_{j_0}}{j_0} - j_0 \frac{\alpha(\alpha-1)}{2} + \alpha \ell_{j_0} + \alpha \\
&= \alpha m - j_0 \frac{\alpha(\alpha-1)}{2} + \alpha.
\end{aligned}
$$

Thus we get a contradiction if the last term is smaller or equal to $\beta n - t$ and hence the first result follows.

Now we consider the case where $\alpha \geq \lfloor m/j_0 \rfloor + 1$. Then

$$
\begin{aligned}
\sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j,\alpha)} i_{j,l} &= j_0 \left(\left\lfloor \frac{m}{j_0} \right\rfloor + \left\lfloor \frac{m}{j_0} \right\rfloor - 1 + \cdots + 1 \right) + (\ell_{j_0} + 1) \left(\left\lfloor \frac{m}{j_0} \right\rfloor + 1 \right) \\
&= j_0 \frac{\lfloor m/j_0 \rfloor (\lfloor m/j_0 \rfloor + 1)}{2} + (\ell_{j_0} + 1) \left(\left\lfloor \frac{m}{j_0} \right\rfloor + 1 \right) \\
&\leq \frac{1}{2} \alpha j_0 \left\lfloor \frac{m}{j_0} \right\rfloor + (\ell_{j_0} + 1) \left(\left\lfloor \frac{m}{j_0} \right\rfloor + 1 \right).
\end{aligned}
$$

Again we get a contradiction if the last term is smaller than or equal to $\beta n - t$ and hence also the second result follows. □

Note that Theorem 3.13 implies that $\beta n - t < \alpha m + 1$ for $\alpha = 1, 2$ (choose $j_0 = 1$) and based on the proof of Theorem 3.13 one can show that $\beta n - t < \alpha m$ for $\alpha \geq 3$ (choose $j_0 = 1$). Thus, as $\beta n - t < \alpha m + 1$, we can w.l.o.g. choose β and n such that $\beta n < \alpha m + 1$ (for $\beta n \geq \alpha m + 1$ we must have $t > 0$, hence we do not exclude any cases by choosing $\beta n < \alpha m + 1$) or if β is such that βn is an integer, we have $\beta \leq \alpha m / n$.

Choosing $j_0 = s$ in Theorem 3.13 and estimating $\ell_{j_0} + 1 \leq j_0$ we obtain the following corollary.

Corollary 3.14. *Assume that $n, m, \alpha \in \mathbb{N}$, $0 < \beta \leq 1$ a real number, and $0 \leq t \leq \beta n$ an integer, such that there exists a $(t, \alpha, \beta, n, m, s)$-net in base b. Then we have*

$$
\beta n - t < \alpha m - s \frac{\alpha(\alpha-1)}{2} + \alpha, \qquad \text{for } m \geq \alpha s,
$$

and

$$
\beta n - t < \frac{1}{2} \alpha m + m + s, \qquad \text{for } m < \alpha s.
$$

Next, we discuss some simple propagation rules to get a better understanding of the parameters of the $(t, \alpha, \beta, n, m, s)$-nets and $(t, \alpha, \beta, \sigma, s)$-sequences.

The following theorem is analogous to [21, Theorem 4.10].

Theorem 3.15. *Let $\mathcal{P}$ be a $(t, \alpha, \beta, n, m, s)$-net in base b and $\mathcal{S}$ be a $(t, \alpha, \beta, \sigma, s)$-sequence in base b. Then we have the following:*

(i) *$\mathcal{P}$ is a $(t', \alpha', \beta', n, m, s)$-net for all $\alpha' \geq 1$ where $\beta' = \beta \min(\alpha, \alpha')/\alpha$ and $t' = \lceil t \min(\alpha, \alpha')/\alpha \rceil$, and S is a $(t', \alpha', \beta', \sigma, s)$-sequence for all $\alpha' \geq 1$ where $\beta' = \beta \min(\alpha, \alpha')/\alpha$ and where $t' = \lceil t \min(\alpha, \alpha')/\alpha \rceil$.*

(ii) *Any $(t, \alpha, \beta, \sigma, s)$-sequence is a $(t, \alpha, \beta, \sigma', s)$-sequence for all $1 \leq \sigma' \leq \sigma$.*

(iii) *Any $(t, \alpha, \beta, n, m, s)$-net is a classical $(m - \lfloor \beta n/\alpha \rfloor + \lceil t/\alpha \rceil, m, s)$-net, and any $(t, \alpha, \beta, \sigma, s)$-sequence with $\alpha = \beta\sigma$ is a classical $(\lceil t/\alpha \rceil, s)$-sequence.*

Proof. To prove the first part we firstly consider the case $\alpha' \geq \alpha$. Let $1 \leq i_{j,\nu_j} < \cdots < i_{j,1}$, $\nu_j \geq 0$, for $j = 1, \ldots, s$, with

$$\sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j, \alpha')} i_{j,l} \leq \beta n - t.$$

As

$$\sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j, \alpha)} i_{j,l} \leq \sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j, \alpha')} i_{j,l}$$

and $\mathcal{P}$ is a $(t, \alpha, \beta, n, m, s)$-net, it follows that $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ contains $b^{m-|\boldsymbol{\nu}|_1}$ points for all admissible $\boldsymbol{a}_{\boldsymbol{\nu}}$ and hence this case follows for nets.

Let now $\alpha' < \alpha$ and assume

$$\sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j, \alpha')} i_{j,l} \leq \beta' n - t' = \frac{\alpha'}{\alpha} \beta n - \left\lceil t \frac{\alpha'}{\alpha} \right\rceil.$$

Since for a fixed j the $i_{j,l}$ are decreasing in l, it follows that

$$\frac{1}{\alpha} \sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j, \alpha)} i_{j,l} \leq \frac{1}{\alpha'} \sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j, \alpha')} i_{j,l}$$

and consequently

$$\sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j, \alpha)} i_{j,l} \leq \frac{\alpha}{\alpha'} (\beta' n - t') \leq \beta n - t.$$

As $\mathcal{P}$ is a $(t, \alpha, \beta, n, m, s)$-net, it follows that $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ contains exactly $b^{m-|\boldsymbol{\nu}|_1}$ points for all admissible $\boldsymbol{a}_{\boldsymbol{\nu}}$ completing the proof for nets. For sequences the result follows from the result for nets and Definition 3.7.

For the second part we have to show that every point set $\boldsymbol{x}_{kb^m}, \ldots, \boldsymbol{x}_{(k+1)b^m-1}$ is a $(t, \alpha, \beta, \sigma' m, m, s)$-net. We know that this point set is a $(t, \alpha, \beta, \sigma m, m, s)$-net from Definition 3.7. As $\sigma' m - t \leq \sigma m - t$ this follows as the condition on the points $\boldsymbol{x}_{kb^m}, \ldots, \boldsymbol{x}_{(k+1)b^m-1}$ can only become weaker, which implies the result.

For the last part we use part (i), which shows that every $(t, \alpha, \beta, n, m, s)$-net $\mathcal{P}$ is also a $(\lceil t/\alpha \rceil, 1, \beta/\alpha, n, m, s)$-net. At the beginning of this section it was shown that Definition 3.4 implies that a $(t, 1, \beta, n, m, s)$-net is also a (t', m, s)-net with $t' = m - \lfloor \beta n \rfloor + t$, hence $\mathcal{P}$ is also a (t', m, s)-net where

$$t' = m - \left\lfloor \frac{\beta}{\alpha} n \right\rfloor + \left\lceil \frac{t}{\alpha} \right\rceil .$$

Now consider a $(t, \alpha, \beta, \sigma, s)$-sequence $\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots$. For any $k \geq 0$ the set of points $\boldsymbol{x}_{kb^m}, \ldots, \boldsymbol{x}_{(k+1)b^m-1}$ forms a $(t, \alpha, \beta, \sigma m, m, s)$-net. Hence the above result implies that this is a classical (t', m, s)-net where

$$t' = m - \left\lfloor \frac{\beta}{\alpha} \sigma m \right\rfloor + \left\lceil \frac{t}{\alpha} \right\rceil = \left\lceil \frac{t}{\alpha} \right\rceil .$$

As $\boldsymbol{x}_{kb^m}, \ldots, \boldsymbol{x}_{(k+1)b^m-1}$ is a classical (t', m, s)-net for all $k \geq 0$, the result follows. □

Remark 3.16. *By Theorem 3.15 a* $(2, \alpha, 1, 6, 3, 2)$*-net,* $\alpha \geq 2$*, is a classical* $(4 - \lfloor \frac{6}{\alpha} \rfloor, 3, 2)$*-net. By the forthcoming Theorem 3.19 the digital* $(2, \alpha, 1, 6 \times 3, 2)$*-net from Remark 2.31 is a* $(2, \alpha, 1, 6, 3, 2)$*-net, hence we have an example of a* $(2, \alpha, 1, 6, 3, 2)$*-net which is a strict* $(1, 3, 2)$*-net. See also Figure 3.1 for an example of a* $(2, \alpha, 1, 6, 3, 2)$*-net which is a* $(0, 3, 2)$*-net.*

In part (iii) of the above theorem we had the restriction that $\alpha = \beta\sigma$. If $\mathcal{S}$ is a $(t, \alpha, \beta, \sigma, s)$-sequence with $\alpha > \beta\sigma$, then we cannot use part (iii) of the above theorem to imply that $\mathcal{S}$ is a classical (t', s)-sequence, as then we would obtain a t'-value of the subnets $\boldsymbol{x}_{kb^m}, \ldots, \boldsymbol{x}_{(k+1)b^m-1}$ which grows with m. Hence we do not obtain a classical sequence in this way. On the other hand, we always have $\alpha \geq \beta\sigma$, as we show in the following theorem.

Theorem 3.17. *Assume that* $t, \alpha, \sigma, s \in \mathbb{N}$*, and* $\beta \in \mathbb{R}$*,* $0 < \beta \leq 1$ *are such that there exists a* $(t, \alpha, \beta, \sigma, s)$*-sequence. Then* $\beta\sigma \leq \alpha$*.*

Proof. Let $\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots$ be a $(t, \alpha, \beta, \sigma, s)$-sequence. Then the set of points $\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{b^m-1}$ forms a $(t, \alpha, \beta, \sigma m, m, s)$-net for all $m > t/(\beta\sigma)$.

Assume to the contrary that $\alpha < \beta\sigma$. As $\beta\sigma m - t < \alpha m + 1$, which was shown after the proof of Theorem 3.13, we can choose an m large enough to obtain a contradiction. Hence $\beta\sigma \leq \alpha$. □

Higher order digital sequences for which $\alpha = \beta\sigma$ are of interest, as the resulting qMC rules achieve the optimal rate of convergence of the integration error for functions with square integrable mixed partial derivatives of order α in each variable, whereas for $\alpha > \beta\sigma$ they do not achieve the optimal rate, see [21]. But for the case $\alpha = \beta\sigma$ we get the following bound on the value of t from Theorem 3.13.

Theorem 3.18. *Assume that $t, \alpha, \sigma, s \in \mathbb{N}$, and $\beta \in \mathbb{R}$, $0 < \beta \leq 1$, are such that $\alpha = \beta\sigma$ and such that there exists a $(t, \alpha, \beta, \sigma, s)$-sequence. Then for all $\alpha \geq 2$ we have*

$$t > s\frac{\alpha(\alpha-1)}{2} - \alpha.$$

Proof. Let $m_0 = \alpha s$. Then the first b^{m_0} points of a $(t, \alpha, \beta, \sigma, s)$-sequence form a $(t, \alpha, \beta, \sigma m_0, m_0, s)$-net. By Corollary 3.14 we obtain that

$$\beta\sigma m_0 - t < \alpha m_0 - s\frac{\alpha(\alpha-1)}{2} + \alpha.$$

By substituting α for $\beta\sigma$ in the last equation we obtain the result. □

The next theorem establishes that a digital $(t, \alpha, \beta, n \times m, s)$-net over $\mathbb{Z}_b$ is a $(t, \alpha, \beta, n, m, s)$-net in base b and analogously for sequences. This also yields explicit constructions of $(t, \alpha, \beta, n, m, s)$-nets and $(t, \alpha, \beta, \sigma, s)$-sequences as digital constructions are known from Subsection 2.3.1, see also [27].

Theorem 3.19. *Let b be a prime, then every digital $(t, \alpha, \beta, n \times m, s)$-net over $\mathbb{Z}_b$ is a $(t, \alpha, \beta, n, m, s)$-net in base b and every digital $(t, \alpha, \beta, \sigma, s)$-sequence over $\mathbb{Z}_b$ is a $(t, \alpha, \beta, \sigma, s)$-sequence in base b.*

Proof. Assume we are given an arbitrary generalized elementary interval

$$J(\boldsymbol{i}_\nu, \boldsymbol{a}_\nu) = \prod_{j=1}^{s} \bigcup_{\substack{a_{j,l}=0 \\ l \in \{1,\ldots,n\} \setminus \left\{i_{j,1},\ldots,i_{j,\nu_j}\right\}}}^{b-1} \left[\frac{a_{j,1}}{b} + \cdots + \frac{a_{j,n}}{b^n}, \frac{a_{j,1}}{b} + \cdots + \frac{a_{j,n}}{b^n} + \frac{1}{b^n}\right),$$

for some given values of $\boldsymbol{\nu}$, $\boldsymbol{i}_\nu$, and $\boldsymbol{a}_\nu$ such that $1 \leq i_{j,\nu_j} < \cdots < i_{j,1}$, $j = 1, \ldots, s$, $\nu_j \geq 0$, and

$$\sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j, \alpha)} i_{j,l} \leq \beta n - t. \tag{3.2}$$

We have to show that $J(\boldsymbol{i}_\nu, \boldsymbol{a}_\nu)$ contains exactly $b^{m-|\nu|_1}$ points of the digital $(t, \alpha, \beta, n \times m, s)$-net, which we denote by $\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{b^m-1}$. Let $\boldsymbol{x}_h = (x_{h,1}, \ldots, x_{h,s})$ and $x_{h,j} = \xi_{h,j,1}b^{-1} + \xi_{h,j,2}b^{-2} + \ldots$ be the b-adic representation of $x_{h,j}$.

Then for each $0 \leq h < b^m$ it follows that $\boldsymbol{x}_h \in J(\boldsymbol{i}_\nu, \boldsymbol{a}_\nu)$ if and only if $\xi_{h,j,k} = a_{j,k}$ for all $k \in \left\{i_{j,1}, \ldots, i_{j,\nu_j}\right\}$ and all $j = 1, \ldots, s$. The value of $\xi_{h,j,k}$ is obtained from the generalized digital construction scheme in the following way: let $C_1, \ldots, C_s$ denote the generating matrices of the digital $(t, \alpha, \beta, n \times m, s)$-net over $\mathbb{Z}_b$. Then by Definition 2.21 $\xi_{h,j,k} = \vec{c}_{j,k}\vec{h}$, where $\vec{c}_{j,k}$ denotes the kth row of C_j.

Let $C = (\vec{c}_{1,i_{1,1}}^\top, \ldots, \vec{c}_{1,i_{1,\nu_1}}^\top, \ldots, \vec{c}_{s,i_{s,1}}^\top, \ldots, \vec{c}_{s,i_{s,\nu_s}}^\top)^\top$ and further we define the vector $\vec{b} = (a_{1,i_{1,1}}, \ldots, a_{1,i_{1,\nu_1}}, \ldots, a_{s,i_{s,1}}, \ldots, a_{s,i_{s,\nu_s}})^\top$. Then by the above it follows that $\boldsymbol{x}_h \in J(\boldsymbol{i}_\nu, \boldsymbol{a}_\nu)$ if and only if $C\vec{h} = \vec{b}$.

We now investigate how many solutions $\vec{h}$ the system of equations $C\vec{h} = \vec{b}$ has. As Equation (3.2) is satisfied, Definition 2.22 implies that the rows of the matrix C are linearly independent. As C has $|\boldsymbol{\nu}|_1$ ($|\boldsymbol{\nu}|_1 \le m$) rows, there are exactly $b^{m-|\boldsymbol{\nu}|_1}$ solutions to this system, and hence $b^{m-|\boldsymbol{\nu}|_1}$ of the $\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{b^m-1}$ fall into $J(\boldsymbol{i}_\nu, \boldsymbol{a}_\nu)$, which shows that every digital $(t, \alpha, \beta, n \times m, s)$-net is also a $(t, \alpha, \beta, n, m, s)$-net.

Now we turn to sequences. Let $\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots$ be a digital $(t, \alpha, \beta, \sigma, s)$-sequence over the finite field $\mathbb{Z}_b$. Let $k \ge 0$ and $m > t/(\beta\sigma)$. Then the point set $\boldsymbol{x}_{\ell b^m}, \ldots, \boldsymbol{x}_{(\ell+1)b^m-1}$ can be obtained from the generalized digital construction scheme with an added digital shift, i.e., there are matrices $C_1, \ldots, C_s \in \mathbb{Z}_b^{n\times m}$ and vectors $\vec{d}_{j,\ell} = (d_{j,1,\ell}, \ldots, d_{j,n,\ell})^\top \in \mathbb{Z}_b^n$, $1 \le j \le s$, which depend on ℓ, such that $\xi_{h,j,k} = \vec{c}_{j,k}\vec{h} + d_{j,k,\ell}$. Thus we have $\vec{c}_{j,k}\vec{h} = \xi_{h,j,k} - d_{j,k,\ell} \in \mathbb{Z}_b$. For some given generalized elementary interval $J(\boldsymbol{i}_\nu, \boldsymbol{a}_\nu)$ we have $\boldsymbol{x}_h \in J(\boldsymbol{i}_\nu, \boldsymbol{a}_\nu)$ if and only if $\vec{c}_{j,k}\vec{h} = a_{j,k} - d_{j,k,\ell}$ for all $k \in \{i_{1,1}, \ldots, i_{1,\nu_1}, \ldots, i_{s,1}, \ldots, i_{s,\nu_s}\}$ and $j = 1, \ldots, s$. Thus the same argument as for nets applies and the result follows. □

We conclude this section by discussing the rate at which the quality parameter t of $(t, \alpha, \beta, \sigma, s)$-sequences increases. The following definition is useful in this discussion.

Definition 3.20. *Let b be a prime. Then let $d_b(\alpha, s)$ denote the smallest value of t such that there exists a digital $(t, \alpha, \beta, \sigma, s)$-sequence over the finite field $\mathbb{Z}_b$ with $\alpha = \beta\sigma$.*

The analogue of Definition 3.20 for classical digital sequences, i.e. the case $\alpha = 1$, has already appeared in [73], see also [74, Definition 8]. For $\alpha = \beta = \sigma = 1$, i.e. digital (t, s)-sequences, see Remark 3.11, it is true that

$$\frac{s}{b-1} - \mathcal{O}(\log s) < d_b(1, s) \le \frac{c}{\log b} s + 1,$$

for all $s \ge 1$, where $c > 0$ is an absolute constant. The lower bound was shown in [93] and also holds for (t, s)-sequences, whereas the upper bound can be found in [73, Theorem 4] and [74, Corollary 1]. Improved results for several special values of b can also be found in [76].

The following theorem now considers the case $\alpha \ge 2$.

Theorem 3.21. *Let b be a prime. Then for all $s \geq 1$ and $\alpha \geq 2$ we have*

$$s\frac{\alpha(\alpha-1)}{2} - \alpha < d_b(\alpha, s) \leq s\alpha^2 \frac{c}{\log b} + \alpha + \alpha \left\lfloor \frac{s(\alpha-1)}{2} \right\rfloor,$$

where $c > 0$ is an absolute constant.

Proof. The lower bound is taken from Theorem 3.18. To prove the upper bound we use Theorem 2.26 with $d = \alpha$ to obtain a digital $(t, \alpha, 1, \alpha, s)$-sequence over $\mathbb{Z}_b$ with

$$t = \alpha t' + \alpha \left\lfloor \frac{s(\alpha-1)}{2} \right\rfloor,$$

where t' is the quality parameter of the classical digital $(t', s\alpha)$-sequence upon which the construction is based. From [73, Theorem 4], [74, Corollary 1] we know that there exist digital (t', s)-sequences for which $t' \leq \frac{c}{\log b} s + 1$. Upon combining the last two formulae, where we replace s with $s\alpha$ in the last formula as we consider digital $(t', s\alpha)$-sequences, the result follows. □

Note that the bounds in Theorem 3.21 also apply to (non-digital) $(t, \alpha, \beta, \sigma, s)$-sequences with $\alpha = \beta\sigma$ and t value as small as possible.

3.4 Characterizing higher order nets

In this section we characterize $(t, \alpha, \beta, n, m, s)$-nets using Weyl sums. The structure of this section is analogous to Subsection 2.4.2, in particular, we generalize Lemmas 2.38 and 2.39 and Corollary 2.40, as we now consider higher order nets.

Lemma 3.22 (c.f. Lemma 2.38). *Let $\mathcal{P} = \{\boldsymbol{x}_h\}_{h=0}^{b^m-1}$ be a $(t, \alpha, \beta, n, m, s)$-net in base $b \geq 2$, where $\alpha \geq 2$ is an integer, β a real number such that $0 < \beta \leq 1$ and $n, m, s \in \mathbb{N}$. Then for all $\boldsymbol{k} \in \mathbb{N}_0^s$ satisfying $0 < \mu_\alpha(\boldsymbol{k}) \leq \beta n - t$ we have*

$$Q_{b^m}(\mathrm{wal}_{\boldsymbol{k}}, \mathcal{P}) = 0,$$

where $\mu_\alpha(.)$ is defined in Equation (2.8)*.*

Proof. Let $\boldsymbol{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ be such that $0 < \mu_\alpha(\boldsymbol{k}) \leq \beta n - t$ (hence $\boldsymbol{k} \neq \boldsymbol{0}$) and for $k_j \neq 0$ let

$$k_j = \kappa_{j,1} b^{i_{j,1}-1} + \cdots + \kappa_{j,\nu_j} b^{i_{j,\nu_j}-1},$$

with $\kappa_{j,l} \in \{1, \ldots, b-1\}$, be the b-adic expansion of k_j, $1 \leq j \leq s$. Then for j for which $k_j \neq 0$ and $x = \sum_{l=1}^{\infty} \xi_l b^{-l} \in [0, 1)$ we have

$$\mathrm{wal}_{k_j}(x) = \exp\left(\frac{2\pi \mathtt{i}}{b}(\kappa_{j,1}\xi_{i_{j,1}} + \cdots + \kappa_{j,\nu_j}\xi_{i_{j,\nu_j}})\right).$$

Hence, if we set $\boldsymbol{i}_{\boldsymbol{\nu}} = (i_{1,1}, \ldots, i_{1,\nu_1}, \ldots, i_{s,1}, \ldots, i_{s,\nu_s})$, which only depends on $\boldsymbol{k}$, then $\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x})$ is constant on generalized elementary intervals $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ of the form given in Equation (3.1). Furthermore, we denote the value of $\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x})$ on $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ by $c_{\boldsymbol{a}_{\boldsymbol{\nu}}}$. As $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$, $\boldsymbol{a}_{\boldsymbol{\nu}} \in \{0, \ldots, b-1\}^{|\boldsymbol{\nu}|_1}$, is a partition of $[0,1)^s$ we obtain

$$\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) = \sum_{\boldsymbol{a}_{\boldsymbol{\nu}} \in \{0,\ldots,b-1\}^{|\boldsymbol{\nu}|_1}} c_{\boldsymbol{a}_{\boldsymbol{\nu}}} \mathbf{1}_{J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})}(\boldsymbol{x}),$$

where $\mathbf{1}_{J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})}$ denotes the characteristic function of $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$.

For $\boldsymbol{k} \neq \boldsymbol{0}$ we have $\int_{[0,1]^s} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} = 0$, and hence it follows that $\sum_{\boldsymbol{a}_{\boldsymbol{\nu}} \in \{0,\ldots,b-1\}^{|\boldsymbol{\nu}|_1}} c_{\boldsymbol{a}_{\boldsymbol{\nu}}} = 0$, as the volume of $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ only depends on $\boldsymbol{\nu}$. Consequently,

$$\begin{aligned} Q_{b^m}(\mathrm{wal}_{\boldsymbol{k}}, \mathcal{P}) &= \sum_{\boldsymbol{a}_{\boldsymbol{\nu}} \in \{0,\ldots,b-1\}^{|\boldsymbol{\nu}|_1}} c_{\boldsymbol{a}_{\boldsymbol{\nu}}} Q_{b^m}(\mathbf{1}_{J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})}, \mathcal{P}) \\ &= \sum_{\boldsymbol{a}_{\boldsymbol{\nu}} \in \{0,\ldots,b-1\}^{|\boldsymbol{\nu}|_1}} c_{\boldsymbol{a}_{\boldsymbol{\nu}}} Q_{b^m}(\mathbf{1}_{J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})} - \lambda_s(J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})), \mathcal{P}). \end{aligned}$$

As $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ is a generalized elementary interval of volume $b^{-|\boldsymbol{\nu}|_1}$ for which by assumption

$$\sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j, \alpha)} i_{j,l} = \mu_\alpha(\boldsymbol{k}) \le \beta n - t,$$

it follows that $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ contains $b^{m-|\boldsymbol{\nu}|_1}$ points of $\mathcal{P}$ and hence

$$Q_{b^m}(\mathbf{1}_{J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})} - \lambda_s(J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})), \mathcal{P}) = \frac{1}{b^m}(b^{m-|\boldsymbol{\nu}|_1} - b^m \lambda_s(J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}}))) = 0$$

as required. □

To establish the converse we need the following lemma, which generalizes [44, Lemma 3(i)] and which can be proven along the same lines as [44, Remark (iv), Lemma 2(i) and Lemma 3(i)].

Lemma 3.23. *For given $\boldsymbol{\nu}$, $\boldsymbol{i}_{\boldsymbol{\nu}}$ and $\boldsymbol{a}_{\boldsymbol{\nu}}$ let*

$$J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}}) = \prod_{j=1}^{s} \bigcup_{\substack{a_{j,l}=0 \\ l \in \{1,\ldots,n\} \setminus \{i_{j,1},\ldots,i_{j,\nu_j}\}}}^{b-1} \left[\frac{a_{j,1}}{b} + \cdots + \frac{a_{j,n}}{b^n}, \frac{a_{j,1}}{b} + \cdots + \frac{a_{j,n}}{b^n} + \frac{1}{b^n}\right)$$

and let $f(\boldsymbol{x}) = \mathbf{1}_{J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})}(\boldsymbol{x}) - \lambda_s(J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}}))$. Define

$$\begin{aligned} \Delta_{\boldsymbol{i}_{\boldsymbol{\nu}}} := \Big\{ & \boldsymbol{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s : k_j = \kappa_{j,1} b^{i_{j,1}-1} + \cdots + \kappa_{j,\nu_j} b^{i_{j,\nu_j}-1}, \\ & \kappa_{j,1}, \ldots, \kappa_{j,\nu_j} \in \{1, \ldots, b-1\} \text{ if } \nu_j > 0 \text{ and } k_j = 0 \text{ for } \nu_j = 0 \Big\}. \end{aligned}$$

Then for all $\boldsymbol{k} \notin \Delta_{\boldsymbol{i}_{\boldsymbol{\nu}}}$ we have $|\widehat{f}(\boldsymbol{k})| = 0$.

The following lemma generalizes [45, Lemma 2], see Lemma 2.39.

Lemma 3.24 (c.f Lemma 2.39). *Let $\mathcal{P} = \{\boldsymbol{x}_h\}_{h=0}^{b^m-1}$ be a finite sequence of b^m points in the s-dimensional unit cube $[0,1)^s$ and suppose that for each $\boldsymbol{k} \in \mathbb{N}_0^s$ satisfying $0 < \mu_\alpha(\boldsymbol{k}) \le \beta n - t$ we have*

$$Q_{b^m}(\mathrm{wal}_{\boldsymbol{k}}, \mathcal{P}) = 0.$$

Then $\mathcal{P}$ is a $(t, \alpha, \beta, n, m, s)$-net in base b.

Proof. Suppose that $J(\boldsymbol{i}_\nu, \boldsymbol{a}_\nu)$ is an arbitrary generalized elementary interval of the form given in Equation (3.1). We define $f(\boldsymbol{x}) = \mathbf{1}_{J(\boldsymbol{i}_\nu, \boldsymbol{a}_\nu)}(\boldsymbol{x}) - \lambda_s(J(\boldsymbol{i}_\nu, \boldsymbol{a}_\nu))$. In order to show that $\mathcal{P}$ is a $(t, \alpha, \beta, n, m, s)$-net in base b, it suffices to prove that $Q_{b^m}(f, \mathcal{P}) = 0$. If $\widehat{\mathbf{1}}_{J(\boldsymbol{i}_\nu, \boldsymbol{a}_\nu)}(\boldsymbol{k})$ denotes the $\boldsymbol{k}$th Walsh coefficient of $\mathbf{1}_{J(\boldsymbol{i}_\nu, \boldsymbol{a}_\nu)}$, then due to Lemma 3.23, for all $\boldsymbol{x} \in [0,1)^s$, we have

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in \Delta_{\boldsymbol{i}_\nu}} \widehat{\mathbf{1}}_{J(\boldsymbol{i}_\nu, \boldsymbol{a}_\nu)}(\boldsymbol{k}) \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x})$$

and hence

$$Q_{b^m}(f, \mathcal{P}) = \sum_{\boldsymbol{k} \in \Delta_{\boldsymbol{i}_\nu}} \widehat{\mathbf{1}}_{J(\boldsymbol{i}_\nu, \boldsymbol{a}_\nu)}(\boldsymbol{k}) Q_{b^m}(\mathrm{wal}_{\boldsymbol{k}}, \mathcal{P}).$$

But $\boldsymbol{k} \in \Delta_{\boldsymbol{i}_\nu}$ implies that $\mu_\alpha(\boldsymbol{k}) = \sum_{j=1}^s \sum_{l=1}^{\min(\nu_j, \alpha)} i_{j,l} \le \beta n - t$, hence $Q_{b^m}(\mathrm{wal}_{\boldsymbol{k}}, \mathcal{P}) = 0$. This implies that $Q_{b^m}(f, \mathcal{P}) = 0$. □

Combining Lemmas 3.22 and 3.24 we obtain the following characterization of $(t, \alpha, \beta, n, m, s)$-nets in terms of Weyl sums (for the Walsh function system).

Theorem 3.25 (c.f Corollary 2.40). *Let $\mathcal{P} = \{\boldsymbol{x}_h\}_{h=0}^{b^m-1}$ be a finite sequence of b^m points in the s-dimensional unit cube $[0,1)^s$. Then $\mathcal{P}$ is a $(t, \alpha, \beta, n, m, s)$-net in base b if and only if for all $\boldsymbol{k} \in \mathbb{N}_0^s$ satisfying $0 < \mu_\alpha(\boldsymbol{k}) \le \beta n - t$ we have*

$$Q_{b^m}(\mathrm{wal}_{\boldsymbol{k}}, \mathcal{P}) = 0.$$

3.5 Randomizing higher order nets

In this section we discuss the randomization of higher order nets. The randomization of higher order nets is important, especially in the context of numerical integration, see Section 5.3: randomizing higher order nets we are able to obtain unbiased estimators of integrals and statistical information about integration errors.

In particular, we consider randomizations using a digital shift, see Subsection 2.6.1, and a digital shift of depth n, see Subsection 2.6.2. We let $\mathcal{P} = \{\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{b^m-1}\}$ be a $(t, \alpha, \beta, n, m, s)$-net in base b, $\boldsymbol{x}_h = (x_{h,1}, \ldots, x_{h,s})$ for $0 \le h < b^m$, and we assume that the b-adic expansion of $x_{h,j}$ is given by $x_{h,j} = \frac{\xi_{h,j,1}}{b} + \cdots + \frac{\xi_{h,j,n}}{b^n} + \frac{\xi_{h,j,n+1}}{b^{n+1}} + \ldots$ for

$0 \le h < b^m$ and $1 \le j \le s$. Also, let $\mathbf{\Delta} = (\Delta_1, \dots, \Delta_s)$, where Δ_j, $1 \le j \le s$, are uniformly distributed in $[0,1)$ and mutually independent. We also consider the b-adic expansion of each coordinate of $\mathbf{\Delta}$, i.e. $\Delta_j = \frac{\Delta_{j,1}}{b} + \frac{\Delta_{j,2}}{b^2} + \dots$ for $1 \le j \le s$.

Regarding the digital shift in base b, the randomly digitally shifted point set $\mathcal{P}_{\mathbf{\Delta}} = \{\boldsymbol{z}_0, \boldsymbol{z}_1, \dots, \boldsymbol{z}_{b^m-1}\}$ is given by

$$\boldsymbol{z}_h = \boldsymbol{x}_h \oplus \mathbf{\Delta} = (z_{h,1}, \dots, z_{h,s}), \quad 0 \le h < b^m,$$

where $\oplus$ is defined component-wise, where for $0 \le h < b^m$ and $1 \le j \le s$

$$z_{h,j} := \frac{\xi_{h,j,1} \oplus \Delta_{j,1}}{b} + \dots + \frac{\xi_{h,j,n} \oplus \Delta_{j,n}}{b^n} + \frac{\xi_{h,j,n+1} \oplus \Delta_{j,n+1}}{b^{n+1}} + \dots$$

and where $\xi_{h,j,l} \oplus \Delta_{j,l} := \xi_{h,j,l} + \Delta_{j,l} \pmod b$ for $l \ge 1$.

Regarding the digital shift in base b of depth n, we choose digits $\Delta_{j,l}$ for $1 \le j \le s$, $1 \le l \le n$ uniformly distributed on $\{0, 1, \dots, b-1\}$ and mutually independent and also choose $\delta_{h,j}$ for $0 \le h < b^m$, $1 \le j \le s$ uniformly distributed on $[0, b^{-n})$ and mutually independent. Consequently, recalling the digital expansion of $x_{h,j}$ we define

$$z_{h,j,l} = \xi_{h,j,l} + \Delta_{j,l} \pmod b$$

for $0 \le h < b^m$, $1 \le j \le s$ and $1 \le l \le n$ and finally set

$$z_{h,j} = \frac{z_{h,j,1}}{b} + \dots + \frac{z_{h,j,n}}{b^n} + \delta_{h,j}, \quad 0 \le h < b^m, 1 \le j \le s,$$

to obtain the point set $\mathcal{P}_{\mathbf{\Delta},\delta} = \{\boldsymbol{z}_0, \boldsymbol{z}_1, \dots, \boldsymbol{z}_{b^m-1}\}$.

The next proposition establishes that each point in $\mathcal{P}_{\mathbf{\Delta}}$ and $\mathcal{P}_{\mathbf{\Delta},\delta}$ is uniformly distributed in $[0,1)^s$, which is useful, as it means that estimators based on $\mathcal{P}_{\mathbf{\Delta}}$ or $\mathcal{P}_{\mathbf{\Delta},\delta}$ will be unbiased.

Proposition 3.26 (c.f. Proposition 2.42 (i), Proposition 2.43 (i))**.** *Let $\mathcal{P}$ be a $(t, \alpha, \beta, n, m, s)$-net in base b and $\mathcal{P}_{\mathbf{\Delta}}$ and $\mathcal{P}_{\mathbf{\Delta},\delta}$ be defined as above. Then each point in $\mathcal{P}_{\mathbf{\Delta}}$ and $\mathcal{P}_{\mathbf{\Delta},\delta}$ is uniformly distributed in $[0,1)^s$.*

Proof. The proof follows immediately from [83, Proposition 3.1]. □

Using Theorem 3.25, one can show that both a digital shift and a digital shift of depth n, preserve the net property.

Proposition 3.27 (c.f. Proposition 2.42 (ii), Proposition 2.43 (ii))**.** *Let $\mathcal{P}$ be a $(t, \alpha, \beta, n, m, s)$-net in base b and $\mathcal{P}_{\mathbf{\Delta}}$ and $\mathcal{P}_{\mathbf{\Delta},\delta}$ be defined as above. Then $\mathcal{P}_{\mathbf{\Delta}}$ is a $(t, \alpha, \beta, n, m, s)$-net in base b with probability 1 and $\mathcal{P}_{\mathbf{\Delta},\delta}$ is a $(t, \alpha, \beta, n, m, s)$-net in base b.*

Proof. Using Theorem 3.25 we need to show that $Q_{b^m}(\mathrm{wal}_{\boldsymbol{k}}, \mathcal{P}_{\boldsymbol{\Delta}}) = 0$ for all $\boldsymbol{k} \in \mathbb{N}_0^s$ for which $0 < \mu_\alpha(\boldsymbol{k}) \leq \beta n - t$. Clearly, for $\boldsymbol{k} \in \mathbb{N}_0^s$ for which $0 < \mu_\alpha(\boldsymbol{k}) \leq \beta n - t$ we have

$$Q_{b^m}(\mathrm{wal}_{\boldsymbol{k}}, \mathcal{P}_{\boldsymbol{\Delta}}) = \frac{1}{b^m} \sum_{h=0}^{b^m-1} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{z}_h) = \frac{1}{b^m} \sum_{h=0}^{b^m-1} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_h)\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{\Delta}) = 0, \tag{3.3}$$

as $\mathcal{P}$ is a $(t, \alpha, \beta, n, m, s)$-net in base b. Equation (3.3) holds with probability 1, as it only holds if infinitely many digits in the b-adic expansion of each coordinate of $\boldsymbol{z}_h$ are different from $b - 1$. This however occurs with probability 1.

The result for $\mathcal{P}_{\boldsymbol{\Delta},\delta}$ is shown in the same way, but we do not need the condition "with probability one", as the digital shift is only applied to the first n digits. □

3.6 Conclusion and future work

In this chapter we introduced higher order nets (higher order sequences, respectively) and showed that they generalize both:

- classical nets (classical sequences, respectively),
- higher order digital nets (higher order digital sequences, respectively).

Furthermore, the rate at which the quality parameter of higher order sequences worsens was studied. Lastly, higher order nets were characterized in terms of Weyl sums and the randomization of higher order nets using a digital shift and a digital shift of depth n was studied.

Having introduced new point sets, two obvious questions arise:

i) how can one construct these new point sets,

ii) what are possible applications for these new point sets?

The first question is answered in Chapter 4, where we provide duality theory and propagation rules for higher order nets, and the second question is answered in Chapter 5, where we apply qMC rules based on higher order nets to numerical integration.

Chapter Four

Duality theory and propagation rules for higher order nets

Before discussing duality theory and propagation rules for higher order nets we firstly motivate the topics.

4.1 Motivation

Dual nets have already appeared in Subsections 2.2.2, 2.2.3, 2.3.1, and 2.3.2 and their importance in the context of numerical integration has already been stressed, see e.g. Subsection 2.7.1 and also Chapters 6 and 7. Duality theory was first introduced in [72] in the context of digital (t, m, s)-nets and consequently generalized to cover digital $(t, \alpha, \beta, n \times m, s)$-nets in [27]; we remark that duality theory for digital (t, s)-sequences was recently considered in [31]. We point out that [27; 72] deal with duality theory in the context of classical digital nets and higher order digital nets.

The analogue of a dual net for non-digital nets is introduced in Section 4.2. Duality theory is often a crucial tool in proving propagation rules, see [27; 72]. In brief, a propagation rule shows how to construct new nets from existing ones; we point out that propagation rules for higher order nets are discussed in detail in Section 4.3. We rely on the duality theory for higher order nets developed in Section 4.2 to establish some of the propagation rules in Section 4.3. In particular, the $(u, u + v)$-construction, the matrix-product construction and the double-m construction are proven using duality theory. Regarding the propagation rules Table 4.1 shows in which case a propagation rule presented in Section 4.3 generalizes an existing rule for digital $(t, \alpha, \beta, n \times m, s)$-nets or (t, m, s)-nets and in which case the analogue for digital $(t, \alpha, \beta, n \times m, s)$-nets and (t, m, s)-nets does not exist.

In Section 4.4 we discuss propagation rules for higher order sequences and use them to establish an explicit bound on the t-value of $(t, \alpha, \beta, \sigma, s)$-sequences satisfying $\alpha = \beta\sigma$.

It is easy to see that the material covered in Sections 4.2 and 4.3 was essentially

	digital $(t, \alpha, \beta, n \times m, s)$-nets	(t, m, s)-nets
The direct product of two nets	Yes	Yes
The $(u, u + v)$-construction	Yes	Yes
The matrix-product construction	Yes	No
A double m construction	Yes	No
A base change rule	Yes	Yes
Pirsic's base change rule	No	Yes
A higher order to higher order construction	Yes	No

Table 4.1. *This table illustrates if the analogue of a propagation rule presented in Section 4.3 exists for digital $(t, \alpha, \beta, n \times m, s)$-nets and (t, m, s)-nets.*

inspired by [27], which discussed duality theory and propagation rules for higher order digital nets. However, we remark that we did not manage to find a non-digital version of Propagation Rule 3.7 in [27] and likewise in [27], there was no digital analogue of Pirsic's base change rule. Finally, propagation rules for higher order digital sequences were not discussed in [27].

4.2 Duality theory

Duality theory, as introduced by Niederreiter and Pirsic [72] (see also [27]), is a helpful tool in the analysis and construction of classical and higher order digital nets. Here we introduce a duality theory for not necessarily digital constructions. The basic tool used in the analysis are Walsh functions in integer base $b \geq 2$ whose definition and basic properties were recalled in Subsection 2.4.1.

Now we turn to duality theory for not necessarily digital nets. Let $\mathcal{K}_{r,b}^s = \{0, \ldots, b^r - 1\}^s$. We also assume there is an ordering of the elements of $\mathcal{K}_{r,b}^s$ which can be arbitrary but needs to be the same in each instance, i.e. let $\mathcal{K}_{r,b}^s = \{\boldsymbol{k}_0, \ldots, \boldsymbol{k}_{b^{sr}-1}\}$. (Note that $|\mathcal{K}_{r,b}^s| = b^{sr}$.) By this we mean that in expressions such as $\sum_{\boldsymbol{k} \in \mathcal{K}_{r,b}^s}$, $(a_{\boldsymbol{k},\boldsymbol{l}})_{\boldsymbol{k},\boldsymbol{l} \in \mathcal{K}_{r,b}^s}$ and $(c_{\boldsymbol{k}})_{\boldsymbol{k} \in \mathcal{K}_{r,b}^s}$ the elements $\boldsymbol{k}$ and $\boldsymbol{l}$ run through the set $\mathcal{K}_{r,b}^s$ always in the same order.

The following $b^{sr} \times b^{sr}$ matrix plays a central role in the duality theory for higher order nets

$$\mathbf{W}_r := \left(\mathrm{wal}_{\boldsymbol{k}}\left(b^{-r}\boldsymbol{l}\right)\right)_{\boldsymbol{k},\boldsymbol{l} \in \mathcal{K}_{r,b}^s}.$$

We call $\mathbf{W}_r$ a Walsh matrix and note that $\mathbf{W}_r$ is symmetric.

In the following we denote by A^* the conjugate transpose of a matrix A over the complex numbers $\mathbb{C}$, i.e. $A^* = \overline{A}^\top$.

Lemma 4.1. *The Walsh matrix $\mathbf{W}_r$ is invertible and its inverse is given by $\mathbf{W}_r^{-1} = \frac{1}{b^{sr}}\mathbf{W}_r^*$.*

Proof. Let $\mathbf{A} = (a_{\boldsymbol{k},\boldsymbol{l}})_{\boldsymbol{k},\boldsymbol{l}\in\mathcal{K}_{r,b}^s} = \mathbf{W}_r \frac{1}{b^{sr}}\mathbf{W}_r^*$. Then using the orthogonality of the Walsh functions we obtain

$$\begin{aligned} a_{\boldsymbol{k},\boldsymbol{l}} &= \frac{1}{b^{sr}} \sum_{\boldsymbol{h}\in\mathcal{K}_{r,b}^s} \mathrm{wal}_{\boldsymbol{k}}\left(b^{-r}\boldsymbol{h}\right) \overline{\mathrm{wal}_{\boldsymbol{l}}\left(b^{-r}\boldsymbol{h}\right)} = \frac{1}{b^{sr}} \prod_{j=1}^{s} \sum_{h=0}^{b^r-1} \mathrm{wal}_{k_j\ominus l_j}(h/b^r) \\ &= \begin{cases} 1 & \text{if } \boldsymbol{k}=\boldsymbol{l}, \\ 0 & \text{if } \boldsymbol{k}\neq\boldsymbol{l}, \end{cases} \end{aligned}$$

where $\boldsymbol{k} = (k_1, \ldots, k_s)$ and $\boldsymbol{l} = (l_1, \ldots, l_s)$ are in $\mathcal{K}_{r,b}^s$. □

Let $b \geq 2$ and $r, N \geq 1$ be integers. For a multiset $\mathcal{P} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$ in $[0,1)^s$ and $\boldsymbol{k} \in \mathcal{K}_{r,b}^s$ we define

$$c_{\boldsymbol{k}} = c_{\boldsymbol{k}}(\mathcal{P}) := \sum_{h=0}^{N-1} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_h)$$

(note that $|c_{\boldsymbol{k}}| \leq N$ and $c_{\boldsymbol{0}} = N$) and the vector

$$\vec{C} = \vec{C}(\mathcal{P}) := (c_{\boldsymbol{k}})_{\boldsymbol{k}\in\mathcal{K}_{r,b}^s}. \tag{4.1}$$

For $\boldsymbol{a} = (a_1, \ldots, a_s) \in \mathcal{K}_{r,b}^s$ define the elementary b-adic interval

$$E_{\boldsymbol{a}} := \prod_{j=1}^{s} \left[\frac{a_j}{b^r}, \frac{a_j+1}{b^r}\right).$$

Lemma 4.2. *We have*

$$\sum_{\boldsymbol{k}\in\mathcal{K}_{r,b}^s} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}\ominus\boldsymbol{y}) = \begin{cases} |\mathcal{K}_{r,b}^s| & \text{if } \boldsymbol{x},\boldsymbol{y}\in E_{\boldsymbol{a}} \text{ for some } \boldsymbol{a}\in\mathcal{K}_{r,b}^s, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. It suffices to consider the one-dimensional case: for $x = \frac{\xi_1}{b} + \frac{\xi_2}{b^2} + \ldots$, $y = \frac{\eta_1}{b} + \frac{\eta_2}{b^2} + \ldots$, and $k = \kappa_0 + \kappa_1 b + \ldots$, we note that

$$\begin{aligned} \sum_{k=0}^{b^r-1} \mathrm{wal}_k(x\ominus y) &= \sum_{\kappa_0=0}^{b-1} \exp\left(\frac{2\pi\mathtt{i}}{b}\kappa_0(\xi_1-\eta_1)\right) \sum_{\kappa_1=0}^{b-1} \exp\left(\frac{2\pi\mathtt{i}}{b}\kappa_1(\xi_2-\eta_2)\right) \\ &\quad \cdots \sum_{\kappa_{r-1}=0}^{b-1} \exp\left(\frac{2\pi\mathtt{i}}{b}\kappa_{r-1}(\xi_r-\eta_r)\right). \end{aligned}$$

Clearly, if $\xi_i - \eta_i = 0$, $i = 1, \ldots, r$, then $\sum_{k=0}^{b^r-1} \mathrm{wal}_k(x\ominus y) = b^r$, but if there exists $i \in \{1, \ldots, r\}$ so that $\xi_i \neq \eta_i$, then $\sum_{k=0}^{b^r-1} \mathrm{wal}_k(x\ominus y) = 0$, and the result follows. □

Let $\boldsymbol{x} \in E_{\boldsymbol{a}}$ for some $\boldsymbol{a} \in \mathcal{K}^s_{r,b}$. Then using Lemma 4.2 we have

$$\begin{aligned}\frac{1}{|\mathcal{K}^s_{r,b}|}\sum_{\boldsymbol{k}\in\mathcal{K}^s_{r,b}} c_{\boldsymbol{k}}\overline{\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x})} &= \frac{1}{|\mathcal{K}^s_{r,b}|}\sum_{\boldsymbol{k}\in\mathcal{K}^s_{r,b}}\sum_{h=0}^{N-1}\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_h\ominus\boldsymbol{x}) \\ &= \sum_{h=0}^{N-1}\frac{1}{|\mathcal{K}^s_{r,b}|}\sum_{\boldsymbol{k}\in\mathcal{K}^s_{r,b}}\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_h\ominus\boldsymbol{x}) \\ &= |\{h : \boldsymbol{x}_h \in E_{\boldsymbol{a}}\}| =: m_{\boldsymbol{a}}.\end{aligned}$$

Definition 4.3. Let $b \geq 2$ and $r, N \geq 1$ be integers, $\mathcal{P} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$ a multiset in $[0,1)^s$ and $\mathcal{K}^s_{r,b} = \{0, \ldots, b^r - 1\}^s$.

i) For $\boldsymbol{a} \in \mathcal{K}^s_{r,b}$ let

$$m_{\boldsymbol{a}} = m_{\boldsymbol{a}}(\mathcal{P}) := |\{h : \boldsymbol{x}_h \in E_{\boldsymbol{a}}\}|$$

and

$$\vec{M} = \vec{M}(\mathcal{P}) := (m_{\boldsymbol{a}})_{\boldsymbol{a}\in\mathcal{K}^s_{r,b}}.$$

Then we call the vector $\vec{M}$ the point set vector (with resolution r).

ii) The vector $\vec{C} = \vec{C}(\mathcal{P})$ from Equation (4.1) is called the dual vector (with respect to the Walsh matrix $\mathbf{W}_r$).

iii) The set

$$\mathcal{D}_r = \mathcal{D}_r(\mathcal{P}) := \{\boldsymbol{k} \in \mathcal{K}^s_{r,b} : c_{\boldsymbol{k}} \neq 0\}$$

is called the dual set (with respect to the Walsh matrix $\mathbf{W}_r$).

The relationship between a point set vector and its dual vector is stated in the following theorem.

Theorem 4.4. *Let $\mathcal{P} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$ be a multiset in $[0,1)^s$, $r \in \mathbb{N}$, $\vec{M}$ the point set vector with resolution r and $\vec{C}$ the dual vector with respect to $\mathbf{W}_r$ defined as above. Then*

$$\frac{1}{|\mathcal{K}^s_{r,b}|}\mathbf{W}_r\vec{C} = \vec{M} \quad \text{and} \quad \vec{C} = \mathbf{W}^*_r\vec{M}. \tag{4.2}$$

Proof. The first result follows from Lemma 4.2 and the second result follows from Lemma 4.1 and the identity $\vec{C} = |\mathcal{K}^s_{r,b}|\mathbf{W}_r^{-1}\vec{M} = \mathbf{W}^*_r\vec{M}$. □

The vector $\vec{C}$ carries sufficient information to construct a point set in the following way: given $\vec{C}$ we can use Theorem 4.4 to determine how many points are to be placed in the interval $E_{\boldsymbol{a}}$, where $\boldsymbol{a} \in \mathcal{K}^s_{r,b}$.

Note that for the $(t, \alpha, \beta, n, m, s)$-net property to hold, it is of no importance where exactly within an interval $E_{\boldsymbol{a}}$, $\boldsymbol{a} \in \mathcal{K}^s_{n,b}$, the points are placed. Hence we can reconstruct a higher order net from a dual vector with respect to $\mathbf{W}_r$ provided that $r \geq \lfloor \beta n \rfloor - t$; this is due to the fact that in order to construct a $(t, \alpha, \beta, n, m, s)$-net, we need information on intervals of volume greater or equal to $b^{-(\lfloor \beta n \rfloor - t)}$. The matrix $\mathbf{W}_r$ provides us with information on such intervals, provided that $r \geq \lfloor \beta n \rfloor - t$. In other words, if one knows the dual vector of a higher order net, then one can use this dual vector to obtain the higher order net via Theorem 4.4 provided that the resolution is bigger than or equal to the strength of the higher order net.

In analogy, the dual spaces of classical digital and higher order digital nets allow us to reconstruct the original point sets, see Definition 2.16 and Equation (2.6). Although $\vec{C}$ is different from the dual spaces of classical digital and higher order digital nets, it contains the same information and can be used in a manner similar to these dual spaces. Below, this will be shown by example of the $(u, u+v)$-construction, the matrix-product construction and the double m construction for higher order nets. In case $\mathcal{P}$ is a digital $(t, \alpha, \beta, n \times m, s)$-net, the dual set defined in Definition 4.3 coincides with the dual space defined in Equation (2.6) intersected with $\mathcal{K}^s_{n,b}$, and if $\mathcal{P}$ is a digital (t, m, s)-net, it coincides with the dual space from Definition 2.16 intersected with $\mathcal{K}^s_{m,b}$.

Although the above results hold for arbitrary point sets, in the following we consider point sets which are higher order nets and show how to relate the quality of a $(t, \alpha, \beta, n, m, s)$-net to its dual set. To this end we need to recall the function μ_α, which was defined in Equation (2.8). As in Chapter 3, for purposes of this chapter, α is used as a net parameter and satisfies $\alpha \in \mathbb{N}$, $\alpha \geq 2$.

For a vector $\boldsymbol{k} = (k_1, \dots, k_s) \in \mathbb{N}_0^s$ we define $\mu_\alpha(\boldsymbol{k}) = \mu_\alpha(k_1) + \cdots + \mu_\alpha(k_s)$ and for a subset $\mathcal{Q}$ of $\mathcal{K}^s_{n,b}$ satisfying $\mathcal{Q} \setminus \{\mathbf{0}\} \neq \emptyset$ and $\alpha \geq 1$ we set

$$\rho_\alpha(\mathcal{Q}) := \min_{\boldsymbol{k} \in \mathcal{Q} \setminus \{\mathbf{0}\}} \mu_\alpha(\boldsymbol{k}).$$

For $\mathcal{Q} \subseteq \{\mathbf{0}\}$ we set $\rho_\alpha(\mathcal{Q}) = r + 1$.

Let $\mathcal{P} = \{\boldsymbol{x}_0, \dots, \boldsymbol{x}_{N-1}\} \subset [0,1)^s$. In the following we examine for which cases we have $\mathcal{D}_r(\mathcal{P}) = \{\mathbf{0}\}$ (note that $\mathbf{0} \in \mathcal{D}_r(\mathcal{P})$ for any point set $\mathcal{P}$ containing at least one point). If $\mathcal{D}_r(\mathcal{P}) = \{\mathbf{0}\}$, then we have $c_{\mathbf{0}} \neq 0$ and $c_{\boldsymbol{k}} = 0$ for all $\boldsymbol{k} \in \mathcal{K}^s_{r,b} \setminus \{\mathbf{0}\}$. By Theorem 4.4 we have $\vec{M}(\mathcal{P}) = c_{\mathbf{0}} b^{-rs}(1, 1, \dots, 1)^\top$, i.e. each box $E_{\boldsymbol{a}}$ contains exactly $c_{\mathbf{0}} b^{-rs}$ points for all $\boldsymbol{a} \in \mathcal{K}^s_{r,b}$ and $\mathcal{P}$ consists of $N = c_{\mathbf{0}}$ points in total. This is the only case for which we have $\mathcal{D}_r(\mathcal{P}) = \{\mathbf{0}\}$.

Conversely, since the number of points in $E_{\boldsymbol{a}}$ must be an integer, it follows that $c_{\mathbf{0}} b^{-rs} \in \mathbb{N}$, i.e. b^{rs} divides $c_{\mathbf{0}}$ and therefore b^{rs} divides N. From this we conclude that if we choose a resolution $r \in \mathbb{N}$ such that $b^{rs} > N$, i.e. $r > \frac{1}{s} \log_b N$, then $\mathcal{D}_r(\mathcal{P}) \neq \{\mathbf{0}\}$.

For a higher order net with $N = b^m$ points this means that we require $r > m/s$.

The following theorem establishes a relationship between $\rho_\alpha(\mathcal{Q})$ and the quality of a $(t, \alpha, \beta, n, m, s)$-net.

Theorem 4.5. *Let $\mathcal{P} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{b^m-1}\} \subset [0,1)^s$ be a multiset. Then $\mathcal{P}$ is a $(t, \alpha, \beta, n, m, s)$-net in base b if and only if $\rho_\alpha(\mathcal{D}_{\lfloor \beta n \rfloor - t}) \geq \lfloor \beta n \rfloor - t + 1$. If $\mathcal{P}$ is a strict $(t_0, \alpha, \beta, n, m, s)$-net in base b, then $\rho_\alpha(\mathcal{D}_{\lfloor \beta n \rfloor - t_0}) = \lfloor \beta n \rfloor - t_0 + 1$.*

Proof. It was shown in Theorem 3.25 that $\mathcal{P}$ is a $(t, \alpha, \beta, n, m, s)$-net in base b if and only if for all $\boldsymbol{k} \in \mathbb{N}_0^s$ satisfying $0 < \mu_\alpha(\boldsymbol{k}) \leq \lfloor \beta n \rfloor - t$ we have $\sum_{h=0}^{b^m-1} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_h) = 0$ and this is equivalent to $\rho_\alpha(\mathcal{D}_{\lfloor \beta n \rfloor - t}) \geq \lfloor \beta n \rfloor - t + 1$, since for $\boldsymbol{k} \in \mathbb{N}_0^s$ with $\mu_\alpha(\boldsymbol{k}) \leq \lfloor \beta n \rfloor - t$ we have $\boldsymbol{k} \in \mathcal{K}^s_{\lfloor \beta n \rfloor - t, b}$.

For the second assertion we assume that $\mathcal{P}$ is a strict $(t_0, \alpha, \beta, n, m, s)$-net in base b, which implies that $\rho_\alpha(\mathcal{D}_{\lfloor \beta n \rfloor - t_0}) \geq \lfloor \beta n \rfloor - t_0 + 1$. Furthermore, we assume that $\rho_\alpha(\mathcal{D}_{\lfloor \beta n \rfloor - t_0}) \geq \lfloor \beta n \rfloor - t_0 + 2 = \lfloor \beta n \rfloor - (t_0 - 1) + 1$. Arguing in the same fashion as in the first part of the proof this is equivalent to $\sum_{h=0}^{b^m-1} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_h) = 0$ for all $\boldsymbol{k}$ satisfying $0 < \mu_\alpha(\boldsymbol{k}) \leq \lfloor \beta n \rfloor - (t_0 - 1)$. Using Theorem 3.25 again this implies that $\mathcal{P}$ is a $(t_0 - 1, \alpha, \beta, n, m, s)$-net in base b, which contradicts the assumption that $\mathcal{P}$ is a strict $(t_0, \alpha, \beta, n, m, s)$-net in base b. $\square$

Let now $\mathcal{P}$ be a strict $(t_0, \alpha, \beta, n, m, s)$-net in base b and $r \geq \lfloor \beta n \rfloor - t_0$. Then $\mathcal{D}_r \supseteq \mathcal{D}_{\lfloor \beta n \rfloor - t_0}$ and $\mathcal{D}_r \setminus \mathcal{D}_{\lfloor \beta n \rfloor - t_0} \subseteq \mathcal{K}^s_{r,b} \setminus \mathcal{K}^s_{\lfloor \beta n \rfloor - t_0, b}$. For any $\boldsymbol{k} \in \mathcal{K}^s_{r,b} \setminus \mathcal{K}^s_{\lfloor \beta n \rfloor - t_0, b}$ we have $\mu_\alpha(\boldsymbol{k}) \geq \lfloor \beta n \rfloor - t_0 + 1$. Theorem 4.5 implies that $\rho_\alpha(\mathcal{D}_{\lfloor \beta n \rfloor - t_0}) = \lfloor \beta n \rfloor - t_0 + 1$ and hence $\rho_\alpha(\mathcal{D}_r) = \rho_\alpha(\mathcal{D}_{\lfloor \beta n \rfloor - t_0}) = \lfloor \beta n \rfloor - t_0 + 1$. In particular, for all $r, r' \geq \lfloor \beta n \rfloor - t_0$ we have

$$\rho_\alpha(\mathcal{D}_r) = \rho_\alpha(\mathcal{D}_{r'}) = \rho_\alpha(\mathcal{D}_n) = \lfloor \beta n \rfloor - t_0 + 1, \tag{4.3}$$

since $n \geq \lfloor \beta n \rfloor - t_0$.

4.3 Propagation rules for $(t, \alpha, \beta, n, m, s)$-nets

In this section we introduce several propagation rules for $(t, \alpha, \beta, n, m, s)$-nets which generalize the analogous results for the digital case given in [27]. Some simple propagation rules for $(t, \alpha, \beta, n, m, s)$-nets and $(t, \alpha, \beta, \sigma, s)$-sequences in base b were already listed in Sections 3.2 and 3.3. For completeness we repeat them here. We also add some further trivial propagation rules in the following list.

Let $\mathcal{P}$ be a $(t, \alpha, \beta, n, m, s)$-net in base b and $\mathcal{S}$ a $(t, \alpha, \beta, \sigma, s)$-sequence in base b. Then we have the following:

(i) $\mathcal{P}$ is a $(t', \alpha, \beta', n, m, s)$-net in base b for all $0 < \beta' \le \beta$ and all $t \le t' \le \beta' n$, and $\mathcal{S}$ is a $(t', \alpha, \beta', \sigma, s)$-sequence in base b for all $0 < \beta' \le \beta$ and all $t \le t'$.

(ii) $\mathcal{P}$ is a $(t', \alpha', \beta', n, m, s)$-net in base b for all $\alpha' \ge 1$ where $\beta' = \beta \min(\alpha, \alpha')/\alpha$ and $t' = \lceil t \min(\alpha, \alpha')/\alpha \rceil$, and $\mathcal{S}$ is a $(t', \alpha', \beta', \sigma, s)$-sequence in base b for all $\alpha' \ge 1$ where $\beta' = \beta \min(\alpha, \alpha')/\alpha$ and $t' = \lceil t \min(\alpha, \alpha')/\alpha \rceil$.

(iii) Consider the point set $\mathcal{P}'$ obtained by truncating each coordinate of each element of $\mathcal{P}$ in its base b representation after n' digits, $1 \le n' \le n$. The resulting point set is a $(t', \alpha, \beta, n', m, s)$-net in base b where $t' = \max(t - \beta(n - n'), 0)$.

(iv) Consider the point set $\mathcal{P}'$ obtained by truncating each coordinate of each element of $\mathcal{P}$ in its base b representation after n digits and adding $n' - n$ extra digits to every coordinate of every element, all of which are zero, where $n' \ge n$. The resulting point set is a $(t, \alpha, \beta', n', m, s)$-net where $\beta' = \beta n / n'$.

(v) The point set obtained by projecting $\mathcal{P}$ onto the coordinates in $\mathfrak{u}$, $\mathfrak{u} \subseteq \{1, \ldots, s\}$, is a $(t_{\mathfrak{u}}, \alpha, \beta, n, m, |\mathfrak{u}|)$-net in base b where $t_{\mathfrak{u}} \le t$.

(vi) Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_{b^r}$ be $(t, \alpha, \beta, n, m, s)$-nets in base b. Then the multiset obtained from the union of the elements of $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_{b^r}$ is a $(t, \alpha, \beta, n, m + r, s)$-net in base b.

(vii) $\mathcal{S}$ is a $(t, \alpha, \beta, \sigma', s)$-sequence for all $1 \le \sigma' \le \sigma$.

(viii) $\mathcal{P}$ is a classical $(m - \lfloor \beta n / \alpha \rfloor + \lceil t/\alpha \rceil, m, s)$-net and $\mathcal{S}$ with $\alpha = \beta\sigma$ is a classical $(\lceil t/\alpha \rceil, s)$-sequence.

We remark that Propagation Rules i) -vi) are analogous to Propagation Rules I-VI in [27] for higher order digital nets.

4.3.1 The direct product of two $(t, \alpha, \beta, n, m, s)$-nets

Let $\mathcal{P}_1 = \{\boldsymbol{x}_h\}_{h=0}^{b^{m_1}-1}$ be a $(t_1, \alpha_1, \beta_1, n_1, m_1, s_1)$-net in base b and $\mathcal{P}_2 = \{\boldsymbol{y}_i\}_{i=0}^{b^{m_2}-1}$ be a $(t_2, \alpha_2, \beta_2, n_2, m_2, s_2)$-net in base b. Based on $\mathcal{P}_1$ and $\mathcal{P}_2$ a new $(t, \alpha, \beta, n, m, s)$-net in base b is formed where $n = n_1 + n_2$, $m = m_1 + m_2$ and $s = s_1 + s_2$. The elements of $\mathcal{P}$ are defined to be the direct product of the points from $\mathcal{P}_1$ and $\mathcal{P}_2$, i.e. $\mathcal{P}$ is the multiset of b^m points

$$(\boldsymbol{x}_h, \boldsymbol{y}_i), \quad \text{for} \quad 0 \le h \le b^{m_1} - 1 \text{ and } 0 \le i \le b^{m_2} - 1,$$

in some order. The following theorem gives information on the t-value of the resulting (t,α,β,n,m,s)-net.

Theorem 4.6. *Let $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}$ be defined as above. Then $\mathcal{P}$ is a (t,α,β,n,m,s)-net in base b where $\alpha=\max(\alpha_1,\alpha_2)$, $\beta=\min(\beta_1,\beta_2)$ and*

$$t=\max(\beta_1 n_1+t_2,\beta_2 n_2+t_1).$$

Proof. Note first that

$$\begin{aligned}\beta n-t &= \beta n_1+\beta n_2-\max(\beta_1 n_1+t_2,\beta_2 n_2+t_1)\\ &= \min(\beta n_1+\beta n_2-\beta_1 n_1-t_2,\beta n_1+\beta n_2-\beta_2 n_2-t_1)\\ &\leq \min(\beta_1 n_1-t_1,\beta_2 n_2-t_2). \end{aligned} \tag{4.4}$$

Let $\nu_1,\dots,\nu_s\geq 0$, and for $1\leq j\leq s$ fix $1\leq i_{j,\nu_j}<\dots<i_{j,1}$ with

$$i_{1,1}+\dots+i_{1,\min(\nu_1,\alpha)}+\dots+i_{s,1}+\dots+i_{s,\min(\nu_s,\alpha)}\leq\beta n-t. \tag{4.5}$$

We need to check that the generalized elementary interval

$$\begin{aligned} J(\boldsymbol{i}_\nu,\boldsymbol{a}_\nu) &= \prod_{j=1}^{s}\bigcup_{\substack{a_{j,l}=0\\ l\in\{1,\dots,n\}\setminus\{i_{j,\nu_j},\dots,i_{j,1}\}}}^{b-1}\left[\frac{a_{j,1}}{b}+\dots+\frac{a_{j,n}}{b^n},\frac{a_{j,1}}{b}+\dots+\frac{a_{j,n}}{b^n}+\frac{1}{b^n}\right)\\ &= \left[\prod_{j=1}^{s_1}\bigcup_{\substack{a_{j,l}=0\\ l\in\{1,\dots,n\}\setminus\{i_{j,\nu_j},\dots,i_{j,1}\}}}^{b-1}\left[\frac{a_{j,1}}{b}+\dots+\frac{a_{j,n}}{b^n},\frac{a_{j,1}}{b}+\dots+\frac{a_{j,n}}{b^n}+\frac{1}{b^n}\right)\right]\times\\ &\quad\left[\prod_{j=s_1+1}^{s}\bigcup_{\substack{a_{j,l}=0\\ l\in\{1,\dots,n\}\setminus\{i_{j,\nu_j},\dots,i_{j,1}\}}}^{b-1}\left[\frac{a_{j,1}}{b}+\dots+\frac{a_{j,n}}{b^n},\frac{a_{j,1}}{b}+\dots+\frac{a_{j,n}}{b^n}+\frac{1}{b^n}\right)\right] \end{aligned}$$

contains $b^{m-|\nu|_1}$ points.

Since $\beta_1,\beta_2\leq 1$ we find from Equations (4.4) and (4.5) that $i_{j,1}\leq n_1$, for $1\leq j\leq s_1$, and $i_{j,1}\leq n_2$, for $j=s_1+1,\dots,s_1+s_2$. Hence the previous expression becomes

$$\begin{aligned} J(\boldsymbol{i}_\nu,\boldsymbol{a}_\nu) &= \left[\prod_{j=1}^{s_1}\bigcup_{\substack{a_{j,l}=0\\ l\in\{1,\dots,n_1\}\setminus\{i_{j,\nu_j},\dots,i_{j,1}\}}}^{b-1}\left[\frac{a_{j,1}}{b}+\dots+\frac{a_{j,n_1}}{b^{n_1}},\frac{a_{j,1}}{b}+\dots+\frac{a_{j,n_1}}{b^{n_1}}+\frac{1}{b^{n_1}}\right)\right]\times\\ &\quad\left[\prod_{j=s_1+1}^{s}\bigcup_{\substack{a_{j,l}=0\\ l\in\{1,\dots,n_2\}\setminus\{i_{j,\nu_j},\dots,i_{j,1}\}}}^{b-1}\left[\frac{a_{j,1}}{b}+\dots+\frac{a_{j,n_2}}{b^{n_2}},\frac{a_{j,1}}{b}+\dots+\frac{a_{j,n_2}}{b^{n_2}}+\frac{1}{b^{n_2}}\right)\right]. \end{aligned}$$

Again from Equations (4.4) and (4.5) we deduce that

$$i_{1,1} + \cdots + i_{1,\min(\nu_1,\alpha)} + \cdots + i_{s_1,1} + \cdots + i_{s_1,\min(\nu_{s_1},\alpha)} \le \beta_1 n_1 - t_1$$

and

$$i_{s_1+1,1} + \cdots + i_{s_1+1,\min(\nu_{s_1+1},\alpha)} + \cdots + i_{s,1} + \cdots + i_{s,\min(\nu_s,\alpha)} \le \beta_2 n_2 - t_2.$$

As $\mathcal{P}_1$ is a $(t_1, \alpha_1, \beta_1, n_1, m_1, s_1)$-net in base b and $\mathcal{P}_2$ a $(t_2, \alpha_2, \beta_2, n_2, m_2, s_2)$-net in base b, it follows that

$$\prod_{j=1}^{s_1} \bigcup_{\substack{a_{j,l}=0 \\ l \in \{1,\ldots,n_1\} \setminus \{i_{j,\nu_j},\ldots,i_{j,1}\}}}^{b-1} \left[\frac{a_{j,1}}{b} + \cdots + \frac{a_{j,n_1}}{b^{n_1}}, \frac{a_{j,1}}{b} + \cdots + \frac{a_{j,n_1}}{b^{n_1}} + \frac{1}{b^{n_1}}\right)$$

contains $b^{m_1 - \sum_{j=1}^{s_1} \nu_j}$ points of $\mathcal{P}_1$ and

$$\prod_{j=s_1+1}^{s} \bigcup_{\substack{a_{j,l}=0 \\ l \in \{1,\ldots,n_2\} \setminus \{i_{j,\nu_j},\ldots,i_{j,1}\}}}^{b-1} \left[\frac{a_{j,1}}{b} + \cdots + \frac{a_{j,n_2}}{b^{n_2}}, \frac{a_{j,1}}{b} + \cdots + \frac{a_{j,n_2}}{b^{n_2}} + \frac{1}{b^{n_2}}\right)$$

contains $b^{m_2 - \sum_{j=s_1+1}^{s} \nu_j}$ points of $\mathcal{P}_2$. By the construction method it follows that $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ contains $b^{m_1+m_2-\sum_{j=1}^{s} \nu_j} = b^{m-|\boldsymbol{\nu}|_1}$ points of $\mathcal{P}$ concluding the proof. □

Remark 4.7. *We remark that Theorem 4.6 can also be proven using duality theory, see [11, Theorem 3.1].*

4.3.2 The $(u, u+v)$-construction

In this subsection we generalize the $(u, u+v)$-construction from coding theory, which seems to stem from [102], to $(t, \alpha, \beta, n, m, s)$-nets. We remark that the $(u, u+v)$-construction has already been used to construct (t, m, s)-nets, see [12], and recently to construct higher order digital nets, see [27]. We rely on Theorem 3.25 to prove the main result of this subsection and now outline the $(u, u+v)$-construction.

Assume we are given a $(t_1, \alpha, \beta_1, n_1, m_1, s_1)$-net $\mathcal{P}_1$ denoted by $\{\boldsymbol{x}_h\}_{h=0}^{b^{m_1}-1}$ and a $(t_2, \alpha, \beta_2, n_2, m_2, s_2)$-net $\mathcal{P}_2$ denoted by $\{\boldsymbol{y}_i\}_{i=0}^{b^{m_2}-1}$, where we assume $s_1 \le s_2$. W.l.o.g. we may assume that $\boldsymbol{x}_h = (x_{h,1}, \ldots, x_{h,s_1})$ with $x_{h,j} = \xi_{h,j,1}/b + \cdots + \xi_{h,j,n_1}/b^{n_1}$ and $\boldsymbol{y}_i = (y_{i,1}, \ldots, y_{i,s_2})$ with $y_{i,j} = \eta_{i,j,1}/b + \cdots + \eta_{i,j,n_2}/b^{n_2}$ (if there are digits $\xi_{h,j,r} \neq 0$ for $r > n_1$ or $\eta_{i,j,r} \neq 0$ for $r > n_2$ we can slightly change $\mathcal{P}_1$ and $\mathcal{P}_2$ by setting $\xi_{h,j,r} = 0$ for $r > n_1$ and $\eta_{i,j,r} = 0$ for $r > n_2$ without changing the $(t_w, \alpha, \beta_w, n_w, m_w, s_w)$-net property of $\mathcal{P}_w$, $w = 1, 2$).

We now define a new point set $\mathcal{P} = (z_h)_{h=0}^{b^{m_1+m_2}-1}$, $\boldsymbol{z}_h = (z_{h,1}, \ldots, z_{h,s_1+s_2})$, consisting of $b^{m_1+m_2}$ points in $[0,1)^{s_1+s_2}$ as follows: first we set

$$\ell := \min(2\beta_1 n_1 - 2t_1 + 1, \beta_2 n_2 - t_2).$$

We recall that the addition modulo b is denoted by $\oplus$ and the subtraction modulo b by $\ominus$ (for short we use $\ominus x := 0 \ominus x$).

Definition 4.8. *We define the following point set $\mathcal{P}$ based on the $(u, u+v)$-construction:*

- *For $j = 1, \ldots, s_1$, $h = 0, \ldots, b^{m_1} - 1$ and $i = 0, \ldots, b^{m_2} - 1$ we set*

$$\begin{aligned} z_{ib^{m_1}+h,j} &= \frac{\xi_{h,j,1} \ominus \eta_{i,j,1}}{b} + \cdots + \frac{\xi_{h,j,\min(\ell,n_1)} \ominus \eta_{i,j,\min(\ell,n_1)}}{b^{\min(\ell,n_1)}} \\ &+ \left(\frac{\xi_{h,j,\ell+1}}{b^{\ell+1}} + \cdots + \frac{\xi_{h,j,n_1}}{b^{n_1}} \right) \mathbf{1}_{n_1 > \ell} \\ &+ \left(\frac{\ominus \eta_{i,j,n_1+1}}{b^{n_1+1}} + \cdots + \frac{\ominus \eta_{i,j,\ell}}{b^{\ell}} \right) \mathbf{1}_{n_1 < \ell}. \end{aligned}$$

- *For $j = s_1 + 1, \ldots, s_1 + s_2$, $h = 0, \ldots, b^{m_1} - 1$ and $i = 0, \ldots, b^{m_2} - 1$ we set*

$$z_{ib^{m_1}+h,j} = y_{i,j-s_1}.$$

Note that for every component of $\boldsymbol{z}_h$ in Definition 4.8 at most the first $\max(n_1, n_2) \le n_1 + n_2 =: n$ digits in its b-adic expansion are non-zero.

In the following we analyze the Weyl sum $Q_{b^{m_1+m_2}}(\mathrm{wal}_{\boldsymbol{k}}, \mathcal{P})$ for $\boldsymbol{k} \in \mathbb{N}_0^{s_1+s_2}$ satisfying $\mu_\alpha(\boldsymbol{k}) \le \ell$. For this analysis we need to recall some notation: for vectors $\boldsymbol{k}, \boldsymbol{l} \in \mathbb{N}_0^s$, $\boldsymbol{k} = (k_1, \ldots, k_s)$, $\boldsymbol{l} = (l_1, \ldots, l_s)$, $\boldsymbol{k} \oplus \boldsymbol{l} := (k_1 \oplus l_1, k_2 \oplus l_2, \ldots, k_s \oplus l_s)$.

We embed a vector $\boldsymbol{u} \in \mathbb{N}_0^{s_1}$ into $\mathbb{N}_0^{s_2}$ by filling up the remaining components with zeros. This vector will be denoted by $(\boldsymbol{u}, \boldsymbol{0}) \in \mathbb{N}_0^{s_2}$. In the following we represent a vector $\boldsymbol{k} \in \mathbb{N}_0^{s_1+s_2}$ in the form $\boldsymbol{k} = (\boldsymbol{u}, (\boldsymbol{u}, \boldsymbol{0}) \oplus \boldsymbol{v})$, where $\boldsymbol{u} \in \mathbb{N}_0^{s_1}$, $\boldsymbol{v} \in \mathbb{N}_0^{s_2}$, i.e. $\boldsymbol{k}$ is the concatenation of the two vectors $\boldsymbol{u} \in \mathbb{N}_0^{s_1}$ and $(\boldsymbol{u}, \boldsymbol{0}) \oplus \boldsymbol{v} \in \mathbb{N}_0^{s_2}$.

Lemma 4.9. *For $\boldsymbol{k} \in \{0, \ldots, b^\ell - 1\}^{s_1+s_2}$ and $\mathcal{P}_1$ and $\mathcal{P}_2$ as defined above and $\mathcal{P}$ as given in Definition 4.8 we have*

$$Q_{b^{m_1+m_2}}(\mathrm{wal}_{\boldsymbol{k}}, \mathcal{P}) = Q_{b^{m_1}}(\mathrm{wal}_{\boldsymbol{u}}, \mathcal{P}_1) Q_{b^{m_2}}(\mathrm{wal}_{\boldsymbol{v}}, \mathcal{P}_2).$$

Proof. For $\boldsymbol{y}_h \in [0,1)^{s_2}$ we denote the projection onto its first s_1 components by $\boldsymbol{y}_h^{(s_1)}$. Then we have

$$\frac{1}{b^{m_1+m_2}} \sum_{h'=0}^{b^{m_1+m_2}-1} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{z}_{h'}) = \frac{1}{b^{m_1+m_2}} \sum_{h=0}^{b^{m_1}-1} \sum_{i=0}^{b^{m_2}-1} \mathrm{wal}_{(\boldsymbol{u},(\boldsymbol{u},\boldsymbol{0}) \oplus \boldsymbol{v})}(\boldsymbol{z}_{ib^{m_1}+h})$$

$$= \frac{1}{b^{m_1+m_2}} \sum_{h=0}^{b^{m_1}-1} \sum_{i=0}^{b^{m_2}-1} \mathrm{wal}_{\boldsymbol{u}}(\boldsymbol{x}_h \ominus \boldsymbol{y}_i^{(s_1)}) \mathrm{wal}_{(\boldsymbol{u},\boldsymbol{0})\oplus \boldsymbol{v}}(\boldsymbol{y}_i)$$
$$= \frac{1}{b^{m_1}} \sum_{h=0}^{b^{m_1}-1} \mathrm{wal}_{\boldsymbol{u}}(\boldsymbol{x}_h) \frac{1}{b^{m_2}} \sum_{i=0}^{b^{m_2}-1} \mathrm{wal}_{\boldsymbol{v}}(\boldsymbol{y}_i).$$

The last two equalities use the assumption that $\boldsymbol{k} \in \{0, \ldots, b^\ell - 1\}^{s_1+s_2}$, which means that for all components of $\boldsymbol{k}$ at most the first ℓ digits in their b-adic expansion are different from zero. □

We need the following lemma, which is [8, Lemma 5].

Lemma 4.10. *For $\alpha \in \mathbb{N}$, $\alpha \geq 2$, and $\boldsymbol{k}, \boldsymbol{l} \in \mathbb{N}_0^s$ we have $\mu_\alpha(\boldsymbol{k} \oplus \boldsymbol{l}) \geq \mu_\alpha(\boldsymbol{k}) - \mu_\alpha(\boldsymbol{l})$.*

The following theorem establishes the main result of this subsection.

Theorem 4.11. *Let $b \in \mathbb{N}$, $b \geq 2$, $\mathcal{P}_1$ be a $(t_1, \alpha, \beta_1, n_1, m_1, s_1)$-net in base b and $\mathcal{P}_2$ be a $(t_2, \alpha, \beta_2, n_2, m_2, s_2)$-net in base b. Then $\mathcal{P}$ as given in Definition 4.8 is a $(t, \alpha, \beta, n, m, s)$-net in base b where $n = n_1 + n_2$, $m = m_1 + m_2$, $s = s_1 + s_2$ and*

$$\beta = \min(\beta_1, \beta_2), \quad t = \beta n - \ell.$$

Proof. We rely on Theorem 3.25 to establish the result, i.e. we need to show that for all $\boldsymbol{k} \in \mathbb{N}_0^{s_1+s_2}$ satisfying $0 < \mu_\alpha(\boldsymbol{k}) \leq \beta n - t$ we have

$$Q_{b^{m_1+m_2}}(\mathrm{wal}_{\boldsymbol{k}}, \mathcal{P}) = 0.$$

For $\boldsymbol{k} \in \mathbb{N}_0^{s_1+s_2}$ satisfying $0 < \mu_\alpha(\boldsymbol{k}) \leq \beta n - t = \ell$ we necessarily have that $\boldsymbol{k} \in \{0, \ldots, b^\ell - 1\}^{s_1+s_2}$. Hence we may use Lemma 4.9 which states that

$$Q_{b^{m_1+m_2}}(\mathrm{wal}_{\boldsymbol{k}}, \mathcal{P}) = Q_{b^{m_1}}(\mathrm{wal}_{\boldsymbol{u}}, \mathcal{P}_1) Q_{b^{m_2}}(\mathrm{wal}_{\boldsymbol{v}}, \mathcal{P}_2).$$

We proceed in a manner very similar to the proof of [93, Theorem 5.3] and distinguish three cases.

Case 1: We firstly assume that $\boldsymbol{v} \neq \boldsymbol{0}$ and $\mu_\alpha(\boldsymbol{k}) \leq \beta n - t$. We want to show that $0 < \mu_\alpha(\boldsymbol{v}) \leq \beta_2 n_2 - t_2$, in which case we obtain $Q_{b^{m_2}}(\mathrm{wal}_{\boldsymbol{v}}, \mathcal{P}_2) = 0$ from Theorem 3.25. As $\boldsymbol{v} \neq \boldsymbol{0}$ we have $\mu_\alpha(\boldsymbol{v}) > 0$. Also using Lemma 4.10

$$\mu_\alpha(\boldsymbol{v}) \leq \mu_\alpha((\boldsymbol{u}, \boldsymbol{0}) \oplus \boldsymbol{v}) + \mu_\alpha(\boldsymbol{u}) = \mu_\alpha(\boldsymbol{k}) \leq \beta n - t \leq \beta_2 n_2 - t_2.$$

Case 2: We now assume that $\boldsymbol{v} = \boldsymbol{0}$, $\boldsymbol{u} \neq \boldsymbol{0}$ and $0 < \mu_\alpha(\boldsymbol{k}) \leq \beta n - t$. We want to show that $0 < \mu_\alpha(\boldsymbol{u}) \leq \beta_1 n_1 - t_1$, in which case we obtain $Q_{b^{m_1}}(\mathrm{wal}_{\boldsymbol{u}}, \mathcal{P}_1) = 0$ from Theorem 3.25. As $\boldsymbol{u} \neq \boldsymbol{0}$ we have $\mu_\alpha(\boldsymbol{u}) > 0$. Also,

$$2(\beta_1 n_1 - t_1) + 1 \geq \beta n - t \geq \mu_\alpha(\boldsymbol{k}) = \mu_\alpha((\boldsymbol{u}, \boldsymbol{0}) \oplus \boldsymbol{v}) + \mu_\alpha(\boldsymbol{u}) = 2\mu_\alpha(\boldsymbol{u}).$$

Hence $\mu_\alpha(\boldsymbol{u}) \le \beta_1 n_1 - t_1$, as $\mu_\alpha(\boldsymbol{u})$ is an integer.

Case 3: We now assume that $\boldsymbol{v} = \boldsymbol{0}$, $\boldsymbol{u} = \boldsymbol{0}$, and $0 < \mu_\alpha(\boldsymbol{k}) \le \beta n - t$. However, as $\boldsymbol{v} = \boldsymbol{0}$ and $\boldsymbol{u} = \boldsymbol{0}$, it follows that $\mu_\alpha(\boldsymbol{k}) = 0$, hence this case need not be considered.

Thus we have $Q_{b^{m_1+m_2}}(\mathrm{wal}_{\boldsymbol{k}}, \mathcal{P}) = 0$ whenever $0 < \mu_\alpha(\boldsymbol{k}) \le \beta n - t$ and this completes the proof. □

Remark 4.12. *Note that we defined the $(u, u+v)$-construction in such a way that it yields the same point set as the $(u, u+v)$-construction for higher order digital nets considered in [27].*

4.3.3 The matrix-product construction

In this subsection, we present the matrix-product construction for $(t, \alpha, \beta, n, m, s)$-nets. The matrix-product construction for classical digital nets was introduced in [71] and generalized to cover higher order digital nets in [27]. Throughout this subsection we assume that b is prime. We firstly introduce matrices which are non-singular by column (NSC), see [13]. Let A be an $M \times M$ matrix over $\mathbb{Z}_b$. For $1 \le l \le M$ let A_l denote the $l \times M$ matrix consisting of the first l rows of A. Furthermore, for $1 \le k_1 < \cdots < k_l \le M$ let $A(k_1, \ldots, k_l)$ denote the $l \times l$ matrix consisting of the columns $k_1, \ldots, k_l$ of A_l.

Definition 4.13. An $M \times M$ matrix A defined over $\mathbb{Z}_b$ is called non-singular by column (NSC) if $A(k_1, \ldots, k_l)$ is non-singular for each $1 \le l \le M$ and $1 \le k_1 < \cdots < k_l \le M$.

It is known that an $M \times M$ NSC matrix over $\mathbb{Z}_b$ exists if and only if $1 \le M \le b$, see [13, Section 3]. For any integer $1 \le M \le b$ an explicit $M \times M$ upper triangular NSC matrix over $\mathbb{Z}_b$ is given in [13, Section 5.2].

For the remainder of this subsection we assume that $A = (A_{k,l})$ is an $M \times M$ upper triangular NSC matrix over $\mathbb{Z}_b$ (upper triangular means that $A_{k,l} = 0$ for all $1 \le l < k \le M$).

We now describe how to construct the point set based on the so-called matrix-product construction.

Let $1 \le s_1 \le \cdots \le s_M$ be integers and define $\sigma_0 := 0$ and $\sigma_k := s_1 + \cdots + s_k$ for $1 \le k \le M$. Let $s := \sigma_M$. For $1 \le k \le M$ let $\mathcal{P}_k = \{\boldsymbol{x}_h^{(k)}\}_{h=0}^{b^{m_k}-1}$, where $\boldsymbol{x}_h^{(k)} = (x_{h,\sigma_{k-1}+1}^{(k)}, \ldots, x_{h,\sigma_k}^{(k)})$ for $0 \le h < b^{m_k}$, be $(t_k, \alpha, \beta_k, n_k, m_k, s_k)$-nets in base b. (As with the $(u, u+v)$-construction one can without loss of generality assume that $x_{h,j}^{(k)} = \xi_{h,j,1}^{(k)}/b + \xi_{h,j,2}^{(k)}/b^2 + \cdots$ where $\xi_{h,j,c}^{(k)} = 0$ for $c > n_k$, as setting the remaining digits to zero does not affect the quality of the net $\mathcal{P}_k$. However, this is not necessary as the results in this subsection also hold otherwise.)

We now define $V = (V_{k,l})_{k,l=1}^M := A^{-1} \in \mathbb{Z}_b^{M \times M}$ and note that V is upper triangular. For

$$h = h_1 + h_2 b^{m_1} + \cdots + h_M b^{m_1+m_2+\cdots+m_{M-1}},$$

with integers $0 \le h_k < b^{m_k}$ (hence $0 \le h < b^m$ where $m = m_1 + \cdots + m_M$) and for $\sigma_{k-1} < j \le \sigma_k$, $k = 1, \ldots, M$, define

$$z_{h,j} := V_{k,k} x_{h_k,j}^{(k)} \oplus \cdots \oplus V_{k,M} x_{h_M,j}^{(M)}, \tag{4.6}$$

where $\oplus$ and also the multiplication are carried out digit-wise modulo b, i.e. $z_{h,j} = \zeta_{h,j,1}/b + \zeta_{h,j,2}/b^2 + \cdots$ where

$$\zeta_{h,j,c} = V_{k,k} \xi_{h_k,j,c}^{(k)} + \cdots + V_{k,M} \xi_{h_M,j,c}^{(M)} \in \mathbb{Z}_b \quad \text{for all } c \ge 1,$$

where $x_{h_l,j}^{(l)} = \xi_{h_l,j,1}^{(l)}/b + \xi_{h_l,j,2}^{(l)}/b^2 + \cdots$ for $k \le l \le M$, where addition and multiplication are carried out in $\mathbb{Z}_b$, and where we assume that for each h and j infinitely many of the digits $\zeta_{h,j,c}$, $c = 1, 2, \ldots$, are different from $b - 1$ (if this is not the case, then for example modifying any of the digits $\zeta_{h,j,c}$, $c = \max_{1 \le k \le M} n_k + 1, \max_{1 \le k \le M} n_k + 2, \ldots$, will solve this problem without affecting the quality of the point set; indeed, the forthcoming Theorem 4.16 will establish that the digits $\zeta_{h,j,c}$, where $c > \min_{1 \le k \le M}(M - k + 1)(\beta_k n_k - t_k)$, can be modified arbitrarily, since they do not influence the quality of the higher order net; this way, for $M = 2$ the $(u, u + v)$-construction can be viewed as a special case of the matrix-product construction).

Analogously to the notation used above we write $\bigoplus_{l=1}^k A_{l,k} \boldsymbol{u}_l^{(k)} = A_{1,k} \boldsymbol{u}_1^{(k)} \oplus \cdots \oplus A_{k,k} \boldsymbol{u}_k^{(k)}$, where the addition and multiplication are carried out digit-wise modulo b.

Now we define $\mathcal{P} = \{\boldsymbol{z}_0, \ldots, \boldsymbol{z}_{b^m-1}\}$ with $m = m_1 + \cdots + m_M$ through $\boldsymbol{z}_h := (z_{h,1}, \ldots, z_{h,s})$ for $0 \le h < b^m$.

Lemma 4.14. *Let $\boldsymbol{d} = (\boldsymbol{d}_1, \ldots, \boldsymbol{d}_M) \in \mathcal{K}_{n,b}^s$ where $\boldsymbol{d}_k \in \mathcal{K}_{n,b}^{s_k}$ and assume that $\boldsymbol{d}_k = \bigoplus_{l=1}^k A_{l,k} \boldsymbol{u}_l^{(k)}$ where for $l \le k$ $\boldsymbol{u}_l^{(k)} = (\boldsymbol{u}_l, \boldsymbol{0}) \in \mathcal{K}_{n,b}^{s_k}$ for some $\boldsymbol{u}_l \in \mathcal{K}_{n,b}^{s_l}$. Then we have*

$$\frac{1}{b^{m_1+m_2+\cdots+m_M}} \sum_{h=0}^{b^{m_1+m_2+\cdots+m_M}-1} \mathrm{wal}_{\boldsymbol{d}}(\boldsymbol{z}_h) = \prod_{r=1}^M \left(\frac{1}{b^{m_r}} \sum_{h_r=0}^{b^{m_r}-1} \mathrm{wal}_{\boldsymbol{u}_r}(\boldsymbol{x}_{h_r}^{(r)}) \right).$$

Proof. Let $\boldsymbol{z}_h = (\boldsymbol{z}_h^{(s_1)}, \ldots, \boldsymbol{z}_h^{(s_M)}) \in [0,1)^{s_1+\cdots+s_M}$ where $\boldsymbol{z}_h^{(s_k)} = (z_{h,\sigma_{k-1}+1}, \ldots, z_{h,\sigma_k}) \in [0,1)^{s_k}$ for $1 \le k \le M$. For $\boldsymbol{d} = (\boldsymbol{d}_1, \ldots, \boldsymbol{d}_M) \in \mathcal{K}_{n,b}^s$ with $\boldsymbol{d}_k \in \mathcal{K}_{n,b}^{s_k}$ we have

$$\sum_{h=0}^{b^{m_1+\cdots+m_M}-1} \mathrm{wal}_{\boldsymbol{d}}(\boldsymbol{z}_h) = \sum_{h=0}^{b^{m_1+\cdots+m_M}-1} \prod_{k=1}^M \mathrm{wal}_{\boldsymbol{d}_k}(\boldsymbol{z}_h^{(s_k)}).$$

By assumption we have $\boldsymbol{d}_k = \bigoplus_{l=1}^{k} A_{l,k}\boldsymbol{u}_l^{(k)}$ where for $l \le k$ $\boldsymbol{u}_l^{(k)} = (\boldsymbol{u}_l, \boldsymbol{0}) \in \mathcal{K}_{n,b}^{s_k}$ for some $\boldsymbol{u}_l \in \mathcal{K}_{n,b}^{s_l}$. Let furthermore $\overline{\boldsymbol{u}}_l = (\boldsymbol{u}_l, \boldsymbol{0}) \in \mathcal{K}_{n,b}^{s}$. Then for each of the above summands we have

$$\begin{aligned}
\prod_{k=1}^{M} \mathrm{wal}_{\boldsymbol{d}_k}(\boldsymbol{z}_h^{(s_k)}) &= \prod_{k=1}^{M} \mathrm{wal}_{\bigoplus_{l=1}^{k} A_{l,k}\boldsymbol{u}_l^{(k)}}(\boldsymbol{z}_h^{(s_k)}) \\
&= \prod_{k=1}^{M} \mathrm{wal}_{\bigoplus_{l=1}^{k} A_{l,k}\boldsymbol{u}_l^{(k)}}(z_{h,\sigma_{k-1}+1}, \ldots, z_{h,\sigma_k}) \\
&= \prod_{k=1}^{M}\prod_{r=k}^{M} \mathrm{wal}_{\bigoplus_{l=1}^{k} A_{l,k}\boldsymbol{u}_l^{(k)}}(V_{k,r}x_{h_r,\sigma_{k-1}+1}^{(r)}, \ldots, V_{k,r}x_{h_r,\sigma_k}^{(r)}) \\
&= \prod_{k=1}^{M}\prod_{r=k}^{M} \mathrm{wal}_{V_{k,r}\left(\bigoplus_{l=1}^{k} A_{l,k}\boldsymbol{u}_l^{(k)}\right)}(x_{h_r,\sigma_{k-1}+1}^{(r)}, \ldots, x_{h_r,\sigma_k}^{(r)}) \\
&= \prod_{k=1}^{M}\prod_{r=k}^{M} \mathrm{wal}_{V_{k,r}\left(\bigoplus_{l=1}^{k} A_{l,k}\boldsymbol{u}_l^{(k)}\right)}(\boldsymbol{x}_{h_r}^{(r)}) \\
&= \prod_{r=1}^{M}\prod_{k=1}^{r} \mathrm{wal}_{V_{k,r}\left(\bigoplus_{l=1}^{k} A_{l,k}\boldsymbol{u}_l^{(k)}\right)}(\boldsymbol{x}_{h_r}^{(r)}) \\
&= \prod_{r=1}^{M} \mathrm{wal}_{\bigoplus_{k=1}^{r} V_{k,r}\left(\bigoplus_{l=1}^{k} A_{l,k}\boldsymbol{u}_l^{(k)}\right)}(\boldsymbol{x}_{h_r}^{(r)}) \\
&= \prod_{r=1}^{M} \mathrm{wal}_{\bigoplus_{k=1}^{r} V_{k,r}\left(\bigoplus_{l=1}^{k} A_{l,k}\overline{\boldsymbol{u}}_l\right)}((\boldsymbol{x}_{h_r}^{(r)}, \boldsymbol{0})),
\end{aligned}$$

where $(\boldsymbol{x}_{h_r}^{(r)}, \boldsymbol{0}) \in [0,1)^s$ is just the concatenation of $\boldsymbol{x}_{h_r}^{(r)} \in [0,1)^{s_r}$ and the $s - s_r$ dimensional zero vector $\boldsymbol{0}$. Since $V = A^{-1}$ we now have

$$\bigoplus_{k=l}^{r} V_{k,r}A_{l,k} = \begin{cases} 1 & \text{if } r = l \\ 0 & \text{if } r \neq l. \end{cases}$$

Hence we obtain $\bigoplus_{k=1}^{r} V_{k,r} \bigoplus_{l=1}^{k} A_{l,k}\overline{\boldsymbol{u}}_l = \bigoplus_{l=1}^{r} \overline{\boldsymbol{u}}_l \bigoplus_{k=l}^{r} V_{k,r}A_{l,k} = \overline{\boldsymbol{u}}_r$ and therefore

$$\prod_{r=1}^{M} \mathrm{wal}_{\bigoplus_{k=1}^{r} V_{k,r}\left(\bigoplus_{l=1}^{k} A_{l,k}\overline{\boldsymbol{u}}_l\right)}((\boldsymbol{x}_{h_r}^{(r)}, \boldsymbol{0})) = \prod_{r=1}^{M} \mathrm{wal}_{\overline{\boldsymbol{u}}_r}((\boldsymbol{x}_{h_r}^{(r)}, \boldsymbol{0})) = \prod_{r=1}^{M} \mathrm{wal}_{\boldsymbol{u}_r}(\boldsymbol{x}_{h_r}^{(r)}).$$

Consequently,

$$\begin{aligned}
\frac{1}{b^{m_1+\cdots+m_M}} \sum_{h=0}^{b^{m_1+\cdots+m_M}-1} \mathrm{wal}_{\boldsymbol{d}}(\boldsymbol{z}_h) &= \frac{1}{b^{m_1+\cdots+m_M}} \sum_{h=0}^{b^{m_1+\cdots+m_M}-1} \prod_{r=1}^{M} \mathrm{wal}_{\boldsymbol{u}_r}(\boldsymbol{x}_{h_r}^{(r)}) \\
&= \prod_{r=1}^{M} \left(\frac{1}{b^{m_r}} \sum_{h_r=0}^{b^{m_r}-1} \mathrm{wal}_{\boldsymbol{u}_r}(\boldsymbol{x}_{h_r}^{(r)}) \right).
\end{aligned}$$

□

For the rest of this subsection we make the convention that

$$\mu_\alpha(\boldsymbol{d}) = \sum_{k=1}^{M} \mu_\alpha(\boldsymbol{d}_k).$$

If $\mu_\alpha(\boldsymbol{d}) > 0$, then there exists at least one integer l so that $\boldsymbol{u}_l \neq \boldsymbol{0}$; the largest integer l so that $\boldsymbol{u}_l \neq \boldsymbol{0}$ is denoted by l^*. We need the following lemma.

Lemma 4.15. *Let $\boldsymbol{d}$ be as in Lemma 4.14 and so that $\mu_\alpha(\boldsymbol{d}) > 0$ and let l^* denote the largest integer l for which $\boldsymbol{u}_l \neq \boldsymbol{0}$. Then we have $\mu_\alpha(\boldsymbol{d}) \geq (M - l^* + 1)\mu_\alpha(\boldsymbol{u}_{l^*})$.*

Proof. The proof follows along the same lines as the proofs of [27, Lemmas 2 and 3]. □

We can now show the main result of this subsection.

Theorem 4.16. *The multiset $\mathcal{P} = \{\boldsymbol{z}_0, \ldots, \boldsymbol{z}_{b^m-1}\}$, where $\boldsymbol{z}_h := (z_{h,1}, \ldots, z_{h,s})$ and the $z_{h,j}$ are given by Equation (4.6), forms a $(t, \alpha, \beta, n, m, s)$-net where $s = s_1 + \cdots + s_M$, $n = \max_{1 \leq k \leq M} n_k$, $m = m_1 + \cdots + m_M$, $\beta = \min(1, \frac{\alpha m}{n})$ and*

$$t \leq \beta n - \min_{1 \leq l \leq M} (M - l + 1)(\beta_l n_l - t_l).$$

Proof. According to Theorem 3.25 it is enough to show that

$$\frac{1}{b^{m_1+m_2+\cdots+m_M}} \sum_{h=0}^{b^{m_1+m_2+\cdots+m_M}-1} \mathrm{wal}_{\boldsymbol{d}}(\boldsymbol{z}_h) = 0$$

for all $\boldsymbol{d} \in \mathbb{N}_0^s$ satisfying $0 < \mu_\alpha(\boldsymbol{d}) \leq \beta n - t$. As $\boldsymbol{d}$ must satisfy $\mu_\alpha(\boldsymbol{d}) \leq \beta n - t$ we may restrict ourselves to $\boldsymbol{d} \in \mathcal{K}_{n,b}^s$ satisfying $0 < \mu_\alpha(\boldsymbol{d}) \leq \beta n - t$. From Lemma 4.14 we know that

$$\frac{1}{b^{m_1+m_2+\cdots+m_M}} \sum_{h=0}^{b^{m_1+m_2+\cdots+m_M}-1} \mathrm{wal}_{\boldsymbol{d}}(\boldsymbol{z}_h) = \prod_{r=1}^{M} \left(\frac{1}{b^{m_r}} \sum_{h_r=0}^{b^{m_r}-1} \mathrm{wal}_{\boldsymbol{u}_r}(\boldsymbol{x}_{h_r}^{(r)}) \right). \tag{4.7}$$

Assume now that $\boldsymbol{d} \in \mathcal{K}_{n,b}^s$ is such that $0 < \mu_\alpha(\boldsymbol{d}) \leq \beta n - t$, then there exists an integer l so that $\mu_\alpha(\boldsymbol{u}_l) > 0$, and as before we denote the largest integer l so that $\mu_\alpha(\boldsymbol{u}_l) > 0$ by l^*. We now use Lemma 4.15 to conclude that

$$\begin{aligned} (M - l^* + 1)(\beta_{l^*} n_{l^*} - t_{l^*}) &\geq \min_{1 \leq l \leq M} (M - l + 1)(\beta_l n_l - t_l) \\ &= \beta n - t \geq \mu_\alpha(\boldsymbol{d}) \geq (M - l^* + 1)\mu_\alpha(\boldsymbol{u}_{l^*}). \end{aligned}$$

Hence we have shown that $0 < \mu_\alpha(\boldsymbol{u}_{l^*}) \leq \beta_{l^*} n_{l^*} - t_{l^*}$ and therefore

$$\frac{1}{b^{m_{l^*}}} \sum_{h_{l^*}=0}^{b^{m_{l^*}}-1} \mathrm{wal}_{\boldsymbol{u}_{l^*}}(\boldsymbol{x}_{h_{l^*}}^{(l^*)}) = 0,$$

i.e. the l^*th factor in Equation (4.7) is zero. □

4.3.4 A double m construction

In this subsection we aim to generalize a propagation rule referred to as "A double m construction" in [27, Section 3.4], which again generalizes a propagation rule from [72] for digital (t, m, s)-nets. We remark that this is the first time that this propagation rule appears in the context of not necessarily digital nets.

Assume we are given a $(t_1, \alpha_1, \beta_1, n, m, s)$-net in base b denoted by $\mathcal{P}_1 = \{\boldsymbol{x}_h\}_{h=0}^{b^m-1}$ and a $(t_2, \alpha_2, \beta_2, n, m, s)$-net in base b denoted by $\mathcal{P}_2 = \{\boldsymbol{y}_i\}_{i=0}^{b^{m-1}}$. For $\boldsymbol{x}_h = (x_{h,1}, \ldots, x_{h,s})$ we write

$$x_{h,j} = \frac{\xi_{h,j,1}}{b} + \cdots + \frac{\xi_{h,j,n}}{b^n}$$

and for $\boldsymbol{y}_i = (y_{i,1}, \ldots, y_{i,s})$ we set

$$y_{i,j} = \frac{\eta_{i,j,1}}{b} + \cdots + \frac{\eta_{i,j,n}}{b^n}.$$

Furthermore, the dual set associated with $\mathcal{P}_1$ is denoted by $\mathcal{D}_n^{(1)}$ and the dual set associated with $\mathcal{P}_2$ by $\mathcal{D}_n^{(2)}$. We are now in a position to define a multiset $\mathcal{P} := \{\boldsymbol{z}_0, \ldots, \boldsymbol{z}_{b^{2m}-1}\}$ as follows: for $h' = hb^m + i$, $0 \le h \le b^m - 1$, $0 \le i \le b^m - 1$, we set

$$z_{h',j} = \frac{\xi_{h,j,1} \oplus \eta_{i,j,1}}{b} + \cdots + \frac{\xi_{h,j,n} \oplus \eta_{i,j,n}}{b^n} + \frac{0 \ominus \eta_{i,j,1}}{b^{n+1}} + \cdots + \frac{0 \ominus \eta_{i,j,n}}{b^{2n}}, \tag{4.8}$$

$h' = 0, \ldots, b^{2m} - 1$, $j = 1, \ldots, s$. We now define a set $\mathcal{N}$, which in the forthcoming Lemma 4.17 is shown to be the dual set of $\mathcal{P}$. Let $\boldsymbol{a}_r = (a_{r,1}, \ldots, a_{r,s}) \in \mathcal{D}_n^{(r)}$, $r = 1, 2$ and define $\boldsymbol{k} = \boldsymbol{k}(\boldsymbol{a}_1, \boldsymbol{a}_2) := (k_1, \ldots, k_s)$ where

$$k_j = a_{1,j} + b^n(a_{1,j} \oplus a_{2,j}), \quad j = 1, \ldots, s,$$

then we set $\mathcal{N} = \left\{\boldsymbol{k}(\boldsymbol{a}_1, \boldsymbol{a}_2) \in \mathcal{K}_{2n,b}^s : \boldsymbol{a}_1 \in \mathcal{D}_n^{(1)}, \boldsymbol{a}_2 \in \mathcal{D}_n^{(2)}\right\}$.

Lemma 4.17. *The set* $\mathcal{N} = \left\{\boldsymbol{k} \in \mathcal{K}_{2n,b}^s : \boldsymbol{a}_1 \in \mathcal{D}_n^{(1)}, \boldsymbol{a}_2 \in \mathcal{D}_n^{(2)}\right\}$ *is the dual set of* $\mathcal{P} = \{\boldsymbol{z}_0, \ldots, \boldsymbol{z}_{b^{2m}-1}\}$ *where* $\boldsymbol{z}_h := (z_{h,1}, \ldots, z_{h,s})$ *and the* $z_{h,j}$ *are given by Equation* (4.8).

Proof. Let $\boldsymbol{k} = (k_1, \ldots, k_s) \in \mathcal{K}_{2n,b}^s$ where $k_j = a_{1,j} + b^n(a_{1,j} \oplus a_{2,j})$, $j = 1, \ldots, s$, and where $\boldsymbol{a}_r = (a_{r,1}, \ldots, a_{r,s}) \in \mathcal{K}_{n,b}^s$, $r = 1, 2$. Clearly,

$$c_{\boldsymbol{k}} = \sum_{h'=0}^{b^{2m}-1} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{z}_{h'}) = \sum_{h=0}^{b^m-1} \sum_{i=0}^{b^m-1} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{z}_{hb^m+i}) = \sum_{h=0}^{b^m-1} \sum_{i=0}^{b^m-1} \prod_{j=1}^{s} \mathrm{wal}_{k_j}(z_{hb^m+i,j}).$$

For brevity, we set $k_j = k_j^{(1)} + b^n k_j^{(2)}$ where $k_j^{(1)}$ and $k_j^{(2)}$ have the b-adic expansions $k_j^{(1)} = \sum_{l=1}^n k_{j,l}^{(1)} b^{l-1}$ and $k_j^{(2)} = \sum_{l=1}^n k_{j,l}^{(2)} b^{l-1}$. Hence,

$$\begin{aligned}
\mathrm{wal}_{k_j}(z_{hb^m+i,j}) &= \exp\left[\frac{2\pi\mathtt{i}}{b}\left(\sum_{l=1}^{n} k_{j,l}^{(1)}(\xi_{h,j,l} \oplus \eta_{i,j,l}) + \sum_{l=n+1}^{2n} k_{j,l-n}^{(2)}(0 \ominus \eta_{i,j,l-n})\right)\right] \\
&= \exp\left[\frac{2\pi\mathtt{i}}{b} \sum_{l=1}^{n} k_{j,l}^{(1)}(\xi_{h,j,l} \oplus \eta_{i,j,l})\right] \exp\left[\frac{2\pi\mathtt{i}}{b} \sum_{l=1}^{n} k_{j,l}^{(2)}(0 \ominus \eta_{i,j,l})\right] \\
&= \mathrm{wal}_{k_j^{(1)}}(x_{h,j} \oplus y_{i,j}) \mathrm{wal}_{k_j^{(2)}}(0 \ominus y_{i,j}) \\
&= \mathrm{wal}_{a_{1,j}}(x_{h,j}) \mathrm{wal}_{a_{1,j}}(y_{i,j}) \mathrm{wal}_{a_{1,j}}(0 \ominus y_{i,j}) \mathrm{wal}_{a_{2,j}}(0 \ominus y_{i,j}) \\
&= \mathrm{wal}_{a_{1,j}}(x_{h,j}) \overline{\mathrm{wal}_{a_{2,j}}(y_{i,j})}
\end{aligned}$$

and further,

$$\begin{aligned} c_{\boldsymbol{k}} &= \sum_{h=0}^{b^m-1} \sum_{i=0}^{b^m-1} \prod_{j=1}^{s} \mathrm{wal}_{a_{1,j}}(x_{h,j}) \overline{\mathrm{wal}_{a_{2,j}}(y_{i,j})} \\ &= \sum_{h=0}^{b^m-1} \sum_{i=0}^{b^m-1} \mathrm{wal}_{\boldsymbol{a}_1}(\boldsymbol{x}_h) \overline{\mathrm{wal}_{\boldsymbol{a}_2}(\boldsymbol{y}_i)} \\ &= \sum_{h=0}^{b^m-1} \mathrm{wal}_{\boldsymbol{a}_1}(\boldsymbol{x}_h) \sum_{i=0}^{b^m-1} \overline{\mathrm{wal}_{\boldsymbol{a}_2}(\boldsymbol{y}_i)} \\ &= \sum_{h=0}^{b^m-1} \mathrm{wal}_{\boldsymbol{a}_1}(\boldsymbol{x}_h) \overline{\sum_{i=0}^{b^m-1} \mathrm{wal}_{\boldsymbol{a}_2}(\boldsymbol{y}_i)}. \end{aligned}$$

If $\boldsymbol{k} \in \mathcal{N}$, then $\boldsymbol{a}_1 \in \mathcal{D}_n^{(1)}$ and $\boldsymbol{a}_2 \in \mathcal{D}_n^{(2)}$, so we have $c_{\boldsymbol{k}} \neq 0$ and hence $\boldsymbol{k}$ is in the dual set of $\mathcal{P}$. If on the other hand $\boldsymbol{k}$ is in the dual set of $\mathcal{P}$, then $c_{\boldsymbol{k}} \neq 0$ and hence $\boldsymbol{a}_1 \in \mathcal{D}_n^{(1)}$ and $\boldsymbol{a}_2 \in \mathcal{D}_n^{(2)}$, so $\boldsymbol{k} \in \mathcal{N}$. □

In order to bound the quality parameter of $\mathcal{P} = \{\boldsymbol{z}_0, \ldots, \boldsymbol{z}_{b^{2m}-1}\}$, we define

$$d = d(\mathcal{D}_n^{(1)}, \mathcal{D}_n^{(2)}) := \max_{1 \le j \le s} \max_{R_j} \max(0, \mu_\alpha(a_{1,j}) - \mu_\alpha(a_{1,j} \oplus a_{2,j})),$$

where R_j is the set of all ordered pairs $(\boldsymbol{a}_1, \boldsymbol{a}_2)$ with $\boldsymbol{a}_r = (a_{r,1}, \ldots, a_{r,s}) \in \mathcal{D}_n^{(r)} \setminus \{\boldsymbol{0}\}$, $a_{1,i} \oplus a_{2,i} = 0$ for $i \neq j$ and $a_{1,j} \oplus a_{2,j} \neq 0$. We define the max over R_j to be zero if R_j is empty. We can now prove the main result of this subsection.

Theorem 4.18. *Let $\mathcal{P}_1$ be a $(t_1, \alpha_1, \beta_1, n, m, s)$-net in base b with dual set $\mathcal{D}_n^{(1)}$ and $\mathcal{P}_2$ be a $(t_2, \alpha_2, \beta_2, n, m, s)$-net in base b with dual set $\mathcal{D}_n^{(2)}$. Let $d = d(\mathcal{D}_n^{(1)}, \mathcal{D}_n^{(2)})$. Then the point set given by Equation* (4.8) *is a $(t, \alpha, \beta, 2n, 2m, s)$-net in base b with $\alpha = \max(\alpha_1, \alpha_2)$, $\beta = \min(\beta_1, \beta_2)$ and*

$$t \le \max(\lfloor 2\beta n \rfloor - n - \lfloor \beta_1 n \rfloor + t_1 + d, \lfloor 2\beta n \rfloor - n - \lfloor \beta_2 n \rfloor + t_2, 0)$$

if $\mathcal{D}_n^{(1)} \cap \mathcal{D}_n^{(2)} = \{\boldsymbol{0}\}$, and

$$t \le \max(\lfloor 2\beta n \rfloor - n - \lfloor \beta_1 n \rfloor + t_1 + d, \lfloor 2\beta n \rfloor - n - \lfloor \beta_2 n \rfloor + t_2, \lfloor 2\beta n \rfloor + 1 - \rho_\alpha(\mathcal{D}_n^{(1)} \cap \mathcal{D}_n^{(2)}), 0)$$

if $\mathcal{D}_n^{(1)} \cap \mathcal{D}_n^{(2)} \neq \{\boldsymbol{0}\}$.

Proof. Clearly, $0 < \beta \le 1$ and $\alpha \ge 1$. We show a lower bound on $\mu_\alpha(\boldsymbol{k})$ for all non-zero vectors $\boldsymbol{k} \in \mathcal{N}$, which by Lemma 4.17 is the dual set of the point set given by Equation (4.8). To this end we use the property that $\rho_\alpha(\mathcal{D}_n^{(r)}) \ge \rho_{\alpha_r}(\mathcal{D}_n^{(r)}) \ge \lfloor \beta_r n \rfloor - t_r + 1$, as $\alpha \ge \alpha_r$, $r = 1, 2$. For $\boldsymbol{k} \in \mathcal{N}$, $\boldsymbol{k} \neq \boldsymbol{0}$, we have $\boldsymbol{k} = \boldsymbol{a}_1 + b^n(\boldsymbol{a}_1 \oplus \boldsymbol{a}_2)$ with $\boldsymbol{a}_1 \in \mathcal{D}_n^{(1)}$ and $\boldsymbol{a}_2 \in \mathcal{D}_n^{(2)}$ (not both of them are zero) and hence

$$\mu_\alpha(\boldsymbol{k}) = \mu_\alpha(\boldsymbol{a}_1 + b^n(\boldsymbol{a}_1 \oplus \boldsymbol{a}_2)).$$

We consider four different cases:

i) If $\boldsymbol{a}_1 = \mathbf{0}$, then $\boldsymbol{a}_2 \neq \mathbf{0}$ and hence

$$\mu_\alpha(\boldsymbol{k}) = \mu_\alpha(b^n \boldsymbol{a}_2) \geq n + \mu_\alpha(\boldsymbol{a}_2) \geq n + \rho_\alpha(\mathcal{D}_n^{(2)}) \geq n + \lfloor \beta_2 n \rfloor - t_2 + 1.$$

ii) If $\boldsymbol{a}_2 = \mathbf{0}$, then $\boldsymbol{a}_1 \neq \mathbf{0}$ and we obtain in a similar manner that

$$\mu_\alpha(\boldsymbol{k}) \geq \mu_\alpha(b^n \boldsymbol{a}_1) \geq n + \rho_\alpha(\mathcal{D}_n^{(1)}) \geq n + \lfloor \beta_1 n \rfloor - t_1 + 1.$$

iii) If $\boldsymbol{a}_1, \boldsymbol{a}_2 \neq \mathbf{0}$, but $\boldsymbol{a}_1 \oplus \boldsymbol{a}_2 = \mathbf{0}$, then $\boldsymbol{a}_1 \in \mathcal{D}_n^{(2)}$, so $\boldsymbol{a}_1 \in \mathcal{D}_n^{(1)} \cap \mathcal{D}_n^{(2)}$. Consequently, if $\mathcal{D}_n^{(1)} \cap \mathcal{D}_n^{(2)} = \{\mathbf{0}\}$, this case is not possible. If $\mathcal{D}_n^{(1)} \cap \mathcal{D}_n^{(2)} \neq \{\mathbf{0}\}$, then

$$\mu_\alpha(\boldsymbol{k}) = \mu_\alpha(\boldsymbol{a}_1) \geq \rho_\alpha(\mathcal{D}_n^{(1)} \cap \mathcal{D}_n^{(2)}).$$

iv) If $\boldsymbol{a}_1, \boldsymbol{a}_2 \neq \mathbf{0}$ and $\boldsymbol{a}_1 \oplus \boldsymbol{a}_2 \neq \mathbf{0}$, then we have

$$\begin{aligned}
\mu_\alpha(\boldsymbol{k}) &= \sum_{j=1}^{s} \mu_\alpha(a_{1,j} + b^n(a_{1,j} \oplus a_{2,j})) \\
&= \sum_{\substack{j=1 \\ a_{j,1} \oplus a_{j,2} \neq 0}}^{s} \mu_\alpha(a_{1,j} + b^n(a_{1,j} \oplus a_{2,j})) + \sum_{\substack{j=1 \\ a_{j,1} \oplus a_{j,2} = 0}}^{s} \mu_\alpha(a_{1,j}) \\
&\geq \sum_{\substack{j=1 \\ a_{j,1} \oplus a_{j,2} \neq 0}}^{s} \mu_\alpha(b^n(a_{1,j} \oplus a_{2,j})) + \sum_{\substack{j=1 \\ a_{j,1} \oplus a_{j,2} = 0}}^{s} \mu_\alpha(a_{1,j}). \qquad (4.9)
\end{aligned}$$

We now distinguish between two sub-cases: firstly, assume that the first sum in Equation (4.9) has at least two terms, then $\mu_\alpha(\boldsymbol{k}) \geq 2n + 2$. Otherwise, it has exactly one term, say for $j = j_0$, which gives a value smaller than $2n + 2$. In this sub-case we have

$$\begin{aligned}
\mu_\alpha(\boldsymbol{k}) &= \mu_\alpha(b^n(a_{1,j_0} \oplus a_{2,j_0})) + \mu_\alpha(\boldsymbol{a}_1) - \mu_\alpha(a_{1,j_0}) \\
&\geq n + \mu_\alpha(\boldsymbol{a}_1) - (\mu_\alpha(a_{1,j_0}) - \mu_\alpha(a_{1,j_0} \oplus a_{2,j_0})) \\
&\geq n + \rho_\alpha(\mathcal{D}_n^{(1)}) - d(\mathcal{D}_n^{(1)}, \mathcal{D}_n^{(2)}) \\
&\geq n + \lfloor \beta_1 n \rfloor - t_1 + 1 - d(\mathcal{D}_n^{(1)}, \mathcal{D}_n^{(2)}).
\end{aligned}$$

Hence combining the four cases, we have

$$\rho_\alpha(\mathcal{N}) \geq \min(n + \lfloor \beta_1 n \rfloor - t_1 + 1 - d(\mathcal{D}_n^{(1)}, \mathcal{D}_n^{(2)}), n + \lfloor \beta_2 n \rfloor - t_2 + 1, \rho_\alpha(\mathcal{D}_n^{(1)} \cap \mathcal{D}_n^{(2)}))$$

if $\mathcal{D}_n^{(1)} \cap \mathcal{D}_n^{(2)} \neq \{\mathbf{0}\}$, and

$$\rho_\alpha(\mathcal{N}) \geq \min(n + \lfloor \beta_1 n \rfloor - t_1 + 1 - d(\mathcal{D}_n^{(1)}, \mathcal{D}_n^{(2)}), n + \lfloor \beta_2 n \rfloor - t_2 + 1)$$

if $\mathcal{D}_n^{(1)} \cap \mathcal{D}_n^{(2)} = \{\mathbf{0}\}$. Now the result follows from Theorem 4.5. □

4.3.5 A base change rule

In this subsection we show how one can obtain a higher order net in base b from a higher order net in base b^L. Thereby we generalize [67, Propagation Rule 7] (see also [27, Propagation Rule XI]) to $(t, \alpha, \beta, n, m, s)$-nets. The proof technique and the construction used in the forthcoming Theorem 4.19 follow [67, Proposition 7] very closely.

Theorem 4.19. *If there exists a $(t, \alpha, \beta, n, m, s)$-net in base b^L with an integer $L \geq 1$, then there exists a $(t, \alpha, \beta, n, mL, sL)$-net in base b.*

Proof. Let $\mathcal{P} = \{\boldsymbol{x}_h\}_{h=0}^{(b^L)^m-1}$ be a $(t, \alpha, \beta, n, m, s)$-net in base b^L. Without loss of generality we may assume that $\boldsymbol{x}_h = (x_{h,1}, \ldots, x_{h,s})$ with

$$x_{h,j} = \sum_{l=1}^{n} \xi_{h,j,l} (b^L)^{-l} \quad \text{for } 0 \leq h \leq (b^L)^m - 1,$$

where all $\xi_{h,j,l} \in \mathbb{Z}_{b^L}$. Let the expansion of $\xi_{h,j,l}$ in base b be

$$\xi_{h,j,l} = \sum_{k=1}^{L} z_{h,l,k}^{(j)} b^{k-1} \quad \text{for } 0 \leq h \leq (b^L)^m - 1, 1 \leq j \leq s, 1 \leq l \leq n,$$

where all $z_{h,l,k}^{(j)} \in \mathbb{Z}_b$. Now we define a multiset $\mathcal{Q} = \{\boldsymbol{w}_0, \ldots, \boldsymbol{w}_{b^{mL}-1}\}$ whose elements are in $[0,1)^{sL}$. The coordinate indices range from 1 to sL, and so we can denote them by $(j-1)L + k$ with $1 \leq j \leq s$ and $1 \leq k \leq L$. Let $w_{h,(j-1)L+k}$ denote the corresponding coordinates of the point $\boldsymbol{w}_h$. To complete the definition of $\mathcal{Q}$, we put

$$w_{h,(j-1)L+k} = \sum_{l=1}^{n} z_{h,l,k}^{(j)} b^{-l} \quad \text{for } 1 \leq j \leq s, 1 \leq k \leq L, 0 \leq h \leq b^{mL} - 1.$$

We will now show that $\mathcal{Q}$ is a $(t, \alpha, \beta, n, mL, sL)$-net in base b. To this end we fix $\boldsymbol{\nu}, \boldsymbol{a}_{\boldsymbol{\nu}}, \boldsymbol{i}_{\boldsymbol{\nu}}$ so that $1 \leq i_{(j-1)L+k,\nu_{(j-1)L+k}} < \cdots < i_{(j-1)L+k,1}$, for $1 \leq k \leq L$ and $1 \leq j \leq s$, and $\sum_{j=1}^{s} \sum_{k=1}^{L} \sum_{l=1}^{\min(\nu_{(j-1)L+k},\alpha)} i_{(j-1)L+k,l} \leq \beta n - t$.

For $\boldsymbol{w}_h$ to be in $J(\boldsymbol{a}_{\boldsymbol{\nu}}, \boldsymbol{i}_{\boldsymbol{\nu}})$, we need

$$w_{h,(j-1)L+k,l} = a_{(j-1)L+k,l} \quad \text{for all } l \in \left\{ i_{(j-1)L+k,\nu_{(j-1)L+k}}, \ldots, i_{(j-1)L+k,1} \right\},$$

which is satisfied if and only if

$$z_{h,l,k}^{(j)} = a_{(j-1)L+k,l} \quad \text{for all } l \in \left\{ i_{(j-1)L+k,\nu_{(j-1)L+k}}, \ldots, i_{(j-1)L+k,1} \right\}.$$

We define $\bigcup_{k=1}^{L} \left\{ i_{(j-1)L+k,\nu_{(j-1)L+k}}, \ldots, i_{(j-1)L+k,1} \right\} = \left\{ e_{j,\widetilde{\nu}_j}, \ldots, e_{j,1} \right\}$ for $1 \leq j \leq s$. For $l \in \left\{ e_{j,\widetilde{\nu}_j}, \ldots, e_{j,1} \right\}$ we set $\widetilde{a}_{j,l} = \sum_{k=1}^{L} a_{(j-1)L+k,l} b^{k-1}$, where unspecified $a_{(j-1)L+k,l}$ are chosen arbitrarily. In fact the number of $a_{(j-1)L+k,l}$ chosen arbitrarily is given by

$$\sum_{j=1}^{s} \sum_{k=1}^{L} (\widetilde{\nu}_j - \nu_{(j-1)L+k}) = L \sum_{j=1}^{s} \widetilde{\nu}_j - \sum_{j=1}^{s} \sum_{k=1}^{L} \nu_{(j-1)L+k}.$$

Hence there are $b^{L\sum_{j=1}^s \widetilde{\nu}_j - \sum_{j=1}^s \sum_{k=1}^L \nu_{(j-1)L+k}}$ generalized elementary intervals of format

$$J(\widetilde{\boldsymbol{a}}, \boldsymbol{e}) = \prod_{j=1}^{s} \bigcup_{\substack{\widetilde{a}_{j,l}=0 \\ l\in\{1,\ldots,n\}\setminus\{e_{j,\widetilde{\nu}_j},\ldots,e_{j,1}\}}}^{b^L-1} \left[\frac{\widetilde{a}_{j,1}}{b^L} + \cdots + \frac{\widetilde{a}_{j,n}}{(b^L)^n}, \frac{\widetilde{a}_{j,1}}{b^L} + \cdots + \frac{\widetilde{a}_{j,n}}{(b^L)^n} + \frac{1}{(b^L)^n}\right)$$

and volume $(b^L)^{-\sum_{j=1}^s \widetilde{\nu}_j}$. However,

$$\sum_{j=1}^{s} \sum_{l=1}^{\min(\widetilde{\nu}_j,\alpha)} e_{j,l} \le \sum_{j=1}^{s}\sum_{k=1}^{L} \sum_{l=1}^{\min(\nu_{(j-1)L+k},\alpha)} i_{(j-1)L+k,l} \le \beta n - t,$$

hence by the (t,α,β,n,m,s)-net property of $\mathcal{P}$, $J(\widetilde{\boldsymbol{a}},\boldsymbol{e})$ contains $(b^L)^{m-\sum_{j=1}^s \widetilde{\nu}_j}$ points and hence $J(\boldsymbol{i}_\nu, \boldsymbol{a}_\nu)$ contains

$$b^{L\sum_{j=1}^s \widetilde{\nu}_j - \sum_{j=1}^s \sum_{k=1}^L \nu_{(j-1)L+k}} (b^L)^{(m-\sum_{j=1}^s \widetilde{\nu}_j)} = b^{Lm-\sum_{j=1}^s \sum_{k=1}^L \nu_{(j-1)L+k}}$$

points of $\mathcal{Q}$, which proves the required result. □

4.3.6 Pirsic's base change rule

In this subsection we present a generalization of Pirsic's base change rule, see [86, Lemma 12] and also [87]. This result shows how to interpret a (t,α,β,n,m,s)-net in base b^L as a $(t',\alpha',\beta',n',m',s)$-net in base $b^{L'}$. Furthermore, we state some special cases, in particular, we show how to interpret a (t,α,β,n,m,s)-net in base b as a $(t',\alpha',\beta',n',m',s)$-net in base $b^{L'}$ and how to interpret a (t,α,β,n,m,s)-net in base b^L as a $(t',\alpha',\beta',n',m',s)$-net in base b.

Theorem 4.20. *Let $n, n', m, m', s, \alpha, L$ and $L' \in \mathbb{N}$, where $\gcd(L, L') = 1$, $mL = m'L'$ and $nL = n'L'$, let $0 < \beta \le 1$ be a real number and $0 \le t \le \beta n$ and βn be integers. Then a $(t, \alpha L', \beta, n, m, s)$-net in base b^L is a $(t', \alpha, \frac{\beta}{L'}, n', m', s)$-net in base $b^{L'}$ where*

$$t' = \min\left(\left\lceil \frac{tL + s\alpha(L-1)L' - \frac{(L'-1)L'}{2} + (-L' \pmod L)\beta n'}{L'(L' + (-L' \pmod L))}\right\rceil, \left\lceil \frac{tL + (s\alpha L' - 1)(L-1) - \frac{(L'-1)L'}{2}}{L'^2}\right\rceil\right).$$

Proof. The proof proceeds as follows: we start with a generalized elementary interval for the point set in base $b^{L'}$, then change this into a generalized elementary interval in base b and consequently rewrite the latter as a union of generalized elementary intervals in base b^L.

Assume we are given an arbitrary generalized elementary interval $J(\boldsymbol{i}_\nu, \boldsymbol{a}_\nu)$ in base $b^{L'}$ for some given values of $\boldsymbol{\nu}$, $\boldsymbol{i}_\nu$, $\boldsymbol{a}_\nu$, such that $\nu_j \ge 0$, $1 \le i_{j,\nu_j} < \cdots < i_{j,1}$, $j = 1, \ldots, s$, and so that for a non-negative integer t''

$$\sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j,\alpha)} i_{j,l} \leq \frac{\beta}{L'} n' - t''. \tag{4.10}$$

Without loss of generality we may assume that there exists at least one ν_j satisfying $\nu_j > 0$, then $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ admits the following representation:

$$J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}}) = \prod_{j=1}^{s} \bigcup_{\substack{a_{j,l}=0 \\ l \in \{1,\ldots,n'\} \setminus \left\{ i_{j,\nu_j},\ldots,i_{j,1} \right\}}}^{b^{L'}-1} \left[\frac{a_{j,1}}{b^{L'}} + \cdots + \frac{a_{j,n'}}{(b^{L'})^{n'}}, \frac{a_{j,1}}{b^{L'}} + \cdots + \frac{a_{j,n'}}{(b^{L'})^{n'}} + \frac{1}{(b^{L'})^{n'}} \right).$$

As $a_{j,l} \in \left\{0, \ldots, b^{L'} - 1\right\}$, it has a b-adic representation of the form $a_{j,l} = a_{j,l,1} + a_{j,l,2} b + \cdots + a_{j,l,L'} b^{L'-1}$ and hence

$$\frac{a_{j,l}}{(b^{L'})^l} = \frac{a_{j,l,L'}}{b^{(l-1)L'+1}} + \cdots + \frac{a_{j,l,2}}{b^{lL'-1}} + \frac{a_{j,l,1}}{b^{lL'}},$$

for $1 \leq l \leq n'$, where $a_{j,l,g} \in \{0, \ldots, b-1\}$. We now set

$$\frac{a_{j,l}}{(b^{L'})^l} = \sum_{k=(l-1)L'+1}^{lL'} \frac{\widetilde{a}_{j,k}}{b^k},$$

i.e. $\widetilde{a}_{j,lL'-g+1} = a_{j,l,g}$, $1 \leq l \leq n'$, $1 \leq g \leq L'$ and $1 \leq j \leq s$. The generalized elementary interval $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ can now be rewritten as a generalized elementary interval in base b,

$$\begin{aligned} J(\widetilde{\boldsymbol{i}}_{\boldsymbol{\nu}}, \widetilde{\boldsymbol{a}}_{\boldsymbol{\nu}}) &= \prod_{j=1}^{s} \bigcup_{\substack{\widetilde{a}_{j,l}=0 \\ l \in \{1,\ldots,n'L'\} \setminus \left\{ \widetilde{i}_{j,\nu_j L'}, \widetilde{i}_{j,\nu_j L'-1},\ldots,\widetilde{i}_{j,1} \right\}}}^{b-1} \left[\frac{\widetilde{a}_{j,1}}{b} + \cdots + \frac{\widetilde{a}_{j,L'}}{b^{L'}} + \cdots + \frac{\widetilde{a}_{j,n'L'}}{b^{n'L'}}, \right. \\ &\qquad \left. \frac{\widetilde{a}_{j,1}}{b} + \cdots + \frac{\widetilde{a}_{j,L'}}{b^{L'}} + \cdots + \frac{\widetilde{a}_{j,n'L'}}{b^{n'L'}} + \frac{1}{b^{n'L'}} \right), \end{aligned}$$

where

$$\widetilde{i}_{j,(k-1)L'+g} = i_{j,k} L' + 1 - g$$

for $1 \leq g \leq L'$ and $1 \leq k \leq \nu_j$. Since $nL = n'L'$, then clearly,

$$J(\widetilde{\boldsymbol{i}}_{\boldsymbol{\nu}}, \widetilde{\boldsymbol{a}}_{\boldsymbol{\nu}}) = \prod_{j=1}^{s} \bigcup_{\substack{\widetilde{a}_{j,l}=0 \\ l \in \{1,\ldots,nL\} \setminus \left\{ \widetilde{i}_{j,\nu_j L'},\ldots,\widetilde{i}_{j,1} \right\}}}^{b-1} \left[\frac{\widetilde{a}_{j,1}}{b} + \cdots + \frac{\widetilde{a}_{j,nL}}{b^{nL}}, \frac{\widetilde{a}_{j,1}}{b} + \cdots + \frac{\widetilde{a}_{j,nL}}{b^{nL}} + \frac{1}{b^{nL}} \right).$$

Now for $1 \leq j \leq s$ and $1 \leq k \leq \nu_j L'$ we define integers $r_{j,k}$ and $e_{j,k}$ such that $0 \leq r_{j,k} < L$ and

$$\widetilde{i}_{j,k} = e_{j,k} L - r_{j,k}.$$

Note that it is possible that $e_{j,k} = e_{j,k'}$ for $k \neq k'$. Let now $\left\{\widetilde{e}_{j,\widetilde{\nu}_j}, \ldots, \widetilde{e}_{j,1}\right\}$ be the set of distinct elements of $\left\{e_{j,\nu_j L'}, \ldots, e_{j,1}\right\}$. Then $\widetilde{\nu}_j \leq \nu_j L'$ and $\left\{e_{j,\nu_j L'}, \ldots, e_{j,1}\right\} = \left\{\widetilde{e}_{j,\widetilde{\nu}_j}, \ldots, \widetilde{e}_{j,1}\right\}$.

Let $\widetilde{\boldsymbol{\nu}} = (\widetilde{\nu}_1, \ldots, \widetilde{\nu}_s)$. For fixed $\widetilde{a}_{j,l}$, where $1 \leq j \leq s$, $\widetilde{e}_{j,k}L - (L-1) \leq l \leq \widetilde{e}_{j,k}L$ and $1 \leq k \leq \widetilde{\nu}_j$, we set

$$\widetilde{\widetilde{a}}_{j,\widetilde{e}_{j,k}} = b^{L-1}\widetilde{a}_{j,\widetilde{e}_{j,k}L-(L-1)} + b^{L-2}\widetilde{a}_{j,\widetilde{e}_{j,k}L-(L-2)} + \cdots + \widetilde{a}_{j,\widetilde{e}_{j,k}L}.$$

Furthermore, for fixed j only $\nu_j L'$ of the $\widetilde{a}_{j,l}$, where $\widetilde{e}_{j,k}L - (L-1) \leq l \leq \widetilde{e}_{j,k}L$ and $1 \leq k \leq \widetilde{\nu}_j$, are specified in $\widetilde{\boldsymbol{a}}_{\boldsymbol{\nu}}$. Hence $J(\widetilde{\boldsymbol{i}}_{\boldsymbol{\nu}}, \widetilde{\boldsymbol{a}}_{\boldsymbol{\nu}})$, and therefore also $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$, is the union of $b^{L\sum_{j=1}^{s}\widetilde{\nu}_j - L'\sum_{j=1}^{s}\nu_j}$ disjoint intervals of the format

$$\prod_{j=1}^{s} \bigcup_{\substack{\widetilde{\widetilde{a}}_{j,l}=0 \\ l \in \{1,\ldots,n\} \setminus \left\{\widetilde{e}_{j,\widetilde{\nu}_j},\ldots,\widetilde{e}_{j,1}\right\}}}^{b^L-1} \left[\frac{\widetilde{\widetilde{a}}_{j,1}}{(b^L)} + \frac{\widetilde{\widetilde{a}}_{j,2}}{(b^L)^2} + \cdots + \frac{\widetilde{\widetilde{a}}_{j,n}}{(b^L)^n}, \frac{\widetilde{\widetilde{a}}_{j,1}}{(b^L)} + \frac{\widetilde{\widetilde{a}}_{j,2}}{(b^L)^2} + \cdots + \frac{\widetilde{\widetilde{a}}_{j,n}}{(b^L)^n} + \frac{1}{(b^L)^n}\right).$$

If we can show that

$$\sum_{j=1}^{s} \sum_{l=1}^{\min(\widetilde{\nu}_j, \alpha L')} \widetilde{e}_{j,l} \leq \beta n - t, \tag{4.11}$$

then each interval contains $(b^L)^{m-|\widetilde{\nu}|_1}$ points and consequently $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ contains

$$(b^L)^{m-|\widetilde{\nu}|_1} b^{|\widetilde{\nu}|_1 L - |\nu|_1 L'} = b^{mL-|\nu|_1 L'} = b^{m'L'-|\nu|_1 L'} = (b^{L'})^{m'-|\nu|_1}$$

points and the proof is complete. Hence $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ contains the right number of points if Equation (4.11) is satisfied or equivalently if

$$\sum_{j=1}^{s} \sum_{l=1}^{\min(\widetilde{\nu}_j, \alpha L')} \widetilde{e}_{j,l} L \leq L(\beta n - t).$$

So $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ still contains the right number of points if

$$\sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j L', \alpha L')} \widetilde{i}_{j,l} + \sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j L', \alpha L')} r_{j,l} \leq L(\beta n - t). \tag{4.12}$$

We now find a bound for $\sum_{j=1}^{s}\sum_{l=1}^{\min(\nu_j L', \alpha L')} \widetilde{i}_{j,l}$:

$$\begin{aligned}
\sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j L', \alpha L')} \widetilde{i}_{j,l} &= \sum_{j=1}^{s} \sum_{l=1}^{L' \min(\nu_j, \alpha)} \widetilde{i}_{j,l} \\
&= \sum_{j=1}^{s} \sum_{k=1}^{\min(\nu_j, \alpha)} \sum_{g=1}^{L'} \widetilde{i}_{j,(k-1)L'+g} = \sum_{j=1}^{s} \sum_{k=1}^{\min(\nu_j, \alpha)} \sum_{g=1}^{L'} (i_{j,k}L' + 1 - g) \\
&= \sum_{j=1}^{s} \sum_{k=1}^{\min(\nu_j, \alpha)} \left[\sum_{g=1}^{L'} i_{j,k}L' - \sum_{g=1}^{L'-1} g\right] \\
&= \sum_{j=1}^{s} \sum_{k=1}^{\min(\nu_j, \alpha)} \left[i_{j,k}L'^2 - \frac{(L'-1)L'}{2}\right]
\end{aligned}$$

$$\leq \sum_{j=1}^{s} \sum_{k=1}^{\min(\nu_j,\alpha)} \left\lceil i_{j,k} L'^2 \right\rceil - \frac{(L'-1)L'}{2}$$

$$\leq \beta n' L' - t'' L'^2 - \frac{(L'-1)L'}{2}, \tag{4.13}$$

where we used Equation (4.10). Combining Equations (4.12) and (4.13), we find that $J(\boldsymbol{i}_\nu, \boldsymbol{a}_\nu)$ contains the right number of points if

$$t'' L'^2 + \frac{(L'-1)L'}{2} - \sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j L', \alpha L')} r_{j,l} \geq tL.$$

That is, we can set

$$t' = \min \left\{ t'' : t'' L'^2 + \frac{(L'-1)L'}{2} - M(t'') \geq tL \right\}, \tag{4.14}$$

where

$$M(t'') = \max \left\{ \sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j L', \alpha L')} (-\widetilde{i}_{j,l} \pmod{L}) \,:\, i_{j,l} \geq 0 \text{ and } \sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j,\alpha)} i_{j,l} \leq \frac{\beta}{L'} n' - t'' \right\},$$

where we recall $\widetilde{i}_{j,l} = \widetilde{e}_{j,l} L - r_{j,l}$, for $1 \leq l \leq \nu_j L'$ and $1 \leq j \leq s$, and $\widetilde{i}_{j,(k-1)L'+g} = i_{j,k} L' + 1 - g$, for $1 \leq g \leq L'$ and $1 \leq k \leq \nu_j$. We now aim to put an upper bound on $\sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j L', \alpha L')} (-\widetilde{i}_{j,l} \pmod{L})$. We have

$$\sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j L', \alpha L')} (-\widetilde{i}_{j,l} \pmod{L}) = \sum_{j=1}^{s} \sum_{k=1}^{\min(\nu_j,\alpha)} \sum_{g=1}^{L'} (-i_{j,k} L' - 1 + g \pmod{L})$$

$$\leq \sum_{j=1}^{s} \sum_{k=1}^{\min(\nu_j,\alpha)} \sum_{g=1}^{L'} (-i_{j,k} L' \pmod{L}) + \sum_{j=1}^{s} \sum_{k=1}^{\min(\nu_j,\alpha)} \sum_{g=1}^{L'} (g - 1 \pmod{L})$$

$$\leq \sum_{j=1}^{s} \sum_{k=1}^{\min(\nu_j,\alpha)} \sum_{g=1}^{L'} (-L' \pmod{L}) i_{j,k} + s\alpha(L-1)L'$$

$$\leq (-L' \pmod{L}) L' \left(\frac{\beta}{L'} n' - t'' \right) + s\alpha(L-1)L'$$

$$= (-L' \pmod{L})(\beta n' - t'' L') + s\alpha(L-1)L'.$$

From Equation (4.14) it follows that

$$t' \leq \min \left\{ t'' \,:\, t'' L'^2 + \frac{(L'-1)L'}{2} - \left((-L' \pmod{L})(\beta n' - t'' L') + (L-1)L'\alpha s \right) \geq tL \right\}.$$

This condition is satisfied for all t'' with

$$t'' \geq \left\lceil \frac{tL + s\alpha(L-1)L' - \frac{(L'-1)L'}{2} + (-L' \pmod{L})\beta n'}{L'(L' + (-L' \pmod{L}))} \right\rceil,$$

which gives the first bound. For the second bound, let

$$t'' = \left\lceil \frac{tL + (s\alpha - 1)(L-1) - \frac{(L'-1)L'}{2}}{L'^2} \right\rceil,$$

then using Equation (4.13), we have

$$\begin{aligned}
\sum_{j=1}^{s}\sum_{l=1}^{\min(\widetilde{v}_j,\alpha L')} \widetilde{e}_{j,l} \le \frac{1}{L}\left(\sum_{j=1}^{s}\sum_{l=1}^{\min(v_jL',\alpha L')} \widetilde{i}_{j,l} + \sum_{j=1}^{s}\sum_{l=1}^{\min(v_jL',\alpha L')} r_{j,l}\right) \\
\le \frac{1}{L}\left(\beta n'L' - t''L'^2 - \frac{(L'-1)L'}{2} + \sum_{j=1}^{s}\sum_{l=1}^{\min(v_jL',\alpha L')} r_{j,l}\right) \\
\le \frac{1}{L}\left(\beta n'L' - t''L'^2 - \frac{(L'-1)L'}{2} + s\alpha(L-1)L'\right) \\
\le \frac{1}{L}\left(\beta n'L' - Lt - (s\alpha L'-1)(L-1) + \frac{(L'-1)L'}{2} - \frac{(L'-1)L'}{2} + s\alpha(L-1)L'\right) \\
= \beta n - t + \frac{L-1}{L}.
\end{aligned}$$

By assumption, βn and $\sum_{j=1}^{s}\sum_{l=1}^{\min(\widetilde{v}_j,\alpha L')} \widetilde{e}_{j,l}$ are integers, hence

$$\sum_{j=1}^{s}\sum_{l=1}^{\min(\widetilde{v}_j,\alpha L')} \widetilde{e}_{j,l} \le \beta n - t,$$

which completes the proof. □

We point out that $\alpha L'$ changes to α in Theorem 4.20. Using Propagation Rule (ii) from this section, we can establish the following corollary to Theorem 4.20, which avoids a change in the parameter α.

Corollary 4.21. *Let $n, n', m, m', s, \alpha, L$ and $L' \in \mathbb{N}$, where $\gcd(L, L') = 1$, $mL = m'L'$ and $nL = n'L'$, let $0 < \beta \le 1$ be a real number and $0 \le t \le \beta n$ and βn be integers. Then a $(t, \alpha, \beta, n, m, s)$-net in base b^L is a $(t', \alpha, \frac{\beta}{L'}, n', m', s)$-net in base $b^{L'}$ where*

$$t' = \min\left(\left\lceil \frac{tL + s\alpha(L-1)L' - \frac{(L'-1)L'}{2} + (-L' \pmod L)\beta n'}{L'(L' + (-L' \pmod L))} \right\rceil, \left\lceil \frac{tL + (s\alpha L'-1)(L-1) - \frac{(L'-1)L'}{2}}{L'^2} \right\rceil\right).$$

However, in some cases it is possible to improve on Corollary 4.21.

Theorem 4.22. *Let $n, n', m, m', s, \alpha, L$ and $L' \in \mathbb{N}$, $L' \ge \alpha$, where $\gcd(L, L') = 1$, $mL = m'L'$ and $nL = n'L'$, let $0 < \beta \le 1$ be a real number and $0 \le t \le \beta n$ and βn be integers. Then a $(t, \alpha, \beta, n, m, s)$-net in base b^L is a $(t', \alpha, \frac{\beta}{\alpha}, n', m', s)$-net in base $b^{L'}$ where*

$$t' = \min\left(\left\lceil \frac{tL + sf(\alpha, L) - \frac{(\alpha-1)\alpha}{2} + (-L' \pmod L)\beta n'}{\alpha(L' + (-L' \pmod L))} \right\rceil, \left\lceil \frac{tL + (s\alpha-1)(L-1) - \frac{(\alpha-1)\alpha}{2}}{\alpha L'} \right\rceil\right),$$

where $f(\alpha, L) = \sum_{l=1}^{\alpha}(l-1 \pmod L)$.

Proof. Using the same definitions as in the proof of Theorem 4.20, we aim to establish that the assumption

$$\sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j,\alpha)} i_{j,l} \leq \frac{\beta}{\alpha} n' - t'',$$

where t'' is a non-negative integer, implies that

$$\sum_{j=1}^{s} \sum_{l=1}^{\min(\widetilde{\nu}_j,\alpha)} \widetilde{e}_{j,l} \leq \beta n - t. \tag{4.15}$$

We proceed in a manner similar to the proof of Theorem 4.20, i.e. $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ contains the right number of points if Equation (4.15) is satisfied which in turn is equivalent to

$$\sum_{j=1}^{s} \sum_{l=1}^{\min(\widetilde{\nu}_j,\alpha)} \widetilde{e}_{j,l} L \leq \beta n L - tL,$$

and hence $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ still contains the right number of points if

$$\sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j L',\alpha)} \widetilde{i}_{j,l} + \sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j L',\alpha)} r_{j,l} \leq \beta n L - tL. \tag{4.16}$$

Next we find a bound for $\sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j L',\alpha)} \widetilde{i}_{j,l}$. Clearly,

$$\begin{aligned} \sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j L',\alpha)} \widetilde{i}_{j,l} &= \sum_{\substack{j=1 \\ \nu_j>0}}^{s} \sum_{l=1}^{\alpha} \widetilde{i}_{j,l} = \sum_{\substack{j=1 \\ \nu_j>0}}^{s} \sum_{l=1}^{\alpha} \left[i_{j,1} L' + 1 - l \right] \\ &= L' \sum_{\substack{j=1 \\ \nu_j>0}}^{s} \sum_{l=1}^{\alpha} i_{j,1} + \sum_{\substack{j=1 \\ \nu_j>0}}^{s} \sum_{l=1}^{\alpha} (1 - l) \\ &= \alpha L' \sum_{\substack{j=1 \\ \nu_j>0}}^{s} i_{j,1} - \sum_{\substack{j=1 \\ \nu_j>0}}^{s} \frac{(\alpha-1)\alpha}{2} \\ &\leq \alpha L' \left(\frac{\beta}{\alpha} n' - t'' \right) - \frac{(\alpha-1)\alpha}{2}. \end{aligned} \tag{4.17}$$

Hence combining Equations (4.16) and (4.17), we find that $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ contains the right number of points if

$$t'' \alpha L' + \frac{(\alpha-1)\alpha}{2} - \sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j L',\alpha)} r_{j,l} \geq tL.$$

We set

$$t' = \min \left\{ t'' \,:\, t'' \alpha L' + \frac{(\alpha-1)\alpha}{2} - M(t'') \geq tL \right\},$$

where

$$M(t'') = \max \left\{ \sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j L',\alpha)} (-\widetilde{i}_{j,l} \pmod{L}) \,:\, i_{j,1} \geq 0, \sum_{\substack{j=1 \\ \nu_j>0}}^{s} i_{j,1} \leq \frac{\beta}{\alpha} n' - t'' \right\}.$$

Now we establish a bound for $\sum_{j=1}^{s}\sum_{l=1}^{\min(\nu_j L',\alpha)}(-\widetilde{i}_{j,l} \pmod{L})$, where we set $f(\alpha,L)=\sum_{l=1}^{\alpha}(l-1 \pmod{L})$. We have

$$\begin{aligned}
&\sum_{j=1}^{s}\sum_{l=1}^{\min(\nu_j L',\alpha)}(-\widetilde{i}_{j,l} \pmod{L})\\
&= \sum_{\substack{j=1\\ \nu_j>0}}^{s}\sum_{l=1}^{\alpha}\left(-i_{j,1}L'-1+l \pmod{L}\right)\\
&\leq \sum_{\substack{j=1\\ \nu_j>0}}^{s}\sum_{l=1}^{\alpha}(-i_{j,1}L' \pmod{L})+\sum_{\substack{j=1\\ \nu_j>0}}^{s}\sum_{l=1}^{\alpha}(l-1 \pmod{L})\\
&\leq (-L' \pmod{L})\alpha\left(\frac{\beta}{\alpha}n'-t''\right)+sf(\alpha,L).
\end{aligned}$$

Hence

$$t'\leq \min\left\{t'' : t''\alpha L'+\frac{(\alpha-1)\alpha}{2}-\left((-L' \pmod{L})(\beta n'-t''\alpha)+sf(\alpha,L)\right)\geq tL\right\},$$

which is satisfied for all t'' with

$$t''\geq\left\lceil\frac{tL+sf(\alpha,L)+(-L' \pmod{L})\beta n'-\frac{(\alpha-1)\alpha}{2}}{\alpha(L'+(-L' \pmod{L}))}\right\rceil.$$

To obtain the second bound, we set

$$t''=\left\lceil\frac{tL+(s\alpha-1)(L-1)-\frac{(\alpha-1)\alpha}{2}}{\alpha L'}\right\rceil.$$

Consequently

$$\begin{aligned}
\sum_{j=1}^{s}\sum_{l=1}^{\min(\widetilde{\nu}_j,\alpha)}\widetilde{e}_{j,l} &\leq \frac{1}{L}\left(\sum_{j=1}^{s}\sum_{l=1}^{\min(\nu_j L',\alpha)}\widetilde{i}_{j,l}+\sum_{j=1}^{s}\sum_{l=1}^{\min(\nu_j L',\alpha)}r_{j,l}\right)\\
&\leq \frac{\alpha L'}{L}\left(\frac{\beta}{\alpha}n'-t''\right)-\frac{(\alpha-1)\alpha}{2L}+\frac{s\alpha(L-1)}{L}\\
&\leq \beta n-t+\frac{L-1}{L},
\end{aligned}$$

hence $\sum_{j=1}^{s}\sum_{l=1}^{\min(\widetilde{\nu}_j,\alpha)}\widetilde{e}_{j,l}\leq\beta n-t$ and the proof is complete. □

In the following corollary, we recover the result due to Pirsic.

Corollary 4.23. *Let m,m',L and $L'\in\mathbb{N}$, $\gcd(L,L')=1$, $mL=m'L'$ and $0\leq t\leq m$ be an integer. Then a (t,m,s)-net in base b^L is a (t',m',s)-net in base $b^{L'}$ with*

$$t'=\min\left(\left\lceil\frac{tL+(-L' \pmod{L})m'}{L'+(-L' \pmod{L})}\right\rceil,\left\lceil\frac{tL+(s-1)(L-1)}{L'}\right\rceil\right).$$

Proof. The proof follows immediately from Theorem 4.22, where we set $\alpha=\beta=1$, $n=m$ and $n'=m'$ and notice that $f(1,L)=0$. □

We again remark that in Theorem 4.20 $\alpha L'$ changes to α. However when considering a base change from b^L to b, there is no need to change α, as the following theorem shows, which can be regarded as a generalization of [27, Theorem 9] and [73, Lemma 9].

Theorem 4.24. *For $L \in \mathbb{N}$ a $(t', \alpha, \beta, n, m, s)$-net in base b^L is a $(t, \alpha, \beta, nL, mL, s)$-net in base b where*

$$t \leq t'L + (s\alpha - 1)(L - 1).$$

Proof. The proof follows from Corollary 4.21 by taking $L' = 1$. □

Finally we study a base change from b to $b^{L'}$, which can be regarded as a generalization of [64, Lemma 2.9].

Theorem 4.25. *Let n, m, s, α, $L' \in \mathbb{N}$, let $0 < \beta \leq 1$ be a real number and $0 \leq t \leq \frac{\beta}{L'}n$ be an integer. Then a $(tL'^2 + \frac{(L'-1)L'}{2}, \alpha L', \beta, nL', mL', s)$-net in base b is a $(t, \alpha, \frac{\beta}{L'}, n, m, s)$-net in base $b^{L'}$.*

Proof. The proof is similar to the proof of Theorem 4.20. □

Furthermore we point out that $\alpha L'$ changes to α in Theorem 4.25. Using Propagation Rule (ii) from this section, we can establish the following corollary to Theorem 4.25, which avoids a change in the parameter α.

Corollary 4.26. *Let $n, m, s, \alpha, L' \in \mathbb{N}$, let $0 < \beta \leq 1$ be a real number and $0 \leq t \leq \frac{\beta}{L'}n$ be an integer. Then a $(tL'^2 + \frac{(L'-1)L'}{2}, \alpha, \beta, nL', mL', s)$-net in base b is a $(t, \alpha, \frac{\beta}{L'}, n, m, s)$-net in base $b^{L'}$.*

However in some cases it is possible to improve on Corollary 4.26.

Theorem 4.27. *Let $L' \geq \alpha$, then a $(t\alpha L' + \frac{(\alpha-1)\alpha}{2}, \alpha, \beta, nL', mL', s)$-net in base b is a $(t, \alpha, \frac{\beta}{\alpha}, n, m, s)$-net in base $b^{L'}$.*

Proof. The proof proceeds along the same lines as the proof of Theorem 4.22. □

4.3.7 A higher order to higher order construction

We next consider a propagation rule which was referred to as "A higher order to higher order construction" in [27]. In Subsection 2.3.1 it was shown how to construct digital $(t, \alpha, \beta, n \times m, s)$-nets from digital (t, m, sd)-nets. Essentially the "higher order to higher order construction" from [27] replaces the digital (t, m, sd)-net with a digital $(t, \alpha, \beta, n \times m, sd)$-net, but makes use of the same construction algorithm. We now show that the same idea can be applied to $(t, \alpha, \beta, n, m, s)$-nets. Assume we are given a multiset $\{\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{b^m-1}\}$ forming a $(t', \alpha', \beta', n, m, sd)$-net in base b. We write $\boldsymbol{x}_h = (x_{h,1}, \ldots, x_{h,sd})$ and $x_{h,j} = \xi_{h,j,1}/b + \xi_{h,j,2}/b^2 + \ldots$ for all $0 \le h \le b^m - 1$ and $1 \le j \le sd$.

Then we construct a multiset $\{\boldsymbol{y}_0, \ldots, \boldsymbol{y}_{b^m-1}\}$ as follows: for $0 \le h < b^m$ we set $\boldsymbol{y}_h = (y_{h,1}, \ldots, y_{h,s})$ in $[0,1)^s$, where for $1 \le j \le s$

$$y_{h,j} = \sum_{l=1}^{n} \sum_{k=1}^{d} \xi_{h,(j-1)d+k,l} b^{-k-(l-1)d}. \tag{4.18}$$

Theorem 4.28. *Let $d \in \mathbb{N}$ and $\{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{b^m-1}\}$ be a $(t', \alpha', \beta', n, m, sd)$-net in base b, where we assume that $\beta' n$ is an integer. Then for any $\alpha \ge 1$, the multiset $\{\boldsymbol{y}_0, \ldots, \boldsymbol{y}_{b^m-1}\}$ defined by Equation* (4.18) *forms a $(t, \alpha, \beta' \min(1, \alpha/(\alpha' d)), dn, m, s)$-net in base b with*

$$t = \left\lceil \min\left(d, \frac{\alpha}{\alpha'}\right) \min\left(\beta' n, t' + \left\lfloor \frac{\alpha' s(d-1)}{2} \right\rfloor\right) \right\rceil.$$

Proof. The case where $\beta' n \le t' + \lfloor \alpha' s(d-1)/2 \rfloor$ is trivial, see Remark 3.9. Hence we assume from now on that $\beta' n > t' + \lfloor \alpha' s(d-1)/2 \rfloor$ and that we deal with an arbitrary generalized elementary interval $J(\boldsymbol{i}_{\boldsymbol{v}}, \boldsymbol{a}_{\boldsymbol{v}})$, for some given values of $\boldsymbol{v}$, $\boldsymbol{i}_{\boldsymbol{v}}$, $\boldsymbol{a}_{\boldsymbol{v}}$, such that $1 \le i_{j,\nu_j} < \cdots < i_{j,1}$, $\nu_j \ge 0$, for $1 \le j \le s$, and

$$\sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j, \alpha)} i_{j,l} \le \beta' \min\left(1, \frac{\alpha}{\alpha' d}\right) dn - t.$$

We need to show that $J(\boldsymbol{i}_{\boldsymbol{v}}, \boldsymbol{a}_{\boldsymbol{v}})$ contains $b^{m-|\boldsymbol{v}|_1}$ points. For $\boldsymbol{y}_h$, $0 \le h \le b^m - 1$, to be in $J(\boldsymbol{i}_{\boldsymbol{v}}, \boldsymbol{a}_{\boldsymbol{v}})$, we need for $0 \le h \le b^m - 1$, $1 \le j \le s$, $1 \le l \le n$, and $1 \le k \le d$

$$\eta_{h,j,(l-1)d+k} = a_{j,(l-1)d+k} \text{ whenever } (l-1)d + k \in \left\{i_{j,\nu_j}, \ldots, i_{j,1}\right\},$$

where $y_{h,j} := \eta_{h,j,1}/b + \cdots + \eta_{h,j,dn}/b^{dn}$. But from the construction method we find that the condition $\eta_{h,j,(l-1)d+k} = a_{j,(l-1)d+k}$ is equivalent to $\xi_{h,(j-1)d+k,l} = a_{j,(l-1)d+k}$. As $\{\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{b^m-1}\}$ forms a $(t', \alpha', \beta', n, m, sd)$-net, we translate the above condition into a condition on a generalized elementary interval of dimension sd. In particular we set

$$a'_{(j-1)d+k,l} = a_{j,(l-1)d+k} \text{ if } (l-1)d + k \in \left\{i_{j,\nu_j}, \ldots, i_{j,1}\right\}.$$

Also for each choice of $1 \le j \le s$ and $1 \le k \le d$ we let $w_{(j-1)d+k}$ denote the largest integer such that there are $e_{(j-1)d+k,1} > \cdots > e_{(j-1)d+k,w_{(j-1)d+k}} > 0$ for which

$$\left\{(e_{(j-1)d+k,u} - 1)d + k : u = 1, \ldots, w_{(j-1)d+k}\right\} \subseteq \left\{i_{j,\nu_j}, \ldots, i_{j,1}\right\},$$

where for $w_{(j-1)d+k} = 0$ we set $\left\{(e_{(j-1)d+k,u} - 1)d + k : u = 1, \ldots, w_{(j-1)d+k}\right\} = \emptyset$. Consequently for dimension $(j-1)d + k$, with $1 \le j \le s$ and $1 \le k \le d$, the digits $a'_{(j-1)d+k,1}, \ldots, a'_{(j-1)d+k,w_{(j-1)d+k}}$ are specified whenever $w_{(j-1)d+k} > 0$. In particular, $w_{(j-1)d+k}$ gives the number of digits in dimension $(j-1)d+k$ that the generalized elementary interval corresponding to the $(t', \alpha', \beta', n, m, sd)$-net contributes to dimension j of the generalized elementary interval corresponding to the $(t, \alpha, \beta' \min(1, \alpha/(\alpha' d)), dn, m, s)$-net. We hence note that

$$\sum_{k=1}^{d} w_{(j-1)d+k} = \nu_j \text{ for } 1 \leq j \leq s \tag{4.19}$$

and obtain the following generalized elementary interval $J(\boldsymbol{e_w}, \boldsymbol{a'_w})$ of dimension sd, where $\boldsymbol{e_w} = (e_{1,w_1}, \ldots, e_{1,1}, \ldots, e_{sd,w_{sd}}, \ldots, e_{sd,1})$ and $\boldsymbol{a'_w} = (a'_{1,w_1}, \ldots, a'_{1,1}, \ldots, a'_{sd,w_{sd}}, \ldots, a'_{sd,1})$:

$$J(\boldsymbol{e_w}, \boldsymbol{a'_w}) = \prod_{j=1}^{sd} \bigcup_{\substack{a'_{j,l}=0 \\ l \in \{1,\ldots,n\} \setminus \{e_{j,w_j},\ldots,e_{j,1}\}}}^{b-1} \left[\frac{a'_{j,1}}{b} + \frac{a'_{j,2}}{b^2} + \cdots + \frac{a'_{j,n}}{b^n}, \frac{a'_{j,1}}{b} + \frac{a'_{j,2}}{b^2} + \cdots + \frac{a'_{j,n}}{b^n} + \frac{1}{b^n} \right).$$

By the $(t', \alpha', \beta', n, m, sd)$-net property of $\{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{b^m-1}\}$, if

$$\sum_{j=1}^{sd} \sum_{l=1}^{\min(w_j, \alpha')} e_{j,l} \leq \beta' n - t', \tag{4.20}$$

then $J(\boldsymbol{e_w}, \boldsymbol{a'_w})$ contains $b^{m - \sum_{j=1}^{sd} w_j} = b^{m - \sum_{j=1}^{s} \nu_j}$ points, where we used Equation (4.19), as required. However, distinguishing the cases $\alpha' d \leq \alpha$ and $\alpha' d > \alpha$, it was shown in [27] that Equation (4.20) holds, which completes the proof. □

Remark 4.29. As in Example 2.27, one can employ a $(0, m, 2)$-net in base b, to show that Theorem 4.28 cannot be improved on in general.

The following corollary generalizes Theorem 2.26, see also Corollary 4.34.

Corollary 4.30. *Let $d \in \mathbb{N}$ and $\{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{b^m-1}\}$ be a (t', m, sd)-net in base b. Then for every $\alpha \geq 1$, the multiset $\{\boldsymbol{y}_0, \ldots, \boldsymbol{y}_{b^m-1}\}$ defined by Equation* (4.18) *forms a $(t, \alpha, \min(1, \frac{\alpha}{d}), dm, m, s)$-net in base b with*

$$t = \min(d, \alpha) \min\left(m, t' + \left\lfloor \frac{s(d-1)}{2} \right\rfloor\right).$$

Proof. The proof follows immediately from Remark 3.10 and by setting $\alpha' = \beta' = 1$ and $n = m$ in Theorem 4.28. □

Theorem 4.28 can be improved on when $\alpha = \alpha'$, which we show in the following proposition.

Proposition 4.31. *Let $\alpha, d \in \mathbb{N}$ and $\{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{b^m-1}\}$ form a (t,α,β,n,m,sd)-net in base b. Then the multiset $\{\boldsymbol{y}_0, \ldots, \boldsymbol{y}_{b^m-1}\}$ defined by Equation (4.18) forms a (t,α,β,n,m,s)-net in base b.*

Proof. Let $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_s) \in \{0, \ldots, nd\}^s$ be given and for $1 \le j \le s$ let $dn > i_{j,1} > \cdots > i_{j,\nu_j} > 0$ be such that

$$\sum_{j=1}^{s} \sum_{l=1}^{\min(\alpha,\nu_j)} i_{j,l} \le \beta n - t.$$

We assume that $\boldsymbol{i}_{\boldsymbol{\nu}} = (i_{1,1}, \ldots, i_{1,\nu_1}, \ldots, i_{s,1}, \ldots, i_{s,\nu_s})$, $\boldsymbol{a}_{\boldsymbol{\nu}} = (a_{1,i_{1,1}}, \ldots, a_{1,i_{1,\nu_1}}, \ldots, a_{s,i_{s,1}}, \ldots, a_{s,i_{s,\nu_s}})$, and a generalized elementary interval

$$J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}}) = \prod_{j=1}^{s} \bigcup_{\substack{a_{j,l}=0 \\ l \in \{1,\ldots,nd\} \setminus \{i_{j,1},\ldots,i_{j,\nu_j}\}}}^{b-1} \left[\frac{a_{j,1}}{b} + \cdots + \frac{a_{j,nd}}{b^{nd}}, \frac{a_{j,1}}{b} + \cdots + \frac{a_{j,nd}}{b^{nd}} + \frac{1}{b^{nd}}\right),$$

where $\left\{i_{j,1}, \ldots, i_{j,\nu_j}\right\} = \emptyset$ in case $\nu_j = 0$ for $1 \le j \le s$, are given.

Let $\boldsymbol{y}_h = (y_{h,1}, \ldots, y_{h,s})$ with $y_{h,j} = \eta_{h,j,1}/b + \eta_{h,j,2}/b^2 + \cdots$. Then $\boldsymbol{y}_h \in J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ if and only if $\eta_{h,j,l} = a_{j,l}$ for all $l \in \{i_{j,1}, \ldots, i_{j,\nu_j}\}$ and $1 \le j \le s$.

We now define a new generalized elementary interval J' of dimensionality sd such that $\boldsymbol{y}_h \in J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ if and only if $\boldsymbol{x}_h \in J'$. To this end, for $1 \le j \le s$ let $a'_{(j-1)d+k,l} = a_{j,(l-1)d+k}$, where $1 \le k \le d$ and $1 \le l \le n$ are such that $(l-1)d + k \in \{i_{j,1}, \ldots, i_{j,\nu_j}\}$. For $1 \le j' \le sd$ we have now specified $a'_{j',i'}$ for certain values of $i' \in \{1, \ldots, n\}$. Let $U_{j'}$ be the set of i' for which $a'_{j',i'}$ is specified, i.e.

$$U_{j'} = \{1 \le i' \le n : (i'-1)d + j' - (j-1)d \in \{i_{j,1}, \ldots, i_{j,\nu_j}\} \text{ for } j = \lceil j'/d \rceil\}.$$

We set $U_{j'} = \{i'_{j',1}, \ldots, i'_{j',\nu'_{j'}}\}$, where we assume that the elements are ordered such that $n \ge i'_{j',1} > \cdots > i'_{j',\nu'_{j'}} > 0$. Define now $\boldsymbol{\nu}' = (\nu'_1, \ldots, \nu'_{sd}) \in \{0, \ldots, n\}^{sd}$, $\boldsymbol{i}'_{\boldsymbol{\nu}'} = (i'_{1,1}, \ldots, i'_{1,\nu'_1}, \ldots, i'_{sd,1}, \ldots, i'_{sd,\nu'_{sd}})$, and $\boldsymbol{a}'_{\boldsymbol{\nu}'} = (a'_{1,i'_{1,1}}, \ldots, a'_{1,i'_{1,\nu'_1}}, \ldots, a'_{sd,i'_{sd,1}}, \ldots, a'_{sd,i'_{sd,\nu'_{sd}}})$. Then $J' = J(\boldsymbol{i}'_{\boldsymbol{\nu}'}, \boldsymbol{a}'_{\boldsymbol{\nu}'})$ has the property that $\boldsymbol{y}_h \in J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ if and only if $\boldsymbol{x}_h \in J(\boldsymbol{i}'_{\boldsymbol{\nu}'}, \boldsymbol{a}'_{\boldsymbol{\nu}'})$.

Note that $\nu'_{(j-1)d+1} + \cdots + \nu'_{(j-1)d+d} = \nu_j$ for $1 \le j \le s$ and therefore $|\boldsymbol{\nu}|_1 = |\boldsymbol{\nu}'|_1$. Thus, if J' contains $b^{m-|\boldsymbol{\nu}'|_1}$ points, then $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ contains $b^{m-|\boldsymbol{\nu}|_1}$ points. The former is the case if $\sum_{j=1}^{sd} \sum_{l=1}^{\min(\alpha,\nu'_j)} i'_{j,l} \le \beta n - t$, which we show in the following.

If $\nu_j < \alpha$, then it can be confirmed that

$$\sum_{j'=(j-1)d+1}^{jd} \sum_{l=1}^{\nu'_{j'}} i'_{j',l} \le \left\lceil \frac{i_{j,1}}{\alpha} \right\rceil + \cdots + \left\lceil \frac{i_{j,\nu_j}}{\alpha} \right\rceil$$

$$\leq \frac{i_{j,1}+\cdots+i_{j,\nu_j}+\nu_j(\alpha-1)}{\alpha}$$
$$\leq i_{j,1}+\cdots+i_{j,\nu_j},$$

since $i_1+\cdots+i_{\nu_j} \geq \frac{\nu_j(\nu_j+1)}{2}$.

If $\nu_j \geq \alpha$, then

$$\sum_{j'=(j-1)d+1}^{jd} \sum_{l=1}^{\min(\nu'_{j'},\alpha)} i'_{j',l} \leq \left\lceil \frac{i_{j,1}}{\alpha} \right\rceil + \cdots + \left\lceil \frac{i_{j,\alpha}}{\alpha} \right\rceil + \left\lceil \frac{i_{j,\alpha}-1}{\alpha} \right\rceil + \cdots + \left\lceil \frac{i_{j,\alpha}-\alpha(d-1)}{\alpha} \right\rceil$$
$$\leq i_{j,1}+\cdots+i_{j,\alpha}.$$

Therefore we have

$$\sum_{j=1}^{sd} \sum_{l=1}^{\min(\alpha,\nu'_j)} i'_{j,l} \leq \sum_{j=1}^{s} \sum_{l=1}^{\min(\alpha,\nu_j)} i_{j,l} \leq \beta n - t.$$

Hence the result follows, since $\{\boldsymbol{x}_0,\ldots,\boldsymbol{x}_{b^m-1}\}$ is a (t,α,β,n,m,sd)-net and therefore J' contains $b^{m-|\nu'|_1}$ points. □

4.4 Propagation rules for $(t,\alpha,\beta,\sigma,s)$-sequences and an application

Using results from Section 4.3, we deduce properties of $(t,\alpha,\beta,\sigma,s)$-sequences in base b.

4.4.1 A higher order to higher order construction

We use the higher order to higher order construction from Subsection 4.3.7 to construct $(t,\alpha,\beta,\sigma,s)$-sequences in base b.

Assume we are given an infinite sequence $\{\boldsymbol{x}_0,\boldsymbol{x}_1,\ldots\}$ forming a $(t',\alpha',\beta',\sigma,sd)$-sequence in base b. We write $\boldsymbol{x}_h=(x_{h,1},\ldots,x_{h,sd})$ and $x_{h,j}=\xi_{h,j,1}/b+\xi_{h,j,2}/b^2+\ldots$ for all $h\geq 0$ and $1\leq j\leq sd$.

Then we construct an infinite sequence $\{\boldsymbol{y}_0,\boldsymbol{y}_1,\ldots\}$ as follows: for $h\geq 0$ we set $\boldsymbol{y}_h=(y_{h,1},\ldots,y_{h,s})$ in $[0,1)^s$ where

$$y_{h,j}=\sum_{l=1}^{\infty}\sum_{k=1}^{d}\xi_{h,(j-1)d+k,l}\,b^{-k-(l-1)d}. \tag{4.21}$$

Theorem 4.32. *Let $\alpha',d,s,\sigma\in\mathbb{N}$, $0<\beta'\leq 1$ be such that $\beta'\sigma$ is an integer and $t'\geq 0$ be an integer. Let $\{\boldsymbol{x}_0,\boldsymbol{x}_1,\ldots\}$ be a $(t',\alpha',\beta',\sigma,sd)$-sequence in base b. Then for any $\alpha\geq 1$, the infinite sequence $\{\boldsymbol{y}_0,\boldsymbol{y}_1,\ldots\}$ defined by Equation (4.21) forms a $(t,\alpha,\beta'\min(1,\alpha/(\alpha'd)),d\sigma,s)$-sequence in base b with*

$$t=\left\lceil \min\left(d,\frac{\alpha}{\alpha'}\right)\left(t'+\left\lfloor\frac{\alpha' s(d-1)}{2}\right\rfloor\right)\right\rceil.$$

Proof. We need to show that for all $k \geq 0$ and $m > \frac{t}{\beta' \min(1, \frac{\alpha}{\alpha' d}) d\sigma}$ the multiset $\{\boldsymbol{y}_{kb^m}, \ldots, \boldsymbol{y}_{(k+1)b^m-1}\}$ forms a $(t, \alpha, \beta' \min(1, \frac{\alpha}{\alpha' d}), d\sigma m, m, s)$-net in base b. It is clear that $m > \frac{t'}{\beta'\sigma}$ and hence $\left\{\boldsymbol{x}_{kb^m}, \ldots, \boldsymbol{x}_{(k+1)b^m-1}\right\}$ forms a $(t', \alpha', \beta', \sigma m, m, sd)$-net in base b. But $\beta'\sigma m$ is an integer, hence by Theorem 4.28 $\left\{\boldsymbol{y}_{kb^m}, \ldots, \boldsymbol{y}_{(k+1)b^m-1}\right\}$ forms a $(t, \alpha, \beta' \min(1, \frac{\alpha}{\alpha' d}), d\sigma m, m, s)$-net in base b where $t \leq \lceil \min(d, \frac{\alpha}{\alpha'})(t' + \lfloor \frac{\alpha' s(d-1)}{2} \rfloor)\rceil$. Consequently Equation (4.21) defines a $(t, \alpha, \beta' \min(1, \alpha/(\alpha' d)), d\sigma, s)$-sequence. □

Remark 4.33. *As in Remark 4.29 and Example 2.27 in Subsection 2.3.1, one can employ a* $(0,2)$*-sequence in base b, to show that Theorem 4.32 cannot be improved on in general.*

The following corollary is similar to Corollary 4.30 in Subsection 4.3.7.

Corollary 4.34. *Let* $\alpha', d, s, \sigma \in \mathbb{N}$, $0 < \beta' \leq 1$ *be such that* $\beta'\sigma$ *is an integer,* $t' \geq 0$ *be an integer and* $\{\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots\}$ *be a* (t', sd)*-sequence in base b. Then for any* $\alpha \geq 1$*, the infinite sequence* $\{\boldsymbol{y}_0, \boldsymbol{y}_1, \ldots\}$ *defined by Equation* (4.21) *forms a* $(t, \alpha, \min(1, \frac{\alpha}{d}), d, s)$*-sequence in base b with*

$$t = \min(d, \alpha) \left(t' + \left\lfloor \frac{s(d-1)}{2} \right\rfloor \right).$$

The following result is analogous to Proposition 4.31.

Proposition 4.35. *Let* $\alpha, d \in \mathbb{N}$ *and* $\{\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots\}$ *be a* $(t, \alpha, \beta, \sigma, sd)$*-sequence in base b. Then the infinite sequence* $\{\boldsymbol{y}_0, \boldsymbol{y}_1, \ldots\}$ *defined by Equation* (4.21) *forms a* $(t, \alpha, \beta, \sigma, s)$*-sequence in base b.*

Proof. We need to show that for $k \geq 0$ and $m > t/(\beta\sigma)$ the multiset $\left\{\boldsymbol{y}_{kb^m}, \ldots, \boldsymbol{y}_{(k+1)b^m-1}\right\}$ forms a $(t, \alpha, \beta, \sigma m, m, s)$-net in base b. But for $k \geq 0$ and $m > t/(\beta\sigma)$, the subsequence $\left\{\boldsymbol{x}_{kb^m}, \ldots, \boldsymbol{x}_{(k+1)b^m-1}\right\}$ forms a $(t, \alpha, \beta, \sigma m, m, sd)$-net in base b, hence by Proposition 4.31 the multiset $\left\{\boldsymbol{y}_{kb^m}, \ldots, \boldsymbol{y}_{(k+1)b^m-1}\right\}$ forms a $(t, \alpha, \beta, \sigma m, m, s)$-net in base b. □

4.4.2 A base reduction rule

We show that a $(t', \alpha, \beta, \sigma, s)$-sequence in base b^L is a $(t, \alpha, \beta, \sigma, s)$-sequence in base b with some quality parameter t. The following theorem generalizes [73, Proposition 4].

Theorem 4.36. *Let* $\sigma, s, \alpha, L \in \mathbb{N}$*, let* $0 < \beta \leq 1$ *be a real number and* $t \geq 0$ *and* $\beta\sigma$ *be integers. A* $(t', \alpha, \beta, \sigma, s)$*-sequence in base* b^L *is a* $(t, \alpha, \beta, \sigma, s)$*-sequence in base b with*

$$t = t'L + (s\alpha - 1 + \beta\sigma)(L - 1).$$

Proof. Let $\{\boldsymbol{x}_0,\boldsymbol{x}_1,\dots\}$ be a $(t',\alpha,\beta,\sigma,s)$-sequence in base b^L and t as above and fix $m > \frac{t}{\beta\sigma}$ and write it in the form $m = pL + r$ with integers p and r such that $0 \le r < L$. Note that $p > \frac{t'}{\beta\sigma}$. For a fixed integer $k \ge 0$ we consider the multiset $\mathcal{P} = \{\boldsymbol{x}_{kb^m},\dots,\boldsymbol{x}_{(k+1)b^m-1}\}$. Then $\mathcal{P}$ can be split up into b^r multisets $\{\boldsymbol{x}_{lb^{pL}},\dots,\boldsymbol{x}_{(l+1)b^{pL}-1}\}$ where $kb^r \le l < (k+1)b^r$. As $p > \frac{t'}{\beta\sigma}$ each of these subsequences forms a $(t',\alpha,\beta,\sigma p,p,s)$-net in base b^L, which by Theorem 4.24 is a $(t'L+(s\alpha-1)(L-1),\alpha,\beta,\sigma pL,pL,s)$-net in base b. A $(t'L+(s\alpha-1)(L-1),\alpha,\beta,\sigma pL,pL,s)$-net in base b is also a $(t'L+(s\alpha-1+\beta\sigma)(L-1),\alpha,\beta,\sigma m,pL,s)$-net in base b, as the strength of the latter is smaller than the strength of the former. An application of Propagation Rule (vi) from the beginning of Section 4.3 shows that $\mathcal{P}$ is a $(t'L+(s\alpha-1+\beta\sigma)(L-1),\alpha,\beta,\sigma m,pL+r,s)$-net in base b and hence a $(t,\alpha,\beta,\sigma m,m,s)$-net in base b. □

4.4.3 A base expansion rule

Here we consider a base change in the opposite direction: we show that a $(t,\alpha,\beta,\sigma,s)$-sequence in base b is a $(t',\alpha',\beta',\sigma,s)$-sequence in base $b^{L'}$. The following theorem generalizes Theorem 4.25 from Subsection 4.3.6 to $(t,\alpha,\beta,\sigma,s)$-sequences (see also [73, Proposition 5]).

Theorem 4.37. *Let $\sigma,s,\alpha,L' \in \mathbb{N}$, let $0<\beta\le 1$ be a real number and $t \ge 0$ and $\beta\sigma$ be integers. Then a $(u,\alpha L',\beta,\sigma,s)$-sequence in base b is a $(t,\alpha,\frac{\beta}{L'},\sigma,s)$-sequence in base $b^{L'}$ with*

$$t = \left\lceil \frac{u}{L'^2} - \frac{(L'-1)}{2L'} \right\rceil .$$

Proof. Denote the $(u,\alpha L',\beta,\sigma,s)$-sequence in base b by $\{\boldsymbol{x}_0,\boldsymbol{x}_1,\dots\}$, which is of course also a $(tL'^2+\frac{(L'-1)L'}{2},\alpha L',\beta,\sigma,s)$-sequence in base b. By Definition 3.7 and Remark 3.9, for all integers $k \ge 0$ and $m \ge 1$ the finite subsequence

$$\{\boldsymbol{x}_{kb^{mL'}},\dots,\boldsymbol{x}_{(k+1)b^{mL'}-1}\} \tag{4.22}$$

forms a $(\min(tL'^2+\frac{(L'-1)L'}{2},\beta\sigma mL'),\alpha L',\beta,\sigma mL',mL',s)$-net in base b. We consider two cases:

i) Assume first that m is such that $tL'^2+\frac{(L'-1)L'}{2} \le \beta\sigma mL'$, then by Theorem 4.25 the multiset given by Equation (4.22) forms a $(t,\alpha,\frac{\beta}{L'},\sigma m,m,s)$-net in base $b^{L'}$. Furthermore, $tL'^2+\frac{(L'-1)L'}{2} \le \beta\sigma mL'$ implies that $t \le \lfloor \frac{\beta}{L'}\sigma m\rfloor$.

ii) Now assume $\beta\sigma mL' < tL'^2+\frac{(L'-1)L'}{2}$. According to Remark 3.9 the multiset given by Equation (4.22) forms a $(\lfloor \frac{\beta}{L'}\sigma m\rfloor,\alpha,\frac{\beta}{L'},\sigma m,m,s)$-net in base $b^{L'}$. Furthermore, $\beta\sigma mL' < tL'^2+\frac{(L'-1)L'}{2}$ implies that $\lfloor \frac{\beta}{L'}\sigma m\rfloor \le t$.

Hence the multiset given by Equation (4.22) forms a $\left(\min(t,\lfloor\frac{\beta}{L'}\sigma m\rfloor),\alpha,\frac{\beta}{L'},\sigma m,m,s\right)$-net in base $b^{L'}$. We conclude that for all m such that $\frac{\beta}{L'}\sigma m > t$ we obtain a $(t,\alpha,\frac{\beta}{L'},\sigma m,m,s)$-net in base $b^{L'}$ and therefore a $(t,\alpha,\frac{\beta}{L'},\sigma,s)$-sequence in base $b^{L'}$. □

We also consider a special case based on Theorem 4.27.

Theorem 4.38. *Let $\sigma,s,\alpha,L' \in \mathbb{N}$, $L' \geq \alpha$, let $0 < \beta \leq 1$ be a real number and $t \geq 0$ and $\beta\sigma$ be integers. Then a $(t\alpha L' + \frac{(\alpha-1)\alpha}{2},\alpha,\beta,\sigma,s)$-sequence in base b is a $(t,\alpha,\frac{\beta}{\alpha},\sigma,s)$-sequence in base $b^{L'}$.*

Proof. We denote the $(t\alpha L' + \frac{(\alpha-1)\alpha}{2},\alpha,\beta,\sigma,s)$-sequence in base b by $\{\boldsymbol{x}_0,\boldsymbol{x}_1,\dots\}$. Then by Definition 3.7 and Remark 3.9, for all integers $k \geq 0$ and $m \geq 1$ the finite subsequence

$$\{\boldsymbol{x}_{kb^{mL'}},\dots,\boldsymbol{x}_{(k+1)b^{mL'}-1}\} \tag{4.23}$$

forms a $(\min(t\alpha L' + \frac{(\alpha-1)\alpha}{2},\beta\sigma mL'),\alpha,\beta,\sigma mL',mL',s)$-net in base b. We consider two cases:

i) Assume that $t\alpha L' + \frac{(\alpha-1)\alpha}{2} \leq \beta\sigma mL'$. Then by Theorem 4.27 the multiset given by Equation (4.23) forms a $(t,\alpha,\frac{\beta}{\alpha},\sigma m,m,s)$-net in base $b^{L'}$. Furthermore, $t\alpha L' + \frac{(\alpha-1)\alpha}{2} \leq \beta\sigma mL'$ implies that $t \leq \lfloor\frac{\beta}{\alpha}\sigma m\rfloor$.

ii) Now assume that $\beta\sigma mL' < t\alpha L' + \frac{(\alpha-1)\alpha}{2}$. According to Remark 3.9 the multiset given by Equation (4.23) forms a $(\lfloor\frac{\beta}{\alpha}\sigma m\rfloor,\alpha,\frac{\beta}{\alpha},\sigma m,m,s)$-net in base $b^{L'}$. Furthermore, $\beta\sigma mL' < t\alpha L' + \frac{(\alpha-1)\alpha}{2}$ implies that $\lfloor\frac{\beta}{\alpha}\sigma m\rfloor \leq t$.

Hence the multiset given by Equation (4.23) forms a $(\min(t,\lfloor\frac{\beta}{\alpha}\sigma m\rfloor),\alpha,\frac{\beta}{\alpha},\sigma m,m,s)$-net in base $b^{L'}$. We conclude that for all m such that $\frac{\beta}{\alpha}\sigma m > t$ we obtain a $(t,\alpha,\frac{\beta}{\alpha},\sigma m,m,s)$-net in base $b^{L'}$ and therefore a $(t,\alpha,\frac{\beta}{\alpha},\sigma,s)$-sequence in base $b^{L'}$. □

4.4.4 An explicit bound for $t_b(\alpha,s)$ for prime powers b

In this subsection the smallest value of t such that there exists a $(t,\alpha,\beta,\sigma,s)$-sequence in base b is studied.

Definition 4.39. For integers $b \geq 2$, $s \geq 1$, and $\alpha \geq 1$ let $t_b(\alpha,s)$ denote the smallest value of t such that there exists a $(t,\alpha,\beta,\sigma,s)$-sequence in base b with $\alpha = \beta\sigma$.

Remark 4.40. In Definition 3.20 the analogous quantity for the digital case was introduced: let b be a prime, then $d_b(\alpha,s)$ denotes the smallest value of t such that there exists a digital $(t,\alpha,\beta,\sigma,s)$-sequence over the finite field $\mathbb{Z}_b$ with $\alpha = \beta\sigma$.

In this case it is known, see Theorem 3.21, that for all $s \geq 1$ and $\alpha \geq 2$ we have

$$s\frac{\alpha(\alpha-1)}{2} - \alpha < d_q(\alpha, s) \leq s\alpha^2 \frac{c}{\log q} + \alpha + \alpha \left\lfloor \frac{s(\alpha-1)}{2} \right\rfloor,$$

where $c > 0$ is an absolute constant. Note that these bounds also apply to (non-digital) $(t, \alpha, \beta, \sigma, s)$-sequences with $\alpha = \beta\sigma$.

The next corollary follows from Theorems 4.36 and 4.37. Setting $\alpha = \beta = \sigma = 1$ and making use of Theorems 4.36 and 4.38, we could even recover [73, Corollary 4].

Corollary 4.41. *For all integers $b \geq 2$, $s \geq 1$, and $\alpha = \beta\sigma \geq 1$ we have*

$$\frac{t_b(\alpha, s) - (s\alpha - 1 + \beta\sigma)(L-1)}{L} \leq t_{b^L}(\alpha, s) \leq \left\lceil \frac{t_b(\alpha L, s) - \frac{(L-1)L}{2}}{L^2} \right\rceil.$$

The next theorem provides an explicit bound for $t_b(\alpha, s)$ for prime powers b; we assume that b is a prime power as we invoke [73, Theorem 5] to complete the proof. Setting $\alpha = \beta = \sigma = 1$, this result recovers [73, Proposition 6].

Theorem 4.42. *For every prime power b we have*

$$t_b(\alpha, s) \leq \frac{2bs\alpha^2}{b-1} - 2\frac{b\alpha^{3/2}s^{1/2}}{(b^2-1)^{1/2}} + 2\alpha \left\lfloor \frac{s(\alpha-1)}{2} \right\rfloor + s\alpha - 1 + \alpha.$$

Proof. We use Theorem 4.36 with $L = 2$ to obtain

$$t_b(\alpha, s) \leq 2t_{b^2}(\alpha, s) + (s\alpha - 1 + \alpha).$$

By Corollary 4.34, where we set $d = \alpha$,

$$t_{b^2}(\alpha, s) \leq \alpha t_{b^2}(1, s\alpha) + \alpha \left\lfloor \frac{s(\alpha-1)}{2} \right\rfloor,$$

where $t_{b^2}(1, s\alpha)$ corresponds to the smallest value of t such that there exists a classical $(t, s\alpha)$-sequence in base b^2. From [73, Theorem 5] we obtain

$$t_{b^2}(1, s\alpha) \leq \frac{bs\alpha}{b-1} - \frac{b(s\alpha)^{1/2}}{(b^2-1)^{1/2}}$$

and the result follows. □

4.5 Conclusion and future work

In this chapter we introduced duality theory for non-digital nets and used it to prove propagation rules for higher order nets. Furthermore, we used some of the propagation rules for higher order nets to establish the corresponding propagation rules for higher order sequences, which were in turn used to establish an explicit bound on the quality parameter of higher order sequences.

Interesting future work includes trying to establish a duality theory for higher order sequences, i.e. trying to generalize the results from [31]. Furthermore, it should be investigated if there are analogues of the following propagation rules for higher order nets:

- Propagation Rule 3.7 in [27],
- the subnet propagation rule, see Lemma 4.4 in [66],
- A Generalized Matrix-Product Construction, see [94].

Chapter Five

Quasi-Monte Carlo rules based on higher order nets

Before discussing qMC rules based on higher order nets, we firstly motivate the chapter.

5.1 Motivation

Numerical integration is a classical application of qMC rules based on both, integration lattices and nets, see e.g. [35; 66; 97]. Hence having introduced a new class of nets, higher order nets, it seems obvious to check how qMC rules based on higher order nets fare when applied to numerical integration. This is the topic of this chapter. In Section 5.2 we apply qMC rules based on higher order nets to numerical integration in $W_{\alpha,s,\gamma}$. We find that integration errors converge at a rate of $\mathcal{O}(N^{-(\alpha-1)}(\log N)^{s\alpha})$, which is not optimal: for functions in $W_{\alpha,s,\gamma}$, e.g. qMC rules based on higher order digital nets perform better, see e.g. Subsection 2.7.1. However, we point out that this result does show that qMC rules based on higher order nets can exploit the smoothness of the integrand under consideration; it remains an interesting open problem to establish whether the convergence rate of $\mathcal{O}(N^{-(\alpha-1)}(\log N)^{s\alpha})$ is best possible for qMC rules based on higher order nets or whether it can be improved on.

In Section 5.3 we consider randomized higher order nets, as introduced in Section 3.5. For qMC rules based on such point sets, we manage to establish that the root mean-square integration error for functions in $W_{\alpha,s,\gamma}$ decays at a rate of $\mathcal{O}(N^{-(\alpha-\frac{1}{2})}(\log N)^{s\alpha/2})$, which is an improvement on the result presented in Section 5.2.

Finally, Table 5.1 shows integration errors and root mean-square integration errors for digital and non-digital higher order nets for functions in $W_{\alpha,s,\gamma}$, where $\alpha \in \mathbb{N}_0$, $\alpha \geq 2$ and $\epsilon > 0$. For an explanation for the discrepancy in convergence rates, see Remark 5.2.

	digital higher order nets	non-digital higher order nets
integration error	$N^{-\alpha+\epsilon}$	$N^{-(\alpha-1)+\epsilon}$
root mean-square integration error	$N^{-\alpha+\epsilon}$	$N^{-(\alpha-1/2)+\epsilon}$

Table 5.1. *Integration and root mean-square integration errors for digital and non-digital higher order nets*

5.2 Integration errors

In this section we establish that qMC rules based on (t,α,β,n,m,s)-nets can exploit the smoothness α of a function $f \in W_{\alpha,s,\gamma}$. The parameter α is used to denote the smoothness of functions in $W_{\alpha,s,\gamma}$, $\alpha \in \mathbb{N}$ and $\alpha \geq 2$. We need to recall some notation: firstly, $[s] = \{1,\ldots,s\}$, and for $\boldsymbol{k} = (k_1,\ldots,k_s) \in \mathbb{N}_0^s$ and for $\emptyset \neq \mathfrak{u} \subseteq [s]$, $\boldsymbol{k}_\mathfrak{u}$ denotes the vector in $\mathbb{N}_0^{|\mathfrak{u}|}$ which consists of all components of $\boldsymbol{k}$ whose indices belong to $\mathfrak{u}$. Furthermore, $(\boldsymbol{k}_\mathfrak{u}, \boldsymbol{0})$ is the vector $\boldsymbol{k}$ with all components whose indices are not in $\mathfrak{u}$ replaced by 0. With this notation we have $\mu_\alpha(\boldsymbol{k}_\mathfrak{u}) = \mu_\alpha(\boldsymbol{k}_\mathfrak{u}, \boldsymbol{0})$ and for a sequence $\boldsymbol{\gamma} = (\gamma_j)_{j\geq 1}$ we write $\gamma_\mathfrak{u} = \prod_{j\in\mathfrak{u}} \gamma_j$.

We need the following lemma.

Lemma 5.1. *Let $\{\boldsymbol{x}_h\}_{h=0}^{b^m-1}$ be a (t,α,β,n,m,s)-net in base b and $f \in W_{\alpha,s,\gamma}$, then*

$$\left| \int_{[0,1]^s} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} - \frac{1}{b^m}\sum_{h=0}^{b^m-1} f(\boldsymbol{x}_h) \right| \leq \|f\|_{W_{\alpha,s,\gamma}} \sum_{\emptyset\neq\mathfrak{u}\subseteq[s]} \gamma_\mathfrak{u} \sum_{\substack{\boldsymbol{k}_\mathfrak{u}\in\mathbb{N}^{|\mathfrak{u}|} \\ \mu_\alpha(\boldsymbol{k}_\mathfrak{u})>\beta n-t}} b^{-\mu_\alpha(\boldsymbol{k}_\mathfrak{u})}. \tag{5.1}$$

Proof. Since $f \in W_{\alpha,s,\gamma}$ and $\{\boldsymbol{x}_h\}_{h=0}^{b^m-1}$ is a (t,α,β,n,m,s)-net, we can write

$$\begin{aligned}
\left| \int_{[0,1]^s} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} - \frac{1}{b^m}\sum_{h=0}^{b^m-1} f(\boldsymbol{x}_h) \right| &= \left| \widehat{f}(\boldsymbol{0}) - \frac{1}{b^m}\sum_{h=0}^{b^m-1}\sum_{\boldsymbol{k}\in\mathbb{N}_0^s} \widehat{f}(\boldsymbol{k})\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_h) \right| \\
= \left| \widehat{f}(\boldsymbol{0}) - \sum_{\boldsymbol{k}\in\mathbb{N}_0^s} \widehat{f}(\boldsymbol{k}) \frac{1}{b^m}\sum_{h=0}^{b^m-1} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_h) \right| &= \left| \sum_{\boldsymbol{k}\in\mathbb{N}_0^s\setminus\{\boldsymbol{0}\}} \widehat{f}(\boldsymbol{k}) \frac{1}{b^m}\sum_{h=0}^{b^m-1} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_h) \right| \\
= \left| \sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^s\setminus\{\boldsymbol{0}\} \\ \mu_\alpha(\boldsymbol{k})>\beta n-t}} \widehat{f}(\boldsymbol{k}) \frac{1}{b^m}\sum_{h=0}^{b^m-1} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_h) \right|, &
\end{aligned} \tag{5.2}$$

where we used Lemma 3.22. Using the triangle inequality, it now follows that

$$\left| \int_{[0,1]^s} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} - \frac{1}{b^m}\sum_{h=0}^{b^m-1} f(\boldsymbol{x}_h) \right| \leq \sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^s\setminus\{\boldsymbol{0}\} \\ \mu_\alpha(\boldsymbol{k})>\beta n-t}} |\widehat{f}(\boldsymbol{k})|$$

$$\leq \|f\|_{W_{\alpha,s,\gamma}} \sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^s\setminus\{\boldsymbol{0}\} \\ \mu_\alpha(\boldsymbol{k})>\beta n-t}} r_{\alpha,s,\gamma}(\boldsymbol{k}) = \|f\|_{W_{\alpha,s,\gamma}} \sum_{\emptyset\neq\mathfrak{u}\subseteq[s]} \gamma_{\mathfrak{u}} \sum_{\substack{\boldsymbol{k}_{\mathfrak{u}}\in\mathbb{N}^{|\mathfrak{u}|} \\ \mu_\alpha(\boldsymbol{k}_{\mathfrak{u}})>\beta n-t}} b^{-\mu_\alpha(\boldsymbol{k}_{\mathfrak{u}})},$$

as required. □

Remark 5.2. Comparing Equations (5.1) and (2.15), we note that in Equation (5.1) the second sum runs over all $\boldsymbol{k}_{\mathfrak{u}} \in \mathbb{N}^{|\mathfrak{u}|}$ for which $\mu_\alpha(\boldsymbol{k}_{\mathfrak{u}}) > \beta n - t$, whereas in Equation (2.15) the corresponding sum is taken over all $\boldsymbol{k}_{\mathfrak{u}}$ in the dual space corresponding to the set $\mathfrak{u}$. We obtain this estimate as we bound the absolute value of the character sum $b^{-m}\sum_{h=0}^{b^m-1} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_h)$ in Equation (5.2) by 1. Given concrete constructions, better estimates of this sum may be obtained, as is the case for higher order digital nets and higher order digital sequences.

To establish the main result of this section, we need the following lemma.

Lemma 5.3. *Let $l \geq 1$ and $\alpha \geq 2$ be integers. Then*

$$\sum_{\substack{k\in\mathbb{N} \\ \mu_\alpha(k)=l}} 1 \leq 2\binom{l+\alpha-1}{\alpha-1}(b-1)^\alpha b^{\lfloor l/\alpha\rfloor}.$$

Proof. For $k \in \mathbb{N}$ let ν_k denote the number of non-zero digits in the base b representation of k. We represent $k \in \mathbb{N}$ as follows

$$k = \kappa_1 b^{a_1-1} + \cdots + \kappa_{\nu_k} b^{a_{\nu_k}-1},$$

where $\kappa_1, \ldots, \kappa_{\nu_k} \in \{1, \ldots, b-1\}$ and $a_1 > \ldots > a_{\nu_k} \geq 1$. We firstly consider those $k \in \mathbb{N}$ for which $\nu_k \leq \alpha$. In that case, we put a bound on the number of k for which $\mu_\alpha(k) = a_1 + \cdots + a_{\nu_k} = l$. Then we have

$$\begin{aligned}
&|\{(a_1,\ldots,a_{\nu_k}) : a_1+\cdots+a_{\nu_k} = l, a_1 > \ldots > a_{\nu_k} \geq 1\}| \\
&\leq |\{(a_1,\ldots,a_{\nu_k}) : a_1+\cdots+a_{\nu_k} = l, a_1 \geq 0, \ldots, a_{\nu_k} \geq 0\}| \\
&\leq |\{(a_1,\ldots,a_\alpha) : a_1+\cdots+a_\alpha = l, a_1 \geq 0, \ldots, a_\alpha \geq 0\}| \\
&= \binom{l+\alpha-1}{\alpha-1},
\end{aligned}$$

where the final equality employs a well-known combinatorial result, see also [21, Lemma 5.2]. The coefficients $\kappa_1, \ldots, \kappa_{\nu_k}$ take values in the set $\{1, \ldots, b-1\}$, hence there are $(b-1)^{\nu_k} \leq (b-1)^\alpha$ possibilities and consequently

$$\sum_{\substack{k\in\mathbb{N} \\ \mu_\alpha(k)=l, \nu_k\leq\alpha}} 1 \leq (b-1)^\alpha \binom{l+\alpha-1}{\alpha-1}.$$

We now consider those k for which $\nu_k > \alpha$. Then

$$k = \kappa_1 b^{a_1-1} + \cdots + \kappa_\alpha b^{a_\alpha - 1} + \kappa_{\alpha+1} b^{a_{\alpha+1}-1} + \cdots + \kappa_{\nu_k} b^{a_{\nu_k}-1},$$

and we put a bound on the number of k for which $\mu_\alpha(k) = a_1 + \cdots + a_\alpha = l$. Now,

$$\begin{aligned} &|\{(a_1,\ldots,a_\alpha) : a_1 + \cdots + a_\alpha = l, a_1 > \cdots > a_\alpha \geq 1\}| \\ &\leq \ |\{(a_1,\ldots,a_\alpha) : a_1 + \cdots + a_\alpha = l, a_1 \geq 0, \ldots, a_\alpha \geq 0\}| \\ &= \ \binom{l+\alpha-1}{\alpha-1}. \end{aligned}$$

Regarding the coefficients, it is clear that $\kappa_1,\ldots,\kappa_{\nu_k} \in \{1,\ldots,b-1\}$, so the first α coefficients, $\kappa_1,\ldots,\kappa_\alpha$, can assume $(b-1)^\alpha$ different values. We now focus on the sum

$$\kappa_{\alpha+1} b^{a_{\alpha+1}-1} + \cdots + \kappa_{\nu_k} b^{a_{\nu_k}-1}, \tag{5.3}$$

where $\kappa_{\alpha+1},\ldots,\kappa_{\nu_k} \in \{1,\ldots,b-1\}$ and $a_{\alpha+1} > \ldots > a_{\nu_k} \geq 1$, and note that the number of different values that the sum in Equation (5.3) can assume is bounded by $b^{a_\alpha - 1}$. But by assumption $a_\alpha + \cdots + a_1 = l$, hence $a_\alpha \leq \lfloor l/\alpha \rfloor$, so we conclude that

$$\sum_{\substack{k\in\mathbb{N} \\ \mu_\alpha(k)=l, \nu_k>\alpha}} 1 \leq (b-1)^\alpha b^{\lfloor l/\alpha \rfloor} \binom{l+\alpha-1}{\alpha-1},$$

and the result follows by summing up the two cases. □

Also we will use the following lemma, which appeared as [61, Lemma 2.18].

Lemma 5.4. *For any $b > 1$ and integers i and $j_0 \geq 0$, we have*

$$\sum_{j=j_0}^{\infty} \binom{j+i-1}{i-1} b^{-j} \leq b^{-j_0} \binom{j_0+i-1}{i-1} \left(1 - \frac{1}{b}\right)^{-i}.$$

The next theorem establishes that qMC rules based on (t,α,β,n,m,s)-nets can exploit the smoothness α of a function $f \in W_{\alpha,s,\gamma}$.

Theorem 5.5. *Let $\{\boldsymbol{x}_h\}_{h=0}^{b^m-1}$ be a (t,α,β,n,m,s)-net in base b and $f \in W_{\alpha,s,\gamma}$. Then*

$$\begin{aligned} &\left| \int_{[0,1]^s} f(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} - \frac{1}{b^m} \sum_{h=0}^{b^m-1} f(\boldsymbol{x}_h) \right| \leq \|f\|_{W_{\alpha,s,\gamma}} \\ &\quad \times b^{-(1-1/\alpha)(\lfloor \beta n - t \rfloor + 1)} \sum_{\emptyset \neq \mathfrak{u} \subseteq [s]} \gamma_{\mathfrak{u}} \left(\frac{b^{1-1/\alpha}(b-1)}{(b^{1-1/\alpha}-1)} \right)^{\alpha|\mathfrak{u}|} \frac{(\lfloor \beta n - t \rfloor + \alpha|\mathfrak{u}|)!}{(|\mathfrak{u}|-1)!(\lfloor \beta n - t \rfloor + 1)!}. \end{aligned}$$

Before proving Theorem 5.5, we present the following remark, which deals with an important special case of the result presented in Theorem 5.5.

Remark 5.6. For $\beta n = \alpha m$, we obtain a convergence rate of the integration error of $N^{-(\alpha-1)}$ multiplied by a $\log N$ factor. This rate, although not optimal, see Subsection 2.7.1, does establish that qMC rules based on $(t, \alpha, \beta, n, m, s)$-nets can exploit the smoothness of functions in $W_{\alpha,s,\gamma}$. This was not possible with the classical concept of (t, m, s)-nets.

We now provide the proof of Theorem 5.5.

Proof. Lemma 5.1 established that

$$\left| \int_{[0,1]^s} f(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} - \frac{1}{b^m} \sum_{h=0}^{b^m-1} f(\boldsymbol{x}_h) \right| \leq \|f\|_{W_{\alpha,s,\gamma}} \sum_{\emptyset \neq \mathfrak{u} \subseteq [s]} \gamma_{\mathfrak{u}} \sum_{\substack{\boldsymbol{k}_{\mathfrak{u}} \in \mathbb{N}^{|\mathfrak{u}|} \\ \mu_\alpha(\boldsymbol{k}_{\mathfrak{u}}) > \beta n - t}} b^{-\mu_\alpha(\boldsymbol{k}_{\mathfrak{u}})}. \tag{5.4}$$

For a given $\emptyset \neq \mathfrak{u} \subseteq [s]$, $|\mathfrak{u}| \geq 2$, we rewrite

$$\sum_{\substack{\boldsymbol{k}_{\mathfrak{u}} \in \mathbb{N}^{|\mathfrak{u}|} \\ \mu_\alpha(\boldsymbol{k}_{\mathfrak{u}}) > \beta n - t}} b^{-\mu_\alpha(\boldsymbol{k}_{\mathfrak{u}})} = \sum_{l=\lfloor \beta n - t \rfloor + 1}^{\infty} b^{-l} \sum_{\substack{\boldsymbol{k}_{\mathfrak{u}} \in \mathbb{N}^{|\mathfrak{u}|} \\ \mu_\alpha(\boldsymbol{k}_{\mathfrak{u}}) = l}} 1.$$

Using Lemma 5.3, we obtain

$$\begin{aligned}
\sum_{\substack{\boldsymbol{k}_{\mathfrak{u}} \in \mathbb{N}^{|\mathfrak{u}|} \\ \mu_\alpha(\boldsymbol{k}_{\mathfrak{u}}) = l}} 1 &= \sum_{l_1 + \cdots + l_{|\mathfrak{u}|} = l} \prod_{j=1}^{|\mathfrak{u}|} \sum_{\substack{k_j \in \mathbb{N} \\ \mu_\alpha(k_j) = l_j}} 1 \\
&\leq \sum_{l_1 + \cdots + l_{|\mathfrak{u}|} = l} \prod_{j=1}^{|\mathfrak{u}|} \left[2 \binom{l_j + \alpha - 1}{\alpha - 1} (b-1)^\alpha b^{\lfloor l_j / \alpha \rfloor} \right] \\
&\leq 2^{|\mathfrak{u}|} (b-1)^{\alpha |\mathfrak{u}|} b^{l/\alpha} \sum_{l_1 + \cdots + l_{|\mathfrak{u}|} = l} \prod_{j=1}^{|\mathfrak{u}|} \binom{l_j + \alpha - 1}{\alpha - 1}.
\end{aligned}$$

For any $1 \leq j \leq |\mathfrak{u}|$ we have $\binom{l_j+\alpha-1}{\alpha-1} \leq (1 + l_j)^{\alpha-1}$. Since $l_1, \ldots, l_{|\mathfrak{u}|} \geq 1$ and $l_1 + \cdots + l_{|\mathfrak{u}|} = l$, for $|\mathfrak{u}| \geq 2$ we have $1 + l_j \leq l$ and therefore $\binom{l_j+\alpha-1}{\alpha-1} \leq l^{\alpha-1}$. If $|\mathfrak{u}| = 1$, then $l_1 = l$ and $\binom{l+\alpha-1}{\alpha-1} \leq l^{\alpha-1}$. Hence we obtain

$$\begin{aligned}
&2^{|\mathfrak{u}|} (b-1)^{\alpha |\mathfrak{u}|} b^{l/\alpha} \sum_{l_1 + \cdots + l_{|\mathfrak{u}|} = l} \prod_{j=1}^{|\mathfrak{u}|} \binom{l_j + \alpha - 1}{\alpha - 1} \\
&\leq 2^{|\mathfrak{u}|} (b-1)^{\alpha |\mathfrak{u}|} b^{l/\alpha} \sum_{l_1 + \cdots + l_{|\mathfrak{u}|} = l} l^{(\alpha-1)|\mathfrak{u}|} \\
&\leq 2^{|\mathfrak{u}|} (b-1)^{\alpha |\mathfrak{u}|} b^{l/\alpha} l^{(\alpha-1)|\mathfrak{u}|} \binom{l + |\mathfrak{u}| - 1}{|\mathfrak{u}| - 1}.
\end{aligned}$$

Consequently

$$\sum_{l=\lfloor \beta n - t \rfloor + 1}^{\infty} b^{-l} \sum_{\substack{\boldsymbol{k}_{\mathfrak{u}} \in \mathbb{N}^{|\mathfrak{u}|} \\ \mu_\alpha(\boldsymbol{k}_{\mathfrak{u}}) = l}} 1 \leq 2^{|\mathfrak{u}|} (b-1)^{\alpha |\mathfrak{u}|} \sum_{l=\lfloor \beta n - t \rfloor + 1}^{\infty} b^{-l} b^{l/\alpha} l^{(\alpha-1)|\mathfrak{u}|} \binom{l + |\mathfrak{u}| - 1}{|\mathfrak{u}| - 1}.$$

Invoking the inequality $l^{(\alpha-1)|\mathfrak{u}|}\binom{l+|\mathfrak{u}|-1}{|\mathfrak{u}|-1} \le \binom{l+\alpha|\mathfrak{u}|-1}{\alpha|\mathfrak{u}|-1}\frac{(\alpha|\mathfrak{u}|-1)!}{(|\mathfrak{u}|-1)!}$, which is easily checked, we get

$$\begin{aligned}
& 2^{|\mathfrak{u}|}(b-1)^{\alpha|\mathfrak{u}|} \sum_{l=\lfloor \beta n - t\rfloor+1}^{\infty} b^{-l} b^{l/\alpha} l^{(\alpha-1)|\mathfrak{u}|}\binom{l+|\mathfrak{u}|-1}{|\mathfrak{u}|-1} \\
\le\ & 2^{|\mathfrak{u}|}(b-1)^{\alpha|\mathfrak{u}|}\frac{(\alpha|\mathfrak{u}|-1)!}{(|\mathfrak{u}|-1)!}\sum_{l=\lfloor \beta n - t\rfloor+1}^{\infty} b^{-(1-1/\alpha)l}\binom{l+\alpha|\mathfrak{u}|-1}{\alpha|\mathfrak{u}|-1} \\
\le\ & 2^{|\mathfrak{u}|}\left(\frac{b^{1-1/\alpha}(b-1)}{(b^{1-1/\alpha}-1)}\right)^{\alpha|\mathfrak{u}|}\frac{(\alpha|\mathfrak{u}|-1)!}{(|\mathfrak{u}|-1)!} b^{-(1-1/\alpha)(\lfloor \beta n - t\rfloor+1)}\binom{\lfloor \beta n - t\rfloor+\alpha|\mathfrak{u}|}{\alpha|\mathfrak{u}|-1},
\end{aligned}$$

where we used Lemma 5.4.

Finally

$$\begin{aligned}
& \sum_{\emptyset\neq\mathfrak{u}\subseteq[s]}\gamma_{\mathfrak{u}}\sum_{\substack{\boldsymbol{k}_{\mathfrak{u}}\in\mathbb{N}^{|\mathfrak{u}|}\\ \mu_\alpha(\boldsymbol{k}_{\mathfrak{u}})>\beta n - t}} b^{-\mu_\alpha(\boldsymbol{k}_{\mathfrak{u}})} \\
\le\ & b^{-(1-1/\alpha)(\lfloor \beta n - t\rfloor+1)} \\
& \times \sum_{\emptyset\neq\mathfrak{u}\subseteq[s]}\gamma_{\mathfrak{u}}2^{|\mathfrak{u}|}\left(\frac{b^{1-1/\alpha}(b-1)}{(b^{1-1/\alpha}-1)}\right)^{\alpha|\mathfrak{u}|}\frac{(\alpha|\mathfrak{u}|-1)!}{(|\mathfrak{u}|-1)!}\binom{\lfloor \beta n - t\rfloor+\alpha|\mathfrak{u}|}{\alpha|\mathfrak{u}|-1} \\
=\ & b^{-(1-1/\alpha)(\lfloor \beta n - t\rfloor+1)} \\
& \times \sum_{\emptyset\neq\mathfrak{u}\subseteq[s]}\gamma_{\mathfrak{u}}2^{|\mathfrak{u}|}\left(\frac{b^{1-1/\alpha}(b-1)}{(b^{1-1/\alpha}-1)}\right)^{\alpha|\mathfrak{u}|}\frac{(\lfloor \beta n - t\rfloor+\alpha|\mathfrak{u}|)!}{(|\mathfrak{u}|-1)!(\lfloor \beta n - t\rfloor+1)!},
\end{aligned}$$

which establishes the result. □

5.3 Mean-square integration errors

Finally, we discuss numerical integration using qMC rules based on $\mathcal{P}_{\boldsymbol{\Delta}}$ and $\mathcal{P}_{\boldsymbol{\Delta},\delta}$, which were defined in Section 3.5 and whose definitions we now recall. We let $\mathcal{P} = \{\boldsymbol{x}_0,\dots,\boldsymbol{x}_{b^m-1}\}$ be a (t,α,β,n,m,s)-net in base b and let $\boldsymbol{\Delta} = (\Delta_1,\dots,\Delta_s)$, where the Δ_j, $1\le j\le s$, are uniformly distributed in $[0,1)$ and mutually independent. Consequently, we obtain $\mathcal{P}_{\boldsymbol{\Delta}}$ by applying $\boldsymbol{\Delta}$ to each point in $\mathcal{P}$, i.e. $\mathcal{P}_{\boldsymbol{\Delta}} = \{\boldsymbol{z}_0,\boldsymbol{z}_1,\dots,\boldsymbol{z}_{b^m-1}\}$ is given by

$$\boldsymbol{z}_h = \boldsymbol{x}_h \oplus \boldsymbol{\Delta}\,,\ 0\le h< b^m\,.$$

Regarding $\mathcal{P}_{\boldsymbol{\Delta},\delta}$, we need to recall the concept of a digital shift in base b of depth n, see also Section 3.5: we choose digits $\Delta_{j,l}$, for $1\le j\le s$, $1\le l\le n$, uniformly distributed on $\{0,1,\dots,b-1\}$ and mutually independent and also choose $\delta_{h,j}$, for $0\le h< b^m$, $1\le j\le s$, uniformly distributed in $[0,b^{-n})$ and mutually independent. Now if $\boldsymbol{x}_h = (x_{h,1},\dots,x_{h,s})$, for $0\le h<b^m$, where

$$x_{h,j} = \frac{\xi_{h,j,1}}{b}+\dots+\frac{\xi_{h,j,n}}{b^n}+\frac{\xi_{h,j,n+1}}{b^{n+1}}+\dots\,,$$

we define

$$z_{h,j,l} = \xi_{h,j,l} + \Delta_{j,l} \pmod b,$$

for $0 \le h < b^m$, $1 \le j \le s$, and $1 \le l \le n$, and finally set

$$z_{h,j} = \frac{z_{h,j,1}}{b} + \cdots + \frac{z_{h,j,n}}{b^n} + \delta_{h,j}\,,\; 0 \le h < b^m\,,\; 1 \le j \le s,$$

to obtain $\mathcal{P}_{\mathbf{\Delta},\delta} = \{z_0, z_1 \ldots, z_{b^m-1}\}$. We recall that all members of $\mathcal{P}_{\mathbf{\Delta}}$ and $\mathcal{P}_{\mathbf{\Delta},\delta}$ are uniformly distributed in $[0,1)^s$ and that $\mathcal{P}_{\mathbf{\Delta}}$ is a (t,α,β,n,m,s)-net with probability 1 and $\mathcal{P}_{\mathbf{\Delta},\delta}$ is a (t,α,β,n,m,s)-net. Using the latter property and Theorem 5.5, we can easily obtain information on

$$\left| Q_{b^m}(f,\widetilde{\mathcal{P}}) - \int_{[0,1]^s} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \right|,\;\; \widetilde{\mathcal{P}} \in \{\mathcal{P}_{\mathbf{\Delta}}, \mathcal{P}_{\mathbf{\Delta},\delta}\}\,, f \in W_{\alpha,s,\gamma},$$

where $Q_{\cdot}(\cdot,\cdot)$ is defined in Definition 2.1. However, studying the root mean-square integration error is interesting, as we can improve on the convergence rate from Theorem 5.5.

Theorem 5.7. *Let $f \in W_{\alpha,s,\gamma}$, $\mathcal{P}$ be a (t,α,β,n,m,s)-net in base b and $\mathcal{P}_{\mathbf{\Delta}} = \{z_0, z_1, \ldots, z_{b^m-1}\}$ be defined as above. Then*

$$\mathbb{E}\left[\left|\frac{1}{b^m}\sum_{h=0}^{b^m-1} f(\boldsymbol{z}_h) - \int_{[0,1]^s} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right|^2\right] \le \|f\|^2_{W_{\alpha,s,\gamma}} b^{-\frac{2\alpha-1}{\alpha}(\lfloor \beta n - t\rfloor + 1)}$$
$$\times \sum_{\emptyset \neq \mathfrak{u} \subseteq [s]} \gamma_{\mathfrak{u}}^2 2^{|\mathfrak{u}|} \left(\frac{b^{\frac{2\alpha-1}{\alpha}}(b-1)}{b^{\frac{2\alpha-1}{\alpha}}-1}\right)^{\alpha|\mathfrak{u}|} \frac{(\lfloor \beta n - t\rfloor + \alpha|\mathfrak{u}|)!}{(|\mathfrak{u}|-1)!(\lfloor \beta n - t\rfloor + 1)!}.$$

Proof. Arguing as in the proof of Lemma 5.1, we get

$$\left|\frac{1}{b^m}\sum_{h=0}^{b^m-1} f(\boldsymbol{z}_h) - \int_{[0,1]^s} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right| = \left|\sum_{\boldsymbol{k}\in\mathbb{N}_0^s\setminus\{\boldsymbol{0}\}} \widehat{f}(\boldsymbol{k}) Q_{b^m}(\mathrm{wal}_{\boldsymbol{k}}, \mathcal{P}_{\mathbf{\Delta}})\right|.$$

Hence

$$\begin{aligned}&\left|\frac{1}{b^m}\sum_{h=0}^{b^m-1} f(\boldsymbol{z}_h) - \int_{[0,1]^s} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right|^2\\ &= \sum_{\boldsymbol{k}\in\mathbb{N}_0^s\setminus\{\boldsymbol{0}\}} |\widehat{f}(\boldsymbol{k})|^2 |Q_{b^m}(\mathrm{wal}_{\boldsymbol{k}}, \mathcal{P}_{\mathbf{\Delta}})|^2\\ &+ \sum_{\substack{\boldsymbol{k},\boldsymbol{l}\in\mathbb{N}_0^s\setminus\{\boldsymbol{0}\}\\ \boldsymbol{k}\neq\boldsymbol{l}}} \widehat{f}(\boldsymbol{k})\overline{\widehat{f}(\boldsymbol{l})} Q_{b^m}(\mathrm{wal}_{\boldsymbol{k}}, \mathcal{P}_{\mathbf{\Delta}}) \overline{Q_{b^m}(\mathrm{wal}_{\boldsymbol{l}}, \mathcal{P}_{\mathbf{\Delta}})}\,.\end{aligned}$$

Clearly

$$|Q_{b^m}(\mathrm{wal}_{\boldsymbol{k}}, \mathcal{P}_{\mathbf{\Delta}})|^2 = \frac{1}{b^{2m}}\sum_{i,h=0}^{b^m-1} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{z}_h \ominus \boldsymbol{z}_i)$$

and

$$Q_{b^m}(\mathrm{wal}_{\boldsymbol{k}}, \mathcal{P}_{\boldsymbol{\Delta}})\overline{Q_{b^m}(\mathrm{wal}_{\boldsymbol{l}}, \mathcal{P}_{\boldsymbol{\Delta}})} = \frac{1}{b^{2m}} \sum_{i,h=0}^{b^m-1} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{z}_h)\overline{\mathrm{wal}_{\boldsymbol{l}}(\boldsymbol{z}_i)}.$$

Consequently using [20, Lemma 6.1], see also [33, Lemma 3], and Theorem 3.25,

$$\begin{aligned}
&\mathbb{E}\left[\left|\frac{1}{b^m}\sum_{h=0}^{b^m-1} f(\boldsymbol{z}_h) - \int_{[0,1]^s} f(\boldsymbol{x})d\boldsymbol{x}\right|^2\right] \\
&= \mathbb{E}\left[\sum_{\boldsymbol{k}\in\mathbb{N}_0^s\setminus\{\boldsymbol{0}\}} |\widehat{f}(\boldsymbol{k})|^2 \frac{1}{b^{2m}} \sum_{i,h=0}^{b^m-1} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{z}_h)\overline{\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{z}_i)}\right] \\
&= \sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^s\setminus\{\boldsymbol{0}\} \\ \mu_\alpha(\boldsymbol{k})>\beta n-t}} |\widehat{f}(\boldsymbol{k})|^2 \frac{1}{b^{2m}} \sum_{i,h=0}^{b^m-1} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_h \ominus \boldsymbol{x}_i) \\
&\le \sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^s\setminus\{\boldsymbol{0}\} \\ \mu_\alpha(\boldsymbol{k})>\beta n-t}} |\widehat{f}(\boldsymbol{k})|^2 \\
&\le \|f\|^2_{W_{\alpha,s,\gamma}} \sum_{\emptyset\neq\mathfrak{u}\subseteq[s]} \gamma_{\mathfrak{u}}^2 \sum_{\substack{\boldsymbol{k}_{\mathfrak{u}}\in\mathbb{N}^{|\mathfrak{u}|} \\ \mu_\alpha(\boldsymbol{k}_{\mathfrak{u}})>\beta n-t}} b^{-2\mu_\alpha(\boldsymbol{k}_{\mathfrak{u}})}.
\end{aligned}$$

The proof can now be completed in the same way as the proof of Theorem 5.5. □

Remark 5.8. Similar to Remark 5.6, we point out that setting $\beta n = \alpha m$ results in a convergence rate of $N^{-(\alpha-\frac{1}{2})}(\log N)^{\frac{s\alpha}{2}}$ for the root mean-square integration error, improving on the convergence rate given in Theorem 5.5. For the point set $\mathcal{P}_{\boldsymbol{\Delta},\delta}$, the same bound as in Theorem 5.7 can be obtained using the same argument but employing [33, Lemma 3] instead of [20, Lemma 6.1].

5.4 Conclusion and future work

In this chapter we applied qMC rules based on higher order nets and randomized higher order nets to numerical integration. We found that the root mean-square integration errors resulting from the qMC rules based on randomized higher order nets decay at a faster rate than the integration errors resulting from the qMC rules based on higher order nets. However, even though qMC rules based on higher order nets and randomized higher order nets can exploit the smoothness of the integrand under consideration, the convergence rates are not optimal for the function space under consideration.

Important future work of course includes trying to establish whether the results presented in this chapter are best possible for qMC rules based on higher order nets or can be improved on.

Chapter Six

Construction algorithms for higher order polynomial lattice point sets

Before discussing construction algorithms for higher order polynomial lattice point sets, we firstly motivate the chapter.

6.1 Motivation

In this chapter we study a special case of higher order digital nets, namely higher order polynomial lattice point sets, which were introduced in Subsection 2.3.2. In particular, we focus on the application of higher order polynomial lattice point sets to numerical integration, see also Subsection 2.7.1.

Applying higher order digital nets to numerical integration, it is important to have good constructions of higher order digital nets. So far, the construction explained in Subsection 2.3.1 allows to construct higher order digital nets using classical digital nets. Furthermore, propagation rules showing how to construct new higher order digital nets from existing ones were presented in [27]; however, even in the latter case, classical digital nets still form the basis of the construction of the existing higher order digital nets.

The aim of this chapter is to provide constructions of higher order digital nets independent of classical digital nets. Furthermore, we present a construction which produces higher order polynomial lattice point sets which perform well when used for numerical integration; this is different from the usual approach to constructing classical digital and higher order digital nets, which is aimed at producing nets exhibiting a low quality parameter t; of course, nets with a more favorable quality parameter result in more favorable bounds on the integration error, see e.g. Subsection 2.7.1. However, the construction of higher order polynomial lattice point sets is completed without any reference to the quality parameter t of the resulting higher order digital net. Higher order polynomial lattice point sets were first introduced in [34] and also studied in [28].

In the latter paper the concept of a figure of merit, see e.g. [35; 66] and Definition 2.19, was extended to cover higher order polynomial lattice point sets, see Definition 2.34; importantly, the existence of higher order polynomial lattice point sets achieving a better bound on the quality parameter t than the higher order digital nets constructed as outlined in Subsection 2.3.1 was established. Unfortunately, no explicit constructions were provided and a computer search would have to be performed to find them. It is to be pointed out that the existence results for the higher order polynomial lattice point sets improve on the construction from Subsection 2.3.1 especially when α and s are large, which are the cases of practical interest.

The above observations motivate the quest for finding an explicit algorithm to construct higher order polynomial lattice point sets. In particular, we consider two cases: firstly, in Section 6.3 we use a component-by-component (CBC) approach, see e.g. [52], to produce higher order polynomial lattice rules achieving the optimal rate of convergence for functions in $W_{\alpha,s,\gamma}$, see Algorithm 6.1 and Theorem 6.4. In Section 6.4 we combine the CBC approach with a "sieve"-type algorithm, see e.g. [36], to construct higher order polynomial lattice rules which automatically adjust themselves to the smoothness of the integrand in terms of the convergence rate of the integration error within a certain (arbitrary high) range, see Algorithm 6.2 and Theorem 6.10. As in [34], we point out that an analogous result for integration lattices is not known. Furthermore, in Section 6.5 we study a special case of higher order polynomial lattice point sets, namely Korobov polynomial lattice point sets as introduced in [34], see also [49]. We find that results analogous to those obtained for higher order polynomial lattice point sets constructed using a CBC and CBC-sieve approach can be established. Finally, we remark that the approaches to constructing higher order polynomial lattice point sets presented in this chapter are direct, i.e. we avoid employing classical digital nets, which the construction from Subsection 2.3.1 relied on.

6.2 Numerical integration using higher order polynomial lattice point sets

In this section we firstly recall higher order polynomial lattice point sets, as introduced in Subsection 2.3.2, and consequently show how to apply them to numerical integration in $W_{\alpha,s,\gamma}$. As in Chapter 5, we remark that in this chapter, α is used to denote the smoothness of functions in $W_{\alpha,s,\gamma}$, $\alpha \in \mathbb{N}$ and $\alpha \geq 2$. Following Subsection 2.3.2, b is prime, $1 \leq m \leq n$, and v_n is the map from $\mathbb{Z}_b((x^{-1}))$ to the interval $[0,1)$ defined by

$$v_n(\sum_{l=w}^{\infty} t_l x^{-l}) = \sum_{l=\max(1,w)}^{n} t_l b^{-l}.$$

For a given dimension $s \geq 1$, let $q_1(x), \ldots, q_s(x) \in \mathbb{Z}_b[x]$, and let $p(x) \in \mathbb{Z}_b[x]$ with $deg(p(x)) = n \geq 1$. Furthermore, for $0 \leq h < b^m$, let $h = h_0 + h_1 b + \cdots + h_{m-1} b^{m-1}$ be the b-adic expansion of h. With each such h we associate the polynomial

$$\overline{h}(x) = \sum_{r=0}^{m-1} h_r x^r \in \mathbb{Z}_b[x].$$

Consequently, we use $S_{p,m,n}(\boldsymbol{q})$, where $\boldsymbol{q} = (q_1(x), \ldots, q_s(x))$, to denote the point set consisting of the b^m points

$$\boldsymbol{x}_h = \left(v_n(\frac{\overline{h}(x) q_1(x)}{p(x)}), \ldots, v_n(\frac{\overline{h}(x) q_s(x)}{p(x)}) \right) \in [0,1)^s,$$

for $0 \leq h < b^m$. The point set $S_{p,m,n}(\boldsymbol{q})$ is called a higher order polynomial lattice point set. In this chapter, we are interested in the worst-case integration error associated with the quasi-Monte Carlo rule $Q_N(\cdot, \mathcal{P})$, where

$$Q_N(f, \mathcal{P}) = \frac{1}{N} \sum_{h=0}^{N-1} f(\boldsymbol{x}_h),$$

for a given point set $\mathcal{P} = \{\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{N-1}\} \subseteq [0,1)^s$. The worst-case integration error for the Walsh space $W_{\alpha,s,\gamma}$ using the quasi-Monte Carlo rule $Q_N(\cdot, \mathcal{P})$ is given by

$$e(Q_N(\cdot, \mathcal{P}), W_{\alpha,s,\gamma}) = \sup_{\substack{f \in W_{\alpha,s,\gamma} \\ \|f\|_{W_{\alpha,s,\gamma}} \leq 1}} |I(f) - Q_N(f, \mathcal{P})|,$$

where $I(f) = \int_{[0,1]^s} f(\boldsymbol{x}) d\boldsymbol{x}$.

The initial error is given by

$$e(Q_0(\cdot, \emptyset), W_{\alpha,s,\gamma}) = \sup_{\substack{f \in W_{\alpha,s,\gamma} \\ \|f\|_{W_{\alpha,s,\gamma}} \leq 1}} |I(f)|.$$

In particular, we are interested in quasi-Monte Carlo rules based on higher order polynomial lattice point sets, and we denote the quasi-Monte Carlo rule based on the higher order polynomial lattice point set $S_{p,m,n}(\boldsymbol{q})$ by $Q_{b^m}(\cdot, S_{p,m,n}(\boldsymbol{q}))$. The next proposition gives information on the worst-case integration error associated with $Q_{b^m}(\cdot, S_{p,m,n}(\boldsymbol{q}))$; we remark that the dual polynomial lattice associated with $S_{p,m,n}(\boldsymbol{q})$ was introduced in Definition 2.33.

Proposition 6.1. *Let b be a prime and $\alpha \geq 2$ an integer. Then the worst-case error of multivariate integration in $W_{\alpha,s,\gamma}$ using the quasi-Monte Carlo rule $Q_{b^m}(\cdot, S_{p,m,n}(\boldsymbol{q}))$ is given by*

$$e_{b^m,\alpha}(\boldsymbol{q}, p) := e(Q_{b^m}(\cdot, S_{p,m,n}(\boldsymbol{q})), W_{\alpha,s,\gamma}) = \sum_{\boldsymbol{k} \in \mathcal{D}'_p(\boldsymbol{q})} r_\alpha(\boldsymbol{\gamma}, \boldsymbol{k}),$$

where $\mathcal{D}'_p(\boldsymbol{q}) = \mathcal{D}_p(\boldsymbol{q}) \setminus \{\boldsymbol{0}\}$, $\mathcal{D}_p(\boldsymbol{q})$ is given by Definition 2.33 and $r_\alpha(\boldsymbol{\gamma}, \boldsymbol{k})$ is given by Equation (2.10).

Proof. Combine Equation (2.16) with the determination of the dual net of $S_{p,m,n}(\boldsymbol{q})$ from [34, Section 4]. □

Finally, the next proposition presents an expression for $e_{b^m,\alpha}(\boldsymbol{q}, p)$ which is computable; of course, such an expression is needed to implement the algorithms presented in this chapter.

Proposition 6.2. *The worst-case error of multivariate integration in $W_{\alpha,s,\gamma}$ associated with the higher order polynomial lattice point set $S_{p,m,n}(\boldsymbol{q})$ satisfies*

$$e_{b^m,\alpha}(\boldsymbol{q}_s, p) = -1 + \frac{1}{b^m} \sum_{h=0}^{b^m-1} \prod_{j=1}^{s} (1 + \gamma_j \omega(x_{h,j}, \alpha)),$$

where, for $x \in [0,1)$, $\omega(x, \alpha) = \sum_{k=1}^{\infty} r_\alpha(1,k) \mathrm{wal}_k(x)$.

Proof. Using Proposition 6.1 and Lemma 2.45, we get

$$\begin{aligned}
e_{b^m,\alpha}(\boldsymbol{q}_s, p) &= \sum_{\boldsymbol{k} \in \mathcal{D}'_p(\boldsymbol{q})} r_\alpha(\boldsymbol{\gamma}, \boldsymbol{k}) \\
&= \sum_{\boldsymbol{k} \in \mathcal{D}'_p(\boldsymbol{q})} r_\alpha(\boldsymbol{\gamma}, \boldsymbol{k}) \sum_{h=0}^{b^m-1} \frac{\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_h)}{b^m} \\
&= \sum_{\boldsymbol{k} \in \mathbb{N}_0^s \setminus \{\boldsymbol{0}\}} r_\alpha(\boldsymbol{\gamma}, \boldsymbol{k}) \sum_{h=0}^{b^m-1} \frac{\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_h)}{b^m} \\
&= -1 + \frac{1}{b^m} \sum_{h=0}^{b^m-1} \sum_{\boldsymbol{k} \in \mathbb{N}_0^s} r_\alpha(\boldsymbol{\gamma}, \boldsymbol{k}) \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_h) \\
&= -1 + \frac{1}{b^m} \sum_{h=0}^{b^m-1} \prod_{j=1}^{s} \left[\sum_{k=0}^{\infty} r_\alpha(\gamma_j, k) \mathrm{wal}_k(x_{h,j}) \right] \\
&= -1 + \frac{1}{b^m} \sum_{h=0}^{b^m-1} \prod_{j=1}^{s} \left(1 + \gamma_j \omega(x_{h,j}, \alpha)\right).
\end{aligned}$$

□

We conclude this section by noting that an efficient implementation of the function $\omega(.,.)$ is presented in [9].

6.3 Component-by-component construction

We propose the following algorithm to construct a higher order polynomial lattice rule achieving the optimal rate of convergence for functions in $W_{\alpha,s,\gamma}$. Unlike the results presented in Section 6.4, we only deal with a fixed α in this section. In Section 6.4, α is allowed to vary within a prespecified range. For ease of notation, we proceed as follows: we use $q = q(x) \in \mathbb{Z}_b[x]$, $p = p(x) \in \mathbb{Z}_b[x]$ and $a = a(x) \in \mathbb{Z}_b[x]$; also, if we consider the polynomial associated with an integer k, we use $\bar{k} = \bar{k}(x) \in \mathbb{Z}_b[x]$.

Furthermore, we denote the worst-case error associated with the higher order polynomial lattice point set $S_{p,m,n}(\boldsymbol{q})$ by $e_{b^m,\alpha}(\boldsymbol{q}, p)$, and we put

$$G_{b,n} := \{q \in \mathbb{Z}_b[x] : \deg(q) < n\}. \tag{6.1}$$

We also make use of the following lemma, which appeared in a weaker and non-explicit form as [34, Lemma 4.2].

Lemma 6.3. *Let $\alpha \geq 2$ be an integer. Then for every $1/\alpha < \lambda \leq 1$ we have*

$$\sum_{l=1}^{\infty} r_\alpha^\lambda(\gamma, l) \leq \gamma^\lambda C_{b,\alpha,\lambda},$$

where

$$\begin{aligned} C_{b,\alpha,\lambda} &:= \widetilde{C}_{b,\alpha,\lambda} + \frac{(b-1)^\alpha}{b^{\lambda\alpha} - b} \prod_{i=1}^{\alpha-1} \frac{1}{b^{\lambda i} - 1}, \\ \widetilde{C}_{b,\alpha,\lambda} &= \begin{cases} \alpha - 1 & \text{if } \lambda = 1, \\ \frac{(b-1)((b-1)^{\alpha-1} - (b^\lambda - 1)^{\alpha-1})}{(b-b^\lambda)(b^\lambda-1)^{\alpha-1}} & \text{if } \lambda < 1. \end{cases} \end{aligned}$$

Furthermore, the series $\sum_{l=1}^{\infty} r_\alpha^\lambda(\gamma, l)$ diverges to $+\infty$ as λ goes to $1/\alpha$ from the right.

Proof. Let $l = \lambda_1 b^{a_1 - 1} + \cdots + \lambda_v b^{a_v - 1}$ where $v \geq 1$, $0 < a_v < \cdots < a_1$ and $\lambda_i \in \{1, \ldots, b-1\}$. We divide the sum over all $l \in \mathbb{N}$ into two parts, namely firstly where $1 \leq v \leq \alpha - 1$ and secondly where $v > \alpha - 1$. For the first part, we have

$$\begin{aligned} & \sum_{v=1}^{\alpha-1} (b-1)^v \sum_{0 < a_v < \cdots < a_1} \frac{1}{b^{\lambda(a_1 + \cdots + a_v)}} \\ &= \sum_{v=1}^{\alpha-1} (b-1)^v \sum_{a_1 = v}^{\infty} \frac{1}{b^{\lambda a_1}} \sum_{a_2 = v-1}^{a_1 - 1} \frac{1}{b^{\lambda a_2}} \cdots \sum_{a_v = 1}^{a_{v-1} - 1} \frac{1}{b^{\lambda a_v}} \\ &\leq \sum_{v=1}^{\alpha-1} \left(\frac{b-1}{b^\lambda - 1} \right)^v = \begin{cases} \alpha - 1 & \text{if } \lambda = 1, \\ \frac{(b-1)((b-1)^{\alpha-1} - (b^\lambda - 1)^{\alpha-1})}{(b-b^\lambda)(b^\lambda-1)^{\alpha-1}} & \text{if } \lambda < 1, \end{cases} \\ &=: \widetilde{C}_{b,\alpha,\lambda}. \end{aligned}$$

For the second part, we have

$$
\begin{aligned}
&(b-1)^\alpha \sum_{0<a_\alpha<\cdots<a_1} \frac{b^{a_\alpha -1}}{b^{\lambda(a_1+\cdots+a_\alpha)}} \\
&= \frac{(b-1)^\alpha}{b} \sum_{a_1=\alpha}^{\infty} \frac{1}{b^{\lambda a_1}} \sum_{a_2=\alpha-1}^{a_1-1} \frac{1}{b^{\lambda a_2}} \cdots \sum_{a_\alpha=1}^{a_{\alpha-1}-1} \frac{b^{a_\alpha}}{b^{\lambda a_\alpha}} \\
&= \frac{(b-1)^\alpha}{b} \sum_{a_\alpha=1}^{\infty} \frac{b^{a_\alpha}}{b^{\lambda a_\alpha}} \sum_{a_{\alpha-1}=a_\alpha+1}^{\infty} \frac{1}{b^{\lambda a_{\alpha-1}}} \cdots \sum_{a_2=a_3+1}^{\infty} \frac{1}{b^{\lambda a_2}} \sum_{a_1=a_2+1}^{\infty} \frac{1}{b^{\lambda a_1}} \\
&= \frac{(b-1)^\alpha}{b} \prod_{i=1}^{\alpha-1} \frac{1}{b^{\lambda i}-1} \sum_{a_\alpha=1}^{\infty} \frac{b^{a_\alpha}}{b^{\lambda a_\alpha}} \frac{1}{b^{\lambda(\alpha-1)a_\alpha}} \\
&= \frac{(b-1)^\alpha}{b^{\lambda\alpha}-b} \prod_{i=1}^{\alpha-1} \frac{1}{b^{\lambda i}-1}.
\end{aligned}
$$

Hence, we have shown that

$$
\begin{aligned}
\gamma^\lambda \frac{(b-1)^\alpha}{b^{\lambda\alpha}-b} \prod_{i=1}^{\alpha-1} \frac{1}{b^{\lambda i}-1} &\le \sum_{l=1}^{\infty} r_\alpha^\lambda(\gamma, l) \\
&\le \gamma^\lambda \left(\widetilde{C}_{b,\alpha,\lambda} + \frac{(b-1)^\alpha}{b^{\lambda\alpha}-b} \prod_{i=1}^{\alpha-1} \frac{1}{b^{\lambda i}-1} \right) =: \gamma^\lambda C_{b,\alpha,\lambda}.
\end{aligned}
$$

As $\frac{(b-1)^\alpha}{b^{\lambda\alpha}-b} \prod_{i=1}^{\alpha-1} \frac{1}{b^{\lambda i}-1} \to +\infty$ whenever $\lambda \to 1/\alpha$ from the right we also obtain the second assertion. □

We now show that a component-by-component approach can be used to construct a higher order polynomial lattice rule that achieves optimal convergence rates for functions in $W_{\alpha,s,\gamma}$. For $1 \le d \le s$, we set $\boldsymbol{q}_d = (q_1, \ldots, q_d)$. Note that we consider the vector $\boldsymbol{q}_d$ instead of $(1, q, \ldots, q^{d-1})$, c.f. [29, Algorithm 4.3], as otherwise the projection onto the first coordinate does not achieve a convergence rate of $b^{-\alpha m}$, see also [34, Remark 2.3]. The component-by-component algorithm for a fixed α is summarized in Algorithm 6.1.

Algorithm 6.1 CBC algorithm for a fixed α

Require: b a prime, $s, m, n \in \mathbb{N}$ and weights $\boldsymbol{\gamma} = (\gamma_j)_{j\ge 1}$.

1: Choose an irreducible polynomial $p \in \mathbb{Z}_b[x]$, with $\deg(p) = n$.
2: **for** $d = 1$ to s **do**
3: find $q_d \in G_{b,n}$ by minimizing $e_{b^m,\alpha}((q_1, \ldots, q_d), p)$ as a function of q_d.
4: **end for**
5: **return** $\boldsymbol{q} = (q_1, \ldots, q_s)$.

Theorem 6.4. *Let b be prime, α, $s, m, n \in \mathbb{N}$, $\alpha \geq 2$, and $p \in \mathbb{Z}_b[x]$ be irreducible with* $\deg(p) = n$. *Suppose $(q_1^*, \ldots, q_s^*) \in G_{b,n}^s$ is constructed using Algorithm 6.1. Then for all $d = 1, \ldots, s$ we have:*

$$e_{b^m,\alpha}((q_1^*, \ldots, q_d^*), p) \leq \frac{1}{b^{\min(\tau m, n)}} \prod_{j=1}^{d} (1 + 3\gamma_j^{1/\tau} C_{b,\alpha,1/\tau})^\tau \quad \forall 1 \leq \tau < \alpha. \tag{6.2}$$

Proof. We firstly show the result for $d = 1$. By Proposition 6.1,

$$e_{b^m,\alpha}(q_1, p) = \sum_{k \in \mathcal{D}'_p(q_1)} r_\alpha(\gamma, k).$$

The algorithm chooses q_1^* as to minimize the worst-case error, so we have

$$e_{b^m,\alpha}(q_1^*, p) \leq e_{b^m,\alpha}(q_1, p), \quad \forall q_1 \in G_{b,n}.$$

Hence, for all $1/\alpha < \lambda \leq 1$

$$e_{b^m,\alpha}(q_1^*, p)^\lambda \leq \frac{1}{b^n} \sum_{q_1 \in G_{b,n}} e_{b^m,\alpha}(q_1, p)^\lambda.$$

Using an argument very similar to the one used in the proof of [34, Proposition 4.3], it can be shown that for all $1/\alpha < \lambda \leq 1$

$$e_{b^m,\alpha}(q_1^*, p)^\lambda \leq \frac{1}{b^n} \sum_{q_1 \in G_{b,n}} e_{b^m,\alpha}(q_1, p)^\lambda \leq \gamma_1^\lambda C_{b,\alpha,\lambda}(b^{-m} + b^{-\lambda n}).$$

Consequently, we have, setting $\tau = 1/\lambda$,

$$\begin{aligned} e_{b^m,\alpha}(q_1^*, p) &\leq (1 + 2\gamma_1^\lambda C_{b,\alpha,\lambda})^{1/\lambda} b^{-\min(m/\lambda, n)} \\ &\leq (1 + 3\gamma_1^{1/\tau} C_{b,\alpha,1/\tau})^\tau b^{-\min(m\tau, n)}. \end{aligned}$$

We now assume that for some $1 \leq d < s$ we have $\boldsymbol{q}_d^* \in G_{b,n}^d$ and

$$e_{b^m,\alpha}(\boldsymbol{q}_d^*, p) \leq b^{-\min(\tau m, n)} \prod_{j=1}^{d} (1 + 3\gamma_j^{1/\tau} C_{b,\alpha,1/\tau})^\tau.$$

Clearly,

$$\begin{aligned} & e_{b^m,\alpha}((\boldsymbol{q}_d^*, q_{d+1}), p) \\ = & \sum_{(\boldsymbol{k}, k_{d+1}) \in \mathcal{D}'_p(\boldsymbol{q}_d^*, q_{d+1})} r_\alpha(\boldsymbol{\gamma}, \boldsymbol{k}) r_\alpha(\gamma_{d+1}, k_{d+1}) \\ = & \sum_{\boldsymbol{k} \in \mathcal{D}'_p(\boldsymbol{q}_d^*)} r_\alpha(\boldsymbol{\gamma}, \boldsymbol{k}) + \sum_{k_{d+1}=1}^{\infty} r_\alpha(\gamma_{d+1}, k_{d+1}) \sum_{\substack{\boldsymbol{k} \in \mathbb{N}_0^d \\ (\boldsymbol{k}, k_{d+1}) \in \mathcal{D}'_p(\boldsymbol{q}_d^*, q_{d+1})}} r_\alpha(\boldsymbol{\gamma}, \boldsymbol{k}) \\ = & e_{b^m,\alpha}(\boldsymbol{q}_d^*, p) + \theta(\boldsymbol{q}_d^*, q_{d+1}), \end{aligned}$$

where we set

$$\theta(\boldsymbol{q}_d^*, q_{d+1}) := \sum_{k_{d+1}=1}^{\infty} r_\alpha(\gamma_{d+1}, k_{d+1}) \sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^d \\ (\boldsymbol{k},k_{d+1})\in\mathcal{D}_p'(\boldsymbol{q}_d^*,q_{d+1})}} r_\alpha(\boldsymbol{\gamma}, \boldsymbol{k}).$$

It follows from Algorithm 6.1 that q_{d+1}^* is chosen in such a way that the worst-case error $e_{b^m,\alpha}((\boldsymbol{q}_d^*, q_{d+1}), p)$ is minimized. Since the only dependence on q_{d+1} is in $\theta(\boldsymbol{q}_d^*, q_{d+1})$, we have $\theta(\boldsymbol{q}_d^*, q_{d+1}^*) \le \theta(\boldsymbol{q}_d^*, q_{d+1})$ for all $q_{d+1} \in G_{b,n}$, which implies that for all $1/\alpha < \lambda \le 1$ we have

$$\begin{aligned}
\theta(\boldsymbol{q}_d^*, q_{d+1}^*)^\lambda &\le \frac{1}{b^n} \sum_{q_{d+1}\in G_{b,n}} \theta(\boldsymbol{q}_d^*, q_{d+1})^\lambda \\
&= \frac{1}{b^n} \sum_{q_{d+1}\in G_{b,n}} \left(\sum_{k_{d+1}=1}^{\infty} r_\alpha(\gamma_{d+1}, k_{d+1}) \sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^d \\ (\boldsymbol{k},k_{d+1})\in\mathcal{D}_p'(\boldsymbol{q}_d^*,q_{d+1})}} r_\alpha(\boldsymbol{\gamma}, \boldsymbol{k}) \right)^\lambda \\
&\le \frac{1}{b^n} \sum_{q_{d+1}\in G_{b,n}} \sum_{k_{d+1}=1}^{\infty} r_\alpha^\lambda(\gamma_{d+1}, k_{d+1}) \left(\sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^d \\ (\boldsymbol{k},k_{d+1})\in\mathcal{D}_p'(\boldsymbol{q}_d^*,q_{d+1})}} r_\alpha(\boldsymbol{\gamma}, \boldsymbol{k}) \right)^\lambda \\
&\le \sum_{\substack{k_{d+1}=1 \\ p|\overline{k}_{d+1}}}^{\infty} r_\alpha^\lambda(\gamma_{d+1}, k_{d+1}) \left(\sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^d \\ \overline{\boldsymbol{k}}\cdot\boldsymbol{q}_d^* \equiv a \pmod{p} \\ \deg(a)<n-m}} r_\alpha(\boldsymbol{\gamma}, \boldsymbol{k}) \right)^\lambda \\
&\quad + \frac{1}{b^n} \sum_{\substack{k_{d+1}=1 \\ p\nmid\overline{k}_{d+1}}}^{\infty} r_\alpha^\lambda(\gamma_{d+1}, k_{d+1}) \sum_{q_{d+1}\in G_{b,n}} \sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^d \\ \overline{\boldsymbol{k}}\cdot\boldsymbol{q}_d^* + \overline{k}_{d+1}q_{d+1} \equiv a \pmod{p} \\ \deg(a)<n-m}} r_\alpha^\lambda(\boldsymbol{\gamma}, \boldsymbol{k}),
\end{aligned}$$

where we used Jensen's inequality, which states that for a sequence (a_k) of non-negative reals we have $(\sum a_k)^\lambda \le \sum a_k^\lambda$, for any $0 < \lambda \le 1$; of course, $\overline{k}$ denotes the polynomial associated with the integer k and $\overline{\boldsymbol{k}}$ the corresponding vector. Now we have

$$\begin{aligned}
&\sum_{\substack{k_{d+1}=1 \\ p|\overline{k}_{d+1}}}^{\infty} r_\alpha^\lambda(\gamma_{d+1}, k_{d+1}) \left(\sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^d \\ \overline{\boldsymbol{k}}\cdot\boldsymbol{q}_d^* \equiv a \pmod{p} \\ \deg(a)<n-m}} r_\alpha(\boldsymbol{\gamma}, \boldsymbol{k}) \right)^\lambda \\
&\qquad \le \frac{\gamma_{d+1}^\lambda C_{b,\alpha,\lambda}}{b^{\lambda n}} \left(1 + \sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^d\setminus\{\boldsymbol{0}\} \\ \overline{\boldsymbol{k}}\cdot\boldsymbol{q}_d^* \equiv a \pmod{p} \\ \deg(a)<n-m}} r_\alpha(\boldsymbol{\gamma}, \boldsymbol{k}) \right)^\lambda
\end{aligned}$$

$$\leq \frac{\gamma_{d+1}^{\lambda} C_{b,\alpha,\lambda}}{b^{\lambda n}} \left(1 + e_{b^m,\alpha}(\boldsymbol{q}_d^*, p)^{\lambda}\right),$$

where we used the following:

$$\sum_{\substack{k=1\\ p|\bar{k}}}^{\infty} r_{\alpha}^{\lambda}(\gamma, k) = \sum_{l=1}^{\infty} r_{\alpha}^{\lambda}(\gamma, b^n l) + \sum_{l=0}^{\infty} \sum_{\substack{k=1\\ p|\bar{k}}}^{b^n-1} r_{\alpha}^{\lambda}(\gamma, k + b^n l).$$

We note that for $l > 0$ we have $r_{\alpha}(\gamma, b^n l) \leq b^{-n} r_{\alpha}(\gamma, l)$. Further, for $1 \leq k < b^n$ the polynomial p never divides $\bar{k}$ since $\deg(p) = n$. Hence

$$\sum_{\substack{k=1\\ p|\bar{k}}}^{\infty} r_{\alpha}^{\lambda}(\gamma, k) = \sum_{l=1}^{\infty} r_{\alpha}^{\lambda}(\gamma, b^n l) \leq b^{-\lambda n} \sum_{l=1}^{\infty} r_{\alpha}^{\lambda}(\gamma, l) \leq \frac{\gamma^{\lambda} C_{b,\alpha,\lambda}}{b^{\lambda n}}.$$

Next, we consider the case where $\bar{k}_{d+1}$ is not a multiple of p. Here we have

$$\begin{aligned}
&\frac{1}{b^n} \sum_{q_{d+1} \in G_{b,n}} \sum_{\substack{k_{d+1}=1\\ p\nmid \bar{k}_{d+1}}}^{\infty} r_{\alpha}^{\lambda}(\gamma_{d+1}, k_{d+1}) \left(\sum_{(\boldsymbol{k}, k_{d+1}) \in \mathcal{D}_p(\boldsymbol{q}_d^*, q_{d+1})} r_{\alpha}(\boldsymbol{\gamma}, \boldsymbol{k}) \right)^{\lambda} \\
\leq\ &\frac{1}{b^n} \sum_{\substack{k_{d+1}=1\\ p\nmid \bar{k}_{d+1}}}^{\infty} r_{\alpha}^{\lambda}(\gamma_{d+1}, k_{d+1}) \sum_{q_{d+1} \in G_{b,n}} \sum_{\substack{\boldsymbol{k} \in \mathbb{N}_0^d \\ \bar{\boldsymbol{k}} \cdot \boldsymbol{q}_d^* + \bar{k}_{d+1} q_{d+1} \equiv a \pmod{p} \\ \deg(a) < n-m}} r_{\alpha}^{\lambda}(\boldsymbol{\gamma}, \boldsymbol{k}).
\end{aligned}$$

Now we obtain

$$\begin{aligned}
&\sum_{q_{d+1} \in G_{b,n}} \sum_{\substack{\boldsymbol{k} \in \mathbb{N}_0^d \\ \bar{\boldsymbol{k}} \cdot \boldsymbol{q}_d^* + \bar{k}_{d+1} q_{d+1} \equiv a \pmod{p} \\ \deg(a) < n-m}} r_{\alpha}^{\lambda}(\boldsymbol{\gamma}, \boldsymbol{k}) \\
= &\sum_{\boldsymbol{k} \in \mathbb{N}_0^d} r_{\alpha}^{\lambda}(\boldsymbol{\gamma}, \boldsymbol{k}) \sum_{\substack{a \in \mathbb{Z}_b[x] \\ \deg(a) < n-m}} \sum_{\substack{q_{d+1} \in G_{b,n} \\ \bar{\boldsymbol{k}} \cdot \boldsymbol{q}_d^* + \bar{k}_{d+1} q_{d+1} \equiv a \pmod{p}}} 1 \\
\leq &\sum_{\boldsymbol{k} \in \mathbb{N}_0^d} r_{\alpha}^{\lambda}(\boldsymbol{\gamma}, \boldsymbol{k}) b^{n-m} \\
\leq &\ b^{n-m} \prod_{j=1}^{d} (1 + C_{b,\alpha,\lambda} \gamma_j^{\lambda}).
\end{aligned}$$

Hence

$$\begin{aligned}
&\frac{1}{b^n} \sum_{\substack{k_{d+1}=1\\ p\nmid \bar{k}_{d+1}}}^{\infty} r_{\alpha}^{\lambda}(\gamma_{d+1}, k_{d+1}) \sum_{q_{d+1} \in G_{b,n}} \sum_{\substack{\boldsymbol{k} \in \mathbb{N}_0^d \\ \bar{\boldsymbol{k}} \cdot \boldsymbol{q}_d^* + \bar{k}_{d+1} q_{d+1} \equiv a \pmod{p} \\ \deg(a) < n-m}} r_{\alpha}^{\lambda}(\boldsymbol{\gamma}, \boldsymbol{k}) \\
\leq &\frac{1}{b^n} \sum_{k_{d+1}=1}^{\infty} r_{\alpha}^{\lambda}(\gamma_{d+1}, k_{d+1}) b^{n-m} \prod_{j=1}^{d} (1 + C_{b,\alpha,\lambda} \gamma_j^{\lambda}) \\
\leq &\frac{1}{b^m} C_{b,\alpha,\lambda} \gamma_{d+1}^{\lambda} \prod_{j=1}^{d} (1 + C_{b,\alpha,\lambda} \gamma_j^{\lambda}).
\end{aligned}$$

Consequently

$$\begin{aligned}\theta(\boldsymbol{q}_d^*, q_{d+1}^*) &\leq \Bigg(\frac{\gamma_{d+1}^\lambda C_{b,\alpha,\lambda}}{b^{\lambda n}}(1+e_{b^m,\alpha}(\boldsymbol{q}_d^*,p)^\lambda) \\ &\quad + \frac{1}{b^m} C_{b,\alpha,\lambda}\gamma_{d+1}^\lambda \prod_{j=1}^d (1+C_{b,\alpha,\lambda}\gamma_j^\lambda)\Bigg)^{1/\lambda} \\ &\leq \gamma_{d+1} C_{b,\alpha,\lambda}^{1/\lambda}\left[\frac{1}{b^{\lambda n}} + e_{b^m,\alpha}(\boldsymbol{q}_d^*,p)^\lambda + \frac{1}{b^m}\prod_{j=1}^d (1+C_{b,\alpha,\lambda}\gamma_j^\lambda)\right]^{1/\lambda}.\end{aligned}$$

We now set $\tau = 1/\lambda$ and use the induction hypothesis to obtain

$$\begin{aligned}\theta(\boldsymbol{q}_d^*, q_{d+1}^*) &\leq \gamma_{d+1} C_{b,\alpha,1/\tau}^\tau \left(\frac{1}{b^{n/\tau}} + e_{b^m,\alpha}(\boldsymbol{q}_d^*,p)^{1/\tau} + \frac{1}{b^m}\prod_{j=1}^d (1+C_{b,\alpha,1/\tau}\gamma_j^{1/\tau})\right)^\tau \\ &\leq \gamma_{d+1} C_{b,\alpha,1/\tau}^\tau \left(\frac{3}{b^{\min(m,n/\tau)}}\prod_{j=1}^d (1+3\gamma_j^{1/\tau} C_{b,\alpha,1/\tau})\right)^\tau \\ &= \frac{3^\tau}{b^{\min(\tau m,n)}}\gamma_{d+1} C_{b,\alpha,1/\tau}^\tau \prod_{j=1}^d (1+3\gamma_j^{1/\tau} C_{b,\alpha,1/\tau})^\tau.\end{aligned}$$

Finally, we have

$$\begin{aligned}e_{b^m,\alpha}(\boldsymbol{q}_{d+1}^*, p) &= e_{b^m,\alpha}(\boldsymbol{q}_d^*,p) + \theta(\boldsymbol{q}_d^*, q_{d+1}^*) \\ &\leq \frac{1}{b^{\min(\tau m,n)}}\prod_{j=1}^d (1+3\gamma_j^{1/\tau} C_{b,\alpha,1/\tau})^\tau \\ &\quad + \frac{3^\tau}{b^{\min(\tau m,n)}}\gamma_{d+1} C_{b,\alpha,1/\tau}^\tau \prod_{j=1}^d (1+3\gamma_j^{1/\tau} C_{b,\alpha,1/\tau})^\tau \\ &= \frac{1}{b^{\min(\tau m,n)}}(1+3^\tau\gamma_{d+1} C_{b,\alpha,1/\tau}^\tau)\prod_{j=1}^d (1+3\gamma_j^{1/\tau} C_{b,\alpha,1/\tau})^\tau \\ &\leq \frac{1}{b^{\min(\tau m,n)}}\prod_{j=1}^{d+1} (1+3\gamma_j^{1/\tau} C_{b,\alpha,1/\tau})^\tau,\end{aligned}$$

where we again used Jensen's inequality. □

From Theorem 6.4 we obtain the following corollary.

Corollary 6.5. *Let b be prime, $p \in \mathbb{Z}_b[x]$ irreducible with $\deg(p) = n$ and $\alpha \geq 2$ an integer. Suppose $\boldsymbol{q}^* \in G_{b,n}^s$ is constructed using Algorithm 6.1.*

- *We have*

$$e_{b^m,\alpha}(\boldsymbol{q}^*, p) \leq \frac{c_{s,\alpha,\gamma,\delta}}{b^{\min((\alpha-\delta)m,n)}} \quad \forall 0 < \delta \leq \alpha - 1,$$

where

$$c_{s,\alpha,\gamma,\delta} := \prod_{j=1}^s \left(1 + 3\gamma_j^{\frac{1}{\alpha-\delta}} C_{b,\alpha,\frac{1}{\alpha-\delta}}\right)^{\alpha-\delta}.$$

- *Suppose* $\sum_{j=1}^{\infty} \gamma_j^{\frac{1}{\alpha-\delta}} < +\infty$, *then* $c_{s,\alpha,\gamma,\delta} \le c_{\infty,\alpha,\gamma,\delta} < +\infty$ *and we have*

$$e_{b^m,\alpha}(\boldsymbol{q}^*, p) \le \frac{c_{\infty,\alpha,\gamma,\delta}}{b^{\min((\alpha-\delta)m,n)}} \quad \forall 0 < \delta \le \alpha - 1.$$

Thus the worst-case error is bounded independently of the dimension.

- *Under the assumption* $A := \limsup_{s\to\infty} \sum_{j=1}^{s} \gamma_j/(\log s) < +\infty$ *we obtain* $c_{s,\alpha,\gamma,(\alpha-1)} \le \widetilde{c}_\eta s^{2C_{b,\alpha,1}(A+\eta)}$ *and therefore*

$$e_{b^m,\alpha}(\boldsymbol{q}^*, p) \le \frac{\widetilde{c}_\eta s^{2C_{b,\alpha,1}(A+\eta)}}{b^m} \quad \forall \eta > 0,$$

where $\widetilde{c}_\eta$ *depends only on* η. *Thus the worst-case error satisfies a bound which depends only polynomially on the dimension.*

Proof. The first part follows from Theorem 6.4 by setting $\tau = \alpha - \delta$. The second and the third part follow from the first part in exactly the same way as in the proof of [29, Corollary 4.5]. □

The above result shows that the worst-case error of a higher order polynomial lattice rule satisfies a bound which simultaneously converges at the almost optimal rate and depends only polynomially (does not depend) on the dimension s (the technical term for such a behavior is polynomial (strong) tractability). Until now it is not known whether this is possible for integration lattices in the sense of [97].

6.4 Optimal convergence rates for a range of smoothness parameters

In this section we construct higher order polynomial lattice rules which are optimal for a range of smoothness parameters; we use α and τ_α to denote the smoothness, where $2 \le \alpha \le \beta$, $1 \le \tau_\alpha < \alpha$.

We set

$$A_{m,n,s,\alpha,p}(\lambda) := \frac{1}{b^{sn}} \sum_{\boldsymbol{q}_s \in G_{b,n}^s} e^{\lambda}_{b^m,\alpha}(\boldsymbol{q}_s, p).$$

Proposition 6.6. *For* $\alpha \ge 2$ *an integer and* $1/\alpha < \lambda \le 1$ *we have*

$$A_{m,n,s,\alpha,p}(\lambda) \le \frac{2}{b^{\min(m,\lambda n)}} \left(-1 + \prod_{j=1}^{s} (1 + \gamma_j^{\lambda} C_{b,\alpha,\lambda}) \right).$$

Proof. Using Proposition 6.1 and Jensen's inequality, we obtain

$$A_{m,n,s,\alpha,p}(\lambda) \le \frac{1}{b^{sn}} \sum_{\boldsymbol{q} \in G_{b,n}^s} \sum_{\boldsymbol{k} \in \mathcal{D}'_p(\boldsymbol{q})} r_\alpha^{\lambda}(\boldsymbol{\gamma}, \boldsymbol{k})$$

$$= \sum_{\boldsymbol{k}\in\mathbb{N}_0^s\setminus\{\boldsymbol{0}\}} r_\alpha^\lambda(\boldsymbol{\gamma},\boldsymbol{k}) \frac{1}{b^{sn}} \sum_{\substack{\boldsymbol{q}\in G_{b,n}^s \\ \overline{\boldsymbol{k}}\cdot\boldsymbol{q}\equiv a \pmod{p} \\ \deg(a)<n-m}} 1. \tag{6.3}$$

In case all components of $\overline{\boldsymbol{k}}$ are multiples of p, every $\boldsymbol{q}$ satisfies the equation $\overline{\boldsymbol{k}}\cdot\boldsymbol{q}\equiv 0 \pmod{p}$ and hence we have

$$\frac{1}{b^{sn}} \sum_{\substack{\boldsymbol{q}\in G_{b,n}^s \\ \overline{\boldsymbol{k}}\cdot\boldsymbol{q}\equiv a \pmod{p} \\ \deg(a)<n-m}} 1 = 1,$$

and the sum over all $\overline{\boldsymbol{k}}$ which satisfy this condition is therefore bounded by

$$\sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^s\setminus\{\boldsymbol{0}\} \\ \overline{\boldsymbol{k}}\equiv\boldsymbol{0} \pmod{p}}} r_\alpha^\lambda(\boldsymbol{\gamma},\boldsymbol{k}) = -1 + \prod_{j=1}^s \sum_{\substack{k=0 \\ p|\overline{k}}}^\infty r_\alpha^\lambda(\gamma_j,k).$$

Now we have

$$\sum_{\substack{k=0 \\ p|\overline{k}}}^\infty r_\alpha^\lambda(\gamma_j,k) = \sum_{l=0}^\infty r_\alpha^\lambda(\gamma_j,b^n l) + \sum_{l=0}^\infty \sum_{\substack{k=1 \\ p|\overline{k}}}^{b^n-1} r_\alpha^\lambda(\gamma_j,k+b^n l).$$

For $l>0$ we have $r_\alpha(\gamma_j,b^n l) \le b^{-n} r_\alpha(\gamma_j,l)$, and further, for $1\le k<b^n$ the polynomial p never divides $\overline{k}$ since $\deg(p)=n$. Hence

$$\sum_{\substack{k=0 \\ p|\overline{k}}}^\infty r_\alpha^\lambda(\gamma_j,k) = 1 + \sum_{l=1}^\infty r_\alpha^\lambda(\gamma_j,b^n l) \le 1 + \frac{1}{b^{\lambda n}} \sum_{l=1}^\infty r_\alpha^\lambda(\gamma_j,l)\,.$$

Therefore,

$$\begin{aligned} \sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^s\setminus\{\boldsymbol{0}\} \\ \overline{\boldsymbol{k}}\equiv\boldsymbol{0} \pmod{p}}} r_\alpha^\lambda(\boldsymbol{\gamma},\boldsymbol{k}) &\le -1 + \prod_{j=1}^s (1 + b^{-\lambda n}\gamma_j^\lambda C_{b,\alpha,\lambda}) \\ &= \sum_{\emptyset\neq\mathfrak{u}\subseteq[s]} b^{-|\mathfrak{u}|\lambda n} \gamma_\mathfrak{u}^\lambda C_{b,\alpha,\lambda}^{|\mathfrak{u}|} \\ &\le \frac{1}{b^{\lambda n}} \left(-1 + \prod_{j=1}^s (1+\gamma_j^\lambda C_{b,\alpha,\lambda})\right). \end{aligned}$$

In case there is at least one component of $\overline{\boldsymbol{k}}$ which is not a multiple of p, we have

$$\frac{1}{b^{sn}} \sum_{\substack{\boldsymbol{q}\in G_{b,n}^s \\ \overline{\boldsymbol{k}}\cdot\boldsymbol{q}\equiv a \pmod{p} \\ \deg(a)<n-m}} 1 = \frac{1}{b^m},$$

because for any choice a there are $b^{n(s-1)}$ solutions $\boldsymbol{q}$ to $\overline{\boldsymbol{k}}\cdot\boldsymbol{q}\equiv a \pmod{p}$ and there are b^{n-m} possible choices for a. Hence this part of Equation (6.3) is bounded by

$$\frac{1}{b^m} \sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^s\setminus\{\boldsymbol{0}\} \\ \overline{\boldsymbol{k}}\not\equiv\boldsymbol{0} \pmod{p}}} r_\alpha^\lambda(\boldsymbol{\gamma},\boldsymbol{k}) \le \frac{1}{b^m} \sum_{\boldsymbol{k}\in\mathbb{N}_0^s\setminus\{\boldsymbol{0}\}} r_\alpha^\lambda(\boldsymbol{\gamma},\boldsymbol{k})$$

$$\leq \frac{1}{b^m}\left(-1+\prod_{j=1}^{s}(1+\gamma_j^{\lambda}C_{b,\alpha,\lambda})\right).$$

Altogether we now obtain that

$$\begin{aligned} A_{m,n,s,\alpha,p}(\lambda) &\leq \left(\frac{1}{b^m}+\frac{1}{b^{\lambda n}}\right)\left(-1+\prod_{j=1}^{s}(1+\gamma_j^{\lambda}C_{b,\alpha,\lambda})\right) \\ &\leq \frac{2}{b^{\min(m,\lambda n)}}\left(-1+\prod_{j=1}^{s}(1+\gamma_j^{\lambda}C_{b,\alpha,\lambda})\right), \end{aligned}$$

as required. □

Let $\alpha \leq \beta$ and set $n = \beta m$. We use ν to denote the equiprobable measure on $G_{b,\beta m}^s$. For $c \geq 1$ and $1 \leq \tau < \alpha \leq \beta$ the following set is introduced:

$$\mathscr{C}_{b,\alpha}(c,\tau) := \left\{\boldsymbol{q} \in G_{b,\beta m}^s : e_{b^m,\alpha}(\boldsymbol{q},p) \leq E_{b,\alpha,\gamma,s,m}(c,\tau)\right\}, \tag{6.4}$$

where

$$E_{b,\alpha,\gamma,s,m}(c,\tau) := \frac{2^\tau c^\tau}{b^{\tau m}}\left(-1+\prod_{j=1}^{s}(1+\gamma_j^{1/\tau}C_{b,\alpha,1/\tau})\right)^{\tau}.$$

Furthermore, let

$$\begin{aligned} \mathscr{C}_{b,\alpha}(c) &:= \bigcap_{1\leq\tau<\alpha}\mathscr{C}_{b,\alpha}(c,\tau) \\ &= \left\{\boldsymbol{q} \in G_{b,\beta m}^s : e_{b^m,\alpha}(\boldsymbol{q},p) \leq E_{b,\alpha,\gamma,s,m}(c,\tau)\ ,\forall 1 \leq \tau < \alpha\right\}. \end{aligned} \tag{6.5}$$

(Note that the intersection $\bigcap_{1\leq\tau<\alpha}\mathscr{C}_{b,\alpha}(c,\tau)$ can be understood as an intersection of finitely many sets since $\mathscr{C}_{b,\alpha}(c,\tau)$ has only finitely many elements.)

Lemma 6.7. *Let $c \geq 1$ and $1 \leq \tau < \alpha \leq \beta$. Then we have*

$$\nu(\mathscr{C}_{b,\alpha}(c,\tau)) > 1 - c^{-1}.$$

Proof. We denote $\overline{\mathscr{C}}_{b,\alpha}(c,\tau) := G_{b,\beta m}^s \setminus \mathscr{C}_{b,\alpha}(c,\tau)$. Then for all $1 \leq \tau < \alpha$ we have

$$\begin{aligned} A_{m,\beta m,s,\alpha,p}(1/\tau) &= \frac{1}{b^{s\beta m}}\sum_{\boldsymbol{q}\in G_{b,\beta m}^s} e_{b^m,\alpha}^{1/\tau}(\boldsymbol{q},p) \\ &> \nu(\overline{\mathscr{C}}_{b,\alpha}(c,\tau))\frac{2c}{b^m}\left(-1+\prod_{j=1}^{s}(1+\gamma_j^{1/\tau}C_{b,\alpha,1/\tau})\right). \end{aligned}$$

Now using Proposition 6.6, we obtain $\nu(\overline{\mathscr{C}}_{b,\alpha}(c,\tau)) < c^{-1}$ and the result follows. □

Lemma 6.8. *Let $c \geq 1$. Then we have*

$$\nu(\mathscr{C}_{b,\alpha}(c)) > 1 - c^{-1}.$$

Proof. Let $1 \leq \tau_* < \alpha$ be such that

$$E_{b,\alpha,\gamma,s,m}(c,\tau_*) = \inf_{1\leq\tau<\alpha} E_{b,\alpha,\gamma,s,m}(c,\tau)$$

(note that by Lemma 6.3 we have $E_{b,\alpha,\gamma,s,m}(c,\tau) \to +\infty$ whenever $\tau \to \alpha^-$ and hence we can find τ_* with the demanded property). Then we have

$$\mathscr{C}_{b,\alpha}(c,\tau_*) \subseteq \bigcap_{1\leq\tau<\alpha} \mathscr{C}_{b,\alpha}(c,\tau) = \mathscr{C}_{b,\alpha}(c)$$

and consequently the result follows from Lemma 6.7. $\square$

If we choose $c = \beta$ in Lemma 6.8, then we obtain $\nu(\mathscr{C}_{b,\alpha}(\beta)) > 1 - \beta^{-1}$ and therefore we have

$$\nu\left(\bigcap_{\alpha=2}^{\beta} \mathscr{C}_{b,\alpha}(\beta)\right) = 1 - \nu\left(\bigcup_{\alpha=2}^{\beta} \overline{\mathscr{C}}_{b,\alpha}(\beta)\right) \geq 1 - \sum_{\alpha=2}^{\beta} \nu(\overline{\mathscr{C}}_{b,\alpha}(\beta)) > 0.$$

Hence we obtain the following theorem, which establishes the existence of a $\boldsymbol{q}^* \in G_{b,\beta m}^s$ which achieves the optimal convergence rate for a range of α's.

Theorem 6.9. *Let $\beta, m, s \in \mathbb{N}$, $\beta \geq 2$, and $p \in \mathbb{Z}_b[x]$ with $\deg(p) = \beta m$. Then there exists a $\boldsymbol{q}^* \in G_{b,\beta m}^s$ such that*

$$e_{b^m,\alpha}(\boldsymbol{q}^*, p) \leq \frac{2^{\tau_\alpha}\beta^{\tau_\alpha}}{b^{\tau_\alpha m}} \left(-1 + \prod_{j=1}^{s}(1 + \gamma_j^{1/\tau_\alpha} C_{b,\alpha,1/\tau_\alpha})\right)^{\tau_\alpha}, \tag{6.6}$$

for all $2 \leq \alpha \leq \beta$ and all $1 \leq \tau_\alpha < \alpha$.

The proof of Theorem 6.9 suggests that in principle we can find a $\boldsymbol{q}^* \in G_{b,\beta m}^s$, which satisfies Equation (6.6) for all $2 \leq \alpha \leq \beta$ and all $1 \leq \tau_\alpha < \alpha$, by using a so-called "sieve algorithm", which works as follows: use a computer search to find $\lfloor (1-\beta^{-1})b^{\beta ms} \rfloor + 1$ of the $b^{\beta ms}$ vectors $\boldsymbol{q}$ in $G_{b,\beta m}^s$ which satisfy

$$e_{b^m,2}(\boldsymbol{q}, p) \leq E_{b,2,\gamma,s,m}(\beta, \tau_2) \quad \forall 1 \leq \tau_2 < 2,$$

and label this set $\mathcal{T}_2$. By Lemma 6.8, we know that at least such a number of vectors exists.

Then proceed by using a computer search to find $\lfloor (1-2\beta^{-1})b^{\beta ms} \rfloor + 1$ vectors $\boldsymbol{q}$ in $\mathcal{T}_2$ which satisfy

$$e_{b^m,3}(\boldsymbol{q}, p) \leq E_{b,3,\gamma,s,m}(\beta, \tau_3) \quad \forall 1 \leq \tau_3 < 3,$$

and label this set $\mathcal{T}_3$. Since

$$\nu\left(\bigcap_{\alpha=2}^{3} \mathscr{C}_{b,\alpha}(\beta)\right) = 1 - \nu\left(\bigcup_{\alpha=2}^{3} \overline{\mathscr{C}}_{b,\alpha}(\beta)\right) \geq 1 - \sum_{\alpha=2}^{3} \nu(\overline{\mathscr{C}}_{b,\alpha}(\beta)) > 1 - \frac{2}{\beta},$$

we know that there are at least $\lfloor (1-2\beta^{-1})b^{\beta ms} \rfloor + 1$ values in $\mathcal{T}_2$ to populate the set $\mathcal{T}_3$.

In the same way, we proceed to construct the sets $\mathcal{T}_4, \dots, \mathcal{T}_\beta$. Theorem 6.9 guarantees that $\mathcal{T}_\beta$ is not empty and we may select $\boldsymbol{q}^*$ to be any vector from $\mathcal{T}_\beta$. This vector satisfies Equation (6.6) for all $2 \le \alpha \le \beta$ and all $1 \le \tau_\alpha < \alpha$.

However, in practice such a search algorithm would not be applicable since it is much too time consuming. For this reason we show in the sequel how the sieve algorithm may be combined with the component-by-component (CBC) algorithm which makes the search algorithm feasible. Such an algorithm, we call it "CBC sieve algorithm", is presented as Algorithm 6.2. We use the following notation, where $2 \le \alpha \le \beta$ and $p \in \mathbb{Z}_b[x]$ with $\deg(p) = \beta m$: for $d = 0$ and $q_1 \in G_{b,\beta m}$ we set

$$\theta_\alpha(0, q_1) := e_{b^m,\alpha}(q_1, p),$$

and for $d \in \mathbb{N}$, $\boldsymbol{q}_d \in G_{b,\beta m}^d$ and $q_{d+1} \in G_{b,\beta m}$ we set

$$\theta_\alpha(\boldsymbol{q}_d, q_{d+1}) := e_{b^m,\alpha}((\boldsymbol{q}_d, q_{d+1}), p) - e_{b^m,\alpha}(\boldsymbol{q}_d, p).$$

Furthermore, for short, we use the notation

$$M_{d,\alpha,\gamma}(\tau) := \frac{1}{b^m} \prod_{j=1}^{d} (1 + 3\beta\gamma_j^{1/\tau} C_{b,\alpha,1/\tau}).$$

Now we prove the following result.

Theorem 6.10. *Let $s, m, \beta \in \mathbb{N}$, $\beta \ge 2$. Then Algorithm 6.2 constructs a vector $\boldsymbol{q}_d^* \in G_{b,\beta m}^d$ such that*

$$e_{b^m,\alpha}(\boldsymbol{q}_d^*, p) \le \frac{1}{b^{\tau_\alpha m}} \prod_{j=1}^{d} (1 + 3\beta\gamma_j^{1/\tau_\alpha} C_{b,\alpha,1/\tau_\alpha})^{\tau_\alpha},$$

for all $1 \le \tau_\alpha < \alpha$ and all $2 \le \alpha \le \beta$.

To prove Theorem 6.10, we introduce the following set: for $\boldsymbol{q}_d \in G_{b,\beta m}^d$ let $\mathscr{F}_{b,\alpha}(c, \boldsymbol{q}_d)$ be the set of all $q_{d+1} \in G_{b,\beta m}$ such that

$$\theta_\alpha(\boldsymbol{q}_d, q_{d+1}) \le \left(3c\gamma_{d+1}^{1/\tau_\alpha} C_{b,\alpha,1/\tau_\alpha} M_{d,\alpha,\gamma}(\tau_\alpha)\right)^{\tau_\alpha}, \tag{6.7}$$

for all $1 \le \tau_\alpha < \alpha$, where $c \ge 1$.

Lemma 6.11. *Let $2 \le \alpha \le \beta$ and $c \ge 1$. Assume that there exists a $\boldsymbol{q}_d \in G_{b,\beta m}^d$ such that*

$$e_{b^m,\alpha}(\boldsymbol{q}_d, p) \le M_{d,\alpha,\gamma}(\tau_\alpha)^{\tau_\alpha}, \tag{6.8}$$

for all $1 \le \tau_\alpha < \alpha$ and all $2 \le \alpha \le \beta$. Then

$$\nu\left(\mathscr{F}_{b,\alpha}(c, \boldsymbol{q}_d)\right) > 1 - c^{-1}.$$

Algorithm 6.2 CBC sieve algorithm for $2 \le \alpha \le \beta$

Require: b a prime, $s, m, \beta \in \mathbb{N}$, $\beta \ge 2$, and $p \in \mathbb{Z}_b[x]$ with $\deg(p) = \beta m$.

Set $\mathcal{T}_{1,d} := G_{b,\beta m}$ for all $1 \le d \le s$ and $\boldsymbol{q}_0^* = 0$.
for $d = 0$ to $s-1$ **do**
  for $\alpha = 2$ to β **do**
    if $d = 0$ **then**
      Form the set $\mathcal{T}_{\alpha,d+1}$ by performing a computer search to find $\lfloor (1-(\alpha-1)\beta^{-1}) b^{\beta m} \rfloor + 1$ elements q satisfying

$$\left\{ q \in \mathcal{T}_{\alpha-1,d+1} : \theta_\alpha(0,q) \le \frac{1}{b^{\tau_\alpha m}} \left(1 + 3\gamma_1^{1/\tau_\alpha} C_{b,\alpha,1/\tau_\alpha}\right)^{\tau_\alpha}, \forall 1 \le \tau_\alpha < \alpha \right\},$$

    else
      Form the set $\mathcal{T}_{\alpha,d+1}$ by performing a computer search to find $\lfloor (1-(\alpha-1)\beta^{-1}) b^{\beta m} \rfloor + 1$ elements q satisfying

$$\left\{ q \in \mathcal{T}_{\alpha-1,d+1} : \theta_\alpha(\boldsymbol{q}_d^*, q) \le \left(3\beta\gamma_{d+1}^{1/\tau_\alpha} C_{b,\alpha,1/\tau_\alpha} M_{d,\alpha,\gamma}(\tau_\alpha)\right)^{\tau_\alpha}, \forall 1 \le \tau_\alpha < \alpha \right\}.$$

    end if
  end for
  Select an arbitrary $q_{d+1}^* \in \mathcal{T}_{\beta,d+1}$.
  Set $\boldsymbol{q}_{d+1}^* = (\boldsymbol{q}_d^*, q_{d+1}^*)$.
end for
return $\boldsymbol{q}^* = \boldsymbol{q}_s^*$.

Proof. From the proof of Theorem 6.4 and using Assumption (6.8), for all $1/\alpha < \lambda \le 1$ we have

$$\begin{aligned} & \frac{1}{b^{\beta m}} \sum_{q_{d+1} \in G_{b,\beta m}} \theta_\alpha(\boldsymbol{q}_d^*, q_{d+1})^\lambda \\ \le\ & \gamma_{d+1}^\lambda C_{b,\alpha,\lambda} \left(\frac{1}{b^{\lambda\alpha m}} + e_{b^m,\alpha}(\boldsymbol{q}_d^*, p)^\lambda + \frac{1}{b^m} \prod_{j=1}^{d} (1 + \gamma_j^\lambda C_{b,\alpha,\lambda}) \right) \\ \le\ & 3\gamma_{d+1}^\lambda C_{b,\alpha,\lambda} M_{d,\alpha,\gamma}(1/\lambda). \end{aligned}$$

From this the result follows in the same way as in the proofs of Lemma 6.7 and Lemma 6.8. □

Now we give the proof of Theorem 6.10.

Proof. We proceed by induction on d and firstly show the result for $d = 1$. Having fixed $d = 1$, we proceed by induction on α. For $q \in \mathscr{C}_{b,\alpha}(\beta)$ (see Equation (6.5)), we have

$$e_{b^m,\alpha}(q,p) \leq \frac{1}{b^{\tau_\alpha m}}(1 + 3\beta\gamma_1^{1/\tau_\alpha} C_{b,\alpha,1/\tau_\alpha})^{\tau_\alpha} \quad \forall 1 \leq \tau_\alpha < \alpha\,,$$

for $2 \leq \alpha \leq \beta$. According to Lemma 6.8, $\nu(\mathscr{C}_{b,\alpha}(\beta)) > 1 - \beta^{-1}$, $2 \leq \alpha \leq \beta$, hence there are $\lfloor (1-\beta^{-1}) b^{\beta m} \rfloor + 1$ elements to populate $\mathcal{T}_{2,1}$. Assume now that for $2 \leq \alpha < \beta$ there are $\lfloor (1-(\alpha-1)\beta^{-1}) b^{\beta m} \rfloor + 1$ elements to populate $\mathcal{T}_{\alpha,1}$, hence $\nu(\mathcal{T}_{\alpha,1}) > 1 - (\alpha-1)\beta^{-1}$. We want to show that

$$\begin{aligned}&\nu\left(\left\{q \in \mathcal{T}_{\alpha,1} : e_{b^m,\alpha+1}(q,p) \leq M_{1,\alpha+1,\gamma}(\tau_{\alpha+1})^{\tau_{\alpha+1}}, \forall 1 \leq \tau_{\alpha+1} < \alpha+1\right\}\right)\\ &\quad > 1 - \alpha\beta^{-1},\end{aligned} \tag{6.9}$$

which implies that there are $\lfloor (1-\alpha\beta^{-1}) b^{\beta m} \rfloor + 1$ elements to populate $\mathcal{T}_{\alpha+1,1}$. But

$$\begin{aligned}&\left\{q \in \mathcal{T}_{\alpha,1} : e_{b^m,\alpha+1}(q,p) \leq M_{1,\alpha+1,\gamma}(\tau_{\alpha+1})^{\tau_{\alpha+1}}, \forall 1 \leq \tau_{\alpha+1} < \alpha+1\right\}\\ &= \mathcal{T}_{\alpha,1} \cap \left\{q \in G_{b,\beta m} : e_{b^m,\alpha+1}(q,p) \leq M_{1,\alpha+1,\gamma}(\tau_{\alpha+1})^{\tau_{\alpha+1}}, \forall 1 \leq \tau_{\alpha+1} < \alpha+1\right\},\end{aligned}$$

hence we get Equation (6.9) from the induction assumption and from Lemma 6.8. Thus we have proven the assertion for $d = 1$.

We now assume that for $1 \leq d < s$ the algorithm has found $\boldsymbol{q}_d^*$ so that

$$e_{b^m,\alpha}(\boldsymbol{q}_d^*, p) \leq M_{d,\alpha,\gamma}(\tau_\alpha)^{\tau_\alpha}, \tag{6.10}$$

for all $1 \leq \tau_\alpha < \alpha$ and all $2 \leq \alpha \leq \beta$, and again we proceed by induction. According to Lemma 6.11, under Assumption (6.10) we have

$$\nu\left(\mathscr{F}_{b,\alpha}(\beta, \boldsymbol{q}_d^*)\right) > 1 - \beta^{-1}, \forall 2 \leq \alpha \leq \beta,$$

hence there are $\lfloor (1-\beta^{-1}) b^{\beta m} \rfloor + 1$ elements to populate $\mathcal{T}_{2,d+1}$. We now assume that for $2 \leq \alpha < \beta$ there are $\lfloor (1-(\alpha-1)\beta^{-1}) b^{\beta m} \rfloor + 1$ elements to populate $\mathcal{T}_{\alpha,d+1}$, hence $\nu(\mathcal{T}_{\alpha,d+1}) > (1-(\alpha-1)\beta^{-1})$.

Since

$$\begin{aligned}&\Bigg\{q \in \mathcal{T}_{\alpha,d+1} : \theta(\boldsymbol{q}_d^*, q) \leq \left(3\beta\gamma_{d+1}^{1/\tau_{\alpha+1}} C_{b,\alpha+1,1/\tau_{\alpha+1}} M_{d+1,\alpha+1,\gamma}(\tau_{\alpha+1})\right)^{\tau_{\alpha+1}},\\ &\qquad \forall 1 \leq \tau_{\alpha+1} < \alpha+1\Bigg\}\\ &= \mathcal{T}_{\alpha,d+1} \cap \mathscr{F}_{b,\alpha+1}(\beta, \boldsymbol{q}_d^*),\end{aligned}$$

we obtain from the inductive hypothesis and from Lemma 6.11 that

$$\nu\Bigg(\Bigg\{q \in \mathcal{T}_{\alpha,d+1} : \theta(\boldsymbol{q}_d^*, q) \leq \Big(3\beta\gamma_{d+1}^{1/\tau_{\alpha+1}} C_{b,\alpha+1,1/\tau_{\alpha+1}} M_{d+1,\alpha+1,\gamma}(\tau_{\alpha+1})\Big)^{\tau_{\alpha+1}}, \\ \forall 1 \leq \tau_{\alpha+1} < \alpha+1\Bigg\}\Bigg) > 1 - \alpha\beta^{-1},$$

which implies that there are $\lfloor (1-\alpha\beta^{-1})b^{\beta m} \rfloor + 1$ elements to populate $\mathcal{T}_{\alpha+1,d+1}$. Therefore Algorithm 6.2 finds a $q_{d+1}^* \in G_{b,\beta m}$ such that

$$\theta_\alpha(\boldsymbol{q}_d^*, q_{d+1}^*) \leq \Big(3\beta\gamma_{d+1}^{1/\tau_\alpha} C_{b,\alpha,1/\tau_\alpha} M_{d,\alpha,\gamma}(\tau_\alpha)\Big)^{\tau_\alpha},$$

for all $1 \leq \tau_\alpha < \alpha$ and all $2 \leq \alpha \leq \beta$.

Using Equation (6.10), we obtain

$$\begin{aligned} e_{b^m,\alpha}((\boldsymbol{q}_d^*, q_{d+1}^*), p) &= e_{b^m,\alpha}(\boldsymbol{q}_d^*, p) + \theta_\alpha((\boldsymbol{q}_d^*, q_{d+1}^*)) \\ &\leq M_{d,\alpha,\gamma}(\tau_\alpha)^{\tau_\alpha}(1 + (3\beta\gamma_{d+1}^{1/\tau_\alpha} C_{b,\alpha,1/\tau_\alpha})^{\tau_\alpha}) \\ &\leq M_{d+1,\alpha,\gamma}(\tau_\alpha)^{\tau_\alpha}, \end{aligned}$$

for all $1 \leq \tau_\alpha < \alpha$ and all $2 \leq \alpha \leq \beta$. □

6.5 Korobov polynomial lattice point sets

In this section we study a special case of higher order polynomial lattice point sets, namely Korobov polynomial lattice point sets. We establish the existence of Korobov polynomial lattice rules which achieve optimal rates of convergence for a range of smoothness parameters and present an algorithm which shows how to construct such Korobov polynomial lattice rules. This algorithm is the same as the "sieve algorithm" discussed in Section 6.4, but due to the structure of Korobov polynomial lattice point sets, the computational cost of such an algorithm is feasible.

We now present the results which are used to establish the existence of a Korobov polynomial lattice rule achieving optimal rates of convergence for a range of smoothness parameters and its construction. For the remainder of this section, we use $\phi(q) := (q, q^2, \ldots, q^s) \pmod{p}$, $q \in G_{b,\beta m}$, to denote the generating vector of the Korobov polynomial lattice point set $S_{p,m,\beta m}(\phi(q))$ and $e_{b^m,\alpha}(\phi(q), p)$ to denote the corresponding worst-case error, $2 \leq \alpha \leq \beta$; we recall that $\alpha \leq \beta$ and $n = \beta m$. As in Section 6.3, we point out that we use generating vectors $\phi(q) := (q, q^2, \ldots, q^s) \pmod{p}$ instead of $(1, q, \ldots, q^{s-1})$, see e.g. [29, Algorithm 4.6], as otherwise the projection onto the first coordinate does not achieve a convergence rate of $b^{-\alpha m}$. We start with the following proposition, which is analogous to Proposition 6.6, where we set

$$\widetilde{A}_{m,n,s,\alpha,p}(\lambda) := \frac{1}{b^n} \sum_{q \in G_{b,n}} e^{\lambda}_{b^m,\alpha}(\phi(q), p) \,.$$

Proposition 6.12. *For $\alpha \geq 2$ and $1/\alpha < \lambda \leq 1$ we have*

$$\widetilde{A}_{m,n,s,\alpha,p}(\lambda) \leq \frac{s+1}{b^{\min(m,\lambda n)}} \left[-1 + \prod_{j=1}^{s} (1 + \gamma_j^{\lambda} C_{b,\alpha,\lambda}) \right].$$

Proof. Using Proposition 6.1 and Jensen's inequality, we obtain

$$\begin{aligned} \widetilde{A}_{m,n,s,\alpha,p}(\lambda) &\leq \frac{1}{b^n} \sum_{q \in G_{b,\beta m}} \sum_{\boldsymbol{k} \in \mathcal{D}'_p(\phi(q))} r_{\alpha}^{\lambda}(\boldsymbol{\gamma}, \boldsymbol{k}) \\ &= \sum_{\boldsymbol{k} \in \mathbb{N}_0^s \setminus \{\boldsymbol{0}\}} r_{\alpha}^{\lambda}(\boldsymbol{\gamma}, \boldsymbol{k}) \frac{1}{b^n} \sum_{\substack{q \in G_{b,\beta m} \\ \overline{\boldsymbol{k}} \cdot \phi(q) \equiv a \pmod{p} \\ \deg(a) < n-m}} 1. \end{aligned} \tag{6.11}$$

In case all components of $\overline{\boldsymbol{k}}$ are multiples of p, every $q \in G_{b,\beta m}$ satisfies the equation $\overline{\boldsymbol{k}} \cdot \phi(q) \equiv 0 \pmod{p}$ and hence we have

$$\frac{1}{b^n} \sum_{\substack{q \in G_{b,\beta m} \\ \overline{\boldsymbol{k}} \cdot \phi(q) \equiv a \pmod{p} \\ \deg(a) < n-m}} 1 = 1,$$

and the sum over all $\overline{\boldsymbol{k}}$ which satisfy this condition is therefore bounded by

$$\sum_{\substack{\boldsymbol{k} \in \mathbb{N}_0^s \setminus \{\boldsymbol{0}\} \\ \overline{\boldsymbol{k}} \equiv \boldsymbol{0} \pmod{p}}} r_{\alpha}^{\lambda}(\boldsymbol{\gamma}, \boldsymbol{k}) \leq \frac{1}{b^{\lambda n}} \left(-1 + \prod_{j=1}^{s} (1 + \gamma_j^{\lambda} C_{b,\alpha,\lambda}) \right),$$

see the proof of Proposition 6.6.

In case there is at least one component of $\overline{\boldsymbol{k}}$ which is not a multiple of p, we have

$$\frac{1}{b^n} \sum_{\substack{q \in G_{b,\beta m} \\ \overline{\boldsymbol{k}} \cdot \phi(q) \equiv a \pmod{p} \\ \deg(a) < n-m}} 1 \leq s b^{-m},$$

because for any choice a there are at most s solutions q to $\overline{\boldsymbol{k}} \cdot \phi(q) \equiv a \pmod{p}$ and there are b^{n-m} possible choices for a. Therefore this part of Equation (6.11) is bounded by

$$\frac{s}{b^m} \left(-1 + \prod_{j=1}^{s} (1 + \gamma_j^{\lambda} C_{b,\alpha,\lambda}) \right).$$

Altogether we now obtain that

$$\widetilde{A}_{m,n,s,\alpha,p}(\lambda) \leq \left(\frac{s}{b^m} + \frac{1}{b^{\lambda n}} \right) \left(-1 + \prod_{j=1}^{s} (1 + \gamma_j^{\lambda} C_{b,\alpha,\lambda}) \right)$$

$$\leq \frac{s+1}{b^{\min(m,\lambda n)}} \left(-1 + \prod_{j=1}^{s}(1+\gamma_j^{\lambda} C_{b,\alpha,\lambda})\right),$$

as required. □

Let ν denote the equiprobable measure on $G_{b,\beta m}$. For $c \geq 1$ and $1 \leq \tau < \alpha \leq \beta$ the following set is introduced:

$$\widetilde{\mathscr{C}}_{b,\alpha}(c,\tau) := \left\{ q \in G_{b,\beta m} : e_{b^m,\alpha}(\phi(q), p) \leq \widetilde{E}_{b,\alpha,\gamma,s,m}(c,\tau) \right\}, \tag{6.12}$$

where

$$\widetilde{E}_{b,\alpha,\gamma,s,m}(c,\tau) := \frac{c^{\tau}(s+1)^{\tau}}{b^{\tau m}} \left(-1 + \prod_{j=1}^{s}(1+\gamma_j^{1/\tau} C_{b,\alpha,1/\tau})\right)^{\tau}.$$

Furthermore, let

$$\begin{aligned} \widetilde{\mathscr{C}}_{b,\alpha}(c) &:= \bigcap_{1\leq\tau<\alpha} \widetilde{\mathscr{C}}_{b,\alpha}(c,\tau) \\ &= \left\{ q \in G_{b,\beta m} : e_{b^m,\alpha}(\phi(q), p) \leq \widetilde{E}_{b,\alpha,\gamma,s,m}(c,\tau) \ \forall 1 \leq \tau < \alpha \right\}. \end{aligned} \tag{6.13}$$

Lemma 6.13. *Let $c \geq 1$ and $1 \leq \tau < \alpha \leq \beta$. Then we have*

$$\nu(\widetilde{\mathscr{C}}_{b,\alpha}(c,\tau)) > 1 - c^{-1}.$$

Proof. The proof follows exactly along the lines of the proof of Lemma 6.7. □

Lemma 6.14. *Let $c \geq 1$. Then we have*

$$\nu(\widetilde{\mathscr{C}}_{b,\alpha}(c)) > 1 - c^{-1}.$$

Proof. The proof follows exactly along the lines of the proof of Lemma 6.8. □

As in Section 6.4, we now introduce a "sieve algorithm", see Algorithm 6.3, which shows how to obtain a generating vector for a Korobov polynomial lattice rule which achieves optimal convergence rates for a range of smoothness parameters. The next theorem shows that Algorithm 6.3 does indeed produce such a vector.

Theorem 6.15. *Let $s, m, \beta \in \mathbb{N}$, $\beta \geq 2$. Then Algorithm 6.3 finds an element $q \in G_{b,\beta m}$ such that*

$$e_{b^m,\alpha}(\phi(q), p) \leq \frac{(s+1)^{\tau_\alpha} \beta^{\tau_\alpha}}{b^{\tau_\alpha m}} \left(-1 + \prod_{j=1}^{s}(1+\gamma_j^{1/\tau_\alpha} C_{b,\alpha,1/\tau_\alpha})\right)^{\tau_\alpha},$$

for all $1 \leq \tau_\alpha < \alpha$ and all $2 \leq \alpha \leq \beta$.

Algorithm 6.3 Korobov sieve algorithm for $2 \leq \alpha \leq \beta$

Require: b a prime, $s, m, \beta \in \mathbb{N}$, $\beta \geq 2$, and $p \in \mathbb{Z}_b[x]$ with $\deg(p) = \beta m$.

1: Set $\mathcal{T}_1 := G_{b,\beta m}$.

2: **for** $\alpha = 2$ to β **do**

3: perform a computer search to find $\lfloor (1 - (\alpha - 1)\beta^{-1}) b^{\beta m} \rfloor + 1$ elements q in $\mathcal{T}_{\alpha-1}$ to populate the set $\mathcal{T}_\alpha$, which is a subset of

$$\left\{ q \in \mathcal{T}_{\alpha-1} : e_{b^m,\alpha}(\phi(q), p) \leq \widetilde{E}_{b,\alpha,\gamma,s,m}(\beta, \tau_\alpha), \forall 1 \leq \tau_\alpha < \alpha \right\}.$$

4: **end for**

5: Select an arbitrary $q^* \in \mathcal{T}_\beta$.

6: **return** q^*.

Proof. We prove the result by induction on α. For $\alpha = 2$, by Lemma 6.14

$$\nu \left(\widetilde{\mathscr{C}}_{b,\alpha}(\beta) \right) > 1 - \beta^{-1}, \forall 2 \leq \alpha \leq \beta,$$

so there are at least $\lfloor (1 - \beta^{-1}) b^{\beta m} \rfloor + 1$ elements to populate the set $\mathcal{T}_2$. We now assume that there are $\lfloor (1 - (\alpha - 1)\beta^{-1}) b^{\beta m} \rfloor + 1$ elements in the set $\mathcal{T}_\alpha$, where $2 \leq \alpha < \beta$, hence $\nu(\mathcal{T}_\alpha) > 1 - (\alpha - 1)\beta^{-1}$. To complete the proof, we show that

$$\begin{aligned} \nu \left(\left\{ q \in \mathcal{T}_\alpha : e_{b^m,\alpha+1}(\phi(q), p) \leq \widetilde{E}_{b,\alpha+1,\gamma,s,m}(\beta, \tau_{\alpha+1}), \forall 1 \leq \tau_{\alpha+1} < \alpha + 1 \right\} \right) \\ > 1 - \alpha\beta^{-1}, \end{aligned} \tag{6.14}$$

which implies that there are $\lfloor (1 - \alpha\beta^{-1}) b^{\beta m} \rfloor + 1$ elements to populate the set $\mathcal{T}_{\alpha+1}$. Since

$$\left\{ q \in \mathcal{T}_\alpha : e_{b^m,\alpha+1}(\phi(q), p) \leq \widetilde{E}_{b,\alpha+1,\gamma,s,m}(\beta, \tau_{\alpha+1}) \, \forall 1 \leq \tau_{\alpha+1} < \alpha + 1 \right\} = \mathcal{T}_\alpha \cap \widetilde{\mathscr{C}}_{b,\alpha+1}(\beta),$$

we obtain Equation (6.14) from the induction assumption and from Lemma 6.14. □

6.6 Conclusion and future work

In this chapter we showed how to construct higher order polynomial lattice rules achieving optimal convergence rates for a given smoothness parameter and for a certain (arbitrary high) range of smoothness parameters.

There are many possible directions in which this work could be continued, to name but one, it would be interesting to apply higher order polynomial lattice point sets to practical problems and to investigate how they compare with integration lattices, [66; 97], and sparse grids, [14; 103].

Chapter Seven

Classical polynomial lattice point sets with small gain coefficients

Before discussing a construction of classical polynomial lattice point sets with small gain coefficients, we firstly motivate the topic.

7.1 Motivation

In this chapter we apply Owen's scrambling, as introduced in Subsection 2.6.3, to classical polynomial lattice point sets, see Subsection 2.2.3. In particular, we are interested in the variance of the estimator

$$Q_{b^m}(f, \mathcal{P}_\pi) = \frac{1}{b^m} \sum_{h=0}^{b^m-1} f(\boldsymbol{y}_h), \tag{7.1}$$

where the point set $\mathcal{P}_\pi = \{\boldsymbol{y}_h\}_{h=0}^{b^m-1}$ is obtained by applying the scrambling algorithm to $\mathcal{P}$, which is a classical polynomial lattice point set. Notice that $Q_{b^m}(f, \mathcal{P}_\pi)$ is an unbiased estimator of $\int_{[0,1]^s} f(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}$, i.e. $\mathbb{E}(Q_{b^m}(f, \mathcal{P}_\pi)) = \int_{[0,1]^s} f(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}$, see Proposition 2.44.

The variance of the estimator given in Equation (7.1) admits the following representation, see [81],

$$\mathrm{Var}(Q_{b^m}(f, \mathcal{P}_\pi)) = \frac{1}{N} \sum_{\boldsymbol{l} \in \mathbb{N}_0^s \setminus \{\boldsymbol{0}\}} \Gamma_{\boldsymbol{l}} \sigma_{\boldsymbol{l}}^2(f), \tag{7.2}$$

where N denotes the number of quadrature points, i.e. $N = b^m$. Equation (7.2) holds for any estimator $Q_{b^m}(f, \mathcal{P}_\pi)$ obtained by applying the scrambling algorithm to a point set $\mathcal{P} = \{\boldsymbol{x}_h\}_{h=0}^{N-1}$ such that $\boldsymbol{x}_h \in [0,1)^s$. Here, the values $\Gamma_{\boldsymbol{l}}$ are the so-called gain coefficients which depend only on the quadrature points and the values $\sigma_{\boldsymbol{l}}(f)$ depend only on the integrand f. They are derived from the crossed and nested ANOVA decomposition of f, see [81], and can be expressed in terms of Haar coefficients of the function f, see [81], or also as a sum of certain Walsh coefficients of f, see [35, Section 13.2]. In this sense, Equation (7.2) shows that $\mathrm{Var}(Q_{b^m}(f, \mathcal{P}_\pi))$ can be expressed as a weighted sum of gain coefficients, where we interpret the $\sigma_{\boldsymbol{l}}^2(f)$ as weights.

In this chapter we consider a space of functions for which $\sigma_{\boldsymbol{l}}(f)$ has a certain rate of decay. More precisely, for $0 < \alpha \leq 1$ we introduce a norm of the form

$$\|f\|_\alpha = \sup_{\boldsymbol{l}\in\mathbb{N}_0^s} b^{\alpha|\boldsymbol{l}|_1}\sigma_{\boldsymbol{l}}(f), \tag{7.3}$$

where $|\boldsymbol{l}|_1 = l_1 + \cdots + l_s$ for $\boldsymbol{l} = (l_1, \ldots, l_s)$, see Equation (2.13). We recall that α is related to the smoothness of f in the following sense:

If $f \in L_2([0,1]^s)$ has bounded variation of order α, then $\|f\|_\alpha < \infty$.

See Corollary 2.41 for details.

From Equation (7.3) we obtain $\sigma_{\boldsymbol{l}}(f) \leq b^{-\alpha|\boldsymbol{l}|_1}\|f\|_\alpha$ and by substituting this formula into Equation (7.2) we get

$$\mathrm{Var}(Q_{b^m}(f, \mathcal{P}_\pi)) \leq \frac{1}{N} \sum_{\boldsymbol{l}\in\mathbb{N}_0^s\setminus\{\boldsymbol{0}\}} \Gamma_{\boldsymbol{l}} b^{-2\alpha|\boldsymbol{l}|_1}\|f\|_\alpha^2. \tag{7.4}$$

To construct classical polynomial lattice point sets of high quality, we use

$$\frac{1}{N} \sum_{\boldsymbol{l}\in\mathbb{N}_0^s\setminus\{\boldsymbol{0}\}} \Gamma_{\boldsymbol{l}} b^{-2\alpha|\boldsymbol{l}|_1} \tag{7.5}$$

as a quality criterion. Notice that the sum in Equation (7.5) only depends on the quadrature points and not on the function f. We show that the sum in Equation (7.5) has a simple closed form for any $0 < \alpha \leq 1$ which can easily be computed if the quadrature points form a classical digital net. The case $\alpha = 0$ needs to be excluded since in this case the sum in Equation (7.5) is infinite.

Our aim is to find classical polynomial lattice point sets for which the weighted sum of gain coefficients given in Equation (7.5) is minimized. It is known from [81; 82; 107] that a digital (t, m, s)-net with a small quality parameter t yields small gain coefficients. In fact, one has $\Gamma_{\boldsymbol{l}} = 0$ for all $\boldsymbol{l} \in \mathbb{N}_0^s \setminus \{\boldsymbol{0}\}$ such that $|\boldsymbol{l}|_1 \leq m - t$. Here, on the other hand, we aim to minimize the sum in Equation (7.5) since it can be used to bound the variance of the estimator, $\mathrm{Var}(Q_{b^m}(f, \mathcal{P}_\pi))$. In other words, we minimize the upper bound on the variance given in Equation (7.4) for all functions f with $\|f\|_\alpha < \infty$ over the class of classical polynomial lattice point sets.

Following [101], we introduce additional parameters $\boldsymbol{\gamma} = (\gamma_j)_{1\leq j\leq s}$ in the norm given in Equation (7.3), see Equation (2.13). In this case, the criterion given in Equation (7.5) depends on the additional parameters $\boldsymbol{\gamma} = (\gamma_j)_{1\leq j\leq s}$, in which case a small quality parameter t does not necessarily yield the smallest possible gain coefficients anymore. Component-by-component constructions, see [100] and also Section 6.3, have proven useful in such a situation since one can then optimize the quadrature points also with respect to the γ_j. This is also the approach taken in this chapter. More precisely, we show that by constructing classical polynomial lattice rules component-by-component one obtains a convergence of

$$\mathrm{Var}(Q_{b^m}(f, \mathcal{P}_\pi)) = O(N^{-(2\alpha+1)+\delta}) \quad \text{for any } \delta > 0.$$

Apart from δ, which can be arbitrarily close to 0, this rate is best possible as shown in Subsection 2.7.2 for a large class of randomized algorithms. Further, if $\sum_{j=1}^{\infty} \gamma_j < \infty$, then the bound given in Equation (7.4) does not depend on the dimension s. This result is stronger than what can be obtained for (t, m, s)-nets, since the increase in the t-value of the form $t \approx s$ prevents one from obtaining a bound independent of the dimension only assuming that $\sum_{j=1}^{\infty} \gamma_j < \infty$. This condition is also necessary, see [108] for a result on a related space. Hence the rules we construct in this chapter are simultaneously optimal in terms of the convergence rate as well as their dependence on the dimension.

We now give the structure of this chapter: in Section 7.2, we discuss the variance of estimators based on scrambled classical polynomial lattice point sets using results from Subsection 2.7.2. In Sections 7.3 and 7.4, we construct classical polynomial lattice point sets for which the variance of the associated estimator converges at a rate of $N^{-(1+2\alpha)+\epsilon}$, for all $\epsilon > 0$, and which are consequently optimal up to powers of a $\log N$ factor for the class of algorithms defined in Subsection 2.7.2. In Section 7.5, we study the implementation of the component-by-component algorithm, in particular, we show how to reduce the computational effort associated with it. This implementation is made use of in Section 7.6, where we compare the performance of the scrambled classical polynomial lattice point sets constructed in Section 7.3 to the performance of scrambled classical digital nets. Finally, Section 7.7 concludes the chapter and ideas for future work are given.

7.2 Estimators based on scrambled classical polynomial lattice point sets

As in Subsection 2.7.2, we discuss in this section the variance of the estimator

$$Q_{b^m}(f, \mathcal{P}_\pi) = \frac{1}{b^m} \sum_{h=0}^{b^m-1} f(\boldsymbol{y}_h), \tag{7.6}$$

where the point set $\mathcal{P}_\pi = \{\boldsymbol{y}_0, \ldots, \boldsymbol{y}_{b^m-1}\}$ is obtained by applying the scrambling algorithm to $\mathcal{P}$, which is a digital (t, m, s)-net over $\mathbb{Z}_b$. Following Subsection 2.2.3, we assume that b is a prime number. We recall Lemma 2.50 and can now produce a result on the variance of an estimator which is based on a scrambled classical polynomial lattice point set, which we denote by the following:

$$Q_{b^m}(f, S_\pi(\boldsymbol{q})) = \frac{1}{b^m} \sum_{h=0}^{b^m-1} f(\boldsymbol{z}_h), \tag{7.7}$$

where the point set $S_\pi(\boldsymbol{q}) = \{\boldsymbol{z}_0, \ldots, \boldsymbol{z}_{b^m-1}\}$ is obtained by applying the scrambling algorithm to the classical polynomial lattice point set $S_{p,m}(\boldsymbol{q})$.

Proposition 7.1. *Let $f \in L_2([0,1]^s)$, $Q_{b^m}(f, S_\pi(\boldsymbol{q}))$ be given by Equation (7.7), $S_{p,m}(\boldsymbol{q})$ be a classical polynomial lattice point set and $S_\pi(\boldsymbol{q})$ be obtained by applying the scrambling algorithm to $S_{p,m}(\boldsymbol{q})$. Then*

$$\operatorname{Var}(Q_{b^m}(f, S_\pi(\boldsymbol{q}))) = \sum_{\emptyset \neq \mathfrak{u} \subseteq [s]} \frac{b^{|\mathfrak{u}|}}{(b-1)^{|\mathfrak{u}|}} \sum_{\boldsymbol{l}_\mathfrak{u} \in \mathbb{N}^{|\mathfrak{u}|}} \frac{\sigma^2_{(\boldsymbol{l}_\mathfrak{u}, \boldsymbol{0})}(f)}{b^{|\boldsymbol{l}_\mathfrak{u}|_1}} |L_{(\boldsymbol{l}_\mathfrak{u}, \boldsymbol{0})} \cap \mathcal{D}_p(\boldsymbol{q})|,$$

where $\mathcal{D}_p(\boldsymbol{q})$ is given by Definition 2.18.

Proof. The result follows from Lemma 2.50 and the fact that if the $C_1, \ldots, C_s$ are the generating matrices of the point set $S_{p,m}(\boldsymbol{q})$, then for any $\boldsymbol{k} \in \mathbb{N}_0^s$ we have

$$C_1^\top tr_m(\vec{k}_1) + \cdots + C_s^\top tr_m(\vec{k}_s) = \vec{0} \Leftrightarrow tr_m(\boldsymbol{k}) \cdot \boldsymbol{q} \equiv 0 \pmod{p},$$

which was first established in [66, Lemma 4.40]. □

We now introduce the integration problem studied in this chapter, which is the same as the one studied in Subsection 2.7.2, in particular, we are interested in the worst-case variance of multivariate integration in $V_{\alpha,s,\gamma}$, as introduced in Subsection 2.5.2, using a scrambled quasi-Monte Carlo point set $\mathcal{P}$:

$$\operatorname{Var}(Q_{b^m}(\cdot, \mathcal{P}_\pi(\boldsymbol{q})), V_{\alpha,s,\gamma}) = \sup_{\substack{f \in V_{\alpha,s,\gamma} \\ \|f\|_\alpha \leq 1}} \operatorname{Var}(Q_{b^m}(f, \mathcal{P}_\pi(\boldsymbol{q}))),$$

where $Q_{b^m}(\cdot, \mathcal{P}_\pi(\boldsymbol{q}))$ denotes the quasi-Monte Carlo rule based on the point set obtained by applying the scrambling algorithm to $\mathcal{P}$. We point out that as in Subsection 2.7.2, α denotes the smoothness of functions in $V_{\alpha,s,\gamma}$ and $0 < \alpha \leq 1$. As before, the quasi-Monte Carlo rule based on the scrambled classical polynomial lattice point set $S_{p,m}(\boldsymbol{q})$ is denoted by $Q_{b^m}(\cdot, S_\pi(\boldsymbol{q}))$ and the associated worst-case variance by $\operatorname{Var}(Q_{b^m}(\cdot, S_\pi(\boldsymbol{q})), V_{\alpha,s,\gamma})$. For $k = \kappa_0 + \kappa_1 b + \cdots + \kappa_{a-1} b^{a-1} \in \mathbb{N}_0$ let

$$r_{\alpha,\gamma}(k) = \begin{cases} 1 & \text{if } k = 0, \\ \gamma \frac{b}{(b-1)b^{\alpha a}} & \text{if } k > 0, \end{cases}$$

and for $\boldsymbol{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ let $r_{\alpha,\gamma}(\boldsymbol{k}) = \prod_{j=1}^{s} r_{\alpha,\gamma_j}(k_j)$; we remind the reader that this function is different from $r_\alpha(\gamma, \boldsymbol{k})$ defined in Equation (2.10).

The next corollary gives a bound on the quantity $\mathrm{Var}(Q_{b^m}(\cdot, S_\pi(\boldsymbol{q})), V_{\alpha,s,\gamma})$.

Corollary 7.2. *Let $0 < \alpha \leq 1$, $\boldsymbol{q} \in \mathbb{Z}_b[x]^s$ be a generating vector for a classical polynomial lattice point set with modulus p, and $\mathrm{Var}(Q_{b^m}(\cdot, S_\pi(\boldsymbol{q})), V_{\alpha,s,\gamma})$ be defined as above. Then*

$$\mathrm{Var}(Q_{b^m}(\cdot, S_\pi(\boldsymbol{q})), V_{\alpha,s,\gamma}) \leq \sum_{\boldsymbol{k} \in \mathcal{D}'_p(\boldsymbol{q})} r_{2\alpha+1,\gamma}(\boldsymbol{k}),$$

where $\mathcal{D}'_p(\boldsymbol{q}) = \mathcal{D}_p(\boldsymbol{q}) \setminus \{\boldsymbol{0}\}$ and $\mathcal{D}_p(\boldsymbol{q})$ is the dual polynomial lattice given by Definition 2.18.

Proof. From Proposition 7.1 and Equation (2.13), we obtain

$$\mathrm{Var}(Q_{b^m}(f, S_\pi(\boldsymbol{q}))) \leq \sum_{\emptyset \neq \mathfrak{u} \subseteq [s]} \frac{b^{|\mathfrak{u}|}}{(b-1)^{|\mathfrak{u}|}} \gamma_{\mathfrak{u}} \sum_{\boldsymbol{l}_{\mathfrak{u}} \in \mathbb{N}^{|\mathfrak{u}|}} \frac{|L_{(\boldsymbol{l}_{\mathfrak{u}},\boldsymbol{0})} \cap \mathcal{D}_p(\boldsymbol{q})|}{b^{|\boldsymbol{l}_{\mathfrak{u}}|_1(2\alpha+1)}}.$$

However,

$$\sum_{\emptyset \neq \mathfrak{u} \subseteq [s]} \frac{b^{|\mathfrak{u}|}}{(b-1)^{|\mathfrak{u}|}} \gamma_{\mathfrak{u}} \sum_{\boldsymbol{l}_{\mathfrak{u}} \in \mathbb{N}^{|\mathfrak{u}|}} \frac{|L_{(\boldsymbol{l}_{\mathfrak{u}},\boldsymbol{0})} \cap \mathcal{D}_p(\boldsymbol{q})|}{b^{|\boldsymbol{l}_{\mathfrak{u}}|_1(2\alpha+1)}} = \sum_{\boldsymbol{k} \in \mathcal{D}'_p(\boldsymbol{q})} r_{2\alpha+1,\gamma}(\boldsymbol{k}), \tag{7.8}$$

which establishes the result. □

The bound presented in Corollary 7.2 is from now on denoted by

$$B(\boldsymbol{q}, \alpha, \gamma) := \sum_{\boldsymbol{k} \in \mathcal{D}'_p(\boldsymbol{q})} r_{2\alpha+1,\gamma}(\boldsymbol{k}). \tag{7.9}$$

This bound plays a crucial role in the CBC algorithm presented in Section 7.3 and also in the construction of Korobov polynomial lattice point sets, see Section 7.4. For this reason, we now present a formula showing how to compute the bound $B(\boldsymbol{q}, \alpha, \gamma)$.

Theorem 7.3. *Let $B(\boldsymbol{q}, \alpha, \gamma)$ be given by Equation* (7.9). *Then the following equality holds:*

$$B(\boldsymbol{q}, \alpha, \gamma) = \frac{1}{b^m} \sum_{h=0}^{b^m-1} \prod_{j=1}^{s} \left(1 + \frac{b}{b-1} \gamma_j \phi(x_{h,j}, \alpha)\right) - 1, \tag{7.10}$$

where $\phi(x, \alpha) = \sum_{l=1}^{\infty} \frac{1}{b^{l(2\alpha+1)}} \sum_{k=b^{l-1}}^{b^l-1} \mathrm{wal}_k(x)$.

Proof. We fix $\boldsymbol{l}_{\mathfrak{u}} \in \mathbb{N}^{|\mathfrak{u}|}$ and let $\mathfrak{u} = (i_1, \ldots, i_{|\mathfrak{u}|})$, then

$$\begin{aligned} |L_{(\boldsymbol{l}_{\mathfrak{u}},\boldsymbol{0})} \cap \mathcal{D}_p(\boldsymbol{q})| &= \sum_{k_{i_1}=0}^{\infty} \cdots \sum_{k_{i_{|\mathfrak{u}|}}=0}^{\infty} \prod_{j \in \mathfrak{u}} \mathbf{1}_{b^{l_j-1} \leq k_j < b^{l_j}} \mathbf{1}_{(\boldsymbol{k}_{\mathfrak{u}},\boldsymbol{0}) \in \mathcal{D}_p(\boldsymbol{q})} \\ &= \sum_{k_{i_1}=b^{l_{i_1}-1}}^{b^{l_{i_1}}-1} \cdots \sum_{k_{i_{|\mathfrak{u}|}}=b^{l_{i_{|\mathfrak{u}|}}-1}}^{b^{l_{i_{|\mathfrak{u}|}}}-1} \mathbf{1}_{(\boldsymbol{k}_{\mathfrak{u}},\boldsymbol{0}) \in \mathcal{D}_p(\boldsymbol{q})} \end{aligned}$$

$$= \sum_{k_{i_1}=b^{l_{i_1}-1}}^{b^{l_{i_1}}-1} \cdots \sum_{k_{i_{|\mathfrak{u}|}}=b^{l_{i_{|\mathfrak{u}|}}-1}}^{b^{l_{i_{|\mathfrak{u}|}}}-1} \frac{1}{b^m} \sum_{h=0}^{b^m-1} \mathrm{wal}_{(\boldsymbol{k}_\mathfrak{u},\boldsymbol{0})}(\boldsymbol{x}_h)$$

$$= \frac{1}{b^m} \sum_{h=0}^{b^m-1} \sum_{k_{i_1}=b^{l_{i_1}-1}}^{b^{l_{i_1}}-1} \cdots \sum_{k_{i_{|\mathfrak{u}|}}=b^{l_{i_{|\mathfrak{u}|}}-1}}^{b^{l_{i_{|\mathfrak{u}|}}}-1} \mathrm{wal}_{(\boldsymbol{k}_\mathfrak{u},\boldsymbol{0})}(\boldsymbol{x}_h)$$

$$= \frac{1}{b^m} \sum_{h=0}^{b^m-1} \prod_{j\in\mathfrak{u}} \sum_{k_j=b^{l_j-1}}^{b^{l_j}-1} \mathrm{wal}_{k_j}(x_{h,j}) \,.$$

Hence

$$\sum_{\boldsymbol{l}_\mathfrak{u}\in\mathbb{N}^{|\mathfrak{u}|}} \frac{|L_{(\boldsymbol{l}_\mathfrak{u},\boldsymbol{0})} \cap \mathcal{D}_p(\boldsymbol{q})|}{b^{|\boldsymbol{l}_\mathfrak{u}|_1(2\alpha+1)}}$$

$$= \sum_{\boldsymbol{l}_\mathfrak{u}\in\mathbb{N}^{|\mathfrak{u}|}} \left(\frac{1}{b^m} \sum_{h=0}^{b^m-1} \prod_{j\in\mathfrak{u}} \sum_{k_j=b^{l_j-1}}^{b^{l_j}-1} \mathrm{wal}_{k_j}(x_{h,j}) \right) \frac{1}{b^{|\boldsymbol{l}_\mathfrak{u}|_1(2\alpha+1)}}$$

$$= \frac{1}{b^m} \sum_{h=0}^{b^m-1} \sum_{\boldsymbol{l}_\mathfrak{u}\in\mathbb{N}^{|\mathfrak{u}|}} \prod_{j\in\mathfrak{u}} \left(\sum_{k_j=b^{l_j-1}}^{b^{l_j}-1} \frac{\mathrm{wal}_{k_j}(x_{h,j})}{b^{l_j(2\alpha+1)}} \right)$$

$$= \frac{1}{b^m} \sum_{h=0}^{b^m-1} \prod_{j\in\mathfrak{u}} \left(\sum_{l_j=1}^{\infty} \sum_{k_j=b^{l_j-1}}^{b^{l_j}-1} \frac{\mathrm{wal}_{k_j}(x_{h,j})}{b^{l_j(2\alpha+1)}} \right)$$

$$= \frac{1}{b^m} \sum_{h=0}^{b^m-1} \prod_{j\in\mathfrak{u}} \phi(x_{h,j},\alpha) \,.$$

Recalling Equation (7.8), we obtain

$$\begin{aligned}
B(\boldsymbol{q},\alpha,\boldsymbol{\gamma}) &= \sum_{\emptyset\neq\mathfrak{u}\subseteq[s]} \frac{b^{|\mathfrak{u}|}}{(b-1)^{|\mathfrak{u}|}} \gamma_\mathfrak{u} \sum_{\boldsymbol{l}_\mathfrak{u}\in\mathbb{N}^{|\mathfrak{u}|}} \frac{|L_{(\boldsymbol{l}_\mathfrak{u},\boldsymbol{0})} \cap \mathcal{D}_p(\boldsymbol{q})|}{b^{|\boldsymbol{l}_\mathfrak{u}|_1(2\alpha+1)}} \\
&= \sum_{\emptyset\neq\mathfrak{u}\subseteq[s]} \frac{b^{|\mathfrak{u}|}}{(b-1)^{|\mathfrak{u}|}} \gamma_\mathfrak{u} \frac{1}{b^m} \sum_{h=0}^{b^m-1} \prod_{j\in\mathfrak{u}} \phi(x_{h,j},\alpha) \\
&= \frac{1}{b^m} \sum_{h=0}^{b^m-1} \sum_{\emptyset\neq\mathfrak{u}\subseteq[s]} \frac{b^{|\mathfrak{u}|}}{(b-1)^{|\mathfrak{u}|}} \gamma_\mathfrak{u} \prod_{j\in\mathfrak{u}} \phi(x_{h,j},\alpha) \\
&= \frac{1}{b^m} \sum_{h=0}^{b^m-1} \left[\sum_{\emptyset\neq\mathfrak{u}\subseteq[s]} \prod_{j\in\mathfrak{u}} \frac{b}{b-1} \gamma_j \phi(x_{h,j},\alpha) \right] \\
&= \frac{1}{b^m} \sum_{h=0}^{b^m-1} \left[\prod_{j=1}^{s} \left(1 + \frac{b}{b-1} \gamma_j \phi(x_{h,j},\alpha) \right) - 1 \right] \\
&= \frac{1}{b^m} \sum_{h=0}^{b^m-1} \prod_{j=1}^{s} \left(1 + \frac{b}{b-1} \gamma_j \phi(x_{h,j},\alpha) \right) - 1 \,.
\end{aligned}$$

□

Next, we obtain a formula for $\phi(x,\alpha)$ as given in Theorem 7.3.

Lemma 7.4. *Let* $x \in [0,1)$, $0 < \alpha \leq 1$ *and*

$$\phi(x,\alpha) = \sum_{l=1}^{\infty} \frac{1}{b^{l(2\alpha+1)}} \sum_{k=b^{l-1}}^{b^l-1} \mathrm{wal}_k(x)\,,$$

then

$$\phi(x,\alpha) = \begin{cases} \frac{(b-1)}{b(b^{2\alpha}-1)} & \text{for } x = 0, \\ \frac{(b-1)}{b(b^{2\alpha}-1)} - \frac{b^{-2\alpha a_0}(b^{2\alpha+1}-1)}{b(b^{2\alpha}-1)} & \text{for } \xi_i = 0, i = 1,\ldots,a_0-1, \xi_{a_0} \neq 0, a_0 \geq 1, \end{cases}$$

where $x = \frac{\xi_1}{b} + \frac{\xi_2}{b^2} + \ldots$.

Proof. We firstly assume that $l \geq 2$, then

$$\begin{aligned} \sum_{k=b^{l-1}}^{b^l-1} \mathrm{wal}_k(x) &= \sum_{k=b^{l-1}}^{b^l-1} \exp(2\pi\mathtt{i}(\xi_1 k_0 + \xi_2 k_1 + \cdots + \xi_l k_{l-1})/b) \\ &= \sum_{k_{l-1}=1}^{b-1} \exp(2\pi\mathtt{i}\xi_l k_{l-1}/b) \prod_{j=0}^{l-2} \sum_{k_j=0}^{b-1} \exp(2\pi\mathtt{i}\xi_{j+1} k_j/b)\,. \end{aligned}$$

But

$$\sum_{k_j=0}^{b-1} \exp(2\pi\mathtt{i}\xi_{j+1} k_j/b) = \begin{cases} 0 & \text{if } \xi_{j+1} \neq 0, \\ b & \text{if } \xi_{j+1} = 0\,, \end{cases}$$

hence $\sum_{k=b^{l-1}}^{b^l-1} \mathrm{wal}_k(x) = 0$ if $\exists i \in \{1,\ldots,l-1\}$ so that $\xi_i \neq 0$. Now assume $\xi_i = 0$ for $i \in \{1,\ldots,l-1\}$. If $\xi_l = 0$, then

$$\sum_{k=b^{l-1}}^{b^l-1} \mathrm{wal}_k(x) = b^{l-1}(b-1),$$

and if $\xi_l \neq 0$, then

$$\sum_{k=b^{l-1}}^{b^l-1} \mathrm{wal}_k(x) = b^{l-1} \sum_{k_{l-1}=1}^{b-1} \exp(2\pi\mathtt{i}\xi_l k_{l-1}/b) = -b^{l-1}\,.$$

Hence

$$\sum_{k=b^{l-1}}^{b^l-1} \mathrm{wal}_k(x) = \begin{cases} 0 & \text{if } \exists i \in \{1,\ldots,l-1\} \text{ so that } \xi_i \neq 0, \\ (b-1)b^{l-1} & \text{if } \xi_i = 0 \text{ for } i \in \{1,\ldots,l\}\,, \\ -b^{l-1} & \text{if } \xi_i = 0 \text{ for } i \in \{1,\ldots,l-1\} \text{ and } \xi_l \neq 0. \end{cases}$$

Of course, for $l = 1$ we obtain

$$\sum_{k_0=1}^{b-1} \mathrm{wal}_{k_0}(x) = \begin{cases} b-1 & \text{if } \xi_1 = 0, \\ -1 & \text{if } \xi_1 \neq 0. \end{cases}$$

Consequently, for $x = 0$ we get

$$\sum_{l=1}^{\infty} \left(\sum_{k=b^{l-1}}^{b^l-1} \mathrm{wal}_k(x) \right) \frac{1}{b^{l(2\alpha+1)}} = \sum_{l=1}^{\infty} (b-1) \frac{b^{l-1}}{b^{l(2\alpha+1)}} = \left(\frac{b-1}{b} \right) \frac{1}{b^{2\alpha}-1}.$$

We now assume that a_0 is the first non-zero digit of x, i.e. $\xi_i = 0$, $i = 1, \ldots, a_0 - 1$, and $\xi_{a_0} \neq 0$. Then

$$\begin{aligned} &\sum_{l=1}^{\infty} \left(\sum_{k=b^{l-1}}^{b^l-1} \mathrm{wal}_k(x) \right) \frac{1}{b^{l(2\alpha+1)}} \\ &= \sum_{l=1}^{a_0-1} \left(\sum_{k=b^{l-1}}^{b^l-1} \mathrm{wal}_k(x) \right) \frac{1}{b^{l(2\alpha+1)}} \\ &\quad + \left(\sum_{k=b^{a_0-1}}^{b^{a_0}-1} \mathrm{wal}_k(x) \frac{1}{b^{a_0(2\alpha+1)}} \right) + \sum_{l=a_0+1}^{\infty} \left(\sum_{k=b^{l-1}}^{b^l-1} \mathrm{wal}_k(x) \right) \frac{1}{b^{l(2\alpha+1)}}, \end{aligned}$$

where for $a_0 = 1$ (i.e. the first digit is not equal to 0) we set

$$\sum_{l=1}^{0} \left(\sum_{k=b^{l-1}}^{b^l-1} \mathrm{wal}_k(x) \right) \frac{1}{b^{l(2\alpha+1)}} = 0.$$

Hence

$$\begin{aligned} &\sum_{l=1}^{\infty} \left(\sum_{k=b^{l-1}}^{b^l-1} \mathrm{wal}_k(x) \right) \frac{1}{b^{l(2\alpha+1)}} \\ &= \sum_{l=1}^{a_0-1} \left(\sum_{k=b^{l-1}}^{b^l-1} \mathrm{wal}_k(x) \right) \frac{1}{b^{l(2\alpha+1)}} + \left(\sum_{k=b^{a_0-1}}^{b^{a_0}-1} \mathrm{wal}_k(x) \right) \frac{1}{b^{a_0(2\alpha+1)}} \\ &= \sum_{l=1}^{a_0-1} (b-1) b^{l-1} \frac{1}{b^{l(2\alpha+1)}} - \frac{b^{a_0-1}}{b^{a_0(2\alpha+1)}} \\ &= \left(\frac{b-1}{b} \right) \left(\frac{b^{-2\alpha a_0} - b^{-2\alpha}}{b^{-2\alpha}-1} \right) - \frac{b^{-1}}{b^{2\alpha a_0}} \\ &= \frac{b-1}{b} \left(\frac{1 - b^{-2\alpha(a_0-1)}}{b^{2\alpha}-1} \right) - \frac{b^{-2\alpha a_0}}{b} \\ &= \frac{b-1}{b(b^{2\alpha}-1)} - \frac{b^{-2\alpha a_0}(b^{2\alpha+1}-1)}{b(b^{2\alpha}-1)}. \end{aligned}$$

Combining the two cases, we obtain the result. □

In the next remark, we show that if we construct a classical polynomial lattice rule which achieves optimal convergence rates for functions in $V_{\alpha,s,\gamma}$, for some given $0 < \alpha \leq 1$, then this classical polynomial lattice rule also achieves optimal convergence rates for functions in $V_{\alpha',s,\gamma'}$, where $\alpha \leq \alpha' \leq 1$. This means that the classical polynomial lattice rule constructed to achieve optimal convergence rates for functions of smoothness α adjusts itself to the optimal rate of convergence, as long as the smoothness α' of the function under consideration satisfies $\alpha' \geq \alpha$.

Remark 7.5. *Assume that for a fixed α, $0 < \alpha \leq 1$, we have constructed a classical polynomial lattice point set $S_{p,m}(\boldsymbol{q})$ such that*

$$B(\boldsymbol{q}, \alpha, \boldsymbol{\gamma}) \leq C_{s,\alpha,\boldsymbol{\gamma},\delta} N^{-(1+2\alpha)+\delta}, \tag{7.11}$$

for all $\delta > 0$, where $C_{s,\alpha,\boldsymbol{\gamma},\delta}$ is permitted to depend on $s, \alpha, \boldsymbol{\gamma}$ and δ. We point out that explicit constructions of classical polynomial lattice point sets satisfying Equation (7.11) are given in Sections 7.3 and 7.4. It follows immediately from Jensen's inequality that

$$B(\boldsymbol{q}, \alpha, \boldsymbol{\gamma})^{\frac{1+2\alpha'}{1+2\alpha}} \geq B(\boldsymbol{q}, \alpha', \boldsymbol{\gamma}^{\frac{1+2\alpha'}{1+2\alpha}}),$$

for all $\alpha \leq \alpha' \leq 1$. Making use of Assumption (7.11), we conclude that

$$B(\boldsymbol{q}, \alpha', \boldsymbol{\gamma}^{\frac{1+2\alpha'}{1+2\alpha}}) \leq C_{s,\alpha,\boldsymbol{\gamma},\delta}^{\frac{1+2\alpha'}{1+2\alpha}} N^{-(1+2\alpha')+\delta\frac{1+2\alpha'}{1+2\alpha}},$$

for all $\delta > 0$. In particular, this observation motivates the construction of classical polynomial lattice point sets for which $\alpha < 1$, as the resulting quasi-Monte Carlo rules still achieve optimal convergence rates for functions in $V_{\alpha',s,\gamma'}$, where $\alpha \leq \alpha' \leq 1$.

7.3 Component-by-component construction

In this section we show how to construct classical polynomial lattice point sets using a component-by-component approach, so that the bound given in Equation (7.9) converges at a rate of $N^{-(1+2\alpha)+\delta}$, for any $\delta > 0$. We remark that in Theorem 2.54 the corresponding result for classical digital nets was proven. A component-by-component (CBC) approach was first considered in [100] in the context of constructing integration lattices. Subsequently, the CBC algorithm has been applied to the construction of classical polynomial lattice point sets in [29], see also Section 6.3.

We use $R_{b,m}$ to denote the set of all non-zero polynomials in $\mathbb{Z}_b[x]$ with degree at most $m-1$, i.e.

$$R_{b,m} := \{q \in \mathbb{Z}_b[x] : deg(q) < m \text{ and } q \neq 0\} .$$

We remark that we could have also used the notation $G^*_{b,n}$, where $G_{b,n}$ was defined in Equation (6.1). It is clear that $|R_{b,m}| = b^m - 1$, and furthermore it follows from

Algorithm 7.1 CBC algorithm

Require: b a prime, $s, m \in \mathbb{N}$ and weights $\boldsymbol{\gamma} = (\gamma_j)_{j \geq 1}$.

Choose an irreducible polynomial $p \in \mathbb{Z}_b[x]$ with $\deg(p) = m$.

Set $q_1 = 1$.

for $d = 2$ to s **do**

  find $q_d \in R_{b,m}$ by minimizing $B((q_1, \ldots, q_d), \alpha, \boldsymbol{\gamma})$ as a function of q_d.

end for

return $\boldsymbol{q} = (q_1, \ldots, q_s)$.

the construction principle described in Subsection 2.2.3 that the polynomial q_j can be restricted to $R_{b,m}$. Algorithm 7.1 gives the CBC algorithm.

The next theorem shows that Algorithm 7.1 indeed constructs a $\boldsymbol{q}_d^* \in R_{b,m}^d$ so that $B((q_1^*, \ldots, q_d^*), \alpha, \boldsymbol{\gamma})$ converges at a rate of $N^{-(1+2\alpha)+\delta}$, for any $\delta > 0$.

Theorem 7.6. *Let b be prime and $p \in \mathbb{Z}_b[x]$ irreducible with $deg(p) = m \geq 1$, $0 < \alpha \leq 1$. Suppose $(q_1^*, \ldots, q_s^*) \in R_{b,m}^s$ is constructed using Algorithm 7.1. Then for all $d = 1, \ldots, s$ we have*

$$B((q_1^*, \ldots, q_d^*), \alpha, \boldsymbol{\gamma}) \leq \frac{1}{(b^m - 1)^{1/\lambda}} \prod_{j=1}^{d} \left[1 + \gamma_j^\lambda C_{b,\alpha,\lambda}\right]^{1/\lambda},$$

for all $\frac{1}{2\alpha+1} < \lambda \leq 1$, where

$$C_{b,\alpha,\lambda} = \max\left(\frac{1}{(b^{2\alpha} - 1)^\lambda}, \frac{(b-1)^{1-\lambda}}{b^{2\alpha\lambda} - b^{1-\lambda}}\right). \tag{7.12}$$

Proof. We remark that the proof of this result is similar to the proof of [29, Theorem 4.4]. The result is proven by induction: for $d = 1$ we recall that $q_1 = 1$, hence

$$\begin{aligned}
B((1), \alpha, \gamma_1) &= \sum_{k \in \mathcal{D}'_p(1)} r_{2\alpha+1,\gamma_1}(k) \\
&= \sum_{l=1}^{\infty} \sum_{k \in L_l} \mathbf{1}_{k \in \mathcal{D}(1)} r_{2\alpha+1,\gamma_1}(k) \\
&= \gamma_1 \frac{b}{b-1} \sum_{l=1}^{\infty} \frac{1}{b^{(2\alpha+1)l}} \sum_{k \in L_l} \mathbf{1}_{k \in \mathcal{D}_p(1)} \\
&= \gamma_1 \frac{b}{b-1} \sum_{l=1}^{m} \frac{1}{b^{(2\alpha+1)l}} \sum_{k \in L_l} \mathbf{1}_{k \in \mathcal{D}_p(1)} \\
&\quad + \gamma_1 \frac{b}{b-1} \sum_{l=m+1}^{\infty} \frac{1}{b^{(2\alpha+1)l}} \sum_{k \in L_l} \mathbf{1}_{k \in \mathcal{D}_p(1)}.
\end{aligned}$$

For $l = 1, \ldots, m$, we get $tr_m(k) \neq 0$, hence $k \notin \mathcal{D}'_p(1)$. Now consider $l \geq m+1$ and $k = cb^m$, where $c \in \mathbb{N}$, then $tr_m(k) = 0$ and hence $k \in \mathcal{D}'_p(1)$. Finally, for $k = k^* + cb^m$, $0 < k^* \leq b^m - 1$, $c \in \mathbb{N}$, we obtain $tr_m(k) \neq 0$, hence $k \notin \mathcal{D}'_p(1)$. Therefore

$$\begin{aligned}
&\gamma_1 \frac{b}{b-1} \sum_{l=1}^{m} \frac{1}{b^{l(2\alpha+1)}} \sum_{k \in L_l} \mathbf{1}_{k \in \mathcal{D}'_p(1)} \\
&\quad + \gamma_1 \frac{b}{b-1} \sum_{l=m+1}^{\infty} \frac{1}{b^{l(2\alpha+1)}} \sum_{\substack{k \in L_l \\ k = cb^m, c \in \mathbb{N}}} \mathbf{1}_{k \in \mathcal{D}'_p(1)} \\
&\quad + \gamma_1 \frac{b}{b-1} \sum_{l=m+1}^{\infty} \frac{1}{b^{l(2\alpha+1)}} \sum_{\substack{k \in L_l \\ k = k^* + cb^m, \\ c \in \mathbb{N}, 0 < k^* \le b^m - 1}} \mathbf{1}_{k \in \mathcal{D}'_p(1)} \\
&= \gamma_1 \frac{b}{b-1} \sum_{l=m+1}^{\infty} \frac{1}{b^{l(2\alpha+1)}} \sum_{\substack{k \in L_l \\ k = cb^m, c \in \mathbb{N}}} 1 \,.
\end{aligned}$$

Also it is clear that

$$\sum_{\substack{k \in L_l \\ k = cb^m, c \in \mathbb{N}}} 1 = b^{l-m-1}(b-1)\,,$$

hence

$$\begin{aligned}
B((1), \alpha, \gamma_1) &= \gamma_1 \frac{b}{b-1} \sum_{l=m+1}^{\infty} \frac{1}{b^{2\alpha l}} \frac{1}{b^m} \frac{b-1}{b} \\
&= \gamma_1 \frac{1}{b^{m(2\alpha+1)}} \frac{1}{b^{2\alpha}-1} \\
&\le \frac{1}{(b^m-1)^{1/\lambda}} \left(\frac{\gamma_1^\lambda}{(b^{2\alpha}-1)^\lambda} \right)^{1/\lambda} \\
&\le \frac{1}{(b^m-1)^{1/\lambda}} \left(1 + \frac{\gamma_1^\lambda}{(b^{2\alpha}-1)^\lambda} \right)^{1/\lambda} \\
&\le \frac{1}{(b^m-1)^{1/\lambda}} \left(1 + \gamma_1^\lambda C_{b,\alpha,\lambda} \right)^{1/\lambda} \,.
\end{aligned}$$

We now assume that we have established the result for $1 \le d < s$, i.e. we have constructed $\boldsymbol{q}_d^*$ so that

$$B(\boldsymbol{q}_d^*, \alpha, \boldsymbol{\gamma}) \le \frac{1}{(b^m-1)^{1/\lambda}} \prod_{j=1}^{d} (1 + \gamma_j^\lambda C_{b,\alpha,\lambda})^{1/\lambda}\,,$$

for all $\frac{1}{2\alpha+1} < \lambda \le 1$, where $C_{b,\alpha,\lambda}$ is given by Equation (7.12). From Equation (7.9) we have

$$\begin{aligned}
B((\boldsymbol{q}_d^*, q_{d+1}), \alpha, \boldsymbol{\gamma}) &= \sum_{\substack{(\boldsymbol{k}, k_{d+1}) \in \mathbb{N}_0^{d+1} \setminus \{\boldsymbol{0}\} \\ tr_m(\boldsymbol{k}, k_{d+1}) \cdot (\boldsymbol{q}_d^*, q_{d+1}) \equiv 0 \pmod{p}}} r_{2\alpha+1,\boldsymbol{\gamma}}(\boldsymbol{k}) r_{2\alpha+1,\gamma_{d+1}}(k_{d+1}) \\
&= B(\boldsymbol{q}_d^*, \alpha, \boldsymbol{\gamma}) + \theta(q_{d+1})\,, \qquad (7.13)
\end{aligned}$$

where we have separated out the $k_{d+1} = 0$ terms, and

$$\theta(q_{d+1}) = \sum_{k_{d+1}=1}^{\infty} r_{2\alpha+1,\gamma_{d+1}}(k_{d+1}) \sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^d: \\ tr_m(\boldsymbol{k})\boldsymbol{q}_d^* = -tr_m(k_{d+1})\cdot q_{d+1} \pmod p}} r_{2\alpha+1,\boldsymbol{\gamma}}(\boldsymbol{k}) .$$

We see from the algorithm that q_{d+1}^* is chosen such that $B((\boldsymbol{q}_d^*, q_{d+1}), \alpha, \boldsymbol{\gamma})$ is minimized. Since the only dependence on q_{d+1} is in $\theta(q_{d+1})$, we have $\theta(q_{d+1}^*) \leq \theta(q_{d+1})$ for all $q_{d+1} \in R_{b,m}$, which implies that for any $\lambda \leq 1$ we have

$$\theta(q_{d+1}^*)^\lambda \leq \theta(q_{d+1})^\lambda \text{ for all } q_{d+1} \in R_{b,m} ,$$

which leads to

$$\theta(q_{d+1}^*) \leq \left(\frac{1}{b^m - 1} \sum_{q_{d+1}\in R_{b,m}} \theta^\lambda(q_{d+1}) \right)^{\frac{1}{\lambda}} .$$

Let $\frac{1}{2\alpha+1} < \lambda \leq 1$, then it follows from Jensen's inequality that

$$\theta^\lambda(q_{d+1}) \leq \sum_{k_{d+1}=1}^{\infty} r^\lambda_{2\alpha+1,\gamma_{d+1}}(k_{d+1}) \sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^d: \\ tr_m(\boldsymbol{k})\cdot\boldsymbol{q}_d^* = -tr_m(k_{d+1})q_{d+1} \pmod p}} r^\lambda_{2\alpha+1,\boldsymbol{\gamma}}(\boldsymbol{k}) .$$

If k_{d+1} is a multiple of b^m, then $tr_m(k_{d+1}) = 0$ and the corresponding term in the sum is independent of q_{d+1}. Should k_{d+1} not be a multiple of b^m, then $tr_m(k_{d+1})$ can take on any value between 1 and $b^m - 1$. However, since $q_{d+1} \neq 0$ and p is irreducible, $tr_m(k_{d+1})q_{d+1}$ is never a multiple of p. Hence

$$\begin{aligned}
&\frac{1}{b^m-1} \sum_{q_{d+1}\in R_{b,m}} \theta^\lambda(q_{d+1}) \\
&\leq \frac{1}{b^m-1} \sum_{q_{d+1}\in R_{b,m}} \sum_{\substack{k_{d+1}=1 \\ b^m | k_{d+1}}}^{\infty} r^\lambda_{2\alpha+1,\gamma_{d+1}}(k_{d+1}) \sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^d: \\ tr_m(\boldsymbol{k})\cdot\boldsymbol{q}_d^* \equiv 0 \pmod p}} r^\lambda_{2\alpha+1,\boldsymbol{\gamma}}(\boldsymbol{k}) \\
&\quad + \frac{1}{b^m-1} \sum_{q_{d+1}\in R_{b,m}} \sum_{\substack{k_{d+1}=1 \\ b^m \nmid k_{d+1}}}^{\infty} r^\lambda_{2\alpha+1,\gamma_{d+1}}(k_{d+1}) \sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^d: \\ tr_m(\boldsymbol{k})\cdot\boldsymbol{q}_d^* \equiv -tr_m(k_{d+1})q_{d+1} \pmod p}} r^\lambda_{2\alpha+1,\boldsymbol{\gamma}}(\boldsymbol{k}) \\
&= \sum_{\substack{k_{d+1}=1 \\ b^m | k_{d+1}}}^{\infty} r^\lambda_{2\alpha+1,\gamma_{d+1}}(k_{d+1}) \sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^d: \\ tr_m(\boldsymbol{k})\cdot\boldsymbol{q}_d^* \equiv 0 \pmod p}} r^\lambda_{2\alpha+1,\boldsymbol{\gamma}}(\boldsymbol{k}) && (7.14) \\
&\quad + \frac{1}{b^m-1} \sum_{\substack{k_{d+1}=1 \\ b^m \nmid k_{d+1}}}^{\infty} r^\lambda_{2\alpha+1,\gamma_{d+1}}(k_{d+1}) \sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^d: \\ \mathrm{tr}_m(\boldsymbol{k})\cdot\boldsymbol{q}_d^* \not\equiv 0 \pmod p}} r^\lambda_{2\alpha+1,\boldsymbol{\gamma}}(\boldsymbol{k}) . && (7.15)
\end{aligned}$$

Regarding the term in Equation (7.14), we obtain

$$\sum_{\substack{k_{d+1}=1 \\ b^m | k_{d+1}}}^{\infty} r^\lambda_{2\alpha+1,\gamma_{d+1}}(k_{d+1}) = \sum_{\substack{k_{d+1}=1 \\ k_{d+1}=cb^m,\, c\in\mathbb{N}}}^{\infty} r^\lambda_{2\alpha+1,\gamma_{d+1}}(k_{d+1})$$

$$
\begin{aligned}
&= \gamma_{d+1}^{\lambda} \frac{b^{\lambda}}{(b-1)^{\lambda}} \sum_{l=m+1}^{\infty} b^{-(2\alpha+1)\lambda l} \sum_{\substack{k \in L_l: \\ k = cb^m, c \in \mathbb{N}}} 1 \\
&= \gamma_{d+1}^{\lambda} \frac{b^{\lambda}}{(b-1)^{\lambda}} \sum_{l=m+1}^{\infty} b^{-(2\alpha+1)\lambda l} b^{l-m-1} (b-1) \,.
\end{aligned}
$$

Furthermore, regarding the term in Equation (7.15), we get

$$
\begin{aligned}
&\frac{1}{b^m - 1} \sum_{\substack{k_{d+1}=1 \\ b^m \nmid k_{d+1}}}^{\infty} r_{2\alpha+1,\gamma_{d+1}}^{\lambda}(k_{d+1}) \\
&= \frac{1}{b^m - 1} \sum_{l=1}^{\infty} \sum_{\substack{k_{d+1} \in L_l: \\ b^m \nmid k_{d+1}}} r_{2\alpha+1,\gamma_{d+1}}^{\lambda}(k_{d+1}) \\
&= \frac{1}{b^m - 1} \sum_{l=1}^{m} \sum_{\substack{k_{d+1} \in L_l: \\ b^m \nmid k_{d+1}}} r_{2\alpha+1,\gamma_{d+1}}^{\lambda}(k_{d+1}) \\
&\quad + \frac{1}{b^m - 1} \sum_{l=m+1}^{\infty} \sum_{\substack{k_{d+1} \in L_l: \\ b^m \nmid k_{d+1}}} r_{2\alpha+1,\gamma_{d+1}}^{\lambda}(k_{d+1}) \\
&= \frac{1}{b^m - 1} \left(\frac{b}{b-1} \right)^{\lambda} \gamma_{d+1}^{\lambda} \sum_{l=1}^{m} \frac{1}{b^{(2\alpha+1)\lambda l}} \sum_{k \in L_l} 1 \\
&\quad + \frac{1}{b^m - 1} \left(\frac{b}{b-1} \right)^{\lambda} \gamma_{d+1}^{\lambda} \sum_{l=m+1}^{\infty} \frac{1}{b^{(2\alpha+1)\lambda l}} \sum_{\substack{k \in L_l \\ k = k^* + cb^m, \\ c \in \mathbb{N}, 0 < k^* \leq b^m - 1}} 1 \\
&= \frac{1}{b^m - 1} \left(\frac{b}{b-1} \right)^{\lambda} \gamma_{d+1}^{\lambda} \sum_{l=1}^{m} \frac{1}{b^{(2\alpha+1)\lambda l}} (b-1) b^{l-1} \\
&\quad + \frac{1}{b^m - 1} \left(\frac{b}{b-1} \right)^{\lambda} \gamma_{d+1}^{\lambda} \sum_{l=m+1}^{\infty} \frac{1}{b^{(2\alpha+1)\lambda l}} (b^m - 1) b^{l-m-1} (b-1) \,.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\frac{1}{b^m - 1} \sum_{q_{d+1} \in R_{b,m}} \theta^{\lambda}(q_{d+1}) \\
&\leq \frac{1}{b^m - 1} \frac{b^{\lambda-1}}{(b-1)^{\lambda-1}} \gamma_{d+1}^{\lambda} \sum_{l=1}^{m} \frac{b^l}{b^{(2\alpha+1)\lambda l}} \sum_{\substack{\boldsymbol{k} \in \mathbb{N}_0^d: \\ tr_m(\boldsymbol{k}) \cdot \boldsymbol{q}_d^* \not\equiv 0 \pmod{p}}} r_{2\alpha+1,\gamma}^{\lambda}(\boldsymbol{k}) \\
&\quad + \left(\frac{b}{b-1} \right)^{\lambda-1} \gamma_{d+1}^{\lambda} \sum_{l=m+1}^{\infty} \frac{1}{b^{(2\alpha+1)\lambda l}} b^{l-m} \sum_{\substack{\boldsymbol{k} \in \mathbb{N}_0^d: \\ tr_m(\boldsymbol{k}) \cdot \boldsymbol{q}_d^* \not\equiv 0 \pmod{p}}} r_{2\alpha+1,\gamma}^{\lambda}(\boldsymbol{k}) \\
&\quad + \left(\frac{b}{b-1} \right)^{\lambda-1} \gamma_{d+1}^{\lambda} \sum_{l=m+1}^{\infty} \frac{1}{b^{(2\alpha+1)\lambda l}} b^{l-m} \sum_{\substack{\boldsymbol{k} \in \mathbb{N}_0^d: \\ tr_m(\boldsymbol{k}) \cdot \boldsymbol{q}_d^* \equiv 0 \pmod{p}}} r_{2\alpha+1,\gamma}^{\lambda}(\boldsymbol{k})
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{b^m-1}\frac{b^{\lambda-1}}{(b-1)^{\lambda-1}}\gamma_{d+1}^{\lambda}\sum_{l=1}^{m}\frac{b^l}{b^{(2\alpha+1)\lambda l}}\left(\sum_{\boldsymbol{k}\in\mathbb{N}_0^d} r_{2\alpha+1,\gamma}^{\lambda}(\boldsymbol{k}) \right.\\
&\qquad \left. - \sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^d:\\ tr_m(\boldsymbol{k})\cdot\boldsymbol{q}_d^*\equiv 0 \pmod p}} r_{2\alpha+1,\gamma}^{\lambda}(\boldsymbol{k})\right)\\
&\quad + \left(\frac{b}{b-1}\right)^{\lambda-1}\gamma_{d+1}^{\lambda}\sum_{l=m+1}^{\infty}\frac{1}{b^{(2\alpha+1)\lambda l}}b^{l-m}\sum_{\boldsymbol{k}\in\mathbb{N}_0^d} r_{2\alpha+1,\gamma}^{\lambda}(\boldsymbol{k})\\
&\le \frac{1}{b^m-1}\left(\frac{b}{b-1}\right)^{\lambda-1}\gamma_{d+1}^{\lambda}\sum_{l=1}^{\infty}\frac{1}{b^{[(2\alpha+1)\lambda-1]l}}\sum_{\boldsymbol{k}\in\mathbb{N}_0^d} r_{2\alpha+1,\gamma}^{\lambda}(\boldsymbol{k})\\
&= \frac{(b-1)^{1-\lambda}\gamma_{d+1}^{\lambda}}{b^m-1}\frac{1}{b^{2\alpha\lambda}-b^{1-\lambda}}\sum_{\boldsymbol{k}\in\mathbb{N}_0^d} r_{2\alpha+1,\gamma}^{\lambda}(\boldsymbol{k})\,.
\end{aligned}
$$

Also

$$
\sum_{\boldsymbol{k}\in\mathbb{N}_0^d} r_{2\alpha+1,\gamma}^{\lambda}(\boldsymbol{k}) = \prod_{j=1}^{d}\left(1+\frac{\gamma_j^{\lambda}(b-1)^{1-\lambda}}{b^{2\alpha\lambda}-b^{1-\lambda}}\right),
$$

therefore

$$
\frac{1}{b^m-1}\sum_{q_{d+1}\in R_{b,m}}\theta^{\lambda}(q_{d+1}) \le \frac{1}{b^m-1}\frac{(b-1)^{1-\lambda}}{b^{2\alpha\lambda}-b^{1-\lambda}}\gamma_{d+1}^{\lambda}\prod_{j=1}^{d}\left(1+\frac{\gamma_j^{\lambda}(b-1)^{1-\lambda}}{b^{2\alpha\lambda}-b^{1-\lambda}}\right),
$$

and consequently

$$
\begin{aligned}
\theta(\boldsymbol{q}_{d+1}^*) &\le \frac{1}{(b^m-1)^{1/\lambda}}\left(\frac{(b-1)^{1-\lambda}}{b^{2\alpha\lambda}-b^{1-\lambda}}\right)^{1/\lambda}\gamma_{d+1}\prod_{j=1}^{d}\left(1+\frac{\gamma_j^{\lambda}(b-1)^{1-\lambda}}{b^{2\alpha\lambda}-b^{1-\lambda}}\right)^{1/\lambda}\\
&\le \frac{1}{(b^m-1)^{1/\lambda}}\left(\frac{(b-1)^{1-\lambda}}{b^{2\alpha\lambda}-b^{1-\lambda}}\right)^{1/\lambda}\gamma_{d+1}\prod_{j=1}^{d}\left(1+\gamma_j^{\lambda}C_{b,\alpha,\lambda}\right)^{1/\lambda}.
\end{aligned}
$$

It now follows from Equation (7.13) and the induction assumption that

$$
\begin{aligned}
B(\boldsymbol{q}_{d+1}^*,\alpha,\gamma) &\le \frac{1}{(b^m-1)^{1/\lambda}}\left(\frac{(b-1)^{1-\lambda}}{b^{2\alpha\lambda}-b^{1-\lambda}}\right)^{1/\lambda}\gamma_{d+1}\prod_{j=1}^{d}\left(1+\gamma_j^{\lambda}C_{b,\alpha,\lambda}\right)^{1/\lambda}\\
&\quad + \frac{1}{(b^m-1)^{1/\lambda}}\prod_{j=1}^{d}(1+\gamma_j^{\lambda}C_{b,\alpha,\lambda})^{1/\lambda}\\
&\le \frac{1}{(b^m-1)^{1/\lambda}}\left(1+\gamma_{d+1}^{\lambda}\frac{(b-1)^{1-\lambda}}{b^{2\alpha\lambda}-b^{1-\lambda}}\right)^{1/\lambda}\prod_{j=1}^{d}\left(1+\gamma_j^{\lambda}C_{b,\alpha,\lambda}\right)^{1/\lambda}\\
&\le \frac{1}{(b^m-1)^{1/\lambda}}\prod_{j=1}^{d+1}\left(1+\gamma_j^{\lambda}C_{b,\alpha,\lambda}\right)^{1/\lambda},
\end{aligned}
$$

which proves the required result.

□

The next result discusses the tractability of Algorithm 7.1.

Corollary 7.7. *Let b be prime, $p \in \mathbb{Z}_b[x]$ irreducible with $deg(p) = m \geq 1$ and $N = b^m$. Suppose $\boldsymbol{q}_s^* \in R_{b,m}^s$ is constructed using Algorithm 7.1. Then we have the following:*

i)

$$B(\boldsymbol{q}_s^*, \alpha, \boldsymbol{\gamma}) \leq c_{s,\alpha,\gamma,\delta}\,(N-1)^{-(2\alpha+1)+\delta}, \text{ for all } 0 < \delta \leq 2\alpha,$$

where

$$c_{s,\alpha,\gamma,\delta} = \prod_{j=1}^{s} \left[1 + \gamma_j^{\frac{1}{2\alpha+1-\delta}} C_{b,\alpha,(2\alpha+1-\delta)^{-1}}\right]^{2\alpha+1-\delta}.$$

ii) Assume

$$\sum_{j=1}^{\infty} \gamma_j^{\frac{1}{2\alpha+1-\delta}} < \infty. \tag{7.16}$$

Then $c_{s,\alpha,\gamma,\delta} \leq c_{\infty,\alpha,\gamma,\delta} < \infty$ and we have

$$B(\boldsymbol{q}_s^*, \alpha, \boldsymbol{\gamma}) \leq c_{\infty,\alpha,\gamma,\delta}(N-1)^{-(2\alpha+1)+\delta}, \text{ for all } 0 < \delta \leq 2\alpha.$$

Thus the bound $B(\boldsymbol{q}_s^*, \alpha, \boldsymbol{\gamma})$ is bounded independently of the dimension.

iii) Under the assumption

$$A := \limsup_{s\to\infty} \frac{\sum_{j=1}^{s} \gamma_j}{\log s} < \infty,$$

we obtain

$$c_{s,\alpha,\gamma,2\alpha} \leq \tilde{c}_\eta s^{\frac{A+\eta}{b^{2\alpha}-1}}$$

and therefore

$$B(\boldsymbol{q}_s^*, \alpha, \boldsymbol{\gamma}) \leq \tilde{c}_\eta s^{\frac{A+\eta}{b^{2\alpha}-1}} (N-1)^{-1},$$

for all $\eta > 0$, where the constant $\tilde{c}_\eta$ only depends on η. Thus the bound $B(\boldsymbol{q}_s^*, \alpha, \boldsymbol{\gamma})$ satisfies a bound which depends only polynomially on the dimension.

Proof. The proof is similar to the proof of [29, Corollary 4.5]. The first part follows by setting $\frac{1}{\lambda} = 2\alpha + 1 - \delta$. Regarding the second part, we note that

$$\begin{aligned}\prod_{j=1}^{s}\left(1 + \gamma_j^{\frac{1}{2\alpha+1-\delta}} C_{b,\alpha,\frac{1}{2\alpha+1-\delta}}\right) &= \exp\left(\sum_{j=1}^{s} \log(1 + \gamma_j^{\frac{1}{2\alpha+1-\delta}} C_{b,\alpha,\frac{1}{2\alpha+1-\delta}})\right) \\ &\leq \exp\left(C_{b,\alpha,\frac{1}{2\alpha+1-\delta}} \sum_{j=1}^{s} \gamma_j^{\frac{1}{2\alpha+1-\delta}}\right),\end{aligned}$$

where we used the fact that $\log(1+x) \leq x$, for all $x \geq 0$. Therefore $c_{\infty,\alpha,\gamma,\delta} < \infty$ provided that Assumption (7.16) is satisfied. Obviously $c_{s,\alpha,\gamma,\delta} \leq c_{\infty,\alpha,\gamma,\delta}$ and so the second part follows. For the third part we observe that $A < \infty$ and therefore for any positive η there exists a positive s_η such that

$$\sum_{j=1}^{s} \gamma_j \leq (A+\eta)\log s\,, \text{ for all } s \geq s_\eta\,.$$

Consequently

$$\begin{aligned} c_{s,\alpha,\gamma,2\alpha} &= \prod_{j=1}^{s}(1+\frac{\gamma_j}{b^{2\alpha}-1}) \\ &= s^{\frac{\sum_{j=1}^{s}\log(1+\frac{\gamma_j}{b^{2\alpha}-1})}{\log s}} \\ &\leq s^{\sum_{j=1}^{s}\frac{\gamma_j}{\log s}\frac{1}{b^{2\alpha}-1}} \\ &\leq s^{\frac{A+\eta}{b^{2\alpha}-1}}\,, \end{aligned}$$

for any $\eta > 0$ and all $s \geq s_\eta$. Hence there exists a constant $\tilde{c}_\eta$ such that

$$c_{s,\alpha,\gamma,2\alpha} \leq \tilde{c}_\eta s^{\frac{A+\eta}{b^{2\alpha}-1}}$$

and the third assertion follows. □

7.4 Korobov polynomial lattice point sets

In this section we construct Korobov polynomial lattice point sets. The ideas underlying the algorithm stem from the construction of integration lattices, see [49]. We remark that the construction of Korobov polynomial lattice point sets has been examined in [29], see also [55] and Section 6.5. The generating vector of the Korobov polynomial lattice point set is denoted by $\psi(q) = (1, q, \ldots, q^{s-1}) \pmod{p}$. As in Section 7.3 we work with the bound $B(\psi(q), \alpha, \gamma)$ and now state the algorithm showing how to construct Korobov polynomial lattice point sets.

Algorithm 7.2 Korobov algorithm

Require: b a prime, $s, m \in \mathbb{N}$ and weights $\gamma = (\gamma_j)_{j\geq 1}$.

1: Choose an irreducible polynomial $p \in \mathbb{Z}_b[x]$ with $\deg(p) = m$.
2: Find $q^* \in R_{b,m}$ by minimizing $B(\psi(q), \alpha, \gamma)$.

We obtain the following bound for $B(\psi(q^*), \alpha, \gamma)$, where q^* is constructed using Algorithm 7.2.

Theorem 7.8. *Let b be prime, $s \geq 2$ and $p \in \mathbb{Z}_b[x]$ irreducible with $deg(p) = m \geq 1$. A minimizer q^* obtained from Algorithm 7.2 satisfies*

$$B(\psi(q^*), \alpha, \gamma) \leq \frac{s^{1/\lambda}}{(b^m - 1)^{1/\lambda}} \prod_{j=1}^{s} \left(1 + \gamma_j^{\lambda} C_{b,\alpha,\lambda}\right)^{1/\lambda},$$

for all $\frac{1}{2\alpha+1} < \lambda \leq 1$, where $C_{b,\alpha,\lambda} > 0$ is given by Equation (7.12).

Proof. The proof is similar to the proof of [29, Theorem 4.7]: for $\frac{1}{2\alpha+1} < \lambda \leq 1$

$$\begin{aligned} M_{s,\alpha}(p) &:= \frac{1}{b^m - 1} \sum_{q \in R_{b,m}} B^{\lambda}(\psi(q), \alpha, \gamma) \\ &= \frac{1}{b^m - 1} \sum_{q \in R_{b,m}} \left[\sum_{\boldsymbol{k} \in \mathcal{D}'_p(\psi(q))} r_{2\alpha+1,\gamma}(\boldsymbol{k}) \right]^{\lambda} \\ &\leq \frac{1}{b^m - 1} \sum_{q \in R_{b,m}} \sum_{\boldsymbol{k} \in \mathcal{D}'_p(\psi(q))} r^{\lambda}_{2\alpha+1,\gamma}(\boldsymbol{k}) \\ &= \frac{1}{b^m - 1} \sum_{\boldsymbol{k} \in \mathbb{N}_0^s \setminus \{\boldsymbol{0}\}} r^{\lambda}_{2\alpha+1,\gamma}(\boldsymbol{k}) \sum_{\substack{q \in R_{b,m} \\ tr_m(\boldsymbol{k}) \cdot \psi(q) \equiv 0 \pmod{p}}} 1 . \end{aligned}$$

Following the proof of [29, Theorem 4.7], we recall that for an irreducible polynomial $p \in \mathbb{Z}_b[x]$ with $deg(p) = m \geq 1$ and a non-zero $(k_1, \ldots, k_s) \in \mathbb{Z}_b^s[x]$ with $deg(k_j) < m$, $j = 1, \ldots, s$, the congruence

$$k_1 + k_2 q + \ldots k_s q^{s-1} \equiv 0 \pmod{p}$$

has no solution if $k_2 = \ldots k_s = 0$, or it has at most $s - 1$ solutions $q \in R_{b,m}$ otherwise. For $\boldsymbol{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$, $\boldsymbol{k} \neq \boldsymbol{0}$, we consider two cases:

i) For $j = 2, \ldots, s$ let $k_j = b^m l_j$, $l_j \geq 0$, and $k_1 \neq b^m l_1$, $l_1 \geq 0$. In this case we have $tr_m(k_j) = 0$ for $2 \leq j \leq s$ and therefore

$$\sum_{\substack{q \in R_{b,m} \\ tr_m(\boldsymbol{k}) \cdot \psi(q) \equiv 0 \pmod{p}}} 1 = 0 .$$

ii) For $j = 2, \ldots, s$ let $k_j = k_j^* + b^m l_j$, $l_j \geq 0$, $0 \leq k_j^* \leq b^m - 1$ and $(k_2^*, \ldots k_s^*) \neq (0, \ldots, 0)$. Then we obtain

$$\sum_{\substack{q \in R_{b,m} \\ tr_m(\boldsymbol{k}) \cdot \psi(q) \equiv 0 \pmod{p}}} 1 \leq s - 1 .$$

Now

$$\begin{aligned} M_{s,\alpha}(p) &\leq \sum_{\boldsymbol{l}\in\mathbb{N}_0^s\setminus\{\boldsymbol{0}\}} r_{2\alpha+1,\boldsymbol{\gamma}}^{\lambda}(\boldsymbol{l}b^m) \\ &\quad + \frac{s-1}{b^m-1}\sum_{k_1=0}^{\infty}\sum_{l_2,\ldots,l_s=0}^{\infty}\sum_{\substack{k_2^*,\ldots,k_s^*=0\\(k_2^*,\ldots,k_s^*)\neq(0,\ldots,0)}}^{b^m-1} r_{2\alpha+1,\gamma_1}^{\lambda}(k_1)\prod_{j=2}^{s} r_{2\alpha+1,\gamma_j}^{\lambda}(k_j^*+l_jb^m)\,. \end{aligned}$$

Regarding the term $\sum_{\boldsymbol{l}\in\mathbb{N}_0^s\setminus\{\boldsymbol{0}\}} r_{2\alpha+1,\boldsymbol{\gamma}}^{\lambda}(\boldsymbol{l}b^m)$, we have

$$\begin{aligned} \sum_{\boldsymbol{l}\in\mathbb{N}_0^s\setminus\{\boldsymbol{0}\}} r_{2\alpha+1,\boldsymbol{\gamma}}^{\lambda}(\boldsymbol{l}b^m) &= \prod_{j=1}^{s}\sum_{l_j=0}^{\infty} r_{2\alpha+1,\gamma_j}^{\lambda}(l_jb^m)-1 \\ &= \prod_{j=1}^{s}\left(1+\sum_{l_j=1}^{\infty} r_{2\alpha+1,\gamma_j}^{\lambda}(l_jb^m)\right)-1 \\ &= \prod_{j=1}^{s}\left(1+\sum_{\substack{k=b^m:\\k=cb^m,c\in\mathbb{N}}}^{\infty} r_{2\alpha+1,\gamma_j}^{\lambda}(k)\right)-1 \\ &= \prod_{j=1}^{s}\left(1+\sum_{l=m+1}^{\infty}\sum_{\substack{k\in L_l:\\k=cb^m,c\in\mathbb{N}}} r_{2\alpha+1,\gamma_j}^{\lambda}(k)\right)-1 \\ &= \prod_{j=1}^{s}\left(1+\sum_{l=m+1}^{\infty}\gamma_j^{\lambda}\left(\frac{b}{b-1}\right)^{\lambda}\frac{1}{b^{(2\alpha+1)\lambda l}}\sum_{\substack{k\in L_l:\\k=cb^m,c\in\mathbb{N}}} 1\right)-1 \\ &= \prod_{j=1}^{s}\left(1+\sum_{l=m+1}^{\infty}\gamma_j^{\lambda}\left(\frac{b}{b-1}\right)^{\lambda}\frac{1}{b^{(2\alpha+1)\lambda l}}b^{l-m-1}(b-1)\right)-1 \\ &= \prod_{j=1}^{s}\left(1+\gamma_j^{\lambda}\left(\frac{b}{b-1}\right)^{\lambda-1}\frac{1}{b^{(2\alpha+1)\lambda m}}\frac{1}{b^{(2\alpha+1)\lambda-1}-1}\right)-1 \\ &= \prod_{j=1}^{s}\left(1+\gamma_j^{\lambda}(b-1)^{1-\lambda}\frac{1}{b^{(2\alpha+1)\lambda m}}\frac{1}{b^{2\alpha\lambda}-b^{1-\lambda}}\right)-1 \\ &\leq \frac{1}{b^{(2\alpha+1)\lambda m}}\prod_{j=1}^{s}\left(1+\gamma_j^{\lambda}\frac{(b-1)^{1-\lambda}}{b^{2\alpha\lambda}-b^{1-\lambda}}\right)\,. \end{aligned}$$

Regarding the term

$$\frac{s-1}{b^m-1}\sum_{k_1=0}^{\infty}\sum_{l_2,\ldots,l_s=0}^{\infty}\sum_{\substack{k_2^*,\ldots,k_s^*=0\\(k_2^*,\ldots,k_s^*)\neq(0,\ldots,0)}}^{b^m-1} r_{2\alpha+1,\gamma_1}^{\lambda}(k_1)\prod_{j=2}^{s} r_{2\alpha+1,\gamma_j}^{\lambda}(k_j^*+l_jb^m),$$

we note that

$$\frac{s-1}{b^m-1}\sum_{k_1=0}^{\infty}\sum_{l_2,\ldots,l_s=0}^{\infty}\sum_{\substack{k_2^*,\ldots,k_s^*=0\\(k_2^*,\ldots,k_s^*)\neq(0,\ldots,0)}}^{b^m-1} r_{2\alpha+1,\gamma_1}^{\lambda}(k_1)\prod_{j=2}^{s} r_{2\alpha+1,\gamma_j}^{\lambda}(k_j^*+l_jb^m)$$

$$= \frac{s-1}{b^m-1}\sum_{k_1=0}^{\infty}\sum_{l_2,\ldots,l_s=0}^{\infty}\sum_{k_2^*,\ldots,k_s^*=0}^{b^m-1} r_{2\alpha+1,\gamma_1}^{\lambda}(k_1)\prod_{j=2}^{s} r_{2\alpha+1,\gamma_j}^{\lambda}(k_j^*+l_jb^m)$$
$$-\frac{s-1}{b^m-1}\sum_{k_1=0}^{\infty}\sum_{l_2,\ldots,l_s=0}^{\infty} r_{2\alpha+1,\gamma_1}^{\lambda}(k_1)\prod_{j=2}^{s} r_{2\alpha+1,\gamma_j}^{\lambda}(l_jb^m)\,.$$

Furthermore

$$\begin{aligned}
\sum_{l=0}^{\infty}\sum_{k^*=0}^{b^m-1} r_{2\alpha+1,\gamma}^{\lambda}(k^*+lb^m) &= \sum_{l=0}^{\infty} r_{2\alpha+1,\gamma}^{\lambda}(l)\\
&= 1+\sum_{l=1}^{\infty} r_{2\alpha+1,\gamma}^{\lambda}(l)\\
&= 1+\sum_{l=1}^{\infty}\sum_{k\in L_l} r_{2\alpha+1,\gamma}^{\lambda}(k)\\
&= 1+\gamma^{\lambda}\left(\frac{b}{b-1}\right)^{\lambda}\sum_{l=1}^{\infty}\frac{1}{b^{(2\alpha+1)\lambda l}}\sum_{k\in L_l}1\\
&= 1+\gamma^{\lambda}\frac{(b-1)^{1-\lambda}}{b^{2\alpha\lambda}-b^{1-\lambda}}\,.
\end{aligned}$$

Hence

$$\begin{aligned}
&\frac{s-1}{b^m-1}\sum_{k_1=0}^{\infty}\sum_{l_2,\ldots,l_s=0}^{\infty}\sum_{k_2^*,\ldots,k_s^*=0}^{b^m-1} r_{2\alpha+1,\gamma_1}^{\lambda}(k_1)\prod_{j=2}^{s} r_{2\alpha+1,\gamma_j}^{\lambda}(k_j^*+l_jb^m)\\
&\quad-\frac{s-1}{b^m-1}\sum_{k_1=0}^{\infty}\sum_{l_2,\ldots,l_s=0}^{\infty} r_{2\alpha+1,\gamma_1}^{\lambda}(k_1)\prod_{j=2}^{s} r_{2\alpha+1,\gamma_j}^{\lambda}(l_jb^m)\\
&= \frac{s-1}{b^m-1}\left(1+\frac{\gamma_1^{\lambda}(b-1)^{1-\lambda}}{b^{2\alpha\lambda}-b^{1-\lambda}}\right)\left[\prod_{j=2}^{s}\left(1+\frac{\gamma_j^{\lambda}(b-1)^{1-\lambda}}{b^{2\alpha\lambda}-b^{1-\lambda}}\right)\right.\\
&\qquad\left.-\prod_{j=2}^{s}\left(1+\frac{\gamma_j^{\lambda}(b-1)^{1-\lambda}}{b^{(2\alpha+1)\lambda m}(b^{2\alpha\lambda}-b^{1-\lambda})}\right)\right]\\
&\le \frac{s-1}{b^m-1}\prod_{j=1}^{s}\left(1+\frac{\gamma_j^{\lambda}(b-1)^{1-\lambda}}{b^{2\alpha\lambda}-b^{1-\lambda}}\right)\,.
\end{aligned}$$

Finally

$$\begin{aligned}
M_{s,\alpha}(p) &\le \frac{1}{b^{(2\alpha+1)\lambda m}}\prod_{j=1}^{s}\left(1+\frac{\gamma_j^{\lambda}(b-1)^{1-\lambda}}{b^{2\alpha\lambda}-b^{1-\lambda}}\right)\\
&\quad+\frac{s-1}{b^m-1}\prod_{j=1}^{s}\left(1+\frac{\gamma_j^{\lambda}(b-1)^{1-\lambda}}{b^{2\alpha\lambda}-b^{1-\lambda}}\right)\\
&\le \frac{s}{b^m-1}\prod_{j=1}^{s}\left(1+\frac{\gamma_j^{\lambda}(b-1)^{1-\lambda}}{b^{2\alpha\lambda}-b^{1-\lambda}}\right)\\
&\le \frac{s}{b^m-1}\prod_{j=1}^{s}\left(1+\gamma_j^{\lambda}C_{b,\alpha,\lambda}\right)
\end{aligned}$$

and the result follows. □

We point out that the bounds in Theorems 7.6 and 7.8 only differ by the additional factor $s^{1/\lambda}$ and remark that the same observation was made in [29] and is also known from the integration lattice case. This leads to the conclusion that the Korobov construction is inferior to the component-by-component construction.

In the next corollary we discuss the tractability of Algorithm 7.2.

Corollary 7.9. *Let b be prime, $s \geq 2$, $p \in \mathbb{Z}_b[x]$ irreducible with $deg(p) = m \geq 1$ and $N = b^m$. Suppose $q^* \in R_{b,m}$ is constructed using Algorithm 7.2. Then we have the following:*

i)

$$B(\psi(q^*), \alpha, \gamma) \leq c_{s,\alpha,\gamma,\delta} s^{2\alpha+1-\delta}(N-1)^{-(2\alpha+1)+\delta}, \text{ for all } 0 < \delta \leq 2\alpha,$$

where

$$c_{s,\alpha,\gamma,\delta} = \prod_{j=1}^{s} \left(1 + \gamma_j^{\frac{1}{2\alpha+1-\delta}} C_{b,\alpha,(2\alpha+1-\delta)^{-1}}\right)^{2\alpha+1-\delta}.$$

ii) Under the assumption

$$A := \lim sup_{s\to\infty} \frac{\sum_{j=1}^{s} \gamma_j}{\log s} < \infty,$$

we obtain

$$c_{s,\alpha,\gamma,2\alpha} \leq \tilde{c}_\eta s^{\frac{A+\eta}{b^{2\alpha}-1}}$$

and therefore

$$B(\psi(q^*), \alpha, \gamma) \leq \tilde{c}_\eta s^{1+\frac{A+\eta}{b^{2\alpha}-1}}(N-1)^{-1},$$

for all $\eta > 0$, where the constant $\tilde{c}_\eta$ only depends on η. Thus the bound $B(\psi(q^*), \alpha, \gamma)$ satisfies a bound which depends only polynomially on the dimension.

Proof. The proof is similar to the proof of Corollary 7.7, see also the proof of [29, Corollary 4.8]. The first part follows immediately from Theorem 7.8 by setting $\frac{1}{\lambda} = 2\alpha + 1 - \delta$. For the second part we observe that $A < \infty$ and therefore for any positive η there exists a positive s_η such that

$$\sum_{j=1}^{s} \gamma_j \leq (A+\eta)\log s\,, \text{ for all } s \geq s_\eta\,.$$

Consequently

$$\begin{aligned} c_{s,\alpha,\gamma,2\alpha} &= \prod_{j=1}^{s}(1 + \frac{\gamma_j}{b^{2\alpha}-1}) \\ &= s^{\sum_{j=1}^{s} \frac{\log(1+\frac{\gamma_j}{b^{2\alpha}-1})}{\log s}} \\ &\leq s^{\frac{1}{b^{2\alpha}-1}\sum_{j=1}^{s}\frac{\gamma_j}{\log s}} \\ &\leq s^{\frac{A+\eta}{b^{2\alpha}-1}}, \end{aligned}$$

for any $\eta > 0$ and all $s \geq s_\eta$. Hence there exists a constant $\tilde{c}_\eta$ such that

$$c_{s,\alpha,\gamma,2\alpha} \leq \tilde{c}_\eta s^{\frac{A+\eta}{b^{2\alpha}-1}}$$

and the second assertion follows. □

7.5 Implementation of the component-by-component algorithm

In this section we show how to implement the CBC algorithm from Section 7.3. Our approach is based on [78; 79], but we simplify the algorithm using ideas from [24]. Following [78; 79], we obtain for $d \geq 2$

$$\begin{aligned} & B(\boldsymbol{q}, \alpha, \boldsymbol{\gamma}) \\ = \ & \frac{1}{b^m} \sum_{h=0}^{b^m-1} \prod_{j=1}^{d} \left(1 + \frac{b}{b-1}\gamma_j \phi(\boldsymbol{x}_{h,j}, \alpha)\right) - 1 \\ = \ & \frac{1}{b^m} \prod_{j=1}^{d} \left(1 + \frac{b}{b-1}\gamma_j \phi(\boldsymbol{x}_{0,j}, \alpha)\right) - 1 + \frac{1}{b^m} \sum_{h=1}^{b^m-1} \boldsymbol{p}_{d-1}(h) \left(1 + \frac{b}{b-1}\gamma_d \phi(\boldsymbol{x}_{h,d}, \alpha)\right), \end{aligned}$$

where we used Theorem 7.3 and

$$\boldsymbol{p}_{d-1}(h) = \prod_{j=1}^{d-1} \left(1 + \frac{b}{b-1}\gamma_j \phi(\boldsymbol{x}_{h,j}, \alpha)\right).$$

Let $\omega\left(\frac{\overline{h}\overline{q}_d}{p}\right) = \phi_\alpha(\boldsymbol{x}_{h,d})$, where $\overline{h}$ and $\overline{q}_d$ denote the polynomials associated with h and q_d and p denotes the polynomial $p = p(x) \in \mathbb{Z}_b[x]$. As in [78; 79], we now introduce the following matrix

$$\Omega_p = \left[\omega\left(\frac{\overline{h}\overline{q}}{p}\right)\right]_{\substack{q=1,\ldots,b^m-1 \\ h=1,\ldots,b^m-1}}, \tag{7.17}$$

i.e. rows are indexed by q and columns by h.

Let $\boldsymbol{p}_{d-1} = (\boldsymbol{p}_{d-1}(1), \ldots, \boldsymbol{p}_{d-1}(b^m-1))^\top$. Following [78; 79], we have an update rule for $\boldsymbol{p}_d$ given by

$$\boldsymbol{p}_d = \operatorname{diag}\left(\left(\mathbf{1}_{(b^m-1)\times(b^m-1)} + \frac{b}{b-1}\gamma_d \Omega_p\right) v_{q_d}\right) \boldsymbol{p}_{d-1},$$

where $\operatorname{diag}(\boldsymbol{x})$ denotes the diagonal matrix with the elements of $\boldsymbol{x}$ on its diagonal and zero elsewhere and where we use v_j to denote a selection vector with 1 in position j and 0 elsewhere.

We now use the notation $B_{d-1} = (B((\boldsymbol{q}_{d-1}, \overline{1}), \alpha, \boldsymbol{\gamma}), \ldots, B((\boldsymbol{q}_{d-1}, \overline{b^m-1}), \alpha, \boldsymbol{\gamma}))^\top$. Then

$$B_{d-1} = \left[-1 + \frac{1}{b^m} \prod_{j=1}^{d} \left(1 + \frac{b}{b-1}\gamma_j \phi(\boldsymbol{x}_{0,j}, \alpha)\right)\right] \mathbf{1}_{(b^m-1)\times 1}$$

$$+\frac{1}{b^m}\left(\mathbf{1}_{(b^m-1)\times(b^m-1)}+\frac{b}{b-1}\gamma_d\Omega_p\right)\boldsymbol{p}_{d-1}$$
$$= \left[-1+\frac{1}{b^m}\prod_{j=1}^{d}\left(1+\frac{b}{b-1}\gamma_j\phi(\boldsymbol{x}_{0,j},\alpha)\right)\right]\mathbf{1}_{(b^m-1)\times 1}$$
$$+\frac{1}{b^m}\sum_{h=1}^{b^m-1}\boldsymbol{p}_{d-1}(h)\mathbf{1}_{(b^m-1)\times 1}+\frac{1}{b^m}\frac{b}{b-1}\gamma_d\Omega_p\boldsymbol{p}_{d-1}.$$

In the next lemma we summarize an observation from [24]. Let

$$\Pi(g)=[\Pi_{k,l}]_{\substack{k=1,\dots,b^m-1\\ l=1,\dots,b^m-1}},$$

where

$$\Pi_{k,l}=\begin{cases}1 & \text{if } \overline{k}(x)\equiv g^l(x) \pmod{p},\\ 0 & \text{otherwise,}\end{cases} \tag{7.18}$$

and

$$\Pi(g^{-1})=\left[\Pi^{-1}_{k,l}\right]_{\substack{k=1,\dots,b^m-1\\ l=1,\dots,b^m-1}},$$

where

$$\Pi^{-1}_{k,l}=\begin{cases}1 & \text{if } \overline{k}(x)\equiv g^{-l}(x) \pmod{p},\\ 0 & \text{otherwise,}\end{cases} \tag{7.19}$$

be two permutation matrices, where g is a primitive element which generates all elements of $(\mathbb{Z}_b[x]/p)^* = \{g^0, g^1, \dots, g^{b^m-1}\}$; such an element g is known to exist since the multiplicative group of every finite field is cyclic. Also, let $t_k = \deg(g^k \pmod{p})$, $k=0,1,\dots,b^m-2$, and set

$$A_3=\left[b^{2\alpha t_{i-j} \pmod{b^m-1}}\right]_{\substack{i=1,\dots,b^m-1\\ j=1,\dots,b^m-1}} \tag{7.20}$$

and note that A_3 is a circulant matrix, which allows us to use Fast Fourier Transforms (FFTs) as in [78; 79]. We now state the lemma.

Lemma 7.10. *Let p be an irreducible polynomial, g a primitive element of $(\mathbb{Z}_b[x]/p)^*$ and $\Pi(g)$, $\Pi(g^{-1})$, A_3 and Ω_p defined as above. Then*

$$\Omega_p=\mathbf{1}_{(b^m-1)\times(b^m-1)}\frac{b-1}{b(b^{2\alpha}-1)}-\frac{b^{2\alpha+1}-1}{b(b^{2\alpha}-1)}b^{-2\alpha m}\Pi(g)A_3\Pi(g^{-1})^\top.$$

Proof. It follows from the definition of $\phi(x,\alpha)$ that

$$\phi\left(v_m\left(\frac{\overline{h}\overline{q}}{p}\right),\alpha\right)=\frac{b-1}{b(b^{2\alpha}-1)}-\frac{(b^{2\alpha+1}-1)b^{-2\alpha a_{0,h,q}}}{b(b^{2\alpha}-1)},$$

where $a_{0,h,q}$ denotes the smallest integer a so that $\xi_{h,q,a}\neq 0$ and where

$$v_m\left(\frac{\overline{h}\overline{q}}{q}\right)=\frac{\xi_{h,q,1}}{b}+\frac{\xi_{h,q,2}}{b^2}+\dots.$$

Hence

$$\Omega_p = \frac{b-1}{b(b^{2\alpha}-1)}\mathbf{1}_{(b^m-1)\times(b^m-1)} - \frac{b^{2\alpha+1}-1}{b(b^{2\alpha}-1)}A_1,$$

where

$$A_1 = \left[b^{-2\alpha a_{0,h,q}}\right]_{\substack{q=1,\dots,b^m-1\\ h=1,\dots,b^m-1}}.$$

Now assume that for $w \in \mathbb{Z}_b[x]$ we have

$$\frac{w(x)}{p(x)} = u_{1,w}x^{-1} + u_{2,w}x^{-2} + \dots, \tag{7.21}$$

where $u_{j,w} \in \mathbb{Z}_b$. Then

$$v_m\left(\frac{\overline{hq}}{p}\right) = u_{1,\overline{hq}}b^{-1} + u_{2,\overline{hq}}b^{-2} + \dots + u_{m,\overline{hq}}b^{-m},$$

hence $a_{0,h,q}$ is the smallest integer a so that $u_{a,\overline{hq}} \neq 0$, $\overline{h}, \overline{q} \in \mathbb{Z}_b[x]$ (note that $p \nmid \overline{h}, \overline{q}$).

The matrix A_2 given by

$$A_2 = \Pi^\top(g)A_1\Pi(g^{-1})$$

is circulant. Indeed it can be checked that

$$A_2 = \left[b^{-2\alpha a_{0,g^{-j},g^i}}\right]_{\substack{i=1,\dots,b^m-1\\ j=1,\dots,b^m-1}}, \tag{7.22}$$

where g^{-j} and g^i in Equation (7.22) denote the integers associated with the polynomials $g^{-j} \pmod p$ and $g^i \pmod p$. We let $a_{0,g^{-j},g^i} = r_{i-j}$ and note that $r_k = r_{k'}$ for $k \equiv k'$ $(\text{mod } b^m - 1)$ as $g^{b^m-1} \equiv 1 \pmod p$, hence

$$A_2 = \left[b^{-2\alpha r_{i-j}}\right]_{\substack{i=1,\dots,b^m-1\\ j=1,\dots,b^m-1}}.$$

The matrix A_2 is circulant and r_k is the smallest integer r such that $u_{r,g^k} \neq 0$, which implies using Equation (7.21) that

$$\deg(g^k \pmod p) = \deg(p) - r_k$$

and consequently

$$r_k = m - \deg(g^k \pmod p).$$

Now denoting

$$t_k = \deg(g^k \pmod p),$$

we get

$$A_2 = b^{-2\alpha m}A_3,$$

where A_3 is given by Equation (7.20), and the result follows. □

Algorithm 7.3 Fast CBC algorithm

Require: b a prime, $s, m \in \mathbb{N}$ and weights $\boldsymbol{\gamma} = (\gamma_j)_{j \geq 1}$.

Choose a primitive polynomial $p \in \mathbb{Z}_b[x]$ with $\deg(p) = m$ and choose $g(x) = x$.

$\boldsymbol{\mu}_0 := \mathbf{1}_{(b^m-1)\times 1}$.

for $d = 1$ to s **do**

$\tilde{B}_d = A_3 \boldsymbol{\mu}_{d-1}$.

$$w_d = \underset{w \in R_{b,m}}{\arg\max}\, \tilde{B}_d(w)\,.$$

$$\boldsymbol{\mu}_d = \operatorname{diag}\left(\mathbf{1}_{1\times(b^m-1)}\left(1 + \frac{\gamma_d}{(b^{2\alpha}-1)}\right) - \gamma_d \frac{(b^{2\alpha+1}-1)b^{-2\alpha m}}{(b-1)(b^{2\alpha}-1)} A_3(w_d,:)\right)\boldsymbol{\mu}_{d-1}\,.$$

end for

return $\boldsymbol{q} = (q_1, \ldots, q_s)$ by mapping back $(w_1, \ldots, w_s)$ using $\Pi(g)$.

Note that if the polynomial p in the lemma above is primitive, then one can choose the primitive element $g(x) = x$. In Algorithm 7.3 we show how to implement the CBC algorithm from Section 7.3. Several remarks regarding Algorithm 7.3 are in order.

Remark 7.11. *As in [78; 79] we search for the minimum in the permuted space, hence we minimize* $\tilde{B}_d = \Pi^\top(g)B_{d-1}$. *However, as in [78; 79] the component* q_d *can be found by mapping back* w_d *using* $\Pi(g)$.

Remark 7.12. *We have* $\boldsymbol{\mu}_d = \Pi^\top(g^{-1})\boldsymbol{p}_d$ *and consequently update* $\boldsymbol{\mu}_d$ *using* $\boldsymbol{\mu}_{d-1}$. *Hence we do not need to permute back and forth, but can complete the algorithm in the permuted space.*

The next corollary gives information on the computational complexity of Algorithm 7.3. We use the notation

$$\boldsymbol{a} = \begin{bmatrix} b^{2\alpha t_0} \\ b^{2\alpha t_1} \\ \vdots \\ b^{2\alpha t_{b^m-2}}\,. \end{bmatrix} \tag{7.23}$$

for the vector generating the circulant matrix A_3 in Lemma 7.10 and Algorithm 7.3.

Corollary 7.13. *Assume that the vector* $\boldsymbol{a}$ *in Equation* (7.23) *has been precomputed and stored using* $\mathcal{O}(b^m)$ *memory. Then Algorithm 7.3 can be completed in time* $\mathcal{O}(sb^m m)$ *and memory* $\mathcal{O}(b^m)$.

Proof. For a proof, see [78; 79] or also [35, Section 10.3]. ∎

7.6 Numerical experiments

In this section we numerically investigate the performance of the CBC algorithm presented in Section 7.3; we rely on Section 7.5 for the implementation of the algorithm. In Tables 7.1 - 7.3 we present values of $B(\boldsymbol{q}, \alpha, \gamma)$ for different choices of α and γ, where $\boldsymbol{q}$ is constructed using Algorithm 7.3.

We compare the performance of the CBC algorithm to the performance of qMC rules based on scrambled classical digital nets. As was done with scrambled classical polynomial lattice point sets in Section 7.2, we can study the variance of the estimator $Q_{b^m}(f, \mathcal{P}_\pi)$ given by Equation (7.6), consider the worst-case variance of multivariate integration in $V_{\alpha,s,\gamma}$ and bound this variance as follows:

$$\mathrm{Var}(Q_{b^m}(\cdot, \mathcal{P}_\pi(\boldsymbol{C})), V_{\alpha,s,\gamma}) \leq \sum_{\boldsymbol{k} \in \mathcal{D}'} r_{2\alpha+1,\gamma}(\boldsymbol{k}), \tag{7.24}$$

where $\boldsymbol{C} = (C_1, \ldots, C_s)$ are the generating matrices of the classical digital net under consideration and $\mathcal{D}' = \mathcal{D} \setminus \{\mathbf{0}\}$, where $\mathcal{D} = \mathcal{D}(C_1, \ldots, C_s)$ is the dual net of the classical digital net. We denote the bound given in Equation (7.24) by $B(\boldsymbol{C}, \alpha, \gamma)$ and remark that $B(\boldsymbol{C}, \alpha, \gamma)$ can also be computed using Equation (7.10), where $\{\boldsymbol{x}_h\}_{h=0}^{b^m-1}$ is the classical digital net generated by $\boldsymbol{C} = (C_1, \ldots, C_s)$.

Consequently we compare the values of $B(\boldsymbol{q}, \alpha, \gamma)$ to the values of $B(\boldsymbol{C}, \alpha, \gamma)$ in Tables 7.1 - 7.3; in each cell the top number corresponds to the CBC construction and the bottom one to the classical digital net. We choose the following classical digital nets: for $s = 1$ we simply choose equidistributed points, $x_h = \frac{h}{b^m}$, $h = 0, \ldots, b^m - 1$, for $s = 5$ we use Pirsic's implementation of Niederreiter-Xing points, [88], and for $s = 50$ and $s = 100$ we use Sobol' points as constructed in [48]; we point out that for the CBC construction we choose $b = 2$ and likewise the classical digital nets under consideration are classical digital nets over $\mathbb{Z}_2$.

The following conclusions can be derived from the tables: for $s = 1$, as expected, we obtain the optimal rate of convergence, $2^{-(2\alpha+1)m}$, and observe the same values for the CBC construction as for the classical digital nets. Regarding the case $s = 5$, the values are comparable, however, the Niederreiter-Xing construction seems to be slightly better than the CBC construction for the examples considered. Finally for $s = 50$ and $s = 100$, the performances of the two methods are again comparable, however, this time the CBC construction seems to outperform the classical digital nets.

	$\alpha = 0.5$				$\alpha = 1$			
$m =$	$s = 1$	$s = 5$	$s = 50$	$s = 100$	$s = 1$	$s = 5$	$s = 50$	$s = 100$
4	3.91e-03	1.46e+00	7.04e+13	7.92e+28	8.14e-05	4.37e-02	1.10e+05	1.95e+11
	3.91e-03	1.48e+00	7.04e+13	7.92e+28	8.14e-05	4.90e-02	1.10e+05	1.95e+11
5	9.77e-04	6.16e-01	3.52e+13	3.96e+28	1.02e-05	1.09e-02	5.52e+04	9.74e+10
	9.77e-04	6.34e-01	3.52e+13	3.96e+28	1.02e-05	1.32e-02	5.52e+04	9.74e+10
6	2.44e-04	2.66e-01	1.76e+13	1.98e+28	1.27e-06	3.45e-03	2.76e+04	4.87e+10
	2.44e-04	2.61e-01	1.76e+13	1.98e+28	1.27e-06	3.17e-03	2.76e+04	4.87e+10
7	6.10e-05	1.08e-01	8.80e+12	9.90e+27	1.59e-07	9.05e-04	1.38e+04	2.44e+10
	6.10e-05	1.04e-01	8.80e+12	9.90e+27	1.59e-07	7.19e-04	1.38e+04	2.44e+10
8	1.53e-05	4.24e-02	4.40e+12	4.95e+27	1.99e-08	2.36e-04	6.90e+03	1.22e+10
	1.53e-05	3.93e-02	4.40e+12	4.95e+27	1.99e-08	1.48e-04	6.90e+03	1.22e+10
9	3.81e-06	1.74e-02	2.20e+12	2.48e+27	2.48e-09	6.10e-05	3.45e+03	6.09e+09
	3.81e-06	1.44e-02	2.20e+12	2.48e+27	2.48e-09	2.86e-05	3.45e+03	6.09e+09
10	9.54e-07	6.41e-03	1.10e+12	1.24e+27	3.10e-10	1.29e-05	1.72e+03	3.04e+09
	9.54e-07	5.21e-03	1.10e+12	1.24e+27	3.10e-10	5.56e-06	1.72e+03	3.04e+09
11	2.38e-07	2.29e-03	5.50e+11	6.19e+26	3.88e-11	2.56e-06	8.62e+02	1.52e+09
	2.38e-07	1.82e-03	5.50e+11	6.19e+26	3.88e-11	1.01e-06	8.62e+02	1.52e+09
12	5.96e-08	8.39e-04	2.75e+11	3.09e+26	4.85e-12	5.03e-07	4.31e+02	7.61e+08
	5.96e-08	6.17e-04	2.75e+11	3.09e+26	4.85e-12	1.78e-07	4.31e+02	7.61e+08
13	1.49e-08	3.09e-04	1.37e+11	1.55e+26	6.06e-13	1.05e-07	2.15e+02	3.81e+08
	1.49e-08	2.06e-04	1.37e+11	1.55e+26	6.06e-13	3.07e-08	2.16e+02	3.81e+08
14	3.73e-09	1.12e-04	6.87e+10	7.74e+25	7.58e-14	2.56e-08	1.08e+02	1.90e+08
	3.73e-09	6.76e-05	6.87e+10	7.74e+25	7.57e-14	5.17e-09	1.08e+02	1.90e+08
15	9.31e-10	3.66e-05	3.44e+10	3.87e+25	9.27e-15	4.98e-09	5.38e+01	9.52e+07
	9.31e-10	2.18e-05	3.44e+10	3.87e+25	9.33e-15	8.54e-10	5.39e+01	9.52e+07
16	2.33e-10	1.29e-05	1.72e+10	1.93e+25	1.22e-15	8.92e-10	2.69e+01	4.76e+07
	2.33e-10	6.94e-06	1.72e+10	1.93e+25	1.11e-15	1.38e-10	2.69e+01	4.76e+07

Table 7.1. *Values of $B(\boldsymbol{q}, \alpha, \boldsymbol{\gamma})$ and $B(\boldsymbol{C}, \alpha, \boldsymbol{\gamma})$ for $\gamma_j = 1$, $j = 1, \ldots, s$, and $\boldsymbol{q}$ constructed using the CBC algorithm; the top number gives the value of $B(\boldsymbol{q}, \alpha, \boldsymbol{\gamma})$, the bottom the value of $B(\boldsymbol{C}, \alpha, \boldsymbol{\gamma})$.*

	$\alpha = 0.5$				$\alpha = 1$			
$m =$	$s = 1$	$s = 5$	$s = 50$	$s = 100$	$s = 1$	$s = 5$	$s = 50$	$s = 100$
4	3.42e-03	4.64e-01	2.03e+01	2.04e+01	7.12e-05	1.47e-02	2.82e-01	2.83e-01
	3.42e-03	4.84e-01	2.04e+01	2.06e+01	7.12e-05	1.83e-02	3.46e-01	3.48e-01
5	8.54e-04	1.87e-01	9.99e+00	1.01e+01	8.90e-06	3.54e-03	1.16e-01	1.17e-01
	8.54e-04	1.95e-01	1.01e+01	1.01e+01	8.90e-06	4.45e-03	1.38e-01	1.39e-01
6	2.14e-04	7.75e-02	4.91e+00	4.95e+00	1.11e-06	1.04e-03	4.78e-02	4.82e-02
	2.14e-04	7.46e-02	4.94e+00	4.98e+00	1.11e-06	9.29e-04	5.30e-02	5.34e-02
7	5.34e-05	2.96e-02	2.40e+00	2.42e+00	1.39e-07	2.54e-04	1.85e-02	1.87e-02
	5.34e-05	2.80e-02	2.44e+00	2.47e+00	1.39e-07	1.97e-04	2.39e-02	2.41e-02
8	1.34e-05	1.17e-02	1.17e+00	1.18e+00	1.74e-08	5.77e-05	7.39e-03	7.45e-03
	1.34e-05	1.01e-02	1.20e+00	1.21e+00	1.74e-08	3.79e-05	1.08e-02	1.09e-02
9	3.34e-06	4.43e-03	5.66e-01	5.71e-01	2.17e-09	1.29e-05	2.84e-03	2.87e-03
	3.34e-06	3.54e-03	5.89e-01	5.95e-01	2.17e-09	6.98e-06	4.94e-03	4.97e-03
10	8.34e-07	1.56e-03	2.72e-01	2.75e-01	2.72e-10	3.03e-06	1.08e-03	1.09e-03
	8.34e-07	1.22e-03	2.88e-01	2.90e-01	2.72e-10	1.28e-06	2.24e-03	2.26e-03
11	2.09e-07	5.45e-04	1.30e-01	1.31e-01	3.40e-11	6.24e-07	4.01e-04	4.06e-04
	2.09e-07	4.10e-04	1.42e-01	1.43e-01	3.40e-11	2.22e-07	9.58e-04	9.66e-04
12	5.22e-08	1.93e-04	6.20e-02	6.26e-02	4.24e-12	1.16e-07	1.49e-04	1.51e-04
	5.22e-08	1.35e-04	6.73e-02	6.80e-02	4.24e-12	3.79e-08	3.59e-04	3.64e-04
13	1.30e-08	7.07e-05	2.94e-02	2.97e-02	5.31e-13	2.48e-08	5.43e-05	5.51e-05
	1.30e-08	4.38e-05	3.33e-02	3.36e-02	5.30e-13	6.34e-09	2.11e-04	2.13e-04
14	3.26e-09	2.27e-05	1.39e-02	1.40e-02	6.62e-14	4.56e-09	1.99e-05	2.02e-05
	3.26e-09	1.40e-05	1.58e-02	1.60e-02	6.62e-14	1.04e-09	8.23e-05	8.31e-05
15	8.15e-10	8.01e-06	6.49e-03	6.57e-03	8.49e-15	1.10e-09	7.15e-06	7.26e-06
	8.15e-10	4.41e-06	7.70e-03	7.78e-03	8.22e-15	1.67e-10	4.44e-05	4.48e-05
16	2.04e-10	2.68e-06	3.02e-03	3.06e-03	9.99e-16	1.81e-10	2.57e-06	2.61e-06
	2.04e-10	1.37e-06	3.76e-03	3.80e-03	8.88e-16	2.64e-11	2.14e-05	2.15e-05

Table 7.2. *Values of $B(\boldsymbol{q},\alpha,\gamma)$ and $B(\boldsymbol{C},\alpha,\gamma)$ for $\gamma_j = 0.875^j$, $j = 1,\ldots,s$, and $\boldsymbol{q}$ constructed using the CBC algorithm; the top number gives the value of $B(\boldsymbol{q},\alpha,\gamma)$, the bottom the value of $B(\boldsymbol{C},\alpha,\gamma)$.*

	$\alpha = 0.5$				$\alpha = 1$			
$m =$	$s = 1$	$s = 5$	$s = 50$	$s = 100$	$s = 1$	$s = 5$	$s = 50$	$s = 100$
4	3.91e-03	2.75e-02	4.75e-02	4.90e-02	8.14e-05	7.68e-04	1.80e-03	1.88e-03
	3.91e-03	3.20e-02	5.97e-02	6.17e-02	8.14e-05	1.27e-03	4.40e-03	4.62e-03
5	9.77e-04	8.98e-03	1.78e-02	1.84e-02	1.02e-05	1.48e-04	4.88e-04	5.20e-04
	9.77e-04	1.25e-02	2.22e-02	2.30e-02	1.02e-05	3.13e-04	1.23e-03	1.30e-03
6	2.44e-04	2.95e-03	6.29e-03	6.56e-03	1.27e-06	3.37e-05	1.20e-04	1.31e-04
	2.44e-04	3.20e-03	7.23e-03	7.66e-03	1.27e-06	3.62e-05	2.47e-04	2.89e-04
7	6.10e-05	8.96e-04	2.22e-03	2.35e-03	1.59e-07	5.05e-06	2.91e-05	3.31e-05
	6.10e-05	1.11e-03	2.65e-03	2.88e-03	1.59e-07	8.38e-06	6.91e-05	9.12e-05
8	1.53e-05	2.96e-04	7.86e-04	8.36e-04	1.99e-08	1.04e-06	6.94e-06	8.16e-06
	1.53e-05	3.44e-04	1.00e-03	1.12e-03	1.99e-08	1.37e-06	2.16e-05	3.52e-05
9	3.81e-06	9.34e-05	2.81e-04	3.02e-04	2.48e-09	1.90e-07	1.67e-06	2.02e-06
	3.81e-06	9.15e-05	3.27e-04	3.64e-04	2.48e-09	1.60e-07	4.70e-06	6.18e-06
10	9.54e-07	2.78e-05	9.54e-05	1.03e-04	3.10e-10	4.07e-08	4.20e-07	5.14e-07
	9.54e-07	2.69e-05	1.19e-04	1.32e-04	3.10e-10	2.49e-08	1.83e-06	2.37e-06
11	2.38e-07	8.95e-06	3.34e-05	3.64e-05	3.88e-11	6.34e-09	9.59e-08	1.21e-07
	2.38e-07	8.25e-06	4.10e-05	4.71e-05	3.88e-11	3.76e-09	3.25e-07	5.44e-07
12	5.96e-08	2.68e-06	1.18e-05	1.30e-05	4.85e-12	1.30e-09	2.36e-08	3.03e-08
	5.96e-08	2.54e-06	1.46e-05	1.68e-05	4.85e-12	6.34e-10	1.16e-07	1.64e-07
13	1.49e-08	8.29e-07	4.05e-06	4.50e-06	6.06e-13	2.04e-10	5.79e-09	7.61e-09
	1.49e-08	7.08e-07	4.69e-06	5.76e-06	6.06e-13	9.21e-11	2.45e-08	5.16e-08
14	3.73e-09	2.50e-07	1.40e-06	1.57e-06	7.58e-14	4.11e-11	1.40e-09	1.88e-09
	3.73e-09	1.97e-07	1.60e-06	1.98e-06	7.57e-14	1.29e-11	7.04e-09	2.04e-08
15	9.31e-10	7.70e-08	4.91e-07	5.54e-07	9.27e-15	6.15e-12	3.45e-10	4.79e-10
	9.31e-10	5.59e-08	5.50e-07	7.09e-07	9.33e-15	1.74e-12	2.52e-09	4.52e-09
16	2.33e-10	2.34e-08	1.71e-07	1.95e-07	1.22e-15	1.00e-12	8.51e-11	1.22e-10
	2.33e-10	1.69e-08	1.89e-07	2.52e-07	1.11e-15	2.70e-13	9.54e-10	1.45e-09

Table 7.3. *Values of $B(\boldsymbol{q},\alpha,\boldsymbol{\gamma})$ and $B(\boldsymbol{C},\alpha,\boldsymbol{\gamma})$ for $\gamma_j = j^{-2}$, $j = 1,\ldots,s$, and $\boldsymbol{q}$ constructed using the CBC algorithm; the top number gives the value of $B(\boldsymbol{q},\alpha,\boldsymbol{\gamma})$, the bottom the value of $B(\boldsymbol{C},\alpha,\boldsymbol{\gamma})$.*

7.7 Conclusion and future work

In this chapter we used the CBC and the Korobov construction to produce classical polynomial lattice point sets which enjoy the following property: for functions of bounded variation of order α, the variance of qMC rules obtained by applying the scrambling algorithm to these classical polynomial lattice point sets decreases at a rate of $N^{-(1+2\alpha)+\delta}$, for all $\delta > 0$. This rate is optimal up to powers of a $\log N$ factor for a large class of randomized algorithms including adaptive ones. Furthermore we discuss the implementation of the CBC algorithm and numerically compare the performance of qMC rules based on scrambled classical polynomial lattice point sets to qMC rules based on scrambled classical digital nets and find that the former outperform the latter for high-dimensional problems.

In future it would be interesting to apply qMC rules based on these scrambled classical polynomial lattice point sets to practical problems. Integrands satisfying the regularity condition assumed in this chapter should be considered, but it would also be very instructive to study integrands which do not, as this could result in new insights on scrambled classical polynomial lattice point sets.

Chapter Eight

Quasi-Monte Carlo rules for the Kou model

Before discussing qMC rules for the Kou model, we firstly motivate the topic.

8.1 Motivation

The aim of this chapter is to demonstrate how qMC rules can be applied to a problem of practical interest, in this case a problem from financial mathematics. Though concerned with numerical integration, this chapter is of a different flavor than Chapters 5, 6 and 7: the latter study theoretical properties of numerical integration problems, whereas the aim of this chapter is to show that the formulation of the integration problem can be non-trivial and having formulated the integration problem, it can be possible to reformulate the problem to make it more amenable to quasi-Monte Carlo rules. We now introduce the problem studied in this chapter: the pricing of financial derivatives in the Kou model, see [50; 51], using qMC rules. QMC rules have been applied successfully to Gaussian models, e.g. [15], and also to some pure-jump Lévy processes, e.g. [2; 42; 57; 60; 90]. Regarding jump-diffusion processes, we refer to [3; 4] and remark that this chapter is based on [4]. Jump-diffusion processes, such as the one underlying the Kou model, can be considered to be interesting models in their own right, but they are also an important class of stochastic processes for finance, as they are used to approximate Lévy processes that cannot be simulated directly, see e.g. [1] for general results and [89] for an application to finance. These considerations highlight the need to develop good qMC rules for the Kou model and jump-diffusion processes.

QMC rules are numerical techniques for high-dimensional integrals, so one needs to formulate the finance problem as an integration problem in order to be able to apply them. We show how to formulate the finance problems pertaining to the Poisson process, compound Poisson process and jump-diffusion process underlying the Kou model as integration problems; this is the first contribution of the chapter. Secondly, for qMC rules, it is desirable to allocate the early dimensions to important variables. For the integration problems relating to the Poisson, compound Poisson and jump-diffusion processes underlying the Kou model, we introduce approaches designed to pack more variance into the opening dimensions than an increment-by-increment approach. We find that for the problems under consideration, these approaches outperform the increment-by-increment approach and we also conclude that qMC rules achieve lower standard errors than Monte Carlo (MC) methods. These findings agree with those from the Gaussian models and the pure-jump Lévy processes.

This chapter is structured as follows: in Section 8.2, we recall properties of qMC rules and the Kou model relevant for this chapter, and we introduce the financial derivative we are interested in for purposes of this chapter, an arithmetic average Asian call option. We find it convenient to introduce ideas in an simple setting first and consequently to extend these ideas to more complicated settings culminating in the Kou model. In Section 8.3, we simplify the setting by considering a compound Poisson process with fixed jump sizes instead of the jump-diffusion underlying the Kou model. The resulting finance problem corresponds to an integration problem and does not need to be reformulated. In Section 8.4, we consider the compound Poisson process underlying the Kou model. We need to reformulate the resulting finance problem in order to obtain an integration problem. In Section 8.5, we extend the reformulated problem from Section 8.4 to deal with the jump-diffusion driving the stock price in the Kou model. Numerical results for the approaches introduced are presented in the relevant sections and Section 8.6 concludes the chapter.

8.2 Quasi-Monte Carlo rules, Kou model, and Asian options

We firstly recall properties of qMC rules relevant to this chapter.

8.2.1 Quasi-Monte Carlo rules for applications

QMC rules were introduced as a numerical technique to approximate high-dimensional integrals. This is in sharp contrast with MC methods, which are used for simulation. To apply MC methods, one needs to formulate the problem under consideration in terms of the variates to be simulated, to apply qMC rules, one needs to formulate the problem in terms of an integral.

A concept crucial in the explanation of the success of qMC rules, when applied to problems of practical interest, is the concept of effective dimension. In order to recall this concept, we firstly introduce the analysis of variance (ANOVA) decomposition.

Definition 8.1. *Let* $f : [0,1]^s \to \mathbb{R}$, $f_{\mathrm{u}}(\boldsymbol{x}_{\mathrm{u}}) = \int_{[0,1]^{s-|\mathrm{u}|}} f(\boldsymbol{x}) d\boldsymbol{x}_{[s]\setminus \mathrm{u}}$ *and* $\mathrm{Var}(f_{\mathrm{u}}) = \int_{[0,1]^{|\mathrm{u}|}} (f_{\mathrm{u}}(\boldsymbol{x}_{\mathrm{u}}) - \overline{f}_{\mathrm{u}})^2 d\boldsymbol{x}_{\mathrm{u}}$, *where* $\overline{f}_{\mathrm{u}} = \int_{[0,1]^{|\mathrm{u}|}} f_{\mathrm{u}}(\boldsymbol{x}_{\mathrm{u}}) d\boldsymbol{x}_{\mathrm{u}}$, $\mathrm{u} \subseteq \{1, 2, \ldots, s\}$. *Then*

i) the following decomposition, referred to as ANOVA decomposition, holds

$$f(\boldsymbol{x}) = \sum_{\mathrm{u} \subseteq \{1,\ldots,s\}} f_{\mathrm{u}}(\boldsymbol{x}_{\mathrm{u}}) \,. \tag{8.1}$$

ii) The decomposition given in Equation (8.1) *is an orthogonal one, i.e.*

$$\mathrm{Var}(f) = \sum_{\mathrm{u} \subseteq \{1,\ldots,s\}} \mathrm{Var}(f_{\mathrm{u}}) \,.$$

We can now define the concept of effective dimension, where we follow [15].

Definition 8.2. *We distinguish between two forms of effective dimension:*

i) The effective dimension of f, in the superposition sense, is the smallest integer d_S such that

$$\sum_{0 < |\mathrm{u}| \le d_S} \mathrm{Var}(f_{\mathrm{u}}) \ge 0.99 \mathrm{Var}(f) \,.$$

ii) The effective dimension of f, in the truncation sense, is the smallest integer d_T such that

$$\sum_{\mathrm{u} \subseteq \{1,\ldots,d_T\}} \mathrm{Var}(f_{\mathrm{u}}) \ge 0.99 \mathrm{Var}(f) \,.$$

The following two remarks regarding Definition 8.2 are in order.

Remark 8.3. *The threshold* 0.99 *is arbitrary, in different settings other values might be preferred.*

Remark 8.4. *QMC rules can be expected to perform well if the effective dimension, in the superposition and truncation sense, is low.*

For purposes of this chapter, the function f in Definitions 8.1 and 8.2 corresponds to the discounted pay-off function of the financial derivative under consideration, i.e. the effective dimension depends on the variance of the discounted pay-off of the financial derivative under consideration. To reduce the effective dimension, following [15], we allocate the early dimensions to variates with high variances. We note that in the theory of qMC rules it is variation rather than variance that is important. However, variance is more easily computed than variation and hence often used as a proxy. Of course this proxy can be unsatisfactory, see e.g. [84].

In order to apply qMC rules, it is necessary to formulate the problem as an integration problem over the unit cube. Consequently, one needs to be able to invert the distributions of the relevant random variables. For this purpose, we introduce the well-known generalized inverse function, see e.g. [40, p. 55].

Definition 8.5. *Let Y be a random variable following distribution F parameterized by $\boldsymbol{x}$, where $\boldsymbol{x}$ can be a vector. Then the generalized inverse function is given by*

$$F^{-1}(u, \boldsymbol{x}) = \inf\{z \in \mathbb{R} : F(z) \geq u\}, \quad u \in [0,1]. \tag{8.2}$$

We point out that all generalized inverse functions needed for this chapter are available in standard computer packages such as MATLAB.

Regarding the qMC point sets, we will make use of classical digital nets, see Definition 2.5. Given a classical digital net over $\mathbb{Z}_b$, $\mathcal{P} = \{\boldsymbol{x}_h\}_{h=0}^{b^m-1}$, $\boldsymbol{x}_h \in [0,1)^s$, $s \in \mathbb{N}$, we will always randomize the points using a digital shift in the same base b, see Section 2.6.1, to compute standard errors. Hence, given q independent vectors $\boldsymbol{\Delta}_j$, $j = 1, \ldots, q$, uniformly distributed in $[0,1]^s$ and denoting the discounted pay-off of the financial derivative by f, $f : [0,1]^s \to \mathbb{R}$, we obtain the point sets $\mathcal{P}_{\boldsymbol{\Delta}_j} = \left\{\boldsymbol{x}_h \oplus \boldsymbol{\Delta}_j\right\}_{h=0}^{b^m-1}$, $j = 1, \ldots, q$, and estimate the price of the derivative using

$$I_{QMC} = \frac{1}{q}\sum_{j=1}^{q} Q_{b^m}(f, \mathcal{P}_{\boldsymbol{\Delta}_j}) = \frac{1}{q}\sum_{j=1}^{q}\frac{1}{b^m}\sum_{h=0}^{b^m-1} f(\boldsymbol{x}_h \oplus \boldsymbol{\Delta}_j), \tag{8.3}$$

where as in Chapters 2, 3, 4 and 5, $\oplus$ means that for each dimension we perform the digit-wise addition modulo b. Standard errors are computed via

$$\sigma_{QMC} = \sqrt{\frac{\sum_{j=1}^{q}(Q_{b^m}(f, \mathcal{P}_{\boldsymbol{\Delta}_j}) - I_{QMC})^2}{q(q-1)}}. \tag{8.4}$$

It is noted that we could have chosen other qMC point sets, such as integration lattices, e.g. [66; 97], too.

For purposes of comparison, we will also look at MC estimators. In this case, we will choose qn points, $\{\boldsymbol{u}_h\}_{h=0}^{qn-1}$, independent and uniformly distributed in $[0,1]^s$, estimate prices of derivatives using

$$I_{MC} = \frac{1}{qn}\sum_{h=0}^{qn-1} f(\boldsymbol{u}_h) \tag{8.5}$$

and compute standard errors using

$$\sigma_{MC} = \sqrt{\frac{\sum_{h=0}^{qn-1}(f(\boldsymbol{u}_h) - I_{MC})^2}{qn(qn-1)}}. \tag{8.6}$$

(Throughout this chapter, we use n to denote the number of quadrature points used, as N is used to denote the Poisson process, see Subsection 8.2.2.) We conclude this subsection by commenting on the variances of I_{QMC} and I_{MC}. For square-integrable functions f, it is well-known that

$$\mathrm{Var}(I_{MC}) = \frac{\mathrm{Var}(f(\boldsymbol{u}_h))}{qn}.$$

Regarding I_{QMC}, if f is square-integrable, an expression for $\mathrm{Var}(I_{QMC})$ in terms of the Walsh coefficients of f is available, see [59] and compare this to Lemma 2.50. However, in general, one cannot draw conclusions regarding convergence rates of variances from this expression. In order to obtain results concerning convergence rates, one needs to make stronger assumptions about the smoothness of f. In particular, if the mixed partial derivatives of f satisfy a Lipschitz condition, then one can obtain a variance bound in $\mathcal{O}(n^{-2}(\log(n))^d)$, see [59] and compare this to Theorem 2.53. We remark that the required smoothness conditions are not satisfied by the functions we deal with in this chapter. For this reason, it is important to be able to investigate standard errors of qMC rules numerically.

8.2.2 Properties of the Kou model

In this subsection we recall some basic properties of the Kou model, see [50; 51]. We assume that we deal with a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which we introduce the Kou jump-diffusion process in Definition 8.6. The following density function f is also used in Definition 8.6:

$$f(x) = p\eta^+ \exp(-\eta^+ x)\mathbf{1}_{x\geq 0} + (1-p)\eta^- \exp(\eta^- x)\mathbf{1}_{x<0}, \quad \forall x \in \mathbb{R}, \tag{8.7}$$

where $0 \leq p \leq 1$, $\eta^+ > 1$, $\eta^- > 0$.

Definition 8.6. *Let $(N_t)_{t\geq 0}$ be a Poisson process with intensity $\lambda > 0$, $(W_t)_{t\geq 0}$ a Brownian motion and $(Y_k)_{k\geq 1}$ independent, identically distributed random variables with density f given by Equation* (8.7)*, $\sigma \in \mathbb{R}^+$, $r \in \mathbb{R}^+$ and*

$$\mu = r - \frac{1}{2}\sigma^2 - \lambda \int_{\mathbb{R}} (\exp(x) - 1) f(x) dx. \tag{8.8}$$

Then

$$Z_t = \mu t + \sigma W_t + \sum_{k=1}^{N_t} Y_k \tag{8.9}$$

is the Kou jump-diffusion process.

We point out that N_t denotes the number of jumps up to time t, while the Y_k represent the jumps. Furthermore, following [50], we remark that the requirement $\eta^+ > 1$ is needed to ensure that $\mathbb{E}\left[e^{Y_k}\right] < +\infty$ and $\mathbb{E}\left[e^{Z_t}\right] < +\infty$; this can be interpreted to mean that the average upward jump cannot exceed 100%.

Remark 8.7. *Let $\nu(dx) = \lambda f(x)dx$ and $a = \int_{\mathbb{R}} x\mathbf{1}_{|x|\leq 1}\nu(dx)$, then $(Z_t)_{t\geq 0}$ is a Lévy process with characteristic triplet $(\mu + a, \sigma^2, \nu)$. It hence follows, see e.g. [17], that $(Z_t)_{t\geq 0}$ enjoys the* *independent increment and stationary increment properties. For more information on filtrations and martingales, the interested reader is referred to [17].*

We are now in a position to introduce the stock price process on $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 8.8. *For any $S_0 \in \mathbb{R}^+$, we set*

$$S_t = S_0 \exp(Z_t)\,, \tag{8.10}$$

where $(Z_t)_{t\geq 0}$ is given by Equation (8.9). Then $(S_t)_{t\geq 0}$ is the stock price process.

Remark 8.9. *Let us denote by $\mathbb{F}^S$ the filtration generated by the process S. It is easy to see that the discounted stock price process $\exp(-rt)S_t$ is a $(\mathbb{P}, \mathbb{F}^S)$-martingale.*

Remark 8.9 suggests the following convention: in what follows, we assume that $\mathbb{P}$ is the probability measure under which option prices are computed. This assumption results in the following definition.

Definition 8.10. *The price at time $t \in [0, T]$ of a possibly path-dependent European option, with pay-off $\psi_S(T)$ at maturity T, is given by*

$$\psi_S(t) = \exp\left(-r(T-t)\right) \mathbb{E}\left[\psi_S(T)|\mathcal{F}_t^S\right]\,. \tag{8.11}$$

Remark 8.11. *For background on how to derive a pricing probability measure $\mathbb{P}$ starting from a real-world probability measure, the interested reader is referred to [50], where Definition 8.10 appears as Proposition 1.*

8.2.3 Asian options

For purposes of this chapter, arithmetic average Asian call options will be used to test the various algorithms proposed in Sections 8.3, 8.4 and 8.5. We consider a fixed time grid $(T_l)_{l=0}^{M}$ and S_{T_l} denotes the value of the share price at time T_l. The pay-off of the arithmetic average Asian call option at maturity $T = T_M$ is given by

$$\left(\sum_{l=1}^{M} \frac{S_{T_l}}{M} - K\right)^{+}\,, \tag{8.12}$$

where M denotes the number of observation dates of the share price used to compute the arithmetic average in Equation (8.12).

8.3 The integration problem for the Poisson process

Instead of dealing with the stock price process given by Equation (8.10), we initially simplify the problem by setting $\sigma = 0$ and $Y_k = c$, $k \in \mathbb{N}$, in Equation (8.9), so that Equation (8.10) simplifies to

$$S_t = S_0 \exp((r - \lambda\zeta)t + cN_t)\,, \tag{8.13}$$

where $\zeta = \exp(c) - 1$. Similar to Remark 8.9, we note that discounting the stock price given by Equation (8.13) yields a martingale. It is clear that the difficulty in integrating functions of the stock price process given by Equation (8.13) pertains to the Poisson process $(N_t)_{t\geq 0}$.

We stipulate throughout this chapter that $t_i = \frac{i}{s}$, $i = 0, \ldots, s$, where $s = 2^d$ for some fixed $d \in \mathbb{N}_0$, denote $I = (t_i)_{i=0}^{s}$ and $J = \{0, 2, \ldots, 2^d\} \subset 2\mathbb{N}_0$ corresponding to the even indices of I. Clearly $t_s = 1$. In this section, we present two approaches for integrating functions of $(N_{t_i})_{i=1}^{s}$ (of course, $N_{t_0} = N_0 = 0$).

8.3.1 Auxiliary results for the Poisson process

Let us first recall some well-known properties of the Poisson process, which will be used when formulating the integration problem.

Lemma 8.12. *Let $(N_t)_{t\geq 0}$ be a Poisson process. Then*

i) *$(N_t)_{t\geq 0}$ is a Lévy process.*

ii) *For any $t > 0$, N_t follows a Poisson distribution with parameter λt. We write $N_t \sim P(\lambda t)$, meaning that for every $k \in \mathbb{N}_0$*

$$\mathbb{P}\left(N_t = k\right) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}\,.$$

iii) *For every $s < t$ and $k \in \mathbb{N}_0$,*

$$\mathbb{P}\left(N_t - N_s = k | N_s\right) = \mathbb{P}\left(N_t - N_s = k\right) = \mathbb{P}\left(N_{t-s} = k\right)\,.$$

iv) *Let $s < u < t$ and $k_1 < k_2$. Then conditional on $N_s = k_1$ and $N_t = k_2$ the increment $N_u - N_s$ has the binomial distribution with parameters $k_2 - k_1$ and $\frac{u-s}{t-s}$, denoted as $Bin(k_2 - k_1, \frac{u-s}{t-s})$.*

Part (iv) of Lemma 8.12 can be seen as an analogue of the Brownian bridge formula for the Poisson process. The Brownian bridge formula has been successfully applied to qMC problems in the area of finance, see e.g. [15], and the concept has been generalized to the case of the variance-gamma model, see e.g. [2; 90]. The usefulness of this formula in the context of integration problems dealing with Poisson processes is also mentioned in [38].

The following lemma and corollary will be used for Fox's greedy rule in Subsection 8.3.3.

Lemma 8.13. *Let $(N_t)_{t\geq 0}$ be a Poisson process and $U_{jk} = \{t_j, t_{j+1}, \ldots, t_{k-1}, t_k\}$, $j,k \in J$, $j<k$. Then*

$$\max_{t\in U_{jk}} \mathbb{E}\left[\mathrm{Var}(N_t|N_{t_j}, N_{t_k})\right] = \mathbb{E}\left[\mathrm{Var}\Big(N_{\frac{t_j+t_k}{2}}|N_{t_j}, N_{t_k}\Big)\right] = \frac{\lambda}{4}(t_k - t_j)\,.$$

Proof. It follows from Lemma 8.12 that conditional on N_{t_j} and N_{t_k}, the increment $N_t - N_{t_j}$, $t_j < t < t_k$, follows a binomial distribution with parameters $(N_{t_k} - N_{t_j})$ and $\frac{t-t_j}{t_k-t_j}$. This implies that the variance is maximized for $t = \frac{t_j+t_k}{2}$, i.e.

$$\max_{t\in U_{jk}} \mathrm{Var}(N_t|N_{t_j}, N_{t_k}) = \mathrm{Var}(N_{\frac{t_j+t_k}{2}}|N_{t_j}, N_{t_k}) = \frac{1}{4}(N_{t_k} - N_{t_j})\,.$$

The result now follows immediately from the properties of the Poisson process. □

The next result is a corollary to Lemma 8.13.

Corollary 8.14. *Let $A = \{u_0, u_1, \ldots, u_l\}$, where $u_0 = 0 < u_1 < \cdots < u_{l-1} < u_l = 1$ and $A \subset \{t_i : i \in J\}$. Let $B = I \setminus A$ be non-empty. Then*

$$\max_{t\in B} \mathbb{E}\left[\mathrm{Var}\left(N_t|N_{u_0}, \ldots, N_{u_l}\right)\right] = \frac{\lambda}{4}\max(u_1 - u_0, \ldots, u_l - u_{l-1})\,.$$

Proof. The proof follows from the independent increment property of the Poisson process N and Lemma 8.13. □

Using the same notation as in Corollary 8.14, we introduce the map $t_F^* = t_F^*(u_1, \ldots, u_l)$ by setting

$$t_F^* = \arg\max_{t\in B} \mathbb{E}\left[\mathrm{Var}\left(N_t|N_{u_1} = n_{u_1}, N_{u_2} = n_{u_2}, \ldots, N_{u_l} = n_{u_l}\right)\right]\,. \tag{8.14}$$

From Corollary 8.14, we see that $t_F^* = \frac{u_{k^*}+u_{k^*+1}}{2}$, where $k^* = k^*(u_1, \ldots, u_l)$ is given by

$$k^* = \arg\max_{k=0,1,\ldots,l-1} (u_{k+1} - u_k)\,. \tag{8.15}$$

Remark 8.15. *From the above definitions, t_F^* is of course non-unique, as the maximum can be attained at more than one point; we make the convention that we choose the smallest value for t_F^*.*

We can now present two approaches for dealing with integration problems pertaining to $(N_{t_i})_{i=1}^{s}$. As shown below, there is no need to reformulate the problem, we can immediately state the approaches.

8.3.2 Increment-by-increment approach

Similar to the standard construction for a Brownian motion, we use an increment-by-increment approach for the construction of the Poisson process in the integration problem. If $N \sim P(\lambda t)$, then we denote the generalized inverse of its distribution by $F_P^{-1}(u, \lambda t)$ and define it using Equation (8.2). We now formulate the integration problem, where $\triangle N_{t_i} = N_{t_i} - N_{t_{i-1}}$ and $\triangle t_i = t_i - t_{i-1}$, $i = 1, \ldots, s$, f is Borel-measurable, $f : \mathbb{R}^s \to \mathbb{R}$, and $u_1, \ldots, u_s$ are independent and uniformly distributed in $[0, 1]$,

$$\begin{aligned}
&\mathbb{E}\left[f(N_{t_1}, N_{t_2}, N_{t_3}, \ldots, N_{t_s})\right] \\
&= \mathbb{E}\left[g(\triangle N_{t_1}, \triangle N_{t_2}, \ldots, \triangle N_{t_s})\right] \\
&= \mathbb{E}\left[g(F_P^{-1}(u_1, \lambda \triangle t_1), F_P^{-1}(u_2, \lambda \triangle t_2), \ldots, F_P^{-1}(u_s, \lambda \triangle t_s))\right] \\
&= \int_{u_1=0}^{1} \cdots \int_{u_s=0}^{1} g(F_P^{-1}(u_1, \lambda \triangle t_1), F_P^{-1}(u_2, \lambda \triangle t_2), \ldots, F_P^{-1}(u_s, \lambda \triangle t_s)) du_s \ldots du_1 \,.
\end{aligned}$$

We will approximate the above integral using a qMC rule based on a classical digital net, $\{\boldsymbol{x}_h\}_{h=0}^{b^m-1}$, where $\boldsymbol{x}_h \in [0, 1)^s$, $h = 0, \ldots, b^m - 1$, $\boldsymbol{x}_{h,j} \in [0, 1)$, $j = 1, \ldots, s$, $h = 0, \ldots, b^m - 1$,

$$\mathbb{E}\left[f(N_{t_1}, \ldots, N_{t_s})\right] \approx \frac{1}{b^m} \sum_{h=0}^{b^m-1} g(F_P^{-1}(\boldsymbol{x}_{h,1}, \lambda \triangle t_1), F_P^{-1}(\boldsymbol{x}_{h,2}, \lambda \triangle t_2), \ldots, F_P^{-1}(\boldsymbol{x}_{h,s}, \lambda \triangle t_s)) \,.$$

A shortcoming of the above approach as with the standard method for constructing a Brownian motion is that for qMC rules, it is desirable to allocate the early dimensions to variates with high variances. However, for equispaced $(t_i)_{i=0}^{s}$, the variances of the variables under consideration are equal. This issue is addressed in the next subsection.

8.3.3 Fox's greedy rule approach

The following generic rule was proposed by Fox, see [38, Section 3.5].

Greedy rule: Give rank 1 to the variable with the highest variance. Assign rank i to the variable which has the highest expectation of conditional variance given the variables with ranks $1, \ldots, i-1$.

The following proposition gives the ordering of the random variables $(N_{t_i})_{i=1}^{s}$ according to the greedy rule.

Proposition 8.16. *Assume that the greedy rule is applied to the* $(N_{t_i})_{i=1}^s$. *Then the random variables* $(N_{t_i})_{i=1}^s$ *are to be ordered as follows:*

$$N_1, N_{\frac{1}{2}}, N_{\frac{1}{4}}, N_{\frac{3}{4}}, N_{\frac{1}{8}}, N_{\frac{3}{8}}, N_{\frac{5}{8}}, N_{\frac{7}{8}}, \ldots, N_{\frac{1}{2^m}}, N_{\frac{3}{2^m}}, \ldots, N_{\frac{2^m-1}{2^m}} .$$

Proof. The fact that rank 1 is given to N_1 follows from Lemma 8.12, we then obtain $t_F^* = t_F^*(1) = \frac{1}{2}$, consequently $t_F^*(\frac{1}{2}, 1) = \frac{1}{4}$ and $t_F^*(\frac{1}{4}, \frac{1}{2}, 1) = \frac{3}{4}$. The remainder follows from using the relevant formulas for t_F^*. □

We note that the ordering of the variates coincides with the ordering produced by the Brownian bridge, e.g. [15], and the Gamma bridge, [2; 90]. It is well-known that both bridge constructions can improve convergence rates when combining it with qMC rules. However, it cannot be expected that bridge constructions always result in an improvement in convergence rates, see e.g. [84].

We now formulate the integration problem for Fox's greedy rule and recall the definition of the generalized inverse of the Poisson distribution; similarly, if $A \sim Bin(n, \frac{1}{2})$, we denote the generalized inverse of its distribution by $F_B^{-1}(u, n)$ and define it using Equation (8.2). Furthermore, we recall Lemma 8.12 (iii) and the convention $t_i = \frac{i}{s}$, $s = 2^d$, $d \in \mathbb{N}_0$. The integration problem is formulated for the special case $d = 2$, we set $N_i = N_{t_i}$, $i = 1, \ldots, s$ and define the four dimensional vector: $N = (N_1, N_2, N_3, N_4)$ and f is a Borel-measurable function, $f : \mathbb{R}^4 \to \mathbb{R}$. The first equality uses the Tower Property, see e.g. [17]:

$$\begin{aligned}
&\mathbb{E}\left[f(N)\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\mathbb{E}\left[f(N)|N_2, N_4\right]|N_4\right]\right] \\
&= \int_{u_1=0}^1 \mathbb{E}\left[\mathbb{E}\left[f(N_1, N_2, N_3, F_P^{-1}(u_1, \lambda t_4))|N_2, N_4 = F_P^{-1}(u_1, \lambda t_4)\right]|F_P^{-1}(u_1, \lambda t_4)\right] du_1 \\
&= \int_{u_1=0}^1 \int_{u_2=0}^1 \mathbb{E}\left[f(N_1, F_B^{-1}(u_2, F_P^{-1}(u_1, \lambda t_4)), N_3, F_P^{-1}(u_1, \lambda t_4))|\right. \\
&\qquad \left. N_2 = F_B^{-1}(u_2, F_P^{-1}(u_1, \lambda t_4)), N_4 = F_P^{-1}(u_1, \lambda t_4)\right] du_2 du_1 \\
&= \int_{u_1=0}^1 \int_{u_2=0}^1 \int_{u_3=0}^1 \int_{u_4=0}^1 f\left(F_B^{-1}(u_3, F_B^{-1}(u_2, F_P^{-1}(u_1, \lambda t_4))), F_B^{-1}(u_2, F_P^{-1}(u_1, \lambda t_4)),\right. \\
&\qquad \left. F_B^{-1}(u_2, F_P^{-1}(u_1, \lambda t_4)) + F_B^{-1}(u_4, F_P^{-1}(u_1, \lambda t_4) - F_B^{-1}(u_2, F_P^{-1}(u_1, \lambda t_4))), F_P^{-1}(u_1, \lambda t_4)\right) \\
&\qquad du_4 du_3 du_2 du_1 .
\end{aligned}$$

Given a classical digital net, $\{\boldsymbol{x}_h\}_{h=0}^{b^m-1}$, where $\boldsymbol{x}_h \in [0,1)^4$, $x_{h,j} \in [0,1)$, $h = 0, \ldots, b^m - 1$, $j = 1, \ldots, 4$, we approximate the above integral by the following qMC rule

$$\begin{aligned}
&\mathbb{E}\left[f(N)\right] \\
&\approx \frac{1}{b^m}\sum_{h=0}^{b^m-1} f\Big(F_B^{-1}(\boldsymbol{x}_{h,3}, F_B^{-1}(\boldsymbol{x}_{h,2}, F_P^{-1}(\boldsymbol{x}_{h,1}, \lambda t_4))), F_B^{-1}(\boldsymbol{x}_{h,2}, F_P^{-1}(\boldsymbol{x}_{h,1}, \lambda t_4)), \\
&\quad F_B^{-1}(\boldsymbol{x}_{h,2}, F_P^{-1}(\boldsymbol{x}_{h,1}, \lambda t_4)) + F_B^{-1}(\boldsymbol{x}_{h,4}, F_P^{-1}(\boldsymbol{x}_{h,1}, \lambda t_4) - F_B^{-1}(\boldsymbol{x}_{h,2}, F_P^{-1}(\boldsymbol{x}_{h,1}, \lambda t_4))), \\
&\quad F_P^{-1}(\boldsymbol{x}_{h,1}, \lambda t_4)\Big) .
\end{aligned}$$

The integration problem for a general s, where $s = 2^d$, $d \in \mathbb{N}_0$, can be derived using a generalization of this argument. We note that probability distributions are dependent on earlier dimensions.

8.3.4 Numerical results for the Poisson process

The approaches illustrated in Subsections 8.3.2 and 8.3.3 are now applied to the finance problem described in Subsection 8.2.3. We set $T_l = \frac{l}{M}$, $l = 0, \ldots, M$, $M = 32$ and $K = 95$ in Equation (8.12) and $S_0 = 100$. The following choice of parameters is taken from [51]:

$$r = 0.05,\ \lambda = 3,\ \eta^+ = 50,\ \eta^- = 25,\ p = 0.3\,. \tag{8.16}$$

For parameter c in Equation (8.13), we use the expected jump size resulting from the parameters in Equation (8.16), specifically,

$$c = \mathbb{E}(Y_k) = p\frac{1}{\eta^+} - (1-p)\frac{1}{\eta^-} = -0.022\,.$$

Table 8.1 used qMC rules based on Sobol′ points, a classical digital net over $\mathbb{Z}_2$, as constructed in [48] with 30 digital shifts for the increment-by-increment approach (IBI) and the greedy rule (GR); for purposes of comparison, the results when applying Monte Carlo methods (MC) to the finance problem are included as well. Using Equation (8.4), we show standard errors of qMC rules employing n Sobol′ points with $q = 30$. The MC results were computed using Equation (8.6) with qn random points.

We see from Table 8.1 that the qMC rules outperform the MC method and that the approach of changing the ordering of the variates marginally outperforms the increment-by-increment approach.

	$n = 256$	$n = 1024$	$n = 4096$	$n = 16384$
MC	0.0247	0.0124	0.0062	0.0031
IBI	0.0042	0.0013	4.7532e-04	1.3746e-04
GR	0.0031	0.0014	2.9688e-04	1.1144e-04

Table 8.1. *Standard errors corresponding to q random shifts of n qMC points and qn MC points for the Poisson process problem*

8.4 The integration problem for the compound Poisson process

In this section, we relax the assumption that jump sizes are constant and we assume, as in [50; 51], that the sizes of the jumps follow an asymmetric double exponential distribution. We still set $\sigma = 0$ in Definition 8.6. This results in the following stock price dynamics:

$$S_t = S_0 \exp\left((r - \lambda\zeta)t + \sum_{k=1}^{N_t} Y_k\right), \tag{8.17}$$

where $\zeta = \mathbb{E}\left[\exp(Y_k)\right] - 1$ and all processes and variates are defined as in Subsection 8.2.2; in particular, $Y_k \sim f$, where the density f is given by Equation (8.7). As noted in Remark 8.9, discounting the share price given by Equation (8.17) yields a martingale.

We define the compound Poisson process $(X_t)_{t\geq 0}$ as follows

$$X_t = \sum_{k=1}^{N_t} Y_k, \tag{8.18}$$

where N_t and Y_k are as in Definition 8.6. In Subsection 8.4.1, we motivate why qMC rules should not be applied to the problem formulation shown in Equation (8.17) and reformulate the problem in such a way that qMC rules can be applied to it.

8.4.1 Problem formulation for quasi-Monte Carlo rules

In order to apply qMC rules, the problem under consideration should exhibit two properties: firstly, the effective dimensionality of the problem should be low and secondly, for a particular sequence of qMC points, a particular dimension should correspond to a particular random variable. Dealing with random variates $(N_{t_i})_{i=1}^{s}$ and $(Y_i)_{i=1}^{N_{t_s}}$ conflicts with both of the above properties: firstly, the dimensionality of the problem becomes very high, in principle even infinite. Secondly, it becomes difficult to allocate a particular random variable to a particular dimension. For these reasons, Equations (8.17) and (8.18) are not useful for purposes of qMC rules. We remark that for purposes of MC methods, Equations (8.17) and (8.18) are appropriate. The following random variables are used in the sequel: let N_t^+ (N_t^- respectively) denote the number of positive (negative respectively) jumps of X on the interval $[0,t]$, i.e.

$$N_t^+ = \sum_{k=1}^{N_t} \mathbf{1}_{\{Y_k>0\}}, \quad N_t^- = \sum_{k=1}^{N_t} \mathbf{1}_{\{Y_k<0\}}.$$

Clearly, $N_t = N_t^+ + N_t^-$. In addition, we denote

$$Y_k^+ = Y_k \mathbf{1}_{\{Y_k>0\}}, \quad Y_k^- = -Y_k \mathbf{1}_{\{Y_k<0\}}$$

and set

$$S_t^+ = \sum_{k=1}^{N_t} Y_k^+, \quad S_t^- = \sum_{k=1}^{N_t} Y_k^-.$$

We are now in a position to state the following lemma, part ii) of which is an observation from [40, p. 139]. The proof of this result is standard and thus it is omitted. Recall that a random variable X follows a Gamma distribution with parameters k, η (for short $X \sim \Gamma(k,\eta)$) if it admits the probability density function

$$f_X(x) = \frac{\eta^k}{\Gamma(k)} x^{k-1} \exp(-\eta x), \quad x \geq 0.$$

Lemma 8.17. *The following properties hold:*

(i) $\left(N_t^+\right)_{t\geq 0}$ *and* $\left(N_t^-\right)_{t\geq 0}$ *are independent Poisson processes with rates* $p\lambda$ *and* $(1-p)\lambda$ *respectively.*

(ii) *Conditional on* $N_t^+ = k$ *(*$N_t^- = l$ *respectively), we have that* $S_t^+ \sim \Gamma(k,\eta^+)$ *(*$S_t^- \sim \Gamma(l,\eta^-)$ *respectively).*

The next result, which is well-known, see e.g. [91, Example 2.38], gives the conditional distribution of the sum of positive jumps and is used in Subsection 8.4.3. Of course, the analogous result holds for the sum of negative jumps. Recall that a random variable X follows a Beta distribution with parameters a, b (for short $X \sim Beta(a,b)$) if it admits the probability density function

$$f_X(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}, \quad 0 \leq x \leq 1, \quad a > 0, \quad b > 0.$$

Lemma 8.18. *Let* $t_i < t_j < t_k$*,* $n_1 < n_2$*, and* $s_1 < s_2$*. Then conditional on* $N_{t_i}^+ = n_1$*,* $N_{t_j}^+ = n_2$*,* $N_{t_k}^+ = n_3$*,* $S_{t_i}^+ = s_1$*, and* $S_{t_k}^+ = s_2$*, the distribution of*

$$\frac{S_{t_j}^+ - S_{t_i}^+}{S_{t_k}^+ - S_{t_i}^+}$$

is Beta $(n_2 - n_1, n_3 - n_2)$.

Furthermore, if $X \sim \Gamma(k,\eta)$, we denote the generalized inverse of its cumulative distribution function by $F_G^{-1}(u,k,\eta)$, $u \in [0,1]$, $k > 0$, $\eta > 0$, and if $X \sim Beta(a,b)$, then we denote the generalized inverse of its cumulative distribution function by $F_{Beta}^{-1}(u,a,b)$, $u \in [0,1]$, $a > 0$, $b > 0$, of course both generalized inverse functions are defined via Equation (8.2). We make the convention that $F_G^{-1}(u,0,\eta) = 0$, $u \in [0,1]$, $\eta > 0$, $F_{Beta}^{-1}(u,0,b) = 0$, $u \in [0,1]$, $b \geq 0$ and $F_{Beta}^{-1}(u,a,0) = 1$, $u \in [0,1]$, $a > 0$. It is easily seen that $X_t = S_t^+ - S_t^-$ for every $t \geq 0$. It is now clear that in order to obtain $(X_{t_i})_{i=1}^s$, it suffices to compute $(N_{t_i}^+)_{i=1}^s$, $(N_{t_i}^-)_{i=1}^s$, $(S_{t_i}^+)_{i=1}^s$, and $(S_{t_i}^-)_{i=1}^s$, and the approaches introduced in the next subsections are formulated in terms of $(N_{t_i}^+)_{i=1}^s$, $(N_{t_i}^-)_{i=1}^s$, $(S_{t_i}^+)_{i=1}^s$, and $(S_{t_i}^-)_{i=1}^s$. The advantage of this formulation is that the problem is now of fixed dimension, $4s$, and each one of the $4s$ dimensions corresponds to a particular random variable. We conclude that we are now in a position to apply qMC rules to the problem and state the increment-by-increment approach.

8.4.2 Increment-by-increment approach

We now formulate the integration problem for the increment-by-increment approach. The following notation is used in the sequel: $\triangle N_{t_i}^+ = N_{t_i}^+ - N_{t_{i-1}}^+$, $\triangle N_{t_i}^- = N_{t_i}^- - N_{t_{i-1}}^-$, $\triangle S_{t_i}^+ = S_{t_i}^+ - S_{t_{i-1}}^+$, $\triangle S_{t_i}^- = S_{t_i}^- - S_{t_{i-1}}^-$, $i = 1,\ldots,s$, where $\triangle N^+ = (\triangle N_{t_1}^+,\ldots,\triangle N_{t_s}^+)$, $\triangle N^- = (\triangle N_{t_1}^-,\ldots,\triangle N_{t_s}^-)$, $\triangle S^+ = (\triangle S_{t_1}^+,\ldots,\triangle S_{t_s}^+)$ and $\triangle S^- = (\triangle S_{t_1}^-,\ldots,\triangle S_{t_s}^-)$.

We obtain

$$\begin{aligned}
&\mathbb{E}\left[f(N_{t_1}^+,\ldots,N_{t_s}^+,N_{t_1}^-,\ldots,N_{t_s}^-,S_{t_1}^+,\ldots,S_{t_s}^+,S_{t_1}^-,\ldots,S_{t_s}^-)\right]\\
&= \mathbb{E}\left[g(\triangle N_{t_1}^+,\ldots,\triangle N_{t_s}^+,\triangle N_{t_1}^-,\ldots,\triangle N_{t_s}^-,\triangle S_{t_1}^+,\ldots,\triangle S_{t_s}^+,\triangle S_{t_1}^-,\ldots,\triangle S_{t_s}^-)\right]\\
&= \mathbb{E}\left[\mathbb{E}\left[g(\triangle N^+,\triangle N^-,\triangle S^+,\triangle S^-)\mid \triangle N^+,\triangle N^-\right]\right]\\
&= \int_{u_1=0}^1 \cdots \int_{u_s=0}^1 \int_{u_{s+1}=0}^1 \cdots \int_{u_{2s}=0}^1 \mathbb{E}\Big[g(F_P^{-1}(u_1,p\lambda\triangle t_1),\ldots,F_P^{-1}(u_s,p\lambda\triangle t_s),\\
&\quad F_P^{-1}(u_{s+1},(1-p)\lambda\triangle t_1),\ldots,F_P^{-1}(u_{2s},(1-p)\lambda\triangle t_s),\triangle S_{t_1}^+,\ldots,\triangle S_{t_s}^+,\\
&\quad \triangle S_{t_1}^-,\ldots,\triangle S_{t_s}^-)\mid \triangle N_{t_1}^+ = F_P^{-1}(u_1,p\lambda\triangle t_1),\ldots,\triangle N_{t_s}^+ = F_P^{-1}(u_s,p\lambda\triangle t_s),\\
&\quad \triangle N_{t_1}^- = F_P^{-1}(u_{s+1},(1-p)\lambda\triangle t_1),\ldots,\triangle N_{t_s}^- = F_P^{-1}(u_{2s},(1-p)\lambda\triangle t_s)\Big]\\
&\quad du_{2s}\ldots du_{s+1}du_s\ldots du_1
\end{aligned}$$

$$
\begin{aligned}
= & \int_{u_1=0}^{1} \cdots \int_{u_s=0}^{1} \int_{u_{s+1}=0}^{1} \cdots \int_{u_{2s}=0}^{1} \int_{u_{2s+1}=0}^{1} \cdots \int_{u_{3s}=0}^{1} \int_{u_{3s+1}=0}^{1} \cdots \int_{u_{4s}=0}^{1} \\
& g(F_P^{-1}(u_1, p\lambda \triangle t_1), \ldots, F_P^{-1}(u_s, p\lambda \triangle t_s), \\
& F_P^{-1}(u_{s+1}, (1-p)\lambda \triangle t_1), \ldots, F_P^{-1}(u_{2s}, (1-p)\lambda \triangle t_s), \\
& F_G^{-1}(u_{2s+1}, F_P^{-1}(u_1, p\lambda \triangle t_1), \eta^+), \ldots, F_G^{-1}(u_{3s}, F_P^{-1}(u_s, p\lambda \triangle t_s), \eta^+), \\
& F_G^{-1}(u_{3s+1}, F_P^{-1}(u_{s+1}, (1-p)\lambda \triangle t_1), \eta^-), \ldots, F_G^{-1}(u_{4s}, F_P^{-1}(u_{2s}, (1-p)\lambda \triangle t_s), \eta^-)) \\
& du_{4s} \ldots du_{3s} \ldots du_{2s} \ldots du_s \ldots du_1 \,.
\end{aligned}
$$

Given a classical digital net, $\{\boldsymbol{x}_h\}_{h=0}^{b^m-1} \in [0,1)^{4s}$, we approximate the above integral by the following qMC rule

$$
\begin{aligned}
& \mathbb{E}\left[f(N_{t_1}^+, \ldots, N_{t_s}^+, N_{t_1}^-, \ldots, N_{t_s}^-, S_{t_1}^+, \ldots, S_{t_s}^+, S_{t_1}^-, \ldots, S_{t_s}^-)\right] \\
\approx \; & \frac{1}{b^m} \sum_{h=0}^{b^m-1} g(F_P^{-1}(\boldsymbol{x}_{h,1}, p\lambda \triangle t_1), \ldots, F_P^{-1}(\boldsymbol{x}_{h,s}, p\lambda \triangle t_s), \\
& F_P^{-1}(\boldsymbol{x}_{h,s+1}, (1-p)\lambda \triangle t_1), \ldots, F_P^{-1}(\boldsymbol{x}_{h,2s}, (1-p)\lambda \triangle t_s), \\
& F_G^{-1}(\boldsymbol{x}_{h,2s+1}, F_P^{-1}(\boldsymbol{x}_{h,1}, p\lambda \triangle t_1), \eta^+), \ldots, F_G^{-1}(\boldsymbol{x}_{h,3s}, F_P^{-1}(\boldsymbol{x}_{h,s}, p\lambda \triangle t_s), \eta^+), \\
& F_G^{-1}(\boldsymbol{x}_{h,3s+1}, F_P^{-1}(\boldsymbol{x}_{h,s+1}, (1-p)\lambda \triangle t_1), \eta^-), \ldots, \\
& F_G^{-1}(\boldsymbol{x}_{h,4s}, F_P^{-1}(\boldsymbol{x}_{h,2s}, (1-p)\lambda \triangle t_s), \eta^-)) \,.
\end{aligned}
$$

A shortcoming of the above approach as with the standard method for constructing a Brownian motion is that no effort is made to pack a lot of variance into the opening dimensions. Computing conditional variances of increments of the Poisson process and of jumps yields the following:

$$
\begin{aligned}
\mathrm{Var}\left(N_{t_i} - N_{t_{i-1}} | N_{t_{i-1}}\right) &= \frac{\lambda}{s} \\
\mathrm{Var}\left(Y_k | N_{t_{i-1}}, N_{t_i}\right) &= \mathrm{Var}\left(Y_k\right) = \frac{2p}{(\eta^+)^2} + \frac{2(1-p)}{(\eta^-)^2} - \left(\frac{p}{\eta^+} - \frac{1-p}{\eta^-}\right)^2 .
\end{aligned}
$$

If $\mathrm{Var}\left(N_{t_i} - N_{t_{i-1}} | N_{t_{i-1}}\right)$ and $\mathrm{Var}\left(Y_i | N_{t_{i-1}}, N_{t_i}\right)$ differed substantially, it would be more desirable from a qMC point of view to allocate the early dimensions to the variables with the largest variances. This observation motivates the following approach.

8.4.3 Compound Poisson bridge approach

The compound Poisson bridge approach is based on Lemmas 8.12 (iv), 8.17 and 8.18. Again, we introduce the integration problem for the simplified case where $s = 4$ and we set $N_i = (N_{t_i}^+, N_{t_i}^-)$, $N = (N_1, N_2, N_3, N_4)$, $S_i = (S_{t_i}^+, S_{t_i}^-)$ and $S = (S_1, \ldots, S_4)$. The ordering of the variates resembles that of the Brownian bridge and the Gamma bridge, hence the nomenclature.

$$\begin{aligned}
&\mathbb{E}\left[f(N^+_{t_1},N^-_{t_1},S^+_{t_1},S^-_{t_1},N^+_{t_2},N^-_{t_2},S^+_{t_2},S^-_{t_2},N^+_{t_3},N^-_{t_3},S^+_{t_3},S^-_{t_3},N^+_{t_4},N^-_{t_4},S^+_{t_4},S^-_{t_4})\right]\\
&= \mathbb{E}\left[f(N,S)\right]\\
&= \mathbb{E}\left[\mathbb{E}\left[f(N,S)|N,S_2,S_4\right]\right]\\
&= \mathbb{E}\left[\mathbb{E}\left[\mathbb{E}\left[f(N,S)|N,S_2,S_4\right]|N_2,S_2,N_4,S_4\right]\right]\\
&= \mathbb{E}\left[\mathbb{E}\left[\mathbb{E}\left[\mathbb{E}\left[f(N,S)|N,S_2,S_4\right]|N_2,S_2,N_4,S_4\right]|N_2,N_4,S_4\right]\right]\\
&= \mathbb{E}\left[\mathbb{E}\left[\mathbb{E}\left[\mathbb{E}\left[\mathbb{E}\left[f(N,S)|N,S_2,S_4\right]|N_2,S_2,N_4,S_4\right]|N_2,N_4,S_4\right]|N_4,S_4\right]\right]\\
&= \mathbb{E}\left[\mathbb{E}\left[\mathbb{E}\left[\mathbb{E}\left[\mathbb{E}\left[\mathbb{E}\left[f(N,S)|N,S_2,S_4\right]|N_2,S_2,N_4,S_4\right]|N_2,N_4,S_4\right]|N_4,S_4\right]|N_4\right]\right].
\end{aligned}$$

We now present the integral. For simplicity, we consider the case $N_2 = N^+_{t_2}$, $N_3 = N^+_{t_3}$, $N_4 = N^+_{t_4}$, $S_2 = S^+_{t_2}$, $S_3 = S^+_{t_3}$, and $S_4 = S^+_{t_4}$:

$$\begin{aligned}
&\mathbb{E}\left[f(N^+_{t_2},S^+_{t_2},N^+_{t_3},S^+_{t_3},N^+_{t_4},S^+_{t_4})\right]\\
&= \int_{u_1=0}^{1}\int_{u_2=0}^{1}\int_{u_3=0}^{1}\int_{u_4=0}^{1}\int_{u_5=0}^{1}\int_{u_6=0}^{1}\\
&\quad f(N^+_2(u_3,u_1),S^+_2(u_4,u_1,u_2,u_3),N^+_3(u_5,u_1,u_3),\\
&\quad S^+_3(u_6,u_1,u_2,u_3,u_4,u_5),N^+_4(u_1),S^+_4(u_2,u_1))du_6du_5du_4du_3du_2du_1\,,
\end{aligned}$$

where

$$\begin{aligned}
N^+_2(u_3,u_1) &= F_B^{-1}(u_3,N^+_4(u_1))\,,\\
N^+_3(u_5,u_1,u_3) &= N^+_2(u_3,u_1)+F_B^{-1}(u_5,N^+_4(u_1)-N^+_2(u_3,u_1))\,,\\
N^+_4(u_1) &= F_P^{-1}(u_1,\lambda t_4)\,,\\
S^+_2(u_4,u_1,u_2,u_3) &= S^+_4(u_2,u_1)F^{-1}_{Beta}(u_4,N^+_2(u_3,u_1),N^+_4(u_1)-N^+_2(u_3,u_1))\,,\\
S^+_3(u_6,u_1,u_2,u_3,u_4,u_5) &= S^+_2(u_4,u_1,u_2,u_3)\\
&\quad +(S^+_4(u_2,u_1)-S^+_2(u_4,u_1,u_2,u_3))F^{-1}_{Beta}(u_6,\\
&\quad N^+_3(u_5,u_1,u_3)-N^+_2(u_3,u_1),N^+_4(u_1)-N^+_3(u_5,u_1,u_3))\,,\\
S^+_4(u_2,u_1) &= F_G^{-1}(u_2,N^+_4(u_1),\eta^+)\,.
\end{aligned}$$

The approximation of the integral, for a qMC rule based on a classical digital net $\{\boldsymbol{x}_h\}_{h=0}^{b^m-1} \in [0,1)^6$, can now be formulated,

$$
\begin{aligned}
&\mathbb{E}\left[f(N,S)\right] \\
&\approx \frac{1}{b^m}\sum_{h=0}^{b^m-1} f(N_2^+(\boldsymbol{x}_{h,3},\boldsymbol{x}_{h,1}), S_2^+(\boldsymbol{x}_{h,4},\boldsymbol{x}_{h,1},\boldsymbol{x}_{h,2},\boldsymbol{x}_{h,3}), N_3^+(\boldsymbol{x}_{h,5},\boldsymbol{x}_{h,1},\boldsymbol{x}_{h,3}), \\
&\quad S_3^+(\boldsymbol{x}_{h,6},\boldsymbol{x}_{h,1},\boldsymbol{x}_{h,2},\boldsymbol{x}_{h,3},\boldsymbol{x}_{h,4},\boldsymbol{x}_{h,5}), N_4^+(\boldsymbol{x}_{h,1}), S_4^+(\boldsymbol{x}_{h,2},\boldsymbol{x}_{h,1}))\,.
\end{aligned}
$$

Algorithm 8.1 shows how to arrive at a function evaluation $f = f(\boldsymbol{x})$, where $\boldsymbol{x}$ is a qMC point, for the general case, where $d \in \mathbb{N}_0$.

Algorithm 8.1 Compound Poisson bridge approach for the compound Poisson process

a) $N_1^+ = F_P^{-1}(\boldsymbol{x}_1, p\lambda)$

b) $N_1^- = F_P^{-1}(\boldsymbol{x}_2, (1-p)\lambda)$

c) $S_1^+ = F_G^{-1}(\boldsymbol{x}_3, N_1^+, \eta^+)$

d) $S_1^- = F_G^{-1}(\boldsymbol{x}_4, N_1^-, \eta^-)$

for $i = 1$ to d **do**

  for $j = 1$ to 2^{i-1} **do**

    $k = 2^{i-1} + j$

  a) $N^+_{\frac{2j-1}{2^i}} = N^+_{\frac{2j-2}{2^i}} + F_B^{-1}\Big(\boldsymbol{x}_{4(k-1)+1}, N^+_{\frac{2j}{2^i}} - N^+_{\frac{2j-2}{2^i}}\Big)$

  b) $N^-_{\frac{2j-1}{2^i}} = N^-_{\frac{2j-2}{2^i}} + F_B^{-1}\Big(\boldsymbol{x}_{4(k-1)+2}, N^-_{\frac{2j}{2^i}} - N^-_{\frac{2j-2}{2^i}}\Big)$

  c) $S^+_{\frac{2j-1}{2^i}} = S^+_{\frac{2j-2}{2^i}} + (S^+_{\frac{2j}{2^i}} - S^+_{\frac{2j-2}{2^i}})F^{-1}_{Beta}(\boldsymbol{x}_{4(k-1)+3}, N^+_{\frac{2j-1}{2^i}} - N^+_{\frac{2j-2}{2^i}}, N^+_{\frac{2j}{2^i}} - N^+_{\frac{2j-1}{2^i}})$

  d) $S^-_{\frac{2j-1}{2^i}} = S^-_{\frac{2j-2}{2^i}} + (S^-_{\frac{2j}{2^i}} - S^-_{\frac{2j-2}{2^i}})F^{-1}_{Beta}(\boldsymbol{x}_{4(k-1)+4}, N^-_{\frac{2j-1}{2^i}} - N^-_{\frac{2j-2}{2^i}}, N^-_{\frac{2j}{2^i}} - N^-_{\frac{2j-1}{2^i}})$

  end for

end for

return $f(N,S)$.

Remark 8.19. *We did not comment on how we arrived at the ordering of the variables at a particular step of the algorithm. In Subsection 8.4.4, we show that the ordering depends on the parameters of the compound Poisson process given by Equation* (8.18)*, so Algorithm 8.1 serves as an illustration, easily modified to fit a particular problem.*

Remark 8.20. *Instead of dealing with* $(N^+_{t_i})_{i=1}^s$ *and* $(N^-_{t_i})_{i=1}^s$*, one could compute* $(N_{t_i})_{i=1}^s$ *and* $(N^+_{t_i})_{i=1}^s$*, consequently recover* $(N^-_{t_i})_{i=1}^s$ *and subsequently proceed as in Algorithm 8.1.*

8.4.4 Numerical results for the compound Poisson process

In this subsection, the algorithms presented in Subsections 8.4.2 and 8.4.3 are applied to the finance problem presented in Subsection 8.2.3. We set $T_l = \frac{l}{M}$, $l = 0, \ldots, M$, $M = 32$ and $K = 95$ in Equation (8.12) and $S_0 = 100$ and use the set of parameters given by Equation (8.16).

Let us comment on the choice of the ordering for the compound Poisson bridge approach. It is well-known, see e.g. [15], that it often pays off to allocate as much variance as possible to the opening dimensions. This is more difficult to implement for approaches such as the compound Poisson bridge approach, as conditional variances, unlike for the Brownian bridge, depend on variates corresponding to earlier dimensions. We hence use the following reasoning to arrive at the ordering scheme used: for variates appearing in step 1 of the compound Poisson bridge approach, unconditional variances for the set of parameters given by Equation (8.16) are easily computed. The variates in step 1 are then ordered based on unconditional variances and the same ordering is used for steps $2, \ldots, s$. Clearly, in our case

$$\mathrm{Var}(N^{-}_{t_s}) > \mathrm{Var}(N^{+}_{t_s}) > \mathrm{Var}(S^{-}_{t_s}) > \mathrm{Var}(S^{+}_{t_s}) \,,$$

so for Step 5 of the compound Poisson bridge approach, we interchange a) and b) and also c) and d). Of course, other methods of ordering the variates in each step are possible and this is an area of future research.

Table 8.2 used qMC rules based on Sobol' points as constructed in [48] with 30 digital shifts for the increment-by-increment approach (IBI) and the compound Poisson bridge approach (CPB). For purposes of comparison, the results when applying Monte Carlo methods (MC) to the finance problem are also included. Using Equation (8.4), we show standard errors of qMC rules employing n Sobol' points with $q = 30$. The MC results were computed using Equation (8.6) with qn random points.

	$n = 256$	$n = 1024$	$n = 4096$	$n = 16384$
MC	0.0453	0.0225	0.0113	0.0056
IBI	0.0237	0.0086	0.0040	0.0013
CPB	0.0098	0.0058	0.0028	0.0012

Table 8.2. *Standard errors corresponding to q random shifts of n qMC points and qn MC points for the compound Poisson process problem*

We can see from Table 8.2 that the Monte Carlo method performs worst for all values of n and that the bridge construction outperforms the increment-by-increment approach.

8.5 The integration problem for the jump-diffusion process

In this section, we deal with the Kou model, as presented in [50; 51] and summarized in Subsection 8.2.2. We recall that the share price is given by

$$S_t = S_0 \exp\left((r - \frac{1}{2}\sigma^2 - \lambda\zeta)t + \sigma W_t + \sum_{i=1}^{N_t} Y_i\right), \tag{8.19}$$

and we set

$$Z_t = \sigma W_t + \sum_{i=1}^{N_t} Y_i\,, \tag{8.20}$$

where all processes and variables are as in Definition 8.6. As mentioned in Subsection 8.4.1, it is undesirable from a qMC point of view to deal with $(N_{t_i})_{i=1}^{s}$ and $(Y_i)_{i=1}^{N_{t_s}}$, so we recall the random variables introduced in Subsection 8.4.1 and extend the increment-by-increment approach presented in Subsection 8.4.2 and the compound Poisson bridge approach described in Subsection 8.4.3 to the jump-diffusion process $(Z_t)_{t\geq 0}$ given by Equation (8.20). Of course, the increment-by-increment approach explained in Subsection 8.4.2 and the compound Poisson bridge approach detailed in Subsection 8.4.3 were based on the modified problem formulation from Subsection 8.4.1.

8.5.1 Increment-by-increment approach

The integration problem for the increment-by-increment approach is now formulated. We recall the short-hand notation from Subsection 8.4.2, set $\triangle W_{t_i} = W_{t_i} - W_{t_{i-1}}$ and $F_N^{-1}(u,\mu,\sigma^2)$ denotes the generalized inverse of the cumulative distribution function of $X \sim N(\mu,\sigma^2)$ evaluated at $u \in [0,1]$ and defined via Equation (8.2). For a Borel-measurable function f, $f : \mathbb{R}^{5s} \to \mathbb{R}$, we use the Tower Property in the following:

$$\begin{aligned}
&\mathbb{E}\left[f(W_{t_1},\dots,W_{t_s},N^+_{t_1},\dots,N^+_{t_s},N^-_{t_1},\dots,N^-_{t_s},S^+_{t_1},\dots,S^+_{t_s},S^-_{t_1},\dots,S^-_{t_s})\right]\\
&=\mathbb{E}\Big[g(\triangle W_{t_1},\dots,\triangle W_{t_s},\triangle N^+_{t_1},\dots,\triangle N^+_{t_s},\triangle N^-_{t_1},\dots,\triangle N^-_{t_s},\\
&\quad \triangle S^+_{t_1},\dots,\triangle S^+_{t_s},\triangle S^-_{t_1},\dots,\triangle S^-_{t_s})\Big]\\
&=\mathbb{E}\Big[\mathbb{E}\Big[g(\triangle W_{t_1},\dots,\triangle W_{t_s},\triangle N^+_{t_1},\dots,\triangle N^+_{t_s},\triangle N^-_{t_1},\dots,\triangle N^-_{t_s},\\
&\quad \triangle S^+_{t_1},\dots,\triangle S^+_{t_s},\triangle S^-_{t_1},\dots,\triangle S^-_{t_s}) \mid \triangle W_{t_1},\dots,\triangle W_{t_s},\triangle N^+_{t_1},\dots,\triangle N^+_{t_s},\\
&\quad \triangle N^-_{t_1},\dots,\triangle N^-_{t_s}\Big]\Big]
\end{aligned}$$

$$
\begin{aligned}
=&\int_{u_1=0}^{1}\cdots\int_{u_s=0}^{1}\int_{u_{s+1}=0}^{1}\cdots\int_{u_{2s}=0}^{1}\int_{u_{2s+1}=0}^{1}\cdots\int_{u_{3s}=0}^{1}\\
&\mathbb{E}\Big[g(F_N^{-1}(u_1,0,\triangle t_1),\ldots,F_N^{-1}(u_s,0,\triangle t_s),\\
&F_P^{-1}(u_{s+1},p\lambda\triangle t_1),\ldots,F_P^{-1}(u_{2s},p\lambda\triangle t_s),\\
&F_P^{-1}(u_{2s+1},(1-p)\lambda\triangle t_1),\ldots,F_P^{-1}(u_{3s},(1-p)\lambda\triangle t_s),\triangle S^+,\triangle S^-)|\\
&\triangle W_{t_1}=F_N^{-1}(u_1,0,\triangle t_1),\ldots,\triangle W_{t_s}=F_N^{-1}(u_s,0,\triangle t_s),\\
&\triangle N_{t_1}^{+}=F_P^{-1}(u_{s+1},p\lambda\triangle t_1),\ldots,\triangle N_{t_s}^{+}=F_P^{-1}(u_{2s},p\lambda\triangle t_s),\\
&\triangle N_{t_1}^{-}=F_P^{-1}(u_{2s+1},(1-p)\lambda\triangle t_1),\ldots,\triangle N_{t_s}^{-}=F_P^{-1}(u_{3s},(1-p)\lambda\triangle t_s)\Big]\\
&du_{3s}\ldots du_{2s+1}du_{2s}\ldots du_{s+1}\ldots du_1\\
=&\int_{u_1=0}^{1}\cdots\int_{u_s=0}^{1}\int_{u_{s+1}=0}^{1}\cdots\int_{u_{2s}=0}^{1}\int_{u_{2s+1}=0}^{1}\cdots\int_{u_{3s}=0}^{1}\int_{u_{3s+1}=0}^{1}\cdots\int_{u_{4s}=0}^{1}\int_{u_{4s+1}=0}^{1}\cdots\int_{u_{5s}=0}^{1}\\
&g(F_N^{-1}(u_1,0,\triangle t_1),\ldots,F_N^{-1}(u_s,0,\triangle t_s),F_P^{-1}(u_{s+1},p\lambda\triangle t_1),\ldots,F_P^{-1}(u_{2s},p\lambda\triangle t_s),\\
&F_P^{-1}(u_{2s+1},(1-p)\lambda\triangle t_1),\ldots,F_P^{-1}(u_{3s},(1-p)\lambda\triangle t_s),\\
&F_G^{-1}(u_{3s+1},F_P^{-1}(u_{s+1},p\lambda\triangle t_1),\eta^+),\ldots,F_G^{-1}(u_{4s},F_P^{-1}(u_{2s},p\lambda\triangle t_s),\eta^+),\\
&F_G^{-1}(u_{4s+1},F_P^{-1}(u_{2s+1},(1-p)\lambda\triangle t_1),\eta^-),\ldots,F_G^{-1}(u_{5s},F_P^{-1}(u_{3s},(1-p)\lambda\triangle t_s),\eta^-))\\
&du_{5s}\ldots du_{4s+1}du_{4s}\ldots du_{3s+1}du_{3s}\ldots du_{2s+1}du_{2s}\ldots du_{s+1}du_s\ldots du_1\,.
\end{aligned}
$$

Given a classical digital net, $\{\boldsymbol{x}_h\}_{h=0}^{b^m-1}\in[0,1)^{5s}$, we approximate the above integral by the following qMC rule

$$
\begin{aligned}
&\mathbb{E}\left[f(W_{t_1},\ldots,W_{t_s},N_{t_1}^{+},\ldots,N_{t_s}^{+},N_{t_1}^{-},\ldots,N_{t_s}^{-},S_{t_1}^{+},\ldots,S_{t_s}^{+},S_{t_1}^{-},\ldots,S_{t_s}^{-})\right]\\
\approx\ &\frac{1}{b^m}\sum_{h=0}^{b^m-1}g(F_N^{-1}(\boldsymbol{x}_{h,1},0,\triangle t_1),\ldots,F_N^{-1}(\boldsymbol{x}_{h,s},0,\triangle t_s),F_P^{-1}(\boldsymbol{x}_{h,s+1},p\lambda\triangle t_1),\ldots,\\
&F_P^{-1}(\boldsymbol{x}_{h,2s},p\lambda\triangle t_s),F_P^{-1}(\boldsymbol{x}_{h,2s+1},(1-p)\lambda\triangle t_1),\ldots,F_P^{-1}(\boldsymbol{x}_{h,3s},(1-p)\lambda\triangle t_s),\\
&F_G^{-1}(\boldsymbol{x}_{h,3s+1},F_P^{-1}(\boldsymbol{x}_{h,s+1},p\lambda\triangle t_1),\eta^+),\ldots,F_G^{-1}(\boldsymbol{x}_{h,4s},F_P^{-1}(\boldsymbol{x}_{h,2s},p\lambda\triangle t_s),\eta^+),\\
&F_G^{-1}(\boldsymbol{x}_{h,4s+1},F_P^{-1}(\boldsymbol{x}_{h,2s+1},(1-p)\lambda\triangle t_1),\eta^-),\ldots,\\
&\qquad\qquad F_G^{-1}(\boldsymbol{x}_{h,5s},F_P^{-1}(\boldsymbol{x}_{h,3s},(1-p)\lambda\triangle t_s),\eta^-))\,.
\end{aligned}
$$

8.5.2 Jump-diffusion bridge approach

The Brownian bridge has been used successfully to pack more variance into the opening dimensions, see e.g. [15], and can hence be expected to have a favorable impact on standard errors.

We will now formulate the integration problem, since the construction and properties of the Brownian bridge are well-known, they are not discussed here. The jump-diffusion bridge approach makes use of the random variables $(W_{t_i})_{i=1}^s$, $(N_{t_i}^+)_{i=1}^s$, $(N_{t_i}^-)_{i=1}^s$, $(S_{t_i}^+)_{i=1}^s$, and $(S_{t_i}^-)_{i=1}^s$, and the ordering of the random variates resembles that of the Brownian bridge, the Gamma bridge, and the compound Poisson bridge. The integration problem is introduced for the simple case $s = 2^2$, so $d = 2$, but consequently we present an algorithm allowing to compute the relevant function evaluations for the general case. We recall the short-hand notation and set $W_i = W_{t_i}$, $W = (W_1, W_2, W_3, W_4)$, $W_{2,4} = (W_2, W_4)$, $N_{2,4} = (N_2, N_4)$, $S_{2,4} = (S_2, S_4)$, and $f = f(W, N, S)$,

$$
\begin{aligned}
&\mathbb{E}\left[f(W,N,S)\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[f(W,N,S)|W,N,S_{2,4}\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\mathbb{E}\left[f|W,N,S_{2,4}\right]|W_{2,4},N_{2,4},S_{2,4}\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\mathbb{E}\left[\mathbb{E}\left[f|W,N,S_{2,4}\right]|W_{2,4},N_{2,4},S_{2,4}\right]|W_{2,4},N_{2,4},S_4\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\mathbb{E}\left[\mathbb{E}\left[\mathbb{E}\left[f|W,N,S_{2,4}\right]|W_{2,4},N_{2,4},S_{2,4}\right]|W_{2,4},N_{2,4},S_4\right]|W_4,N_4,S_4\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\mathbb{E}\left[\mathbb{E}\left[\mathbb{E}\left[\mathbb{E}\left[f|W,N,S_{2,4}\right]|W_{2,4},N_{2,4},S_{2,4}\right]|W_{2,4},N_{2,4},S_4\right]|W_4,N_4,S_4\right]|W_4,N_4\right]\right] .
\end{aligned}
$$

The integration problem is now presented. For simplicity, we consider the case $N_2 = N_{t_2}^+$, $N_3 = N_{t_3}^+$, $N_4 = N_{t_4}^+$, $S_2 = S_{t_2}^+$, $S_3 = S_{t_3}^+$, and $S_4 = S_{t_4}^+$, now

$$
\begin{aligned}
&\mathbb{E}\left[f(W_{t_2}, N_{t_2}^+, S_{t_2}^+, W_{t_3}, N_{t_3}^+, S_{t_3}^+, W_{t_4}, N_{t_4}^+, S_{t_4}^+)\right] \\
&= \int_{u_1=0}^1 \int_{u_2=0}^1 \int_{u_3=0}^1 \int_{u_4=0}^1 \int_{u_5=0}^1 \int_{u_6=0}^1 \int_{u_7=0}^1 \int_{u_8=0}^1 \int_{u_9=0}^1 \\
&\quad f(W_2(u_4,u_1), N_2^+(u_5,u_2), S_2^+(u_6,u_2,u_3,u_5), W_3(u_7,u_1,u_4), N_3^+(u_8,u_2,u_5), \\
&\quad S_3^+(u_9,u_2,u_3,u_5,u_6,u_8), W_4(u_1), N_4^+(u_2), S_4^+(u_3,u_2)) \\
&\quad du_9 du_8 du_7 du_6 du_5 du_4 du_3 du_2 du_1 \,,
\end{aligned}
$$

where

$$
\begin{aligned}
W_2(u_4,u_1) &= F_N^{-1}(u_4, \frac{1}{2}W_4(u_1), \frac{1}{4}) \,, \\
W_3(u_7,u_1,u_4) &= F_N^{-1}(u_7, \frac{1}{2}(W_2(u_4,u_1) + W_4(u_1)), \frac{1}{8}) \,, \\
W_4(u_1) &= F_N^{-1}(u_1, 0, t_4) \,,
\end{aligned}
$$

and $N_2^+(u_5,u_2)$, $N_3^+(u_8,u_2,u_5)$, $N_4^+(u_2)$, $S_2^+(u_6,u_2,u_3,u_5)$, $S_3^+(u_9,u_2,u_3,u_5,u_6,u_8)$, and $S_4^+(u_3,u_2)$ are defined as in Subsection 8.4.3. We now formulate the approximation of the integral for a qMC rule based on a classical digital net $\{\boldsymbol{x}_h\}_{h=0}^{b^m-1} \in [0,1)^9$,

$$\begin{aligned}\mathbb{E}[f] &\\ \approx\ & \frac{1}{b^m}\sum_{h=0}^{b^m-1} f(W_2(\boldsymbol{x}_{h,4},\boldsymbol{x}_{h,1}), N_2^+(\boldsymbol{x}_{h,5},\boldsymbol{x}_{h,2}), S_2^+(\boldsymbol{x}_{h,6},\boldsymbol{x}_{h,2},\boldsymbol{x}_{h,3},\boldsymbol{x}_{h,5}), W_3(\boldsymbol{x}_{h,7},\boldsymbol{x}_{h,1},\boldsymbol{x}_{h,4}),\\ & N_3^+(\boldsymbol{x}_{h,8},\boldsymbol{x}_{h,2},\boldsymbol{x}_{h,5}), S_3^+(\boldsymbol{x}_{h,9},\boldsymbol{x}_{h,2},\boldsymbol{x}_{h,3},\boldsymbol{x}_{h,5},\boldsymbol{x}_{h,6},\boldsymbol{x}_{h,8}), W_4(\boldsymbol{x}_{h,1}), N_4^+(\boldsymbol{x}_{h,2}), S_4^+(\boldsymbol{x}_{h,3},\boldsymbol{x}_{h,2}))\,.\end{aligned}$$

Algorithm 8.2 shows how to arrive at a function evaluation $f = f(\boldsymbol{x})$, where $\boldsymbol{x}$ is a qMC point, for the general case, where $d \in \mathbb{N}_0$.

Algorithm 8.2 Jump-diffusion bridge approach for the jump-diffusion process

1: a) $W_1 = F_N^{-1}(\boldsymbol{x}_1, 0, 1)$

b) $N_1^+ = F_P^{-1}(\boldsymbol{x}_2, p\lambda)$

c) $N_1^- = F_P^{-1}(\boldsymbol{x}_3, (1-p)\lambda)$

d) $S_1^+ = F_G^{-1}(\boldsymbol{x}_4, N_1^+, \eta^+)$

e) $S_1^- = F_G^{-1}(\boldsymbol{x}_5, N_1^-, \eta^-)$

2: **for** $i = 1$ to d **do**

3: **for** $j = 1$ to 2^{i-1} **do**

4: $k = 2^{i-1} + j$

5: a) $W_{\frac{2j-1}{2^i}} = F_N^{-1}\left(\boldsymbol{x}_{5(k-1)+1}, \frac{1}{2}(W_{\frac{2j-2}{2^i}} + W_{\frac{2j}{2^i}}), \frac{1}{2^{i+1}}\right)$

b) $N^+_{\frac{2j-1}{2^i}} = N^+_{\frac{2j-2}{2^i}} + F_B^{-1}\left(\boldsymbol{x}_{5(k-1)+2}, N^+_{\frac{2j}{2^i}} - N^+_{\frac{2j-2}{2^i}}\right)$

c) $N^-_{\frac{2j-1}{2^i}} = N^-_{\frac{2j-2}{2^i}} + F_B^{-1}\left(\boldsymbol{x}_{5(k-1)+3}, N^-_{\frac{2j}{2^i}} - N^-_{\frac{2j-2}{2^i}}\right)$

d) $S^+_{\frac{2j-1}{2^i}} = S^+_{\frac{2j-2}{2^i}} + (S^+_{\frac{2j}{2^i}} - S^+_{\frac{2j-2}{2^i}})F_{Beta}^{-1}\left(\boldsymbol{x}_{5(k-1)+4}, N^+_{\frac{2j-1}{2^i}} - N^+_{\frac{2j-2}{2^i}}, N^+_{\frac{2j}{2^i}} - N^+_{\frac{2j-1}{2^i}}\right)$

e) $S^-_{\frac{2j-1}{2^i}} = S^-_{\frac{2j-2}{2^i}} + (S^-_{\frac{2j}{2^i}} - S^-_{\frac{2j-2}{2^i}})F_{Beta}^{-1}\left(\boldsymbol{x}_{5(k-1)+5}, N^-_{\frac{2j-1}{2^i}} - N^-_{\frac{2j-2}{2^i}}, N^-_{\frac{2j}{2^i}} - N^-_{\frac{2j-1}{2^i}}\right)$

6: **end for**

7: **end for**

8: **return** $f(W, N, S)$.

Remark 8.21. *We could have used the principal components construction, see e.g. [40], to construct the Brownian motion, instead of the Brownian bridge. The corresponding modification of Algorithm 8.2 is rather obvious and not presented here.*

Remark 8.22. *Within steps* $1, \dots, s$ *of Algorithm 8.2, the ordering of the variables was chosen without comment and it is not claimed that the particular ordering presented in Algorithm 8.2* *is always the optimal choice. This issue is addressed in Subsection 8.5.3, where we show that the values of the parameters in Equation (8.19) influence the ordering.*

8.5.3 Numerical results for the jump-diffusion process

The algorithms presented in Subsections 8.5.1 and 8.5.2 are now applied to the finance problem introduced in Subsection 8.2.3. We set $T_l = \frac{l}{M}$, $l = 0, \ldots, M$, $M = 32$ and $K = 95$ in Equation (8.12) and $S_0 = 100$. The following choice of parameters is taken from [51]:

$$\sigma = 0.2,\ r = 0.05,\ \lambda = 3,\ \eta^+ = 50,\ \eta^- = 25,\ p = 0.3\,.$$

Regarding the ordering of the variates in Algorithm 8.2, we proceed as in Subsection 8.4.4. Clearly, we now have

$$\mathrm{Var}(N^-_{t_d}) > \mathrm{Var}(W_{t_d}) > \mathrm{Var}(N^+_{t_d}) > \mathrm{Var}(S^-_{t_d}) > \mathrm{Var}(S^+_{t_d})\,,$$

so the ordering becomes c), a), b), e), d) in Step 5 of Algorithm 8.2. As in Subsections 8.3.4 and 8.4.4, Table 8.3 used qMC rules based on Sobol' points as constructed in [48] with 30 digital shifts for the increment-by-increment approach (IBI) and the jump-diffusion bridge approach (JDB). For purposes of comparison, the results when applying Monte Carlo methods (MC) to the finance problem are included as well. Using Equation (8.4), we show standard errors of qMC rules employing n Sobol' points with $q = 30$. The MC results were computed using Equation (8.6) with qn random points.

	$n = 256$	$n = 1024$	$n = 4096$	$n = 16384$
MC	0.1149	0.0574	0.0289	0.0144
IBI	0.0467	0.0150	0.0070	0.0043
JDB	0.0303	0.0080	0.0029	0.0007

Table 8.3. *Standard errors corresponding to q random shifts of n qMC points and qn MC points for the jump-diffusion process problem*

The results in Table 8.3 show that the order in which the variates are simulated does matter and that the qMC rules outperform the MC method.

Remark 8.23. *Inspecting Tables 8.2 and 8.3, we note that the standard errors corresponding to the jump-diffusion bridge approach eventually drop below the standard errors corresponding* *to the compound Poisson bridge approach. This might seem curious at first, however, we point out that the drifts in Equations* (8.17) *and* (8.19) *differ. The drift in Equation* (8.19) *is smaller, which, leaving the other components of the problem unchanged, can be expected to facilitate the problem and makes the observed standard errors plausible.*

8.6 Conclusion and future work

The chapter firstly showed how to reformulate the finance problems relating to the compound Poisson process and the jump-diffusion process underlying the Kou model as integration problems so that qMC rules can be applied. Furthermore, the numerical results reported suggest the following:

- algorithms packing more variance into the opening dimensions outperform the increment-by-increment approach,
- quasi-Monte Carlo rules outperform Monte Carlo method.

As exemplified above, the ordering of the variates within the individual steps of the algorithms depends on the parameters of the relevant stochastic processes, and many variations based on the algorithms presented in this chapter are possible, which is an area of future research.

Bibliography

[1] S. Asmussen, and J. Rosiński, Approximations of small jumps of Lévy processes with a view towards simulation, J. Appl. Probab., 38, 482–493, 2001.

[2] A. N. Avramidis, and P. L'Ecuyer, Efficient Monte Carlo and quasi-Monte Carlo option pricing under the variance-gamma model, Management Sci., 52, 1930–1944, 2006.

[3] J. Baldeaux, Quasi-Monte Carlo for finance beyond Black-Scholes, ANZIAM J., 50, C884–C897, 2008.

[4] J. Baldeaux, Quasi-Monte Carlo methods for the Kou model, Monte Carlo Methods Appl., 14, 281–302, 2008.

[5] J. Baldeaux, and J. Dick, QMC rules of arbitrary high order: reproducing kernel Hilbert space approach, Constr. Approx., 30, 495–527, 2009.

[6] J. Baldeaux, and J. Dick, A construction of polynomial lattice rules with small gain coefficients, submitted, 2010.

[7] J. Baldeaux, J. Dick, J. Greslehner, and F. Pillichshammer, Construction algorithms for higher order polynomial lattice rules, submitted, 2010.

[8] J. Baldeaux, J. Dick, and P. Kritzer, On the approximation of smooth functions using generalized digital nets, J. Complexity, 25, 544–567, 2009.

[9] J. Baldeaux, J. Dick, G. Leobacher, D. Nuyens, and F. Pillichshammer, Fast construction of higher order polynomial lattice rules, in preparation.

[10] J. Baldeaux, J. Dick, and F. Pillichshammer, A characterization of higher order nets using Weyl sums and its applications, Unif. Distrib. Theory, 5, 133–155, 2010.

[11] J. Baldeaux, J. Dick, and F. Pillichshammer, Duality theory and propagation rules for higher order nets, submitted, 2010.

[12] J. Bierbrauer, Y. Edel, and W. Ch. Schmid, Coding-theoretic constructions for (t, m, s)-nets and ordered orthogonal arrays, J. Combin. Des., 10, 403–418, 2002.

[13] T. Blackmore, and G. H. Norton, Matrix-product codes over $\mathbb{F}_q$, Applicable Algebra Eng. Comm. Comput., 12, 477–500, 2001.

[14] H.-J. Bungartz, and M. Griebel, Sparse grids, Acta Numer., 13, 147–269, 2004.

[15] R. E. Caflisch, W. J. Morokoff, and A. B. Owen, Valuation of mortgage backed securities using Brownian bridges to reduce effective dimension, J. Comput. Finance, 1, 27–46, 1997.

[16] H. E. Chrestenson, A class of generalized Walsh functions, Pacific J. Math., 5, 17–31, 1955.

[17] R. Cont, and P. Tankov, *Financial modelling with jump processes*, Chapman & Hall, London, 2004.

[18] L. L. Cristea, J. Dick, G. Leobacher, and F. Pillichshammer, The tent transformation can improve the convergence rate of quasi-Monte Carlo algorithms using digital nets, Numer. Math., 105, 413–455, 2007.

[19] J. Dick, On the convergence rate of the component-by-component construction of good lattice rules, J. Complexity, 20, 493–522, 2004.

[20] J. Dick, Explicit constructions of quasi-Monte Carlo rules for the numerical integration of high-dimensional periodic functions, SIAM J. Numer. Anal., 45, 2141–2176, 2007.

[21] J. Dick, Walsh spaces containing smooth functions and quasi-Monte Carlo rules of arbitrary high order, SIAM J. Numer. Anal., 46, 1519 –1553, 2008.

[22] J. Dick, The decay of the Walsh coefficients of smooth functions, Bull. Aust. Math. Soc., 80, 430–453, 2009.

[23] J. Dick, On quasi-Monte Carlo rules achieving higher order convergence, In: P. L'Ecuyer, and A. B. Owen (eds.), *Monte Carlo and quasi-Monte Carlo methods 2008*, 73–96, Springer, Berlin, 2009.

[24] J. Dick, On the fast component-by-component algorithm for polynomial lattice rules, Available at: http://quasirandomideas.wordpress.com/2009/12/31/fast-cbc-for-polynomial-lattice-rules, Posted on December 31st, 2009, Last accessed February 2nd, 2010.

[25] J. Dick, and J. Baldeaux, Equidistribution properties of generalized nets and sequences, In: P. L'Ecuyer, and A. B. Owen (eds.), *Monte Carlo and quasi-Monte Carlo methods 2008*, 305–322, Springer, Berlin, 2009.

[26] J. Dick, and M. Gnewuch, Embedding theorems for fractional spaces and numerical integration, in preparation.

[27] J. Dick, and P. Kritzer, Duality theory and propagation rules for generalized digital nets, Math. Comp., 79, 993–1017, 2010.

[28] J. Dick, P. Kritzer, F. Pillichshammer, and W. Ch. Schmid, On the existence of higher order polynomial lattices based on a generalized figure of merit, J. Complexity, 23, 581–593, 2007.

[29] J. Dick, F. Y. Kuo, F. Pillichshammer, and I. H. Sloan, Construction algorithms for polynomial lattice rules for multivariate integration, Math. Comp., 74, 1895–1921, 2005.

[30] J. Dick, G. Leobacher, and F. Pillichshammer, Randomized Smolyak algorithms based on digital sequences for multivariate integration, IMA J. Numer. Anal., 27, 655–674, 2007.

[31] J. Dick, and H. Niederreiter, Duality for digital sequences, J. Complexity, 25, 406–414, 2009.

[32] J. Dick, and F. Pillichshammer, Multivariate integration in weighted Hilbert spaces based on Walsh functions and weighted Sobolev spaces, J. Complexity, 21, 149–195, 2005.

[33] J. Dick, and F. Pillichshammer, On the mean square weighted $\mathcal{L}_2$ discrepancy of randomized digital (t, m, s)-nets over $\mathbb{Z}_2$, Acta Arith., 117, 371–403, 2005.

[34] J. Dick, and F. Pillichshammer, Strong tractability of multivariate integration of arbitrary high order using digitally shifted polynomial lattice rules, J. Complexity, 23, 436–453, 2007.

[35] J. Dick, and F. Pillichshammer, *Digital nets and sequences*, Cambridge University Press, Cambridge, to appear, 2010.

[36] J. Dick, F. Pillichshammer, and B. J. Waterhouse, The construction of good extensible rank-1 lattices, Math. Comp., 77, 2345–2373, 2008.

[37] H. Faure, Discrépance de suites associées à un système de numération (en dimension s), Acta Arith., 41, 337–351, 1982.

[38] B. Fox, *Strategies for quasi-Monte Carlo*, Kluwer Academic Publishers, Boston, 1999.

[39] K. K. Frolov, Upper bounds for the errors of quadrature formulae on classes of functions, Dokl. Akad. Nauk SSSR, 231, 818–821, 1976. (In Russian)

[40] P. Glasserman, *Monte Carlo methods in financial engineering*, Springer, New York, 2004.

[41] J. M. Hammersley, Monte Carlo methods for solving multivariate problems, Ann. New York Acad. Sci., 86, 844–874, 1960.

[42] J. Hartinger, and M. Predota, Simulation methods for valuing Asian option prices in a hyperbolic asset price model, IMA J. Manag. Math., 14, 65–81, 2003.

[43] S. Heinrich, F. J. Hickernell, and R. X. Yue, Optimal quadrature for Haar wavelet spaces, Math. Comp., 73, 259–277, 2004.

[44] P. Hellekalek, General discrepancy estimates: the Walsh function system, Acta Arith., 67, 209–218, 1994.

[45] P. Hellekalek, Digital (t, m, s)-nets and the spectral test, Acta Arith., 105, 197–204, 2002.

[46] F. J. Hickernell, Quadrature error bounds with applications to lattice rules, SIAM J. Numer. Anal., 33, 1995–2016, 1996.

[47] F. J. Hickernell, and R. X. Yue, The mean square discrepancy of scrambled (T, S)-sequences, SIAM J. Numer. Anal., 38, 1089–1112, 2000.

[48] S. Joe, and F. Y. Kuo, Remark on algorithm 659: implementing Sobol's quasirandom sequence generator, ACM Trans. Math. Software, 29, 49–57, 2003.

[49] N. M. Korobov, Properties and calculation of optimal coefficients, Dokl. Akad. Nauk SSSR, 132, 1009-1012, 1960. (In Russian)

[50] S. G. Kou, A jump diffusion model for option pricing, Management Sci., 48, 1086–1101, 2002.

[51] S. G. Kou, and H. Wang, Option pricing under a double exponential jump diffusion model, Management Sci., 50, 1178–1192, 2004.

[52] F. Y. Kuo, Component-by-component constructions achieve the optimal rate of convergence for multivariate integration in weighted Korobov and Sobolev spaces, J. Complexity, 19, 301–320, 2003.

[53] F. Y. Kuo, W. T. M. Dunsmuir, I. H. Sloan, M. P. Wand, and R. S. Womersley, Quasi-Monte Carlo for highly structured generalised response models, Methodol. Comput. Appl. Probab., 10, 239–275, 2008.

[54] G. Larcher, Digital point sets: analysis and application, In: P. Hellekalek, and G. Larcher (eds.), *Random and quasi-random point sets*, Springer Lecture Notes in Statistics, 138, 167–222, Springer, New York, 1998.

[55] G. Larcher, A. Lauss, H. Niederreiter, and W. Ch. Schmid, Optimal polynomials for (t, m, s)-nets and numerical integration of multivariate Walsh series, SIAM J. Numer. Anal., 33, 2239 – 2253, 1996.

[56] G. Larcher, H. Niederreiter, and W. Ch. Schmid, Digital nets and sequences constructed over finite rings and their application to quasi-Monte Carlo integration, Monatsh. Math., 121, 231–253, 1996.

[57] G. Larcher, M. Predota, and R. F. Tichy, Arithmetic average options in the hyperbolic model, Monte Carlo Methods Appl., 9, 227–239, 2003.

[58] G. Larcher, and C. Traunfellner, On the numerical integration of Walsh series by number-theoretic methods, Math. Comp., 63, 277–291, 1994.

[59] P. L'Ecuyer, and C. Lemieux, Recent advances in randomized quasi-Monte Carlo methods, In: M. Dror, P. L'Ecuyer, and F. Szidarovski (eds.), *Modeling uncertainty: an examination of stochastic theory, methods, and applications*, 419–474, Kluwer Academic Publishers, Boston, 2002.

[60] G. Leobacher, Stratified sampling and quasi-Monte Carlo simulation of Lévy processes, Monte Carlo Methods Appl., 12, 231–238, 2006.

[61] J. Matoušek, *Geometric discrepancy*, Algorithms and Combinatorics, 18, Springer, Berlin, 1999.

[62] W. J. Morokoff, and R. E. Caflisch, Quasi-Monte Carlo integration, J. Comput. Phys., 122, 218–230, 1995.

[63] H. Niederreiter, Low-discrepancy point sets, Monatsh. Math., 102, 155–167, 1986.

[64] H. Niederreiter, Point sets and sequences with small discrepancy, Monatsh. Math., 104, 273–337, 1987.

[65] H. Niederreiter, Low-discrepancy point sets obtained by digital constructions over finite fields, Czechoslovak Math. J., 42, 143–166, 1992.

[66] H. Niederreiter, *Random number generation and quasi-Monte Carlo methods*, CBMS-NSF Series in Applied Mathematics, 63, SIAM, Philadelphia, 1992.

[67] H. Niederreiter, Constructions of (t, m, s)-nets, In: H. Niederreiter, and J. Spanier (eds.), *Monte Carlo and quasi-Monte Carlo methods 1998*, 70–85, Springer, Berlin, 2000.

[68] H. Niederreiter, High-dimensional numerical integration, In: J. M. Hill, and R. Moore (eds.), *Applied mathematics entering the 21st century: invited talks from the ICIAM 2003 congress*, 337–351, SIAM, Philadelphia, 2004.

[69] H. Niederreiter, Constructions of (t, m, s)-nets and (t, s)-sequences, Finite Fields Appl., 11, 578–600, 2005.

[70] H. Niederreiter, Nets, (t,s)-sequences, and codes, In: A. Keller, S. Heinrich, and H. Niederreiter (eds.), *Monte Carlo and quasi-Monte Carlo methods 2006*, 83–100, Springer, Berlin, 2008.

[71] H. Niederreiter, and F. Özbudak, Matrix-product constructions of digital nets, Finite Fields Appl., 10, 464–479, 2004.

[72] H. Niederreiter, and G. Pirsic, Duality for digital nets and its applications, Acta Arith., 97, 173–182, 2001.

[73] H. Niederreiter, and C. P. Xing, Low-discrepancy sequences and global function fields with many rational places, Finite Fields Appl., 2, 241–273, 1996.

[74] H. Niederreiter, and C. P. Xing, Quasirandom points and global function fields, In: S. Cohen, and H. Niederreiter (eds.), *Finite fields and applications*, London Mathematical Society Lecture Note Series, 233, 269–296, Cambridge University Press, Cambridge, 1996.

[75] H. Niederreiter, and C. P. Xing, Global function fields with many rational places and their applications, In: G. L. Mullen, and P. J.-S. Shiue (eds.), *Finite fields: theory, applications, and algorithms (Waterloo, ON, 1997)*, Contemp. Math., 225, 87–111, Amer. Math. Soc., Providence, 1999.

[76] H. Niederreiter, and C. P. Xing, *Rational points on curves over finite fields: theory and applications*, London Mathematical Society Lecture Note Series, 285, Cambridge University Press, Cambridge, 2001.

[77] E. Novak, *Deterministic and stochastic error bounds in numerical analysis*, Lecture Notes in Mathematics, 1349, Springer, Berlin, 1988.

[78] D. Nuyens, and R. Cools, Fast algorithms for component-by-component constructions of rank-1 lattice rules in shift-invariant reproducing kernel Hilbert spaces, Math. Comp., 75, 903–920, 2006.

[79] D. Nuyens, and R. Cools, Fast component-by-component construction, a reprise for different kernels, In: H. Niederreiter, and D. Talay (eds.), *Monte Carlo and quasi-Monte Carlo methods 2004*, 373–388, Springer, Berlin, 2006.

[80] A. B. Owen, Randomly permuted (t,m,s)-nets and (t,s)-sequences, In: H. Niederreiter, and P. J.-S. Shiue (eds.), *Monte Carlo and quasi-Monte Carlo methods in scientific computing*, 299–317, Springer, New York, 1995.

[81] A. B. Owen, Monte Carlo variance of scrambled net quadrature, SIAM J. Numer. Anal., 34, 1884–1910, 1997.

[82] A. B. Owen, Scrambled net variance for integrals of smooth functions, Ann. Statist., 25, 1541–1562, 1997.

[83] A. B. Owen, Variance with alternative scramblings of digital nets, ACM Trans. Modeling and Computer Simulation, 13, 363–378, 2003.

[84] A. Papageorgiou, The Brownian bridge does not offer a consistent advantage in quasi-Monte Carlo integration, J. Complexity, 18, 171–186, 2002.

[85] S. Paskov, and J. F. Traub, Faster valuation of financial derivatives, J. Portfolio Management, 22, 113–120, 1995.

[86] G. Pirsic, Embedding theorems and numerical integration of Walsh series over groups, PhD thesis, University of Salzburg, Austria, 1997.

[87] G. Pirsic, Base changes for (t, m, s)-nets and related sequences, Sitzungsber. Österr. Akad. Wiss. Math.-Naturw. Kl. II, 208, 115–122, 1999.

[88] G. Pirsic, A software implementation of Niederreiter-Xing sequences, In: K. T. Fang, F. J. Hickernell, and H. Niederreiter (eds.), *Monte Carlo and quasi-Monte Carlo methods 2000*, 434–445, Springer, Berlin, 2002.

[89] S. Rasmus, S. Asmussen, and M. Wiktorsson, Pricing of some exotic options with NIG-Lévy input, Lecture Notes in Computer Science, 3039, 795–802, Springer, Berlin, 2004.

[90] C. Ribeiro, and N. Webber, Valuing path-dependent options in the variance-gamma model by Monte Carlo with a gamma bridge, J. Comput. Finance, 7, 81–100, 2004.

[91] S. M. Ross, *Introduction to probability models*, sixth edition, Academic Press, San Diego, 1997.

[92] W. Ch. Schmid, Improvements and extensions of the "Salzburg tables" by using irreducible polynomials, In: H. Niederreiter, and J. Spanier (eds.), *Monte Carlo and quasi-Monte Carlo methods 1998*, 436–447, Springer, Berlin, 2000.

[93] R. Schürer, Ordered orthogonal arrays and where to find them, PhD thesis, University of Salzburg, Austria, 2006, Available at: `http://mint.sbg.ac.at/rudi/projects/corrected_diss.pdf`, Last accessed March 28th, 2010.

[94] R. Schürer, and W. Ch. Schmid, MinT - new features and new results, In: P. L'Ecuyer, and A. B. Owen (eds.), *Monte Carlo and quasi-Monte Carlo methods 2008*, 171–189, Springer, Berlin, 2009.

[95] I. F. Sharygin, A lower estimate for the error of quadrature formulas for certain classes of functions, Ž. Vyčisl. Mat. i Mat. Fiz., 3, 370–376, 1963. (In Russian)

[96] M. M. Skriganov, Coding theory and uniform distributions, Algebra i Analiz, 13, 191–239, 2001.

[97] I. H. Sloan, and S. Joe, *Lattice methods for multiple integration*, Oxford Science Publications, New York, 1994.

[98] I. H. Sloan, F. Y. Kuo, and S. Joe, Constructing randomly shifted lattice rules in weighted Sobolev spaces, SIAM J. Numer. Anal., 40, 1650–1665, 2002.

[99] I. H. Sloan, F. Y. Kuo, and S. Joe, On the step-by-step construction of quasi-Monte Carlo integration rules that achieve strong tractability error bounds in weighted Sobolev spaces, Math. Comp., 71, 1609–1640, 2002.

[100] I. H. Sloan, and A. V. Reztsov, Component-by-component construction of good lattice rules, Math. Comp., 71, 263–273, 2002.

[101] I. H. Sloan, and H. Woźniakowski, When are quasi-Monte Carlo algorithms efficient for high dimensional integrals?, J. Complexity, 14, 1–33, 1998.

[102] N. J. A. Sloane, and D. S. Whitehead, A new family of single-error correcting codes, IEEE Trans. Inform. Theory, 16, 717–719, 1970.

[103] S. A. Smolyak, Quadrature and interpolation formulas for tensor products of certain classes of functions, Dokl. Akad. Nauk SSSR, 4, 240–243, 1963. (In Russian)

[104] I. M. Sobol′, Distribution of points in a cube and approximate evaluation of integrals, Ž. Vyčisl. Mat. i Mat. Fiz., 7, 784–802, 1967. (In Russian)

[105] J. L. Walsh, A closed set of normal orthogonal functions, Amer. J. Math., 45, 5–24, 1923.

[106] R. X. Yue, Variance of quadrature over scrambled unions of nets, Statist. Sinica, 9, 451–473, 1999.

[107] R. X. Yue, and F. J. Hickernell, The discrepancy and gain coefficients of scrambled digital nets, J. Complexity, 18, 135–151, 2002.

[108] R. X. Yue, and F. J. Hickernell, Strong tractability of integration using scrambled Niederreiter points, Math. Comp., 74, 1871–1893, 2005.

[109] R. X. Yue, and S. S. Mao, On the variance of quadrature over scrambled nets and sequences, Statist. Probab. Lett., 44, 267–280, 1999.