Exact Simulation of Occupation Times

Roman N. Makarov and Karl Wouterloot

Abstract A novel algorithm for the exact simulation of occupation times for Brownian processes and jump-diffusion processes with finite jump intensity is constructed. Our approach is based on sampling from the distribution function of occupation times of a Brownian bridge. For more general diffusions we propose an approximation procedure based on the Brownian bridge interpolation of sample paths. The simulation methods are applied to pricing occupation time derivatives and quantile options under the double-exponential jump-diffusion process and the constant elasticity of variance (CEV) diffusion model.

1 Introduction

Consider a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$. For a stochastic process $\mathbf{S} = (S_t)_{t\geq 0}$, adapted to the filtration $(\mathscr{F}_t)_{t\geq 0}$, the occupation times below and above level $L \in \mathbb{R}$ from time 0 to $T > 0$ are respectively defined as follows:

$$A_T^{L,-}(\mathbf{S}) \equiv \int_0^T \mathbf{1}_{S_t \leq L}\,\mathrm{d}t \ \ \text{(below } L\text{)} \ \ \text{and} \ \ A_T^{L,+}(\mathbf{S}) \equiv \int_0^T \mathbf{1}_{S_t > L}\,\mathrm{d}t \ \ \text{(above } L\text{)}. \qquad (1)$$

The occupations times $A_T^{L,\pm}$ are nonnegative quantities and satisfy $A_T^{L,+} + A_T^{L,-} = T$. We will also use the notation $A_{[u,v]}^{L,+} \equiv \int_u^v \mathbf{1}_{S_t > L}\,\mathrm{d}t$ and $A_{[u,v]}^{L,-} \equiv \int_u^v \mathbf{1}_{S_t \leq L}\,\mathrm{d}t$ to denote the occupation times on an arbitrary time interval, $[u,v]$, $0 \leq u < v$.

Note that a strictly monotone transformation of a process does not change the distribution of occupation times. Suppose the process $\mathbf{X} = (X_t)_{t\geq 0}$ is obtained by applying a strictly monotone mapping X to the process $\mathbf{S}$, i.e. $X_t = \mathsf{X}(S_t)$ for $t \geq 0$. Then, $A_t^{L,\pm}(\mathbf{S}) \stackrel{d}{=} A_t^{\ell,\pm}(\mathbf{X})$, for every $t > 0$, where $\ell = \mathsf{X}(L)$. In the paper, we consider

Department of Mathematics, Wilfrid Laurier University, Waterloo, Ontario, Canada, e-mail: rmakarov@wlu.ca and e-mail: kwouterloot@gmail.com

two asset pricing models that can be mapped to other simpler organized processes. In particular, Kou's model (Section 3) is an exponential Lévy process; the CEV diffusion model (Section 4) is a power-type transformation of the CIR model.

There have been numerous papers published on the distribution of occupation times for Brownian motion with and without drift. By using the Feynman-Kac formula, the joint density function of the occupation time and terminal asset value was obtained in [14] and [19] (see also [5]). A similar approach was used in [13] to derive the distribution function of the occupation time for a standard Brownian bridge from 0 to 0. Analytical pricing formulae for occupation time derivatives under the constant elasticity of variance (CEV) diffusion models are obtained in [18]. However, a numerical implementation of those results is difficult.

In this paper, we generalize the result of [13]. For one important case, we are able to express the cumulative distribution functions (c.d.f.'s) of occupation times in terms of the error function and elementary functions. This result allows us to apply the inverse c.d.f. method for the efficient Monte Carlo simulation of occupation times for various (jump-)diffusion processes.

Consider a market consisting of three securities: a risk-free bond with the price process $(B_t = B_0 \mathrm{e}^{rt})_{t\geq 0}$, a risky asset with the price process $(S_t)_{t\geq 0} \in \mathbb{R}_+ \equiv [0,\infty)$, and an occupation-time-related option contingent upon the asset. There are a large number of different derivatives whose payoff functions depend on occupation times of an asset price process. In this paper, we are interested in claims $f^{\pm}$ whose payoff is of the form $f^{\pm} = f(S_T, A_T^{L,\pm})$, for some function $f : \mathbb{R}_+ \times [0,T] \to \mathbb{R}_+$.

Assume there exists an equivalent probability measure (e.m.m. for short) $\widetilde{\mathbb{P}}$ such that the discounted asset price process $(\mathrm{e}^{-rt} S_t)_{t\geq 0}$ is a $\widetilde{\mathbb{P}}$-martingale. The arbitrage free price processes $(V_t^{f,\pm})_{0\leq t\leq T}$ of the claims $f^{\pm}$ are thus defined by

$$V_t^{f,\pm} = \mathrm{e}^{-r(T-t)} \widetilde{\mathbb{E}} \left[f(S_T, A_T^{L,\pm}) \mid \mathscr{F}_t \right]. \tag{2}$$

Step options were first proposed as an alternative to barrier options in [19]. The payoff functions of the proportional step call and step put options are respectively given by $f^{\text{call}}_{\text{step}}(S_T, A_T) = (S_T - K)^+ \mathrm{e}^{-\rho A_T}$ and $f^{\text{put}}_{\text{step}}(S_T, A_T) = (K - S_T)^+ \mathrm{e}^{-\rho A_T}$, where $\rho \geq 0$, and the occupation time A_T in these formulae is given by (1).

As one can see, the payoff function of a step option works under the same principles as knock-and-out barrier options, but with less risk. If a step down option is purchased, the holder's payout will be discounted by the occupation time below L, provided that the process $\mathbf{S}$ does hit L before time T. Letting $\rho \to 0+$, a step option becomes a vanilla European option. Letting $\rho \to \infty$, the payoff of a step option becomes that of a barrier option, since $\lim_{\rho\to\infty} \exp\left(-\rho A_T^{L,-}\right) = \mathbf{1}_{\inf_{0\leq t\leq T} S_t > L}$ a.s..

The payoff functions of the fixed-strike call and floating strike put α-quantile options are respectively defined by $(M_T^{\alpha}(\mathbf{S}) - K)^+$ and $(M_T^{\alpha}(\mathbf{S}) - S_T)^+$, where $M_T^{\alpha}(\mathbf{S}) \equiv \inf\{L : A_T^{L,-} \geq \alpha T\}$ is known as the α-quantile ($0 < \alpha < 1$). The α-quantile options may be viewed as a generalization of lookback options.

There is a remarkable relationship between the α-quantile of a Lévy process and the distribution of the maximum and minimum values of the process obtained in [11]. Let $(X_t)_{t\ge 0}$ and $(Y_t)_{t\ge 0}$ be independent copies of a process $\mathbf{X}$ with stationary and independent increments and with $X_0 = Y_0 = 0$. Then, there is the following equivalence in distribution:

$$\begin{pmatrix} X_t \\ M_t^{\alpha}(\mathbf{X}) \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} X_{\alpha t} + Y_{(1-\alpha)t} \\ \sup_{0\le s\le \alpha t} X_s + \inf_{0\le s\le (1-\alpha)t} Y_s \end{pmatrix}. \tag{3}$$

In this paper, we consider the simulation of occupation times and quantiles for jump-diffusion models and nonlinear solvable diffusions. The main application is the pricing of occupation-time and α-quantile options. As a result, we obtain efficient Monte Carlo algorithms for two asset pricing models, namely, the Kou jump-diffusion model [16] and the CEV diffusion model [10]. Our approach can be easily extended to other Lévy models with finite jump intensity as well as to other solvable state-dependent volatility diffusion models [7].

2 Occupation Times of a Brownian Bridge

Let $(W_t^x)_{t\ge 0}$ denote the Brownian motion starting at $x \in \mathbb{R}$. The Brownian bridge $W_{[0,T]}^{x,y}$ from x to y over $[0,T]$ is defined by $W_{[0,T]}^{x,y}(t) \stackrel{d}{=} \{W_t^x \mid W_T^x = y\}$, $0 \le t \le T$.

Theorem 1. *The c.d.f.* $F_\ell^+(\tau; y) \equiv \mathbb{P}\left\{A_1^{\ell,+}(W_{[0,1]}^{0,y}) \le \tau\right\}$, $0 < \tau < 1$, *of the occupation time above level ℓ for a Brownian bridge from* 0 *to y over* $[0,1]$ *is given by the following cases.*

Case (I) *For $y \le \ell$ and $\ell \ge 0$,*

$$F_\ell^+(\tau; y) = 1 - \frac{2\sqrt{\tau}}{\pi} e^{\frac{y^2}{2}} \int_\tau^1 e^{-\frac{(2\ell-y)^2}{2(1-u)}} \frac{\sqrt{u-\tau}}{u^2\sqrt{1-u}}\, du \tag{4}$$

$$= 1 - (1-\tau) e^{-\frac{b}{\tau} + \frac{y^2}{2}} \left(e^b (2b+1)\, \mathrm{erfc}\,(\sqrt{b}) - 2\sqrt{\frac{b}{\pi}} \right), \tag{5}$$

where $b = \frac{2(\ell - y/2)^2 \tau}{1-\tau}$.

Case (II) *For* $0 \le \ell < y$,

$$F_\ell^+(\tau; y) = \int_0^\tau \frac{(\tau - u) e^{\frac{y^2}{2} - \frac{\ell^2}{2(1-u)} - \frac{(y-\ell)^2}{2u}}}{\sqrt{2\pi}(u(1-u))^{\frac{3}{2}}} \times \left(\frac{\ell(y-\ell)^2}{u} - \frac{(y-\ell)^2 \ell^2}{1-u} + y - 2\ell \right) du. \tag{6}$$

Case (III) *For* $\ell < 0$, $F_\ell^+(\tau; y) = 1 - F_{-\ell}^+(1-\tau; -y)$.

Proof. The case with $x = y = 0$ was done in [13]. For the general case, the argument is almost exactly the same. First, we consider Case (I). Let $f^{t,x}(\tau|y)$ denote the p.d.f. of $A_t^{\ell,+}(W^x)$ conditional on $W_t^x = y$, $0 \le \tau \le t$, $x, y \in \mathbb{R}$. Note that $f^{1,0}(\tau|y) = \frac{\partial}{\partial \tau} F_\ell^+(\tau; y)$. The Fourier transform and double Laplace transform of the joint p.d.f. for $A_t^{\ell,+}$ and W_t^x is $\frac{u(x;p,\lambda,\beta)}{\sqrt{2\pi}} \equiv \mathscr{F}_y\left[\mathscr{L}_t\left[\mathscr{L}_\tau[f^{t,x}(\tau|y)\frac{1}{\sqrt{2\pi t}}\mathrm{e}^{-(y-x)^2/2t};\beta];\lambda\right];p\right]$. By the Feynman-Kac formula, u is a unique solution to $\frac{1}{2}u''(x) - (\lambda + \beta \mathbf{1}_{x>\ell})u(x) = -\mathrm{e}^{ipx}$, subject to conditions $u(\ell-) = u(\ell+)$ and $u'(\ell-) = u'(\ell+)$. From [13], when $x = 0$ we have that $u(0) = \frac{-4\beta\left(\sqrt{2(\lambda+\beta)}+ip\right)}{(2\lambda+p^2)(2(\lambda+\beta)+p^2)} \frac{\exp\left(-\ell\sqrt{2\lambda}+ip\ell\right)}{\sqrt{2(\lambda+\beta)}+\sqrt{2\lambda}} + \frac{2}{2\lambda+p^2}$. Applying the inverse Fourier transform, we obtain that

$$\begin{aligned}\mathscr{L}_t\left[\mathscr{L}_\tau\left[f^{t,0}(\tau|y)\frac{\mathrm{e}^{-\frac{y^2}{2t}}}{\sqrt{2\pi t}};\beta\right];\lambda\right] &= \mathscr{F}_p^{-1}\left[\frac{u(0;p,\lambda,\beta)}{\sqrt{2\pi}};y\right] \\ &= \frac{\mathrm{e}^{-y\sqrt{2\lambda}}}{\sqrt{2\lambda}} - \frac{\sqrt{\lambda+\beta}-\sqrt{\lambda}}{\sqrt{\lambda+\beta}+\sqrt{\lambda}}\mathrm{e}^{(y-2\ell)\sqrt{2\lambda}}.\end{aligned} \tag{7}$$

Taking the inverse Laplace transform of both sides of (7), we obtain

$$1 - \mathscr{L}_\tau\left[f^{1,0}(\tau|y);\beta\right] = \mathrm{e}^{\frac{y^2}{2}} \int_0^1 \frac{\mathrm{e}^{-\frac{\beta u}{2}} I_1\left(\frac{\beta u}{2}\right)\mathrm{e}^{-\frac{(2\ell-y)^2}{2(1-u)}}}{u\sqrt{1-u}}\mathrm{d}u. \tag{8}$$

Integration by parts gives $1 - \mathscr{L}_\tau\left[f^{1,0}(\tau|y);\beta\right] = \beta\mathscr{L}_\tau\left[1 - F_\ell^+(\tau;y);\beta\right]$. Applying the identity $I_1(z/2) = \frac{2z\mathrm{e}^{z/2}}{\pi}\int_0^1 \sqrt{v(1-v)}\mathrm{e}^{-zv}\mathrm{d}v$ in (8), changing the order of integration, and changing variables $uv = \tau$, we obtain equation (4) by uniqueness of the Laplace transform. Changing variables $u = \frac{\tau+\tau x^2}{1+\tau x^2}$ and simplifying the integral obtained, we arrive at (5).

The proof of Case (II) follows a similar argument. From [13], we obtain the formula for $u(0)$. Taking the inverse Fourier transform and then the double inverse Laplace transform, we obtain (6). The derivation can be done by using tables of Fourier transform and Laplace transform pairs and the shift theorem. Case (III) follows by symmetry of the Brownian motion. □

Here, the complementary error function, denoted *erfc*, is defined as

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty \mathrm{e}^{-u^2}\,\mathrm{d}u.$$

Since $A_t^{\ell,-} + A_t^{\ell,+} = t$ holds for every $t \ge 0$, the c.d.f. F_ℓ^- of the occupation time $A_1^{\ell,-}(W_{[0,1]}^{0,y})$ is given by $F_\ell^-(\tau) = 1 - F_\ell^+(1-\tau)$, $0 \le \tau \le 1$. Note that if $x = y = \ell = 0$, then $A_1^{0,\pm}(W_{[0,1]}^{0,0}) \sim \text{Uniform}(0,1)$ (see [5]).

The c.d.f.'s $F_\ell^\pm$ for an arbitrary time interval of length T can be obtained from the respective c.d.f.'s for the time interval of length 1 thanks to the property

$$\mathbb{P}\left\{A_T^{\ell,\pm} \le t \mid W_0 = x, W_T = y\right\} = \mathbb{P}\left\{A_1^{\frac{\ell}{\sqrt{T}},\pm} \le \frac{t}{T} \mid W_0 = \frac{x}{\sqrt{T}}, W_1 = \frac{y}{\sqrt{T}}\right\}.$$

By using the symmetry properties of the Brownian motion, we can evaluate the c.d.f. of $A_T^{\ell,\pm}(W_{[0,T]}^{x,y})$ for the general case with arbitrary x, y, and ℓ. The following equalities in distribution are valid:

$$A_T^{\ell,\pm}\left(W_{[0,T]}^{x,y}\right) \stackrel{d}{=} A_T^{\ell,\mp}\left(W_{[0,T]}^{2\ell-x,2\ell-y}\right) \stackrel{d}{=} A_T^{\ell-x,\pm}\left(W_{[0,T]}^{0,y-x}\right) \stackrel{d}{=} A_T^{-\ell,\mp}\left(W_{[0,T]}^{-x,-y}\right).$$

In Theorem 1, we obtain the c.d.f. of the occupation time above level ℓ for a Brownian motion pinned at points x and y at times 0 and 1, respectively. In practice, the c.d.f. for the case where both x and y lie on one side with respect to the level ℓ can be computed more easily than for the other case. For example, if $x = 0$, $y \le \ell$, and $\ell \ge 0$, then the c.d.f. of $A_1^{\ell,+} = A_1^{\ell,+}(W_{[0,1]}^{0,y})$ given in (5) is expressed in terms of the complimentary error function, which is fast and easy to compute. Therefore, one can use the inverse c.d.f. method to draw the occupation time $A_1^{\ell,+}$. Note that there is a non-zero probability that the Brownian bridge $W_{[0,1]}^{0,y}$ with $y < \ell$ does not cross level $\ell > 0$. Thus, the probability $\mathbb{P}\{A_1^{\ell,+} = 0\}$ is not zero in this case. From (5), we obtain $\mathbb{P}\{A_1^{\ell,+} = 0\} = F_\ell^+(0;y) = 1 - \mathrm{e}^{-2\ell(\ell-y)}$, which is also the probability that the Brownian bridge $W_{[0,1]}^{0,y}$ does not hit the level ℓ.

We also need to consider the other case where the Brownian motion is pinned at points x and y that lie on the opposite sides of the barrier ℓ. For example, if $x = 0$ and $0 \le \ell < y$, then c.d.f. of $A_1^{\ell,+}$ is given by the integral in (6), which is computationally expensive to evaluate during the simulation process when parameters randomly change. To overcome this difficulty, we propose a two-step procedure. First, we sample the first hitting time $\tau_\ell \in (0,T)$ at the barrier ℓ of the Brownian bridge $W_{[0,T]}^{x,y}$, where $x < \ell < y$ or $y < \ell < x$. Then, we sample the occupation time of the Brownian bridge from ℓ to y over $[\tau_\ell, T]$. Since the new bridge starts at the level ℓ, the c.d.f. of the occupation time can be reduced to the integral in (5). Recall that the first hitting time (f.h.t. for short) τ_ℓ of a diffusion process $(X_t)_{t\ge0}$ with almost surely continuous paths is defined by $\tau_\ell(x) = \inf\{t > 0 : X_t = \ell \mid X_0 = x\}$. The c.d.f. F_ℓ^τ of the f.h.t. τ_ℓ, $\ell > 0$, of the Brownian bridge $W_{[0,T]}^{0,y}$, $\ell < y$, is given entirely in terms of error functions, which are quick to compute. It has the following form for $0 < t < T$ (see [4]):

$$\begin{aligned} F_\ell^\tau(t;y) &= \mathbb{P}\{\tau_\ell \le t \mid W_0 = 0, W_T = y\} = \mathbb{P}\{\max_{0\le s\le t} W_s \ge \ell \mid W_0 = 0, W_T = y\} \\ &= \frac{1}{2}\mathrm{e}^{-\frac{2\ell(\ell-y)}{T}}\,\mathrm{erfc}\left(\frac{\ell T - (2\ell - y)t}{\sqrt{2}}\right) + \frac{1}{2}\,\mathrm{erfc}\left(\frac{\ell T - yt}{\sqrt{2tT(T-t)}}\right). \end{aligned} \tag{9}$$

We obtain Algorithm 1 for sampling $A_T^{\ell,\pm}$, where we assume $\ell \ge x$. In the case of $\ell < x$, one can use the equality in distribution: $A_T^{\ell,\pm}(W_{[0,T]}^{x,y}) \stackrel{d}{=} T - A_T^{-\ell,\pm}(W_{[0,T]}^{-x,-y})$.

Algorithm 1 Sampling occupation times $A_T^{\ell,\pm}$ for a Brownian Bridge $W_{[0,T]}^{x,y}$

input x, y, $T > 0$, $\ell \geq x$
set $y_1 \leftarrow \frac{y-x}{\sqrt{T}}$, $\ell_1 \leftarrow \frac{\ell-x}{\sqrt{T}}$
if $y_1 \leq \ell_1$ **then**
  sample $U \sim \text{Uniform}(0,1)$
  set $A \leftarrow \sup\{t \in [0,1] : F_{\ell_1}^{+}(t; y_1) < U\}$
  set $A_T^{\ell,+} = A \cdot T$, $A_T^{\ell,-} = T - A_T^{\ell,+}$
else
  sample i.i.d. $U, V \sim \text{Uniform}(0,1)$
  set $\tau \leftarrow \sup\{t \in [0,1] : F_{\ell_1}^{\tau}(t; y_1) < V\}$
  set $y_2 \leftarrow \frac{y_1-\ell_1}{\sqrt{1-\tau}}$
  set $A \leftarrow 1 - \sup\{t \in [0,1] : F_0^{+}(t; -y_2) < U\}$
  set $A_T^{\ell,+} \leftarrow A \cdot (1-\tau) \cdot T$, $A_T^{\ell,-} \leftarrow T - A_T^{\ell,+}$
end if
return $A_T^{\ell,\pm}$

Note that the bridge distribution of a Brownian motion with drift, $\{W_t + \nu t, t \geq 0\}$, is the same as that of a standard Brownian motion. Thus, the distributions of occupation times will not change with introducing a non-zero drift (see [5]).

3 Pricing Occupation Time Options under a Jump Diffusion

In this section, we propose an algorithm for the exact simulation of occupation times of a Lévy process that has a Gaussian component and a jump component of compound Poisson type. Suppose the stock price is governed by the following dynamics:

$$\frac{\mathrm{d}S_t}{S_{t-}} = \nu\,\mathrm{d}t + \sigma\,\mathrm{d}W_t + \mathrm{d}\left(\sum_{i=1}^{N_t}(V_i - 1)\right), \quad S_{t=0} = S_0 > 0, \tag{10}$$

where ν and σ are constants, $(W_t)_{t\geq 0}$ is a standard Brownian motion, $(N_t)_{t\geq 0}$ is a Poisson process with arrival rate λ, and $\{V_i\}_{i=1,2,\ldots}$ is a sequence of independent identically distributed (i.i.d.) random variables. We assume that (W_t), (N_t), and $\{V_i\}$ are jointly independent.

As an example, we consider Kou's double exponential jump diffusion model [16], where the random variables $Y_i = \ln(V_i)$ follows a double exponential distribution with the p.d.f. $f_Y(y) = p\eta_+ \mathrm{e}^{-\eta_+ y}\mathbf{1}_{y\geq 0} + (1-p)\eta_- \mathrm{e}^{-\eta_-|y|}\mathbf{1}_{y<0}$, where $\eta_+ > 1$, $\eta_- > 0$, $p \in [0,1]$. There are two types of jumps in the process: upward jumps (with occurrence probability p and average jump size $\frac{1}{\eta_+}$) and downward jumps (with occurrence probability $1-p$ and average jump size $\frac{1}{\eta_-}$). Both types of jumps are exponentially distributed.

The stochastic differential equation (s.d.e. for short) in (10) can be solved analytically. Under an e.m.m. $\widetilde{\mathbb{P}}$, we have that $\nu = r - \lambda\zeta$, where $\zeta = \widetilde{\mathbb{E}}[\mathrm{e}^Y - 1]$ is given

Algorithm 2 Simulation of a sample path, occupation times, and extremes for a jump-diffusion model (S_t)

input: moments of jumps $\mathscr{T}_1 < \ldots < \mathscr{T}_N$ on $[0,T]$ and values $\{X_{k-}, X_k\}_{k=1,\ldots,N}$,
where $X_k = X(\mathscr{T}_k)$, $X_{k-} = X(\mathscr{T}_k -)$, and $X(t) \equiv \frac{1}{\sigma} \ln S_t$
set $m_0^X \leftarrow 0,\ M_0^X \leftarrow 0,$
for n **from** 1 **to** N **do**
  sample $A_n = A^{\ell,+}_{[\mathscr{T}_{n-1},\mathscr{T}_n]}\left(W^{X_{n-1},X_{n-}}_{[\mathscr{T}_{n-1},\mathscr{T}_n]}\right)$
  sample $U_n, V_n \sim \text{Uniform}(0,1)$
  set $m_n^X \leftarrow \min\{m_{n-1}^X, X_{n-1} + \frac{1}{2}\left(B_n - \sqrt{B_n^2 - 2\Delta\mathscr{T}_n \ln U_n}\right)\}$
  set $M_n^X \leftarrow \max\{M_{n-1}^X, X_{n-1} + \frac{1}{2}\left(B_n + \sqrt{B_n^2 - 2\Delta\mathscr{T}_n \ln V_n}\right)\}$
end for
set $S_T \leftarrow S_0 \mathrm{e}^{\sigma X_N},\ m_T \leftarrow S_0 \mathrm{e}^{\sigma m_N^X},\ M_T \leftarrow S_0 \mathrm{e}^{\sigma M_N^X}$
set $A_T^{L,+} \leftarrow \sum_{n=1}^N A_n,\ A_T^{L,-} \leftarrow T - A_T^{L,+}$
return S_T and only one of $A_T^{L,\pm}$, m_T, M_T

by $\zeta = \frac{p\eta_+}{\eta_+ - 1} + \frac{(1-p)\eta_-}{\eta_- + 1} - 1$, and $S_t = S_0 \exp\left((r - \frac{\sigma^2}{2} - \lambda\zeta)t + \sigma W_t + \sum_{i=1}^{N_t} Y_i\right)$ (see [17]). Note that $\widetilde{\mathbb{P}}$ can be obtained by using the Esscher transform.

A Lévy process with finite jump intensity behaves like a Brownian motion between successive jumps. The simulation scheme is well known (e.g., see [9]). First, we sample the time and size of each jump occurred on $[0,T]$. Second, we sample the Brownian increment for each time-interval between successive jumps. The only addition to this scheme is the sampling of occupation times. As a result, we obtain Algorithm 2. We can also sample the minimum value m_T and the maximum value M_T of a Lévy sample path. These values are used for pricing α-quantile options thanks to the property in (3). Notice that Algorithm 2 is implemented in a way so that it allows the user to sample the extreme values and the occupation times from their correct marginal distributions, but with an improper joint distribution. Therefore, only one quantity from the list $\{m_T, M_T, A_T^{L,\pm}\}$ can be used after each execution of Algorithm 2. This is sufficient for our applications. To sample an α-quantile option payoff, the user needs to run the algorithm twice to obtain independent sample values of the maximum and minimum. It is possible to sample m_T and M_T from their joint distribution, but the joint distribution of occupation times and extremes for a Brownian bridge is not available to the best of our knowledge.

4 Pricing Occupation Time Options under the CEV Model

Simulation of path-dependent variables such as the running minimum/maximum and occupation times is a challenging computational problem for general stochastic processes. In the case of Brownian motion (and its derivatives) with or without a compound Poisson component, exact simulation algorithms can be constructed by using the Brownian bridge interpolation. This procedure suggests an approximation for more general diffusions.

Consider a discrete-time skeleton of a sample path. Its continuous-time approximation can be obtained by interpolating over each subinterval using independent Brownian bridges. Such an approach can be used to approximately simulate the minimum and maximum and barrier crossing probabilities (see [1, 3, 15]), however resulting estimates of path-dependent quantities are biased. We apply this idea to approximately simulate occupation times of the constant elasticity of variance (CEV) diffusion for which an exact path sampling algorithm is available in [21].

4.1 Exact Simulation of the CEV Process.

The CEV diffusion $\mathbf{S} = (S_t)_{t\geq 0} \in \mathbb{R}_+$ follows $\mathrm{d}S_t = \nu S_t\,\mathrm{d}t + \delta S_t^{\beta+1}\,\mathrm{d}W_t$, $S_{t=0} = S_0 > 0$, where $\delta > 0$ and $\nu \in \mathbb{R}$. Under the e.m.m. $\widetilde{\mathbb{P}}$, we have that $\nu = r$. Here we assume that $\beta < 0$, hence the boundary $s = 0$ of the state space $[0,\infty)$ is regular. Here we consider the case where the endpoint $s = 0$ is a killing boundary. Let τ_0 denote the first hitting time at zero. We assume that $S_t = 0$ for all $t \geq \tau_0$.

The CEV process is a transformation of the Cox-Ross-Ingersoll (CIR) diffusion model $\mathbf{X} = (X_t)_{t\geq 0}$ that follows $\mathrm{d}X_t = (\lambda_0 - \lambda_1 X_t)\,\mathrm{d}t + 2\sqrt{X_t}\,\mathrm{d}W_t$ (see [5]). Indeed, by using Itô's formula, it is easy to show that the mapping $\mathsf{X}(s) \equiv (\delta|\beta|)^{-2} s^{-2\beta}$ (which is strictly increasing since $\beta < 0$) transforms a CEV process into an CIR process with $\lambda_0 = 2 + \frac{1}{\beta}$ and $\lambda_1 = 2\nu\beta$, i.e. $X_t = \mathsf{X}(S_t)$. Moreover, the CIR process can be obtained by a scale and time transformation of the square Bessel (SQB) process. Also note that the radial Ornstein-Uhlenbeck (ROU) process $\mathbf{Z} = (Z_t)_{t\geq 0}$, obeying the s.d.e. $\mathrm{d}Z_t = \left(\frac{\lambda_0 - 1}{2Z_t} - \frac{\lambda_1 Z_t}{2}\right)\mathrm{d}t + \mathrm{d}W_t$, can be obtained by taking the square root of the CIR process, i.e. $Z_t = \sqrt{X_t}$.

The literature on simulating the CIR and other related processes is rather extensive (e.g., see [15] and references therein). However, most of existing algorithms either are approximation schemes or deal with the case without absorption at zero. In [21], a general exact sampling method for Bessel diffusions is presented. The sampling method allows one to exactly sample a variety of diffusions that are related to the SQB process through scale and time transformations, change of variables, and by change of measure. These include the CIR, CEV, and hypergeometric diffusions described in [7]. The paths of the CEV and CIR processes can be sampled simultaneously at time moments $\{t_i\}_{i=0}^N$, $0 = t_0 < t_1 < \ldots < t_N$ conditional on $S_{t=0} = S_0$ as outlined below.

1. Apply Algorithm 3 to sample a path of the SQB process $\mathbf{Y}$ with index $\mu = \frac{1}{2\beta}$, at time points $\{u_i = u(t_i; \lambda_1 = 2\nu\beta)\}_{i=0}^N$ conditional on $Y_0 = \mathsf{X}(S_0)$. Here we define $u(t;\lambda_1) = \frac{\mathrm{e}^{\lambda_1 t} - 1}{\lambda_1}$ if $\lambda_1 \neq 0$, and $u(t;\lambda_1) = t$ if $\lambda_1 = 0$.
2. Use the scale and time transformation to obtain sample paths of the CIR model $\mathbf{X}$ as follows: $X_{t_i} \equiv \mathrm{e}^{\lambda_1 t_i} Y_{u_i}$ for each $i = 0, 1, \ldots, N$.

3. Transform by using the mapping $S_{t_i} = \mathsf{X}^{-1}(X_{t_i})$, $i = 1, \ldots, N$, to obtain a discrete-time sample path of the CEV process $\mathbf{S}$.

Algorithm 3 Simulation of an SQB sample path

The sequential sampling method conditional on the first hitting time at zero, τ_0, for modelling an SQB process with absorption at the origin (see [21]).

input $Y_0 > 0$, $0 = u_0 < u_1 < \ldots < u_N$, $\mu < 0$
sample $G \sim \text{Gamma}(|\mu|, 1)$; **set** $\tau_0 \leftarrow \frac{Y_0}{2G}$
for n **from** 1 **to** N **do**
  if $u_n < \tau_0$ **then**
    sample $P_n \sim \text{Poisson}\left(\frac{Y_{u_{n-1}}(\tau_0 - u_n)}{2(\tau_0 - u_{n-1})(u_n - u_{n-1})}\right)$
    sample $Y_{u_n} \sim \text{Gamma}\left(P_n + |\mu| + 1, \frac{\tau_0 - u_{n-1}}{(\tau_0 - u_n)(u_n - u_{n-1})}\right)$
  else
    set $Y_{u_n} \leftarrow 0$
  end if
end for
return $(Y_0, Y_{u_1}, \ldots, Y_{u_N})$

4.2 Simulation of Occupation Times for the CEV Processes.

The CEV process $\mathbf{S}$ can be obtained by applying a monotone transformation to the ROU process $\mathbf{Z}$ and vice versa. Indeed, $Z_t = \sqrt{\mathsf{X}(S_t)}$, $t \geq 0$. The diffusion coefficient of the s.d.e. describing the ROU process equals one. Therefore, (Z_t) can be well approximated by a drifted Brownian motion on short time intervals. If (Z_t) is pinned at times t_{i-1} and t_i that are close enough together, the process will behave like a Brownian motion pinned at the same times. Therefore, on short time intervals $[t_1, t_2]$, the occupation times of the CEV process conditional on $S_{t_i} = s_i > 0$, $i = 1, 2$, can be approximated by occupation times of a Brownian bridge, i.e.

$$\left(A^{L,\pm}_{[t_1,t_2]}(\mathbf{S}) \mid S_{t_1} = s_1, S_{t_2} = s_2\right) \stackrel{d}{=} \left(A^{\ell,\pm}_{[t_1,t_2]}(\mathbf{Z}) \mid Z_{t_1} = z_1, Z_{t_2} = z_2\right)$$
$$\stackrel{d}{\approx} \left(A^{\ell,\pm}_{[t_1,t_2]}(\mathbf{W}) \mid W_{t_1} = z_1, W_{t_2} = z_2\right),$$

where $\ell = \sqrt{\mathsf{X}(L)}$, $L > 0$, and $z_i = \sqrt{\mathsf{X}(s_i)}$, $i = 1, 2$. Note that numerical tests demonstrate that the Brownian bridge interpolation procedure produces more accurate estimates of occupation times if it is applied to the ROU process rather than the CEV diffusion. Alternatively, one can use a piecewise-linear approximation of continuous-time sample paths of the ROU process to approximate occupation times. The latter approach can also be used to compute α-quantiles of a sample path.

A more rigorous stochastic analysis of such approximation approaches is the matter of our future research.

Since the origin is an absorbing boundary, occupation times only need to be simulated until the maturity T or τ_0, whichever comes first. For arbitrary $T > 0$ and $L > 0$, we have $A_T^{L,+}(\mathbf{S}) = A_{T\wedge\tau_0}^{L,+}(\mathbf{S})$ and $A_T^{L,-}(\mathbf{S}) = T - A_T^{L,+}(\mathbf{S})$. Our strategy for the approximate sampling of occupation times $A_{T\wedge\tau_0}^{L,\pm}(\mathbf{S})$ for the CEV process works as follows.

1. For a given time partition $\{0 = t_0 < t_1 < \ldots < t_N = T \wedge \tau_0\}$, draw a sample CEV path, $S_{t_1}, \ldots, S_{t_N}$, conditional on S_0 and $\tau_0 = \tau_0(S_0)$.
2. Obtain the respective sample path of the ROU process by using the transformation $Z_{t_i} = \sqrt{\mathsf{X}(S_{t_i})}$ for each $i = 0, 1, \ldots, N$.
3. Sample the occupation times of $A_{[t_{i-1},t_i]}^{\ell,\pm}$ for the Brownian bridge from $Z_{t_{i-1}}$ to Z_{t_i} over $[t_{i-1}, t_i]$ for each $i = 1, \ldots, N$. Here, $\ell = \sqrt{\mathsf{X}(L)}$.
4. Obtain the approximation: $A_{t_N}^{L,\pm}(\mathbf{S}) \approx \sum_{i=1}^N A_{[t_{i-1},t_i]}^{\ell,\pm}$.

4.3 The First Hitting Time Approach

There is another approach that can speed up the pricing of occupation time options. Suppose $S_0 > L$ and consider an option whose payoff depends on $A_T^{L,-}$. By using the fact that the events $\{A_T^{L,-} = 0\}$ and $\{\tau_L > T\}$, where $\tau_L = \tau_L(S_0)$ is the first hitting time down at L, are equivalent, we can rewrite the no-arbitrage price of the option as follows:

$$\begin{aligned} \mathrm{e}^{-rT}\widetilde{\mathbb{E}}\left[f(S_T, A_T^{L,-})\right] &= \mathrm{e}^{-rT}\widetilde{\mathbb{E}}\left[f(S_T, 0)\mathbf{1}_{A_T^{L,-}=0}\right] + \mathrm{e}^{-rT}\widetilde{\mathbb{E}}\left[f(S_T, A_T^{L,-})\mathbf{1}_{A_T^{L,-}>0}\right] \\ &= \mathrm{e}^{-rT}\widetilde{\mathbb{E}}\left[f(S_T, 0)\mathbf{1}_{\tau_L > T}\right] + \mathrm{e}^{-rT} p_T \widetilde{\mathbb{E}}\left[f(S_T, A_T^{L,-}) \mid \tau_L \le T\right]. \end{aligned} \tag{11}$$

where the probability $p_T = \mathbb{P}\{\tau_L < T\} = \mathbb{P}\{A_T^{L,-} > 0\}$ can be computed by using results of [20]. Notice that the first term in (11) is the no-arbitrage price for a down-and-out barrier option. The analytical price of the down-and-out barrier option under the CEV model is well known (see [12]). Thus, the first term in (11) can be computed analytically, while the second term can be estimated by the Monte Carlo method.

First, we sample the first hitting time down τ_L with the condition $\tau_L \le T$. The c.d.f. of the first hitting time down is given by the spectral expansion (see [20]). It is computationally expensive to evaluate such an expansion, thus the c.d.f. of τ_L should be computed once on a fine partition of $[0, T]$ and stored in memory. After that, the inverse c.d.f. method is applied to sample τ_L conditional on $\{\tau_L \le T\}$. Second, we sample $A_T^{L,-}$. Since the process $\mathbf{S}$ first hits the level L at τ_L, the only time that $\mathbf{S}$ can spend below L occurs after τ_L. Therefore, the process need not be sampled on the interval $[0, \tau_L]$, since we only need the occupation time below L and the terminal asset price to compute the payoff of an option. Alternatively, one can use the f.h.t.

approach to speed up the sampling of the occupation times thanks to the following property: $A^{L,-}_{[0,T]}(S_{t=0}=S_0) \stackrel{d}{=} \mathbf{1}_{\tau_L \le T} \times A^{L,-}_{[\tau_L,T]}(S_{t=\tau_L}=L)$.

5 Numerical Results

As a test case for Algorithm 2, prices of some proportional step down options with payoffs depending on $A^{L,-}_T$ and α-quantile options were computed. First, we consider pricing under Kou's model. The parameters used in simulations were $S_0 = 100$, $T = 1$ (years), $r = 0.05$, $\sigma = 0.3$, $\lambda = 3$, $p = 0.5$, $\eta_+ = 30$, $\eta_- = 20$, $\rho = 1$, and $L = 102$. Monte Carlo *unbiased* estimates of proportional step option prices were computed for a range of strikes with $N = 10^6$ trials; the results are given in Table 1. In all tables below, s_N denotes the sample standard deviation of the Monte Carlo estimate. Also, all of the simulations in this section were implemented in MATLAB® 7.10.0, and they were run on a Intel Pentium® 4 1.60GHz processor with 3GB of RAM.

Table 1 The Monte Carlo unbiased estimates of proportional step down call option prices under Kou's model are tabulated for various K and S_0. The parameters are $T = 1$, $r = 0.05$, $\sigma = 0.2$, $\lambda = 3$, $p = 0.5$, $\eta_+ = 30$, $\eta_- = 20$, $\rho = 1$, $L = 102$, and $N = 10^6$.

	$S_0 = 100$		$S_0 = 105$	
K	Estimate $\pm s_N$	Exact	Estimate $\pm s_N$	Exact
90	13.7715 ± 0.0173	13.81883	19.0374 ± 0.0207	19.04025
100	9.3901 ± 0.0147	9.42438	13.4581 ± 0.0181	13.45927
110	5.9558 ± 0.0120	5.97929	8.9005 ± 0.0152	8.90134

Simulations of α-quantiles under Kou's model were performed using the exact sampling algorithm. Monte Carlo *unbiased* estimates of fixed strike α-quantile option prices were obtained for various values of K and σ from $N = 10^6$ trials. The other model parameters used in these simulations are $S_0 = 100$, $T = 1$, $r = 0.05$, $\lambda = 3$, $p = 0.6$, $\eta_+ = 34$, $\eta_- = 34$, and $\alpha = 0.2$. The results of these simulations are given in Table 2. Tables 1 and 2 contain the exact prices of the occupation time options taken from [6]. We can observe the perfect agreement between the Monte Carlo estimates and the exact values.

Monte Carlo *biased* estimates of proportional step down option prices under the CEV model were also computed. This was done using the exact CEV path sampling algorithm together with the Brownian bridge approximation or the piecewise linear path interpolation. To reduce the variance, the estimator of a standard European option price was used as a control variate. The Monte Carlo estimates of option prices are compared with the analytical estimates obtained in [8]. Recall that the origin is an absorbing boundary for the CEV model. If the asset price process hits the zero boundary before the maturity date, then the asset goes to bankruptcy and

Table 2 The Monte Carlo unbiased estimates of α-Quantile call option prices for various K and σ under Kou's model are tabulated for $\alpha = 0.2$. The parameters used are $S_0 = 100$, $T = 1$, $r = 0.05$, $\lambda = 3$, $p = 0.6$, $\eta_+ = 34$, $\eta_- = 34$, and $N = 10^6$.

	$\sigma = 0.2$		$\sigma = 0.3$	
K	Estimate $\pm s_N$	Exact	Estimate $\pm s_N$	Exact
90	6.9982 ± 0.0074	6.98492	6.7290 ± 0.0092	6.72912
100	2.0793 ± 0.0043	2.08466	2.6993 ± 0.0060	2.69358
110	0.3666 ± 0.0018	0.37724	0.8643 ± 0.0034	0.86545

a derivative on the asset becomes worthless. Thus, the payoff function is given by

$$f^{\pm}_{\text{step}}(A^{L,\pm}_T(\mathbf{S}), S_T) = \mathrm{e}^{-\rho A^{L,\pm}_T(\mathbf{S})} f(S_T) \mathbf{1}_{T<\tau_0}.$$

The estimates were obtained from averaging over $N = 10^6$ samples, with a time step of $\Delta t = 0.05$. These approximate prices are given in Table 3, for a range of K, and with $\rho = 0.5$, $L = 90$ and $T = 1$. The CEV model parameters used in all simulations were $\delta = 2.5$, $\beta = -0.5$, and $r = 0.1$.

Table 3 The Monte Carlo biased estimates of proportional step down call and put prices under the CEV model using the Brownian bridge interpolation method are tabulated for various values of K. The parameters used are $S_0 = 100$, $T = 1$, $r = 0.1$, $\delta = 2.5$, $\beta = -0.5$, $\rho = 0.5$, $L = 90$, $\Delta t = 0.05$, and $N = 10^6$.

	Step Calls		Step Puts	
K	Estimate $\pm s_N$	Exact	Estimate $\pm s_N$	Exact
90	20.9950 ± 0.0009	20.9976	2.0416 ± 0.0006	2.0382
100	14.8192 ± 0.0006	14.8207	4.1988 ± 0.0010	4.1938
110	9.8621 ± 0.0004	9.8633	7.5750 ± 0.0017	7.5689

The Brownian bridge interpolation and linear interpolation sampling methods were compared for various values of Δt. To do this, Monte Carlo estimates of the step call prices were computed using both of these sampling methods for a range of Δt and with $N = 5 \cdot 10^6$ trials. The results of these simulations are shown in Table 4. As is seen from the table, the Brownian bridge interpolation method works quite accurately even for $\Delta t = 0.5$ (i.e., a sample skeleton only consists of two points).

Finally, the first hitting time method was used to price the proportional step down put option under the CEV model. These simulations used the exact CEV path sampling algorithm along with the Brownian bridge approximation method for $N = 10^6$ trials and with $\Delta t = 0.05$. The prices were computed for a range of K, and they are given in Table 5 along with their standard errors. As is seen from the table, the cost of the f.h.t. method is twice less than that of the regular algorithm. The cost of a MCM algorithm is defined as a product of the sample variance and the computational time.

Table 4 The Monte Carlo biased estimates of proportional step call prices under the CEV model obtained with the use of the Brownian bridge approximation and linear interpolation methods are compared for decreasing time steps Δt. The parameter values used in the simulations are $S_0 = 100$, $T = 1$, $r = 0.1$, $\delta = 2.5$, $\beta = -0.5$, $\rho = 0.5$, $L = 90$, $K = 100$, $N = 5 \cdot 10^6$. The analytical estimate of the option price is 14.8207.

	Bridge Interpolation		Linear Interpolation	
Δt	Estimate $\pm s_N$	Time (sec)	Estimate $\pm s_N$	Time (sec)
0.5	14.8184 ± 0.0003	19 160	14.8451 ± 0.0004	7 105
0.25	14.8191 ± 0.0003	38 815	14.7906 ± 0.0004	15 685
0.1	14.8194 ± 0.0003	78 770	14.7931 ± 0.0003	41 835
0.05	14.8191 ± 0.0006	142 150	14.8046 ± 0.0003	76 560

Table 5 The Monte Carlo biased estimates of proportional step down put option prices under the CEV model using the Brownian bridge approximation are tabulated for various values of K. Also, the regular path sampling algorithm (a) is compared to the accelerated first hitting time sampling algorithm (b). The other parameters used are $S_0 = 100$, $T = 1$, $r = 0.1$, $\delta = 2.5$, $\beta = -0.5$, $\rho = 0.5$, $L = 90$, $\Delta t = 0.05$, and $N = 10^6$.

K	Estimate (a) $\pm s_N$	Estimate (b) $\pm s_N$	Exact
90	2.0394 ± 0.0006	2.0407 ± 0.0004	2.0382
100	4.1927 ± 0.0010	4.1974 ± 0.0008	4.1938
110	7.5616 ± 0.0017	7.5730 ± 0.0012	7.5689
	(Time is 28430 sec)	(Time is 26875 sec)	

6 Conclusions

In this paper, we study the simulation of occupation times of Brownian processes, jump diffusions, and state-dependent volatility diffusion models. An efficient algorithm for the exact sampling of occupation times of a Brownian bridge is presented. It is used for the exact simulation of occupation times for Kou's jump-diffusion model. We apply this method to pricing occupation time derivatives. Also, a similar algorithm is designed for pricing quantile options. The sampling method is efficient and can be extended to general Lévy processes. It works for any finite activity process provided that an exact path simulation algorithm is available. Infinite activity Lévy processes can be treated by replacing small jumps with a diffusion term (e.g., see [2]).

By using the Brownian bridge interpolation of a general diffusion process, we obtain an approximate sampling algorithm for occupation times of the CEV diffusion model. The prices of proportional step options are computed by the Monte Carlo method. The approach can be extended to other types of occupation time derivatives and also to other solvable diffusion models (e.g., see [7]).

Acknowledgements The first author acknowledges the support of the Natural Sciences and Engineering Research Council of Canada (NSERC) for a Discovery Research Grant. We thank the two anonymous reviewers for their comments, which helped to improve the paper.

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