Abstract

We show how, for all dimensions and signatures, the most general first-order linear symmetry operator for the massive Dirac equation is given in terms of Killing-Yano tensors. In the massless case the Killing-Yano condition is relaxed to the conformal Killing-Yano generalisation.

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1 Introduction

Kamran and McLenaghan [1] showed that the most general first-order linear symmetry operator for the massless Dirac equation on a four-dimensional Lorentzian manifold can be made from a conformal Killing vector and a conformal Killing-Yano tensor. In the more restrictive case of a Killing vector, or Killing-Yano tensor, this operator is a symmetry operator of the massive Dirac equation as was previously found by Carter and McLenaghan [2].
The conformal Killing-Yano equation is invariant under Hodge duality. Thus in four dimensions the only conformal Killing-Yano tensors, apart from conformal Killing vectors and their duals, are of degree two. In four dimensions the self-dual and anti-self-dual 2-forms correspond (projectively) to semi-spinors. Thus there are a number of special features of four dimensions that are not present in general, and it is not immediately obvious how these results on Dirac symmetry operators generalise. Benn and Charlton [3] showed that conformal Killing-Yano tensors (of any degree) give rise to Dirac symmetry operators on spin manifolds of arbitrary dimension and signature. They did not, however, show that all first-order linear symmetry operators were thus given. Here we show that this is in fact the case: that is, for arbitrary spin manifolds the most general first-order linear Dirac symmetry operator is given in terms of conformal-Killing-Yano tensors. Thus Kamran and McLenaghan’s results extend to the general case.

2 Results

We will generally follow the conventions and notation of [3], to which the reader is referred. In particular we will identify the space of differential forms with sections of the Clifford bundle, and will juxtapose symbols to denote the Clifford product.

Initially we consider the case of an even-dimensional spin manifold. Suppose that we have some operator $L$ that is a symmetry operator for the massive Dirac equation. That is, $L$ commutes with the Dirac operator $D$. Then $L$ can be decomposed into even and odd parts,

$$L_\pm = \frac{1}{2} \left( L \pm zLz^{-1} \right)$$

where $z$ is the volume $n$-form. These parts separately commute with $D$. It will prove convenient to work with the graded commutator, and to this end we let $L'_\pm = L_\pm z$ so that $[L'_\pm, D] = 0$ where the bracket denotes the graded commutator. Thus given any operator that commutes with the Dirac operator we have a $\mathbb{Z}_2$-homogeneous operator which has vanishing graded commutator with $D$, and vice versa.

For the case of the massless Dirac equation an operator $L$ is a symmetry operator if there is an operator $R$ such that $DL = RD$. Again it proves convenient to write this equivalently in terms of the graded commutator as

$$[D, L] = MD .$$

(Such an $L$ is usually said to ‘$R$-commute’ with $D$, the letter $R$ being employed instead of $M$. We use the latter to avoid any possible confusion with the curvature tensor.) Thus (1) is the requirement that $L$ be a Dirac symmetry operator, with the stronger requirement that $M$ be zero in the massive case. If we can find any $L$ satisfying (1) then, in the massless case, we can always add a multiple of $D$ to $L$ and still satisfy (1) (for some
different $M$ of course). We will exploit this freedom, and show that in fact $M$ can be taken to be of order zero, as is often assumed in the four-dimensional case [1, 6, 5].

In even dimensions the complex Clifford algebra is a total matrix algebra and so any linear transformation on spinors is effected by the Clifford action. Thus the most general first-order linear operator is of the form

$$ L = \omega^a \nabla X_a + \Omega. $$

(2)

In view of the preceding remarks, in seeking symmetry operators we may as well assume that $\omega^a$ and $\Omega$ are $\mathbb{Z}_2$-homogeneous, that is, made up from sums of forms of either even or odd degree, (but inhomogeneous with respect to the $\mathbb{Z}$-grading of the exterior algebra). If $L$ is restricted to be first order then the operator $M$ in (1) cannot be of order greater than one and we write it similarly as

$$ M = 2m^a \nabla X_a + m, $$

(3)

where the factor of 2 proves convenient later.

We may then express the graded commutator as

$$ [D, L] = X^b \lrcorner \omega^a \left( \nabla^2 (X_a, X_b) + \nabla^2 (X_b, X_a) \right) $$

$$ + (e^a (\nabla X_a \omega)^b + 2X^b \lrcorner \Omega) \nabla X_b $$

$$ - (e^b \wedge \omega^a R(X_a, X_b) - e^a \nabla X_a \Omega). $$

(4)

For (1) to hold we separately equate terms of order 2, 1 and 0.

Equating second-order terms gives

$$ X^b \lrcorner \omega^a + X^a \lrcorner \omega^b = m^a e^b + m^b e^a. $$

(5)

We will show that (5) implies that $m^a$ can be expressed in terms of $\omega^a$ in such a way as to show that a multiple of $D$ can be added to $L$ to make $m^a$ vanish. Multiplying (5) on the right by $e_a$ gives

$$ (X^a \lrcorner \omega^b + X^b \lrcorner \omega^a) e_a = (n + 2) m^b - m^b e_a e^b; $$

and (5) gives $x^a \lrcorner \omega_a = m^a e_a$ and so we have

$$ (n + 2) m^b = (X^a \lrcorner \omega^b + X^b \lrcorner \omega^a) e_a + (X^a \lrcorner \omega_a) e^b. $$

(6)

Multiplying (6) on the left by $e_b$ gives (after some manipulation)

$$ e_b m^b = -X_a \lrcorner \eta \omega^a, $$

(7)

where the linear operator $\eta$ is defined by

$$ \eta \phi(p) = (-1)^p \phi(p). $$
for a $p$-form $\phi(p)$. If we multiply (5) on the left by $e_a$ and use (7) we get
\[
(n - 2\pi)m^b = (\pi + 1)\eta\omega^b - X^bJ(e_a\eta\omega^a) + (X_a\omega^a)e^b,
\]
where the linear operator $\pi$ is defined by
\[
\pi\phi(p) = p\phi(p).
\]
Adding (6) and (8) then gives
\[
(n + 1 - \pi)m^b = e^b \wedge \eta(X_a\omega^a).
\]
We have shown that (9) is a necessary condition for (5). We will now show that this is sufficient to enable a multiple of $D$ to be added to $L$ in order to give $m^a = 0$. Let $\hat{L} = L + \alpha D$ for some form $\alpha$. Then $\hat{\omega}^a = \omega^a + \alpha e^a$ and we will have $\hat{m}^b = 0$ if we can choose $\alpha$ such that
\[
(n - \pi)\eta\alpha + X_a\omega^a = 0.
\]
Clearly we can choose $\alpha$ to satisfy (10) (leaving the $n$-form component of $\alpha$ arbitrary). Note that the resulting $\hat{L}$ is $\mathbb{Z}_2$-homogeneous of the same degree as $L$. From now on we will assume that an appropriate multiple of $D$ has been added to $L$ such that the first-order term in $M$ is zero. Then (5) is equivalent to
\[
\omega^a = X^a\omega
\]
for some form $\omega$, ($\omega$ is $\mathbb{Z}_2$-homogeneous, but can be $\mathbb{Z}$-inhomogeneous).

Now we return to (1) and equate first-order terms. Putting (11) into (4) gives
\[
\nabla X_b\omega - X_b\{e^a\nabla X_a\omega - 2\Omega\} = me_b.
\]
We can write this as
\[
\nabla X_b\omega = e_b \wedge \eta m + X_b\{s\}
\]
where
\[
s = d\omega - d^*\omega - 2\Omega - \eta m.
\]
(We have used the relations between covariant and exterior derivatives, $d = e^a \wedge \nabla X_a$ and $d^* = -X_a\nabla X_a$.) Taking the exterior product of (12) with $e^b$ gives
\[
\pi s = d\omega
\]
and hence
\[
(X_b\{s\})_p(p) = \frac{1}{p + 1}X_b\{d\omega(p)\}.
\]
Contracting (12) with $X^b$ gives
\[
(n - \pi)\eta m = -d^*\omega
\]
and hence
\[(e_b \wedge \eta_m)_p = -\frac{1}{n + 1 - p} e_b \wedge d^p \omega(p) .\]
Thus (12) requires that each $p$-form part of $\omega$ separately satisfies
\[\nabla_{X_b} \omega(p) = \frac{1}{p + 1} X_b \omega(p) - \frac{1}{n + 1 - p} e_b \wedge d^p \omega(p)\]
which is the conformal Killing-Yano equation.

Now we still have to ensure that the degree zero term in (1) vanishes. However, at this point we make contact with [3]. We have shown that if $L$ satisfies (1) we can always add a multiple of $D$ to $L$ such that $L$ is a sum of terms of the form $L = 2K_\omega$ where $\omega$ is a conformal Killing-Yano form and
\[K_\omega = X^a \omega_\nabla X_a + \frac{p}{p + 1} d\omega - \frac{n - p}{n + 1 - p} d^p \omega .\]
This is the expression introduced in [3]. It was there shown how the integrability conditions for the conformal Killing-Yano equation ensure that the zeroth-order term in (1) does indeed vanish.

Now we consider the odd-dimensional case. In this case the complex Clifford algebra is the sum of two simple ideals, each simple ideal being isomorphic to the even sub-algebra [4]. Thus if the operator $L$ in (2) acts on semi-spinors we may assume that the forms $\{ \omega^a \}$ and $\Omega$ are even. The argument then proceeds as before and we conclude that the symmetry operator may be made from an odd conformal Killing-Yano form. (Duality invariance of the conformal Killing-Yano equation means that we could equally well choose an even conformal Killing-Yano form.) Thus in odd dimensions too any first-order symmetry operator must be as given in [3].

### 3 Future directions

Historically Killing tensors were introduced by formally generalising Killing’s equation. The totally-symmetric generalisations, Killing tensors, have a certain prominence in that they generalise the property of Killing vectors of having vanishing Schouten bracket with the metric tensor [7], the Schouten bracket being a generalisation of the Lie bracket. Although the importance of Killing-Yano tensors has been known for some time, the results of this paper give them an increased prominence: they correspond exactly to first-order symmetry operators of the Dirac equation.

One obvious direction in which to try and generalise the results of this paper is by considering second-order (and higher) symmetry operators. (This has been done (in four
dimensions) by [5].) One reason for doing this is to consider the algebra of symmetry operators. The (graded) commutator of first-order Dirac symmetry operators is (in general) second order. If we knew how to write all such second order operators then it might prove possible to embed the Killing-Yano forms into a (graded) Lie algebra of generalised Killing tensors in such a way that we have an algebra homomorphism with the algebra of symmetry operators.

Formally the Dirac operator resembles the Hodge-deRham operator. Nonetheless, the obvious formal extension of the Dirac symmetry operators to operators on differential forms does not (save for flat space) give a symmetry operator for the Hodge-deRham operator [3]. We are currently working on constructing the most general Hodge-deRham symmetry operator.

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References


