PROOF IN MATHEMATICS

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Proof is what makes mathematics different from other sciences. In physics, or biology, or economics, results come only after hard work with observations and experiments. In mathematics, there is proof instead. Let us start with an example. How can you add up the numbers 1 to 100 (in some sensible way that gives some understanding)? Write down:

$$1 + 2 + 3 + 4 + \ldots + 98 + 99 + 100$$

Now write underneath them the same numbers, backwards:

$$1 + 2 + 3 + 4 + \ldots + 98 + 99 + 100$$

$$100 + 99 + 98 + 97 + \ldots + 3 + 2 + 1$$

Now add each number in the top row to the one directly below it. It is clear that each pair adds up to 101. There are 100 pairs. So the sum of all the numbers written down is 100×101. This is twice the sum of the numbers 1 to 100 (since we wrote that sum down twice). So

$$1 + 2 + 3 + 4 + \ldots + 98 + 99 + 100$$

$$= \tfrac{1}{2} \times 100 \times 101$$

which is 5050. This answer has thus been *proved* correct.

A proof has three obvious advantages over simply getting the answer by putting the numbers into a calculator.

1. You *understand* the whole process completely, and hence get that sense of intellectual power which is one of the payoffs of studying mathematics.

2. As a result of your understanding, you have *certainty* that the result is correct. There is no prospect of a mistake in entering the numbers, or of the kind of measurement and observational errors that plague the other sciences.

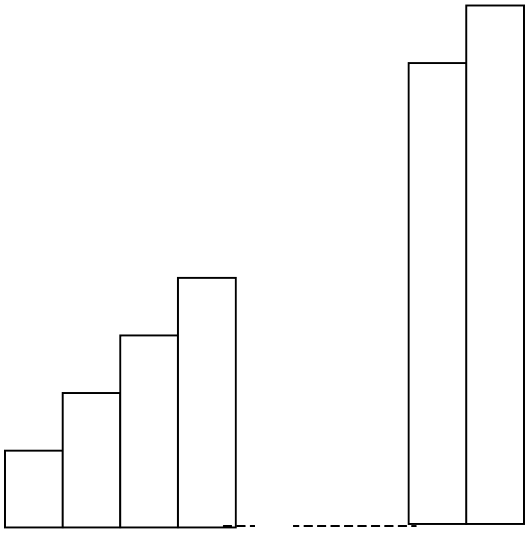
3. Also because you understand, you can *generalise*: it is clear that the fact that the last number was 100 had nothing to do with it. It is just as easy to prove, with the same method, that

$$1 + 2 + 3 + 4 + \ldots + n = \tfrac{1}{2}n\,(n + 1).$$

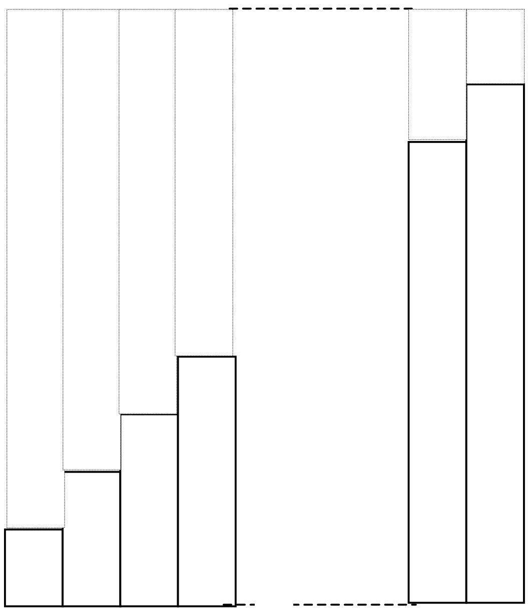
If anything, the problem is easier for arbitrary n than for 100, since there is no chance of being distracted by any particular facts about the number 100. So an proving an infinite number of facts can be easier than proving one. (And then you might think about whether it matters that each number is *one* more than the last: would the same method work if the numbers were, say, *three* apart? What if the first number wasn't 1?)

Proof by pictures

'A picture is worth a thousand words', the saying goes. It certainly takes as much disk space, if you want to store it. As computer power and disk space have become cheaper, computing has become more graphics-oriented: yesterday, bar charts, tomorrow, moving pictures to visualise large data sets. At the same time, psychologists have become more interested in the brain's interpretation of pictorial information. The combination has begun to direct attention to the role of diagrams in understanding mathematics, especially in grasping proof. Let's go back to the example of adding the numbers 1 to *n*. It is natural to draw the problem thus:



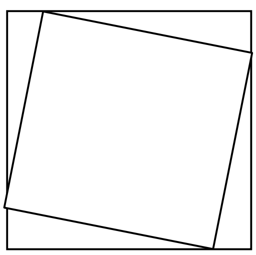
Now, imagine taking a copy of this diagram out of the page, turning it over, and placing it above the original diagram, in the position pictured below with the dotted boundary:



It is clear that the total area of the rectangle formed is $n\,(n + 1)$, so that the sum of the numbers 1 to n is half this.

It is also clear that this proof is just a geometrical version of the symbolic proof above: it might be easier to grasp the picture intuitively, but it would be simple to translate it into words and symbols, if desired.

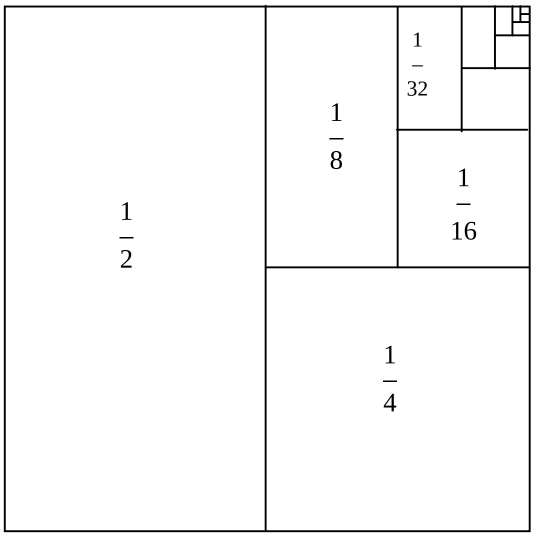
In the following example, though, it is not clear how to translate into symbols. This is an ancient Indian proof of Pythagoras' theorem:



(Explain carefully how it *does* prove Pythagoras' theorem).

Here is another one, which probably *could* be translated into words, but not without a good deal of trouble. It is a proof that

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots = 1.$$



If still pictures are so good, might moving ones be even better? Somebody thinks so. The *New Scientist* of 3 August 1991 prints a picture said to be a still from the world's first video proof (running time 15 minutes). Computer games experts, take note.

The formal sciences

It used to be that mathematics was clearly distinct from the "experimental" sciences. The idea was that proof was confined to mathematics, because mathematics dealt with "abstract" things like numbers or sets. Physics or biology, on the other hand, dealt with real things, and so had to observe and experiment. Consequently, they lacked the absolute certainty of proved results. Einstein said,

> As far as the propositions of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality.

His remark expresses well the physicist's or engineer's view of mathematics as a grab-bag of "formulas" or "methods", a kind of "theoretical juice extractor", not a science that is *about* something.

This picture has been disturbed in the last fifty years or so by the appearance of a large number of new sciences, lying somewhere between mathematics and the older natural sciences. They are called variously the "formal" sciences, or the "mathematical" sciences, or the "sciences of complexity" or "sciences of the artificial". Here are the names of some of them:

- Operations research

- Computer science
- Artificial intelligence
- systems and control / optimization
- (descriptive) statistics / pattern recognition / signal processing / numerical taxonomy
- Network analysis / traffic flow
- Queueing theory
- Information theory
- Game theory / mathematical economics
- Mathematical ecology
- Mathematical linguistics
- Self-organizing systems / theory of cellular automata / artificial life / non-equilibrium thermodynamics
- Mathematical modelling / computer simulation

Proof has spilled over from mathematics into these formal sciences. To give a flavour of what is in them, and how they prove things about the world, let us look at an old example. It is due to Euler in the mid-eighteenth century. Then an isolated example, it is now regarded as the first study in the topology of networks. The citizens of Königsberg noticed that it seemed to be impossible to walk across all seven bridges over the River Pregel, without walking across at least one of them twice. The bridges linked two banks of the river and two islands, thus:

(Draw a few paths, and convince yourself that whatever you do with six bridges, you cannot get to the seventh bridge without crossing one of those you have crossed earlier). Euler proved their conjecture correct, using the simple idea that if one enters and leaves a land area, one uses up two of the bridges. Thus, all the land areas (except the two chosen for the start and finish) must have an even number of bridges leaving them, or there will necessarily be bridges left over, no matter what route is chosen. But all four land areas have an odd number of bridges leaving them, so a path going across all the bridges exactly once is impossible. Notice also that it is possible to solve the problem without any ingenuity at all, by simply checking by computer whether all the possible paths which do not go over any bridge twice (there are certainly less than a few thousand of these) go over all bridges once. It is a less exciting method, but the result is exactly the same: it proves an impossibility about an actual physical thing, resulting from its structure. No idealization is needed to get a mathematical result.

Further reading:

J. Franklin & A. Daoud, *Introduction to Proofs in Mathematics* (Prentice Hall, Sydney, 1988).

W. Bown, ‘New-wave mathematics’, *New Scientist* **131** (3 August 1991): pp. 31-35.

L. Euler, ‘The Königsberg bridges’, translated in *Scientific American* **189** (1), (July 1953): pp. 66-70, reprinted in *Mathematics in the Modern World* (Freeman, San Francisco, 1968), pp. 141-144.