THE SYDNEY SCHOOL
AN ARISTOTELIAN REALIST PHILOSOPHY OF MATHEMATICS

INTRODUCTION

Mathematics is a science of the real world, just as much as biology or sociology are. Where biology studies living things and sociology studies human social relations, mathematics studies the structural aspects of things, or patterns, or complexity. A typical mathematical truth is that there are six different pairs in four objects:

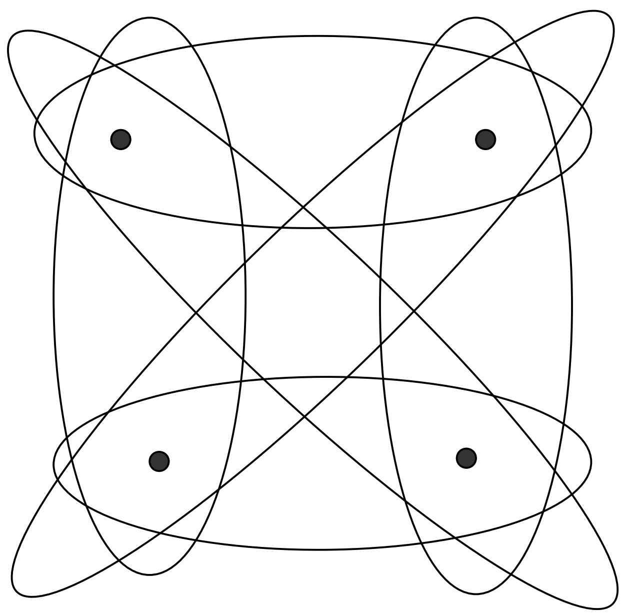

Fig 1 There are 6 different pairs in 4 objects

The objects may be of any kind, physical, mental or abstract. The mathematical statement does not refer to any properties of the objects, but only to patterning of the parts in the complex of the four objects. If that seems to us less a solid truth about the real world than the causation of flu by viruses, that is simply due to our blindness about relations, or tendency to regard them as somehow less real than things and properties. But relations (for example, relations of equality between parts of a structure) are as real as colours or causes. There is therefore nothing to be said for Bertrand Russell's dictum that "mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true."[1] Nor is there anything in the view of engineers that mathematics is no more than a grab-bag of methods and formulas, a "theoretical juice-extractor" for deriving one substantial truth from others.

This perspective raises a number of questions, which are pursued in the following essays.

First, what exactly does "structure" or "pattern" mean? These are somewhat vague words, and the various attempts to develop a structuralist philosophy of mathematics (listed in the bibliography) have either not addressed the definitional problem, or have looked for some kind of sets or other abstract objects to be "structures". One should look instead for a characterisation of what properties of things are structural. That is done in the Introduction.

The next question concerns the necessity of mathematical truths, from which follows the possibility of having certain knowledge of them. Philosophies of mathematics have generally been either empiricist in the style of Mill and Lakatos, denying the necessity and certainty of mathematics, or admitting necessity but denying mathematics a direct application to the real world (for different reasons in the case of Platonism, formalism and logicism). An Aristotelian philosophy of mathematics, however, like the present one, finds necessity in truths directly about the real world. Examples, and discussion of how this is possible, are found in the essay `Mathematical necessity and reality'. The situation in the wider mathematical or formal sciences such as operations research, where the combination of necessity and reality is in some ways clearer than in mathematics proper, is described in `The formal sciences discover the philosophers' stone'. The interview `Philosophy, mathematics and structure' gives a lighter introduction to the same themes. The second part contains a number of brief essays on the implications of such a philosophy for mathematics, in such areas as the impact of computing, the unifying role of complexity and modelling in the applications of mathematics, and the role of proof in knowledge of mathematical necessity. Formalists encouraged a view of mathematics as a manipulation of symbols, and logicists emphasised derivations from arbitrary axioms, which led to a certain aridity in the teaching of mathematics and in the public's view of it. A structuralist view will reinstate mathematics in its deserved place as one of civilisation's prime grips on reality.

The fact that mathematical truths may often be proved does not exclude the possibility that there should be experimental evidence for them. Some conjectures have good evidence for them, and it is that evidence that justifies the effort of trying to prove them. The third section

[1] B. Russell, *Mysticism and Logic* (London, 1917), p. 75.

surveys this topic, in the essay `Non-deductive logic in mathematics'. The existence of experimental evidence in *mathematics*, where truths are necessary, shows the need to revive Keynes' view that probability is, at least sometimes, a matter of pure logic, a kind of partial implication which holds between hypotheses and the evidence for them. This view is defended in the essay `Resurrecting logical probability', where it is contrasted with the more popular frequentist and subjective Bayesian views of probability.

The objectivity of mathematics has always been a mainstay of objectivist views in ethics and elsewhere. Easy relativisms about ethical or scientific truth, that assert that "it's only your opinion" or "the opinions of all tribes are historically constructed and so equally valid", have had difficulty explaining away the absolute objectivity of mathematical truths. But the objectivity of mathematics has usually been supported by a Platonist philosophy, according to which our knowledge acquires its certainty through contact with another world of numbers and "abstract" objects. That philosophy is not plausible, nor is it well adapted to supporting the parallel with objectivity in, say, ethics, which does not appear to need contact with other worlds. As argued in `Last bastion of reason' and `On the parallel between mathematics and ethics', the parallel is better served by an Aristotelian philosophy that sees both mathematics and ethics as dealing in real relations like equality. The introduction of some calculation into ethics to quantify rights is suggested in `Accountancy as computational casuistics'. The volume concludes with discussion of two issues in the philosophy of religion of a mathematical nature, Pascal's wager and Leibniz's theory that this is the best of all possible worlds.

INTRODUCTION

Consider six points, with each pair joined by a line. The lines are all coloured, in one of two colours (represented by dotted and undotted lines in the figure). Then there must exist a triangle of one colour (that is, three points such that all three of the lines joining them have the same colour).

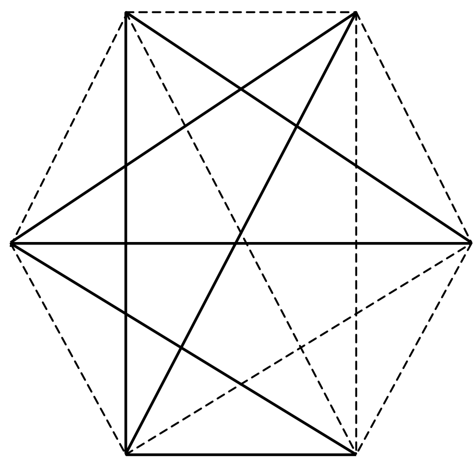

Fig 1

Here is the proof: Take one of the points, and call it O. Then of the five lines from that point to the others, at least three must have the same colour, say colour A. Consider the three points at the end of those lines. If any two of them are joined by a line of colour A, then they and O form an A-colour triangle. But if not, then the three points must all be joined by B-colour lines, so there is a B-colour triangle. So there is always a single-coloured triangle.

QED. Notice that this is not really about geometry. The “points” could be any things whatever, physical or mental, electronic or astral. The “lines” could be any relations between them, and the “colours” any division of the “lines” into two kinds. So it is a truth of a very high level of abstractness, in one sense, but a truth about any possible arrangement of any real things.

Bare hands. No axioms. No calculations. Pure understanding. *That* is mathematics.

Real mathematics has been done here with a very small array of concepts. Over and above purely logical concepts, there are only the concepts of same and different, part and whole. The points are different from one another, and there is some respect in which they differ. That is all that is needed for the problem to be described and the proof to begin. As is well-known, statements about small whole numbers can also be paraphrased using only these notions: "there are two points" is equivalent to "there is a point A and a point B and A is different from B." The reasoning in the proof also requires no new notions: when the five lines from O are considered, three of them must be of the same kind (Let the first line be of kind A; then the second is either of kind A or B; then the third is also either A or B …)

The theory that mathematics is the science of structure (or pattern, or arrangement) has been proposed many times. While is has had some intuitive appeal, it has had the disadvantage that the meaning of "structure" has been left vague. In the light of the example, it is now possible to remedy that defect. A property is *purely structural* if it can be defined wholly in terms of the concepts same and different, and part and whole (and purely logical concepts). Mathematics is then the study of purely structural properties.

Let us take some examples of mathematical concepts, and see how they are purely structural. We have just seen how to deal with *two*, using Bertrand Russell's classic analysis. To be *symmetrical* (with the simplest kind of symmetry) is to consist wholly of two parts which are the same (in some respect). *Continuity* is a little harder, but the work has been done by mathematicians with a slightly different point of view. As is well-known, the standard definition of continuity is expressible in set theory using the axioms of general topology. A function is continuous if the inverse image of any open part is also open. (The definition usually says "subset" rather than "part", but the subtle difference between subsets and parts is not relevant here.) This definition has meaning when it is specified which parts (of the whole "space") are open, and that may be done arbitrarily, subject to the restrictions in the axioms of a topological space (for example, that the intersection of two open parts are open). It is true that this definition of continuity is rather far from the initial one, depending on the intuition of movements in space, that a continuous function is one that makes no "sudden jumps", but the genius of the definition is exactly its ability to cash out these spatial and dynamic notions in purely structural terms.

Readers with some familiarity with mathematical work on the foundations of mathematics will note that something similar can be done with all the major reductions of areas of mathematics to set theory. Groups, vector spaces and so on can all be "constructed" in set theory, in the sense of being regarded as sets of sets of … sets. But in these reductions, the sets at the bottom level serve merely as a general-purpose abstract material to make mathematical objects out of, while the sets at the higher level are chosen to imitate a pre-existing structure that the mathematician has an external understanding of. Let us see how this works in the best-known example, the construction of the sequence of natural numbers out of the empty set. According to one such construction, the natural numbers are

$$\varnothing, \quad \{\varnothing\}, \quad \{\{\varnothing\}\}, \quad \{\{\{\varnothing\}\}\}, \quad \ldots$$

while according to another they are

$$\varnothing, \quad \{\varnothing\}, \quad \{\varnothing, \{\varnothing\}\}, \quad \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}, \quad \ldots$$

There is no reason to prefer one of these to the other, or to many other alternatives. So, what is essential to the numbers must be what the two constructions have in common, namely having the structure of an infinite sequence.[2] Once that is admitted, one knows which alternatives would be acceptable and which not. The only reason for starting with the empty set is that minimalism is the name of the game, so the sequence

virtue, {virtue}, {{virtue}}, …

would be acceptable, but

1 o'clock, 2 o'clock, 3 o'clock, …

would not be, since it comes back to the beginning at the thirteenth step.

Similar remarks apply to the other classical constructions of mathematical objects out of sets, such as the construction of real numbers as sets of Cauchy sequences of rationals (the rationals themselves having been already constructed out of the natural numbers), or the construction of abstract groups as sets satisfying certain properties. In each case, a prior knowledge of the structure aimed at guides the choice of sets in the constructions, while what the actual construction shows is that nothing more than purely abstract materials are needed, that is, that nothing over and above purely structural properties is needed in explicating the mathematical concepts in question.

Let us take another well-known example. Every law of propositional calculus can be translated simply into a law of set theory. For example, the distributive law of propositions:

p and (*q* or *r*) is logically equivalent to (*p* and *q*) or (*p* and *r*)

(for any propositions *p*, *q* and *r*) corresponds to the distributive law of sets:

$$P \cap (Q \cup R) = (P \cap Q) \cup (P \cap R)$$

(for any sets P, Q and R). The correspondence can be expressed by saying that both propositions and sets form the structure of a complemented distributed Boolean lattice (the axioms of a complemented distributed Boolean lattice simply lay down basic principles from which the laws

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[2] P. Benacerraf, `What numbers could not be', *Philosophical Review* 74 (1965): 495-512.

such as the distributive law follow.) Propositions and sets are very different things, but the mathematical structures created by them and their relations are isomorphic. All there is to be said mathematically about either one follows from the structure they have in common.

Finally let us take two examples with a more applied flavour. It is common to draw graphs of how one quantity varies with another: profits over time, velocity with time, heat with distance. For example,

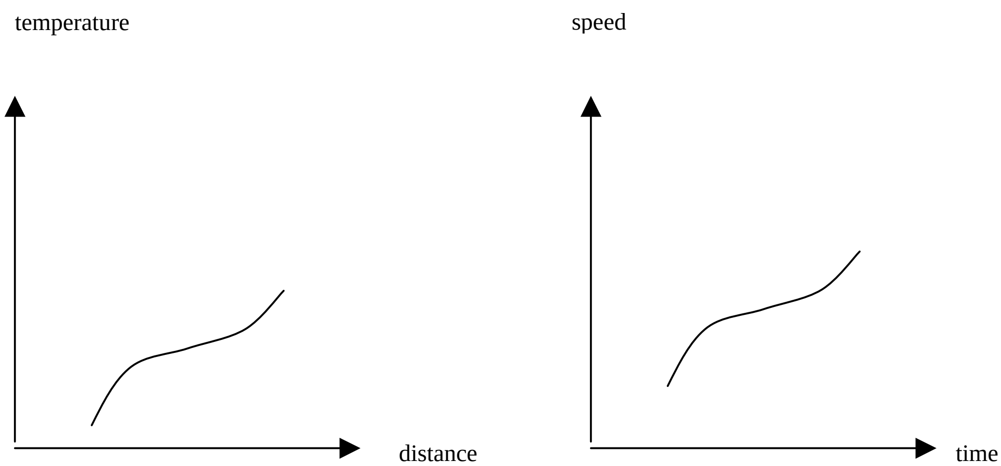

Fig 2

The graphs assert that the variation of a certain temperature with distance (along a rod, say) is literally identical to the variation of a certain speed with time. It is possible for a line (any of the axes) to represent indifferently distance, temperature, time and speed because all these have the same structure: the one-dimensional structure of the continuum[3] (which can be cashed out in purely structural terms, as the "construction" of the real numbers from sets of natural numbers shows.)

The last example introduces the long-running theme of the interaction between local and global structure. Suppose that a large population grows steadily at 4% a year, or that a bank account grows at 4% a year compound interest, compounded daily. Those statements give the local structure of the situation, how the quantity (of population or money) at one time relates to

[3] That is, they have this structure locally; though globally there are certain differences, in that, for example, speed has a natural zero while time does not.

the population a short time earlier. (In the bank case, the amount of money at one day is exactly $(1 + \frac{4}{36500})$ times the amount the day before, while in the population case this is true approximately.) The global structure, how the interactions of the local structure “add up” over time, is the characteristic exponential growth curve.

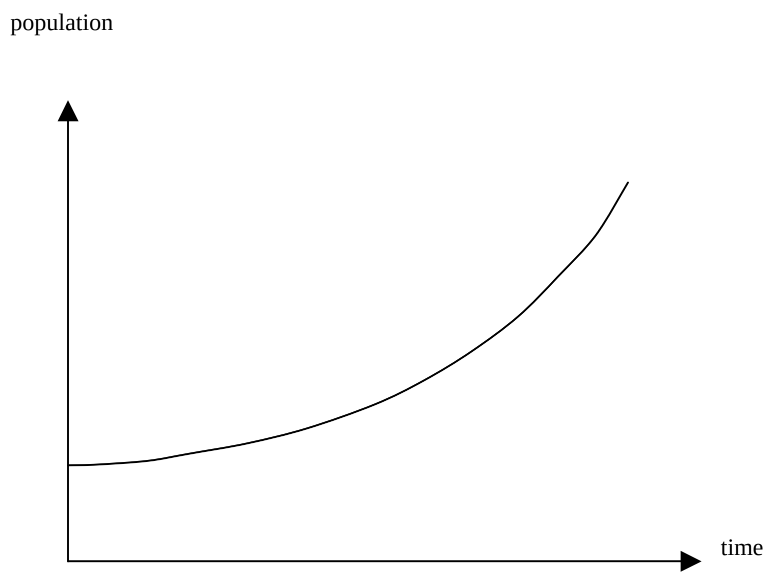

Fig 3

Where does this leave us with epistemology, in particular, the question of why mathematics is, or seems to be, known with certainty and without the need to do any empirical research? The example with which we started shows how it is possible. The result is necessary. That follows from the proof. Our certainty of it comes from our ability to have the steps of the proof in mind, and the fact that the notions which the proof deals in are merely part and whole, same and different, which are fully understood without special empirical investigations. Let us follow these assertions through in an even simpler example. We can fully understand the truth of $2 \times 3 = 3 \times 2$ by considering the diagram:

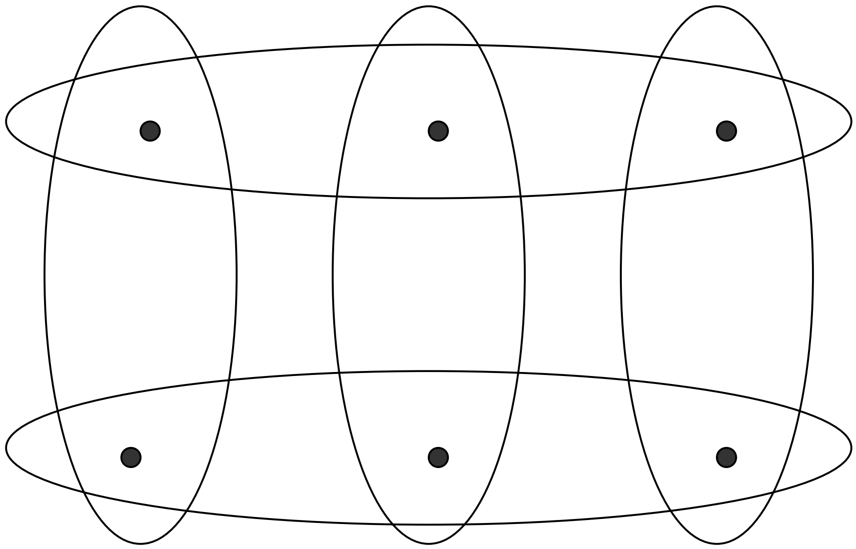

Fig 4

The function of the diagram is simply to organise the six items in two different ways, allowing us to understand that dividing them into two parts of three items each is the same, in total, as dividing them into three parts of two items each. Any derivation of this truth from axioms, or any checking of it in many cases, is superfluous, since its necessity is demonstrated in something the mind can grasp in full.

The fundamental reason why a mind or a computer can do mathematics with no outside resources, but not physics, is that minds and computers use mental representations that contain parts and hence can literally instantiate structural universals. Aristotle's theory that "the soul is in a way all things" — that the mind knows universals by literally being them — cannot be true of universals like "hot'', but it can be true of "six" or "symmetric". Whatever the internal representation of the diagram above, the mind plainly has the ability to set aside six items corresponding to the dots, and to divide them into parts in different ways. The same is true, in principle, of a computer simulation of, for example, the weather. The internal model literally has (some of, or an approximation of some of) the structure of the actual spatiotemporal development of the real weather. That is what allows facts about the real weather to be read off from the model.