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DIAGRAMMATIC REASONING AND MODELLING IN THE IMAGINATION: THE SECRET WEAPONS OF THE SCIENTIFIC REVOLUTION

Tartaglia's Italian Euclid of 1543 is geometry in the narrow sense. But the big two books of 1543, Copernicus' *De revolutionibus* and Vesalius' *De humani corporis fabrica* are also geometry, if a slightly wider sense of the term is allowed. Though Copernicus writes on physics, he does not speak of forces, energies, masses or the like: there are only the appearances of the heavens from certain points of view. Though Vesalius is biology, there is little physiology, or mechanical analogy, or discussion of causes: the emphasis is on how parts of the body look from suitable points of view. But the three books share more than just pictures, and it is this extra element that is the focus of this article. Euclid's *Elements* is not a picture book of shapes. The point of Euclid is to *reason* about the diagrams, and expose the necessary interrelations of the spatial parts. So it is with Copernicus and Vesalius. The text of Copernicus is an exercise in reasoning about which geometrical scheme will best fit the sequences of spatial points recorded in the astronomical tables. Vesalius uses the best of the discoveries of artists to make easy for the reader inference about how the systems of the body look in isolation, and in relation to one another. The difference between a Vesalian diagram and a photograph is exactly that the former allows one to work out structural facts which are almost invisible in the photograph. The plates also allow muscles in the partially dissected cadaver to be drawn with the natural tensions of the living body—obviously impossible with a photograph.

An exclusively geometrical focus persisted in science in the work of Kepler, in both his ellipse theory and his Platonic solids theory of the planets[1] (as well as in his optical theory). Galileo's first success, the *Sidereus nuncius*, is, amongst other things, about inferring the shape of the moon's surface from optical considerations of light and shadow.[2] Galileo's famous saying that the universe is written in the language of mathematics, which when quoted in

G. Freeland and A. Corones (eds.), 1543 and All That, 53 – 115

isolation makes us think, for example, '$s = \frac{1}{2} gt^2$', continues in the original, 'its characters are triangles, circles, and other geometrical figures, without which it is humanly impossible to understand a single word of it'.[3] Descartes still believes that the way forward for science is to ignore all properties of matter except the purely geometrical, but he is already starting to be old-fashioned. By then the next generation had begun to invest science with the many things beyond pure geometry that we now regard as essential to it, and which it was the achievement of the Scientific Revolution to put together—experiment, for example, and forces, and algebraic formulas.

Since *a* Scientific Revolution was plainly under way by 1610, and since at that time it had achieved almost nothing except in geometry, the thesis, 'The Scientific Revolution was kick-started by geometry' is well-supported. This article is intended to explain this fact, by tracing how medieval and Renaissance reasoning with diagrams, both physical and mental, trained Europeans to think adequately to do science.

Part 1 deals with physical diagrams, part 2 with mental ones. The two parts are not as distinct as may appear at first glance, since diagrams are pictures that are especially simplified to cause a mental construct stripped of irrelevancies, while an imagined diagram is imagined as *like* a physical picture.

1. DIAGRAMMATIC REASONING

1.1 Image and Diagram

First, a little precision is desirable on the entities like 'diagrams' and 'images'. This is especially necessary because of the very wide and imprecise usage of the word 'image'. 'Image' is commonly used as a dead metaphor, meaning something like 'implicit theory', so that a phrase like 'the medieval image of the world' has no specially visual or spatial content.[4] Or take titles which do claim to refer to pictures, like, 'Images of the other in incunabular woodcuts: Renaissance constructions of ethnicity and gender'. The post-modernist *frisson* makes the author's commitments clear enough, but in the process neglects certain necessary distinctions among 'images'. These are the distinctions that are common ground in artificial vision research and satellite image processing, where 'images' are classified according to the amount of cognitive processing incorporated into them. One distinguishes at least the following three stages:

1. The raw image, such as a photograph, or the moving dots on a TV screen. Or perhaps the Shroud of Turin, to take a medieval example.

2. A line drawing, like a Dürer rabbit. The edges and regions are explicitly identified, but there is as yet no labelling of the objects in the scene.

3. A map or plan. Here there are symbols attached, identifying some line as a road, or some dot as a church. The symbols are on the map in places that preserve the spatial relations between the entities symbolised—all the spatial relations if the map is 'to scale', or only some of them in examples like the usual plans of the London Underground, where only spatial order is preserved.

Beyond these three there are purely textual descriptions of space, as with a list of objects in a scene, or a description of how to get to the East. The representation of information can be compressed and informative, but also subject to alarming errors of scale, of the kind made famous by Columbus and the Children's Crusade.

'Diagram' is a wider term than 'map' or 'plan', in that what a diagram represents need not be spatial. A diagram is a picture, in which one is intended to perform inference about the thing pictured, by mentally following around the parts of the diagram. The famous saying, 'A picture is worth a thousand words' is true especially of diagrams, those pictures streamlined for inference by removal of irrelevancies (such as, usually, shading and real colour). (A photograph may not be worth any words at all, as a beginning student of biology realises when confronted with slides of the view down a microscope.) 'Worth a thousand words' for what purpose? As a store of information, from which inference can proceed.[5]

Consider the common medieval diagram of the Wheel of Fortune[6] (fig. 1 shows Villard de Honnecourt's especially abstract representation of it[7]). One is supposed to start at any of the positions (most naturally the top), then follow the circle clockwise to infer the order of events, and eventually return to the starting point. The (spatial) circle of the diagram thus represents time in the cycle of events. The Wheel is in fact a particularly bad diagram of fortune, because of the incorrect inferences it encourages. For example, it suggests that reversals of fortune happen inevitably, and even at inevitable intervals, as the

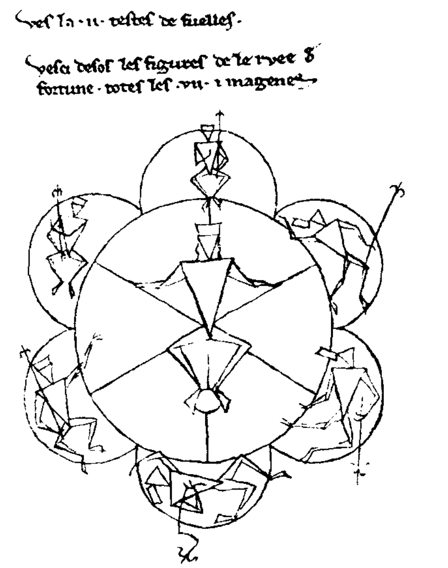

Figure 1. ' *Villard's Wheel of Fortune. The Sketchbook of Villard de Honnecourt.*

Wheel turns inexorably. Seeing fortune in terms of the Wheel must, then, inhibit any thinking of it in terms of randomness, where the time until a reversal is completely unknown, as it is in real life. From the present point of view, this simply casts light on what a diagram is: it represents spatially the structure, or what is believed to be the structure, of something, in a way that facilitates the reaching of conclusions about it. A diagram thus contrasts with a single drawn figure (of a saint, for example), in which there is no intention that one should follow from part to part and infer anything. It also contrasts with such things as Islamic geometrical decoration, where there are complex arrangements of geometrical parts, but parts that have no meaning.

A diagram of higher quality, where the inference is good, is the Square of Opposition in logic (fig. 2). Here, lines in space represent logical relations between propositions. The Square is apparently due to Apuleius of Madaura, in the second century A.D. The original text contains not a diagram, but a description of how to draw one.[8] The logical relation between any proposition and any other can be read off immediately from the diagram.

Before going further, it will be useful to have a rough classification of the kinds of pictorial representation, which will help organise the topics to follow. Let us represent the classification itself with the usual diagram, a tree (fig. 3).

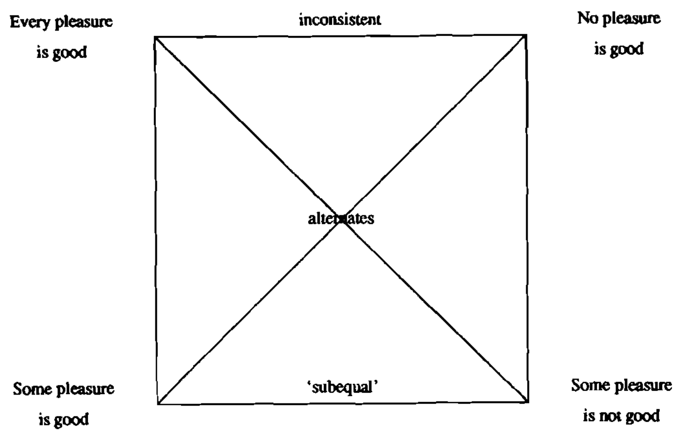

Figure 2. *Apuleius' Square of Opposition.*

The dotted line indicates the connection that gives perspective its peculiar fascination: by drawing something according to geometrical rules (and filling in with colours) one has something recognisably like a photograph. (On the controversy as to whether images literally resemble their objects, the present article accepts the affirmative side.[9]) On the connections between geometrical diagrams and perspective, more will be said later.

Medieval texts are often rich in diagrams, even if that is not always clear from modern editions which criminally leave them out.[10] Our survey begins from the bottom left of the tree below.

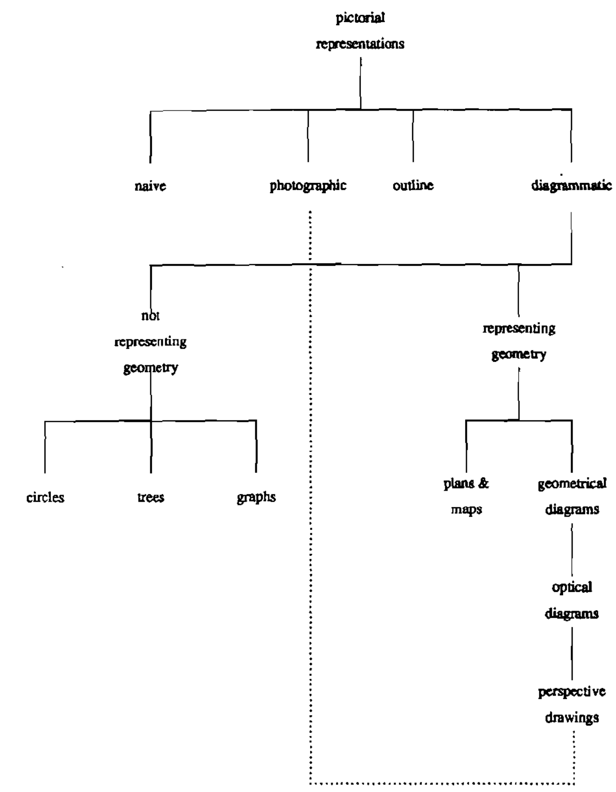

Figure 3. *Classification of pictures.*

1.2 Circle Diagrams

The Wheel of fortune is far from the only circle diagram common in medieval texts. Isidore of Seville's *Etymologies*, the standard medieval encyclopedia, is sometimes called the *Liber rotarum* on account of the number of *rotae*, or wheel diagrams. Among the most popular were ones linking the four elements and the four humours, and those linking the zodiac, months and seasons.[11]

There are a few medieval diagrams in which the wheels are physically separate pieces joined to the page with string, so that the rotation can be done in reality and not just in the imagination. A fourteenth century book on divination includes two geared wheels; the smaller one is rotated on a peg, and the larger one comes to rest at a number which is interpreted according to the instructions in the book. Ramon Lull's *Art* (a method for demonstratively confuting the heathen and recovering Jerusalem) used rotating discs (as well as various trees).[12] The various complicated wheels illustrating parallels between the seven virtues, the seven vices, the seven Beatitudes and the seven gifts of the Holy Spirit are perhaps better imagined than pictured.[13] The English phrase 'by rote' probably derives from the ubiquity of wheel diagrams in medieval education (or if not, it is from the French 'route', which refers to another diagrammatic way of organising facts).[14]

1.3 The Rank Growth Of Trees

One of the most widely visible of medieval diagrams, in the full sense of something geometrical on which one performs inference by following the diagram around, was the Jesse tree. A Jesse tree is the family tree of Christ, going back to Jesse, represented in stained glass or some other medium. Typically, it is rather abbreviated, but some examples have up to fifty personages in the leaves.[15] The reference is to *Isaiah* 11:1, 'A shoot springs from the stock of Jesse, a scion thrusts from his roots'. Whether an actual family tree is envisaged in the original is not entirely clear—the Bible is not rich in visual imagery, except for the apocalyptic books. The ancient sources mention family trees displayed by Roman families in their homes,[16] but no examples are known. Medieval texts, on the other hand, are full of them. There are family trees of kings;[17] also of gods.[18]

The legal world was also familiar with family trees. A kind of generic family tree was common in legal texts to illustrate family relationships, and the degrees of consanguinity to be inferred from them; an example available to almost everyone literate is in Book 9 of Isidore of Seville's *Etymologies*.[19] The same information on family relationships can be depicted by quartering a coat of arms. Sir Anthony Wagner's *Historic Heraldry of Britain* explains the complexity of the inferences involved in interpreting arms:

> A husband impales his wife's Arms with his own, unless she be her father's heir or coheir in blood (that is to say, if she has no brothers), when he displays them on an escutcheon of pretence, superimposed in the centre of his own shield. In the latter case only, the children acquire a right to quarter the Arms of their mother's family (and any quarterings previously acquired in the same way by that family) with their father's. In this way a shield with many quarterings indicates a succession of heiress marriages, so that the bearer represents in blood all the families whose Arms he quarters.[20]

These Gothic extravagances belong especially to the very late medieval period examined in Huizinga's *Waning of the Middle Ages*. Huizinga acutely observes how the excessive growth of trees is part of 'symbolism in its decline', and tends to substitute for serious causal thought:

> The world unfolds like a vast whole of symbols, like a cathedral of ideas. It is the most richly rhythmical conception of the world, a polyphonous expression of eternal harmony ... All notions of one thing proceeding from another took the naive form of procreation or ramification. The image of a tree or pedigree sufficed to represent any relations of origin or cause. An *arbor de origine juris et legum*, for example, classified all law in the form of a tree with numerous branches ... From the causal point of view, symbolism appears as a sort of short-circuit of thought ... all mental association based on any casual similitude whatever will immediately set up the idea of an essential and mystic connexion.[21]

Still, there is structure other than the causal, which can be worthwhile but difficult to investigate. Some trees, at least, summarise genuine information. One is the logical diagram known as Porphyry's Tree, which classifies the kinds of being. In accordance with Stigler's Law of Eponymy,[22] this is not due to Porphyry, but to one of his Latin translators, at the latest Boethius.[23]

A tree diagram (fig. 4) which, unlike Porphyry's Tree, is for the classification of something more or less useful, is the division of the kinds of proportion in Pacioli's *Summa de arithmetica*, of 1494.[24] Pacioli notes that the diagram continues indefinitely off the bottom of the page. The modern mathematician is unlikely to find any but the top few levels of any assistance in understanding ratios, but the basic plan is reasonable enough.

Some historical perspective may be gained by noting that diagrams of essentially the same type have since proved widely useful in the sciences. 'Tree of life' diagrams are important in Darwin's early notebooks, for example.[25] In this century, there are theorems in logic like 'the propositional calculus is a distributive lattice', 'diagram-chasing' in category theory,[26] and 'semantic net' or 'frame' representations in artificial intelligence.[27] 'Venn' diagrams (due to Euler[28]) rely on the fact that the transitive relation of set inclusion is represented geometrically by the obviously transitive relation of containment of areas.

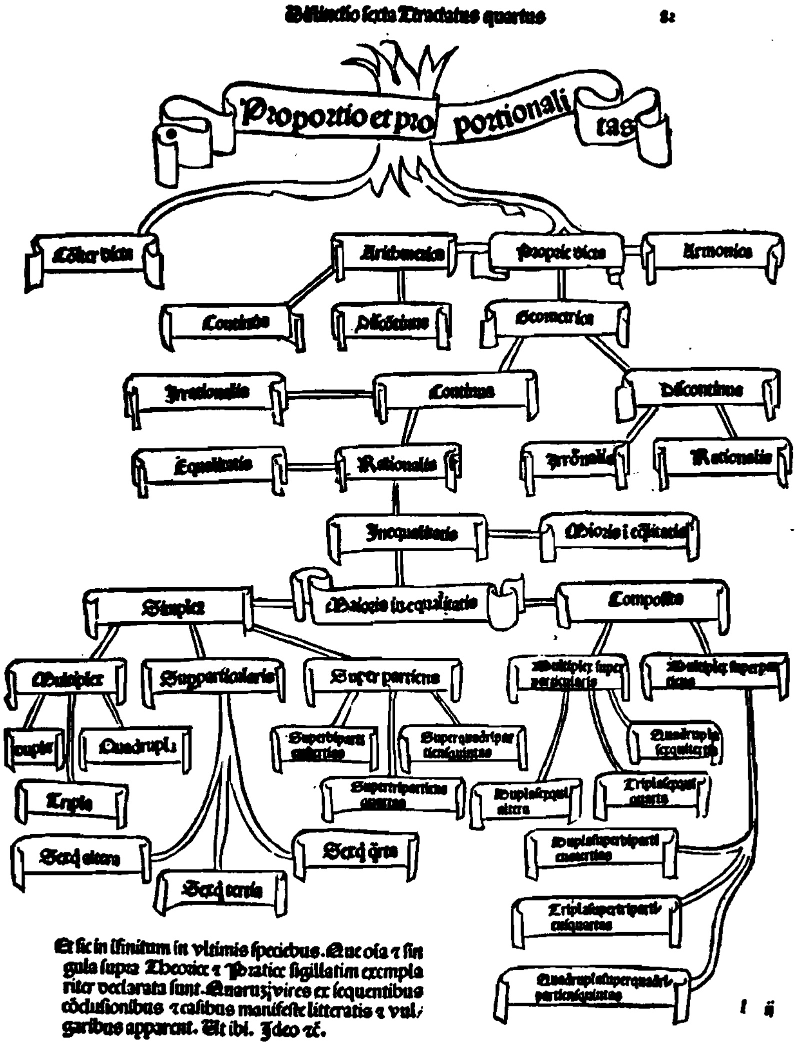

Figure 4. *Pacioli's Classification of Ratios.*

Networks of nodes and connections, which are in general more complicated than trees in that they contain cycles, can be very useful in analysing telecommunications links, games and so on. The first mathematically sophisticated use of such a network was in Vandermonde's study of the knight's tour in chess, in 1771.[29] He used the nodes to represent positions of the knight, and connected with a line those nodes which can be reached from one another by a legal knight's move. It is clear, then, that the medievals were essentially correct in seeing tree diagrams as suitable for the representation of abstract structure.

1.4 Diagrams Of Everything

Space is three-dimensional. Diagrams can use all three dimensions, if someone is prepared to pay for the extra cost. If the money is there, they can also be very big. The iconography of the medieval church is inventive in detail, but standardised in plan: the cycles of frescos are a diagram of the history of the universe, also of the life of Christ, also of the pilgrimage of the individual soul. The cycles end at the back wall, on reaching which one is intended to impose a logical IF-THEN-ELSE structure: IF the soul chooses virtue, THEN go to the top part of the wall, and view the delights of paradise; ELSE, see the lurid detail below.

It is possible to use spatial relations to suggest to the viewer parallels between different stories: between the life of Christ and the life of the Virgin, for example, whose cycles sometimes appear on the same wall at different levels. The Church of St Francis at Assisi uses the same device to suggest parallels between the life of Christ and that of St Francis.[30]

One cannot avoid noticing that Dante's heaven and hell are topographically very complex, compared with the simple Biblical entities of the same name, or even with the most elaborate Biblical described spaces, the Ark and the New Jerusalem.[31]

More abstruse possible structural parallels, such as the Globe Theatre's alleged representation of the universe,[32] remain somewhere in the limbo between the speculative and the proved. It can at least be said that the medieval and Renaissance mind would have regarded a complex building that did not represent the zodiac, or the virtues, or the macrocosm, or all of these at once, as a crying waste of representational possibilities.

When funds did not permit grandiose construction, there was always the human body available for moralising as a diagram or 'microcosm' of the universe.[33]

1.5 Graphs

Graphs, as in graphs of yearly profits, daily temperatures and the like, are one of the few mathematical inventions since ancient times that are needed to read the newspapers. The important thing in reasoning with a graph is that at least one of the dimensions should represent a quantity that is not spatial. Most commonly, the horizontal axis is reserved for time, and the vertical axis for some quantity that varies with respect to time, such as profits, temperature, or the distance travelled by a falling body. Graphs appear to be an invention of Oresme, about 1350, earlier diagrams that look something like graphs being only maps of purely spatial quantities. His horizontal axes can represent time or space, and his vertical axes any quantity that varies in intensity, such as velocity, intensity of colour, or joy. But his graphs have no scales on the axes: he is solely concerned with the differences in the shapes of different graphs.[34] A printed edition of 1486 is liberal with the diagrams.[35]

Consider a graph with time on the horizontal axis and pitch (of sound) on the vertical axis. Discretize both axes; that is, cut up time and pitch into suitable intervals. Then one has, in effect, musical notation. The staff, invented in about the eleventh century, provides a scale for the vertical axis.[36] Despite Oresme's musical interests, it seems unlikely that he saw any connection between musical notation and graphs; if he had, he would surely have put scales on the axes of his graphs.

The usual 'Renaissance' gap in the development of science is evident with graphs, which appear to have had no use until 1600. Then their time came (surely it is the Wheel). Galileo's discovery of the uniform acceleration of free fall involved essential reference to a graph. At first, he drew a diagram of an inclined plane which pictured only spatial quantities. In such a diagram, there is no place for the time variable, making it difficult to reason about time. Galileo was misled by it into thinking that the speed of a falling body is proportional to the distance it has travelled from rest. Later, he drew almost the same diagram, but with one axis now representing *time*, and arrived at the correct conclusion: that speed is proportional to the time from rest, and hence distance travelled proportional to the square of the time.[37]

While Galileo is not thought to have read Oresme, the evidence is overwhelming that he did read certain printed books which incorporated Oresme's ideas. In particular, one can find antecedents of Galileo's graphical proof of the 'Merton mean speed theorem' (that in uniformly accelerated motion, the distance travelled is equal to the distance that would be travelled by a body with uniform velocity equal to half the maximum velocity of the original accelerated body[38]).

1.6 Tables

Academics, especially at exam time, become very aware of the importance of 'setting out', that is, the arrangement of pieces of information in a spatial pattern that allows the information to be grasped easily. Simple uses of space like indentation of new paragraphs, blank lines between sections, justification of type, headings in large type (or illuminated capitals) can reduce the reader's cognitive load remarkably.[39] The main principle is that a spatial division, such as a blank line, should correspond to a major division in the text's meaning. Similar considerations apply to tables of figures. While a table is not precisely a kind of diagram, it is not unlike the 'negative' of a tree diagram: blanks, rather than lines, represent the relations between parts.

A page from a book of accounts, such as that of Impyn's textbook,[40] is notable in several geometrical ways. It is not all text; in fact half of it is empty space. The text comes in aligned blocks, and (besides the sums of money) there are also important non-text items, like lines, crossings-out, and marginal numbers indexing where the entries in the journal have been transferred into the more permanent ledger. All of these things might be different; if they were, making the inferences about the state of the owner's finances would be much more difficult. (These inventions are not sixteenth century; they are all visible in fourteenth-century accounts.[41])

The neat tables of the accountants can give the impression that Renaissance statistical information is generally arranged to allow perspicuous inference about it. This is far from true. Nef records the experience of trying to get a picture of the increase of coal shipments in England in the period after 1550.[42] The Public Record Office contains hundreds of port books with entries about coal shipments, but separating them from those dealing with glass, salt and so on is an exercise for the modern researcher. That is, the records are a mass of items of information, but contain no suitable setting out of what it means; of statistical summaries, in modern language. The modern user of spreadsheets will notice that the account books, themselves much more sophisticated in these matters than the public records, contain places for 'totals', but none for 'averages'. And no bar graphs or pie charts, either.

Tables of compound interest were produced by the fourteenth century Italian merchants,[43] and for some time thereafter were regarded generally as trade secrets. Stevin produced the first printed tables.[44]

Astronomy was, as usual, rather in advance. The Alfonsine tables are ruled very neatly.[45] Astronomical tables differ from accounts in that they are projections onto the discrete realm, so to speak, of something continuous. Inference with them needs to keep in mind that they tabulate only a succession of points in a motion that is really continuous. Such inference is essential to Napier's invention of logarithms, about 1600. One can learn logarithms from a purely numerical point of view, as a set of rules about how to manipulate numerals, but that was not how Napier thought. Trained in spherical trigonometry for the calculation of astronomical tables, he actually defined logarithms in terms of moving points on continuous scales.[46]

The spatial organisation of text to facilitate a grasp of its meaning will shade off into considerations of punctuation, which improved markedly in late medieval times,[47] and such matters as decimals and algebraic notation in mathematics (to both of which Stevin made notable contributions). Interesting as these matters are, they would take us too far afield.

We now move on to diagrams whose geometry represents something which is literally geometrical itself. The material here is generally better known than are the more abstract kinds of diagrams just considered. Here, we will briefly survey the field, calling attention to the reasoning processes needed to interpret the pictures.

1.7 Scientific Illustrations

Drawings of machines are particularly interesting, from the point of view of reasoning, because the viewer has to infer how the machine *works*. To work is, among other things, to change over time, and neither time nor change can appear in the diagram. (In principle, it is possible to draw a series of diagrams representing the phases, as is often done in showing how a car piston works, or as the Bayeux tapestry shows the Battle of Hastings; pre-modern machine diagrams do not attempt this, and in any case, one must understand a single diagram of such a series reasonably well before being able to infer how it changes to the next.) Nor can the diagram explicitly show forces and their transmission, or the direction in which a part is intended to move. The difficulty of the exercise is shown by the fact that it is usually impossible to discover how a machine works from a photograph of it.

An added difficulty with interpreting diagrams of machines is that one has to infer their 3D structure from a 2D picture. Techniques of perspective drawing were useful, after their invention, but are neither necessary nor sufficient for a clear diagram. They are not necessary, since there are other ways of including all the essential information, like drawing sections, or flat diagrams that one imagines folded. They are not sufficient, since a perspective drawing that is of the outside of a machine lacks the working parts entirely, while a perspective projection of everything will normally be too cluttered to understand.

Ancient and medieval machine diagrams have been unfairly disparaged because the conventions by which they solve these problems differ from the modern ones. The few surviving ancient diagrams of pulleys, war machines and the like, are reasonably easy to interpret, though they are neither exactly perspective nor plan views: each part is shown from its own most natural viewpoint.[48] For the modern viewer, the convention is initially confusing, but not hard to learn. It simply requires some mental origami to rotate the parts suitably. Islamic and medieval drawings use the same convention,[49] and it is only replaced by more modern perspective-based techniques in the fifteenth century.[50]

Medical illustrations do not have the problem of the time dimension (until one begins to think of the heart as a pump). But a body is much harder to understand by looking at it than a machine is, being more complex, and largely invisible when in working order. So medical illustrations have serious problems in showing internals, and simplifying enough to show the main structure, or separate different kinds of structure. Here the convention of cut away diagrams was essential. The perfection of Leonardo and Vesalius[51] should not blind us from perceiving that the essentials are present in Guido da Vigevano's *Anathomia*, of the 1340s.[52] There seems to be some connection with anatomical models.[53]

Printing obviously has something to do with the spread of scientific illustrations. But the excitement over Gutenberg (or Koster, or whoever) can make one forget the fact (which of course one knows, when reminded) that he did not invent printing itself, but only printing with movable type (an invention which itself needs—indeed consists in—a basic act of spatial imagination).

Movable type is only useful for printing text, especially in alphabetic languages. Printing of pictures came first, and was reasonably common in Europe from about 1400.[54] Again, the path to the perfection of the art in Dürer is one of evolution, not revolution.

Edgerton, in arguing for the importance of scientific illustration in the Scientific Revolution, usefully juxtaposes some European machine diagrams with Chinese copies of them. At first glance, the copies are reasonable, but it is soon clear that the Chinese artist has misunderstood the diagrams at exactly the places where inference is required. Where a rope carrying tension is shown in a cut away section, and a continuation is shown in another part of the diagram, for example, the Chinese copy does not have the two parts collinear.[55] Mahoney, replying to Edgerton's thesis, points to the crucial role of algebra in the work of, for example, Huygens and Newton.[56] True as that is, Mahoney's examples are from a different period. The later phase of the Scientific Revolution is indeed algebraic, but the earlier one is diagrammatic.

Modern research on cross-cultural psychology has tended to confirm that there are major differences in how cultures perceive pictures, although research has concentrated on cultures that differ from the Western more than does the Chinese.[57]

1.8 Plan and Elevation

A building poses different drafting problems to a machine. It typically has more detail, but it is conveniently hollow. More importantly, most buildings are approximately rectangular prisms, so most of the information on their 3D shape can be incorporated in three 2D diagrams, their 'plan' and two 'elevations'.

Not much is known about ancient plans, though there are such things as scale drawings of Greek temples, and construction lines can sometimes be seen on the actual buildings.[58] Not a great deal is known about medieval building plans either, but there is enough to show that reasoning with plans was a normal part of cathedral building. There is a suggestion that making inferences about a building from plan and elevation was part of the famous 'secrets' of the masons.[59] It is interesting that the Rheims Palimpsest, of about 1250, which is one of the very few surviving early plans, sketches one half of the elevation in detail, and the other only roughly. The artist knows about using symmetry for inference.[60]

The Renaissance classic on architecture is Alberti's *On Building*. It well explains the difference between a plan and a picture:

> The difference between the drawings of a painter and those of the architect is this: the former takes pains to emphasize the relief of objects in paintings with shading and diminishing lines and angles; the architect rejects shading, but takes his projections from the ground plan, and, without altering the lines and by maintaining the true angles, reveals the extent and shape of each elevation and side—he is the one who desires his work to be judged not by deceptive appearance but according to certain calculated standards.[61]

That is, the inference in the case of plans is performed consciously.

The requirements of plan and elevation are, it must be emphasised, opposite in a way Alberti does not mention to those of a sketch which 'looks like' its object. The plan and elevation of a cube are both just squares, which are not very informative about the three-dimensional structure. To get a sketch of a cube that does convey the shape at a glance one should take a view from a 'generic' angle; that is, one that bears no special relationship to the edges of the cube.[62] It is notable that the stylised pictures of buildings and cities in the backgrounds of Byzantine and early medieval paintings take such a generic view (without being very expert in getting the angles right).[63]

1.9 Maps and Projections

A map, like an anatomical diagram, will simplify, select and label to facilitate inference (whether true or false).[64]

Medieval *mappaemundi* are more like diagrams, in the sense of the London Underground map, than maps drawn to scale, or according to a definite projection.[65] Some of them are combined with Wheels of Fortune, to illustrate conditions on earth,[66] and various other purposes can dominate the purely spatial information. Matthew Paris's itinerary map of the journey to the Holy Land is spatial, but essentially one-dimensional: it draws a road with icons of towns indicating the stages of the journey.[67]

Conceiving of a map as unlike a simple picture suggests the idea of adding something to it to indicate such spatial relations as direction. Latitude and longitude were reasonably familiar to the Arabs and medievals, but more in connection with the time differences between places than in drawing maps. Grids on maps are first found in diagrams of the planetary motions against the star background.[68]

The idea of a projection is especially important for inference from maps. The maker of a world map must understand some way of projecting round onto flat, and the user must understand it well enough to infer back from flat to round. Roger Bacon explains the need for a projection:

> Since these climates [i.e., zones] and the famous cities in them cannot be clearly understood by mere words, our sense must be aided by a figure. In the first place, then, I shall give a drawing of this quarter with its climates, and I shall mark the famous cities in their localities by their distance from the equinoctial circle, which is called the latitude of the city or region; and by the distance from the west or east, which is called the longitude of the region.[69]

There follows a scheme of projection where the spacing of parallels decreases systematically towards the Pole. Projections became much better understood with the rediscovery of Ptolemy's work in the fifteenth century,[70] and in the sixteenth, Mercator discovered his famous projection. Apart from aggrandising the circumpolar powers, its advantage is that of allowing the direction between any two places to be inferred directly from the map.

The inferring of directions is also the point of the fourteenth-century portolan charts, which were practical aids for navigation. These include accurate maps of the Mediterranean coast, criss-crossed by many rhumb lines, which one follows with the eye to work out the directions between the important locations. Some also have compass roses and scales.[71] Navigation involves difficult geometrical reasoning, as one must mentally co-ordinate three different spaces: that of a chart, that of an instrument, and that of the surrounding actual space. A great deal of effort was expended on the subject, in view of the costs of mistakes.

Local maps for legal and administrative purposes were occasionally found in the middle ages, but seem not to have been the normal thing until after 1500.[72] Hence, it seems that in general the ability to read maps and plans belonged only to certain specialised professions before about 1500, but became more general thereafter. Alberti describes how to construct a plan of Rome using an odometer and triangulation; surveying as a standard practice is also common after 1500, but not before.[73]

1.10 Instruments

Diagrams are not necessarily drawn on paper. For purposes of use, it may be better to inscribe them on something more durable, like metal. For making the inferences, it may be useful to include moving parts. The most impressive of the common instruments at the time was the astrolabe, useful for measuring anything from the position at sea to heights of towers. It is an analogue computing device, in the same sense as a slide rule is: it represents various continuously varying quantities by lengths on the instrument, and makes inferences back from the instrument to the quantities represented.[74]

One can have diagrams of the universe that incorporate the time dimension by having moving parts. There is a single ancient example, the Antikythera mechanism,[75] and various medieval 'equatories'.[76] If the mechanism is automated, it will become more and more like a clock. The earliest mechanical clocks of which there are reasonably complete descriptions are the complicated mid-fourteenth century planetary models of Richard of Wallingford and Giovanni de Dondi, which also happen to tell the time. The image of the universe as a clock, which we think of as so much an emblem of the Scientific Revolution, is an idea of Oresme[77] (though the phrase *mundi machina* is already in Sacrobosco's *Sphere*[78]).

Clocks provide the public with extensive training in spatial reasoning, since they pose a difficult three-way co-ordination problem. The circular space of the dial, actual (linear) time, and the numerals must be mentally identified. The addition of a minute hand makes the problem even worse, since the dial must be read as two spaces, one for each hand. No wonder it takes forever to teach a child to tell the time.[79]

All these diagrams are circular, as is one of Oughtred's original slide rules of 1632, and Galileo's 'geometrical and military compass', which is an analogue calculating device more than a compass.[80] 'Mathematics' did not have the suggestion of an abstract subject divorced from reality that it has today, but was much more closely associated with practical skills with a variety of instruments.[81]

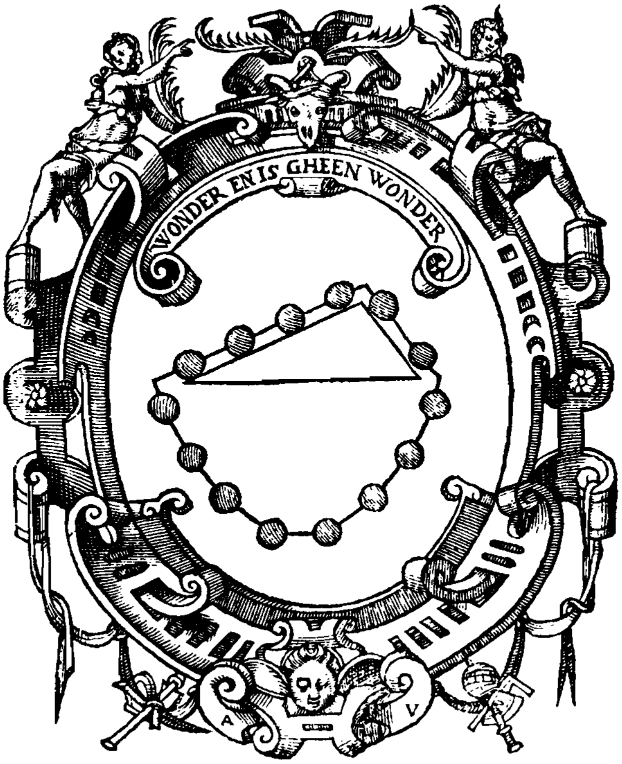

Figure 5. *Stevin's Wreath of Spheres.*

1.11 Forces

A central place in the medieval applications of geometry was held by the science of weights, as classically expounded by Jordanus Nemorarius in his thirteenth-century treatise. His derivation of the law of the lever is interesting in that, unlike Archimedes, he relies on 'virtual displacements': he demonstrates that weights are in equilibrium by considering the movements they would undergo if they were not. The movements with which he calculates exist only in the imagination.[82]

It is the science of weights that includes one of the earliest actual discoveries of the Scientific Revolution, a successful application of reasoning with a diagram. In 1586 Stevin printed the 'Wreath of Spheres' diagram (fig. 5) as the title page of his *Elements of the Art of Weighing*.[83] The text which decorates it says, approximately, 'Wonderful, but not incomprehensible', which it is. It is clear that the circle of balls does not tend to rotate either clockwise or anticlockwise. But the balls hanging below the horizontal line are in equilibrium. One could cut them in the middle, and allow them to hang down, or even remove them altogether, without disturbing the balls resting on the two inclined planes. These upper balls, then, are at rest. The numbers of balls on each side of the apex are in inverse proportion to the sine of the angles at which the planes are inclined. One has derived, therefore, the law of the inclined plane, or, equivalently, the resolution of forces into components.[84] From the point of view of this paper, it is important that this, probably the only significant discovery of the early Scientific Revolution concerning forces, is expressed as pure geometry.

There seem to be no medieval or early modern diagrams that represent forces by arrows (even in the sections on stresses in beams in Galileo's *Two New Sciences* where one is almost forced to imagine arrows[85]). Nevertheless, someone who can design a flying buttress must have some mental representation of the direction of stresses;[86] likewise Brunelleschi in understanding that the hemispherical dome of the Cathedral of Florence could be constructed without scaffolding, a feat impossible with an arch.[87] Cannon and crossbows redirect forces, but it may have been possible to design and use them without imagining the forces. On the other hand, it is surely impossible to tack into the wind[88] without keeping constantly in mind the relations between the wind direction and the angle of the sails.

1.12 Euclid's Geometry

The history of geometry, in the formal sense, is a well-worn topic, and here we may just select a few aspects that are especially relevant to the thesis of the article.

As is well known, Euclid's reasoning cannot be carried out purely in terms of propositions, but relies essentially on the diagrams.[89] Kant is only the most famous of many who have been greatly impressed by the indispensability of 'constructions' in geometrical reasoning.[90] In Book I, proposition 1 Euclid draws a line, and then two circles with centres on the ends of the line and with radius equal to the length of the line, as in fig. 6. He next considers the points

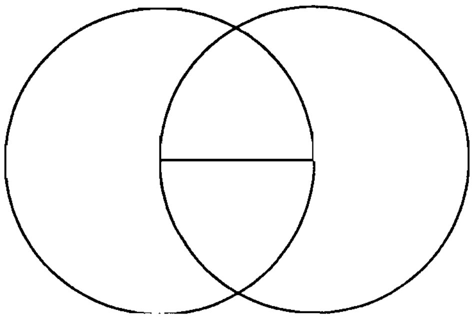

Figure 6. *Euclid I.1.*

where the circles interest. He does not use any axiom to justify the fact that they do intersect, but simply infers that from the diagram.

The intrinsic importance of whatever is in the early propositions of Euclid is magnified a thousandfold, as far as the history of ideas is concerned, by the central place the *Elements* has played in Western education since about 1200.[91] The pedagogical profession, to its lasting credit, held firm on Euclid (at least until around 1960), despite the immense consumer resistance that has left us the phrase *pons asinorum* (Euclid I.5).[92] A whole civilisation followed Euclid across that narrow bridge, into a bright new land of expanded horizons of the intellect.

The transferability of the skills learned with Euclid is of course an issue for debate, as it is when any modern mathematics educator demands more money on the grounds that 'mathematics teaches you to think'. Nevertheless, those involved seem to have had little doubt about the usefulness of geometrical training. Ramelli, for example, in his profusely illustrated book of 1588 on ingenious machines, praises Euclid at length; a modern commentator cannot understand the point of this, when Ramelli does not actually use any Euclidean theorems.[93] The answer must be, as it is to the modern question about the usefulness of training in abstract mathematics, that the more intelligent the pupil, the more transferable his skills, while even the most intelligent of pupils cannot be expected to recapitulate the history of mathematical discovery on his own. (Interestingly, the case for the usefulness of training in logic is much weaker.[94] While it is not impossible that training in formal logic should prove useful—as happened in the 1940s during the development of computers—there seems no reason to think that the immense effort the medievals put into formal logic had any input into the Scientific Revolution.)

Nevertheless, while there is a certain amount of agreement that the West's possession of Euclid was important,[95] there is room for confusion on what that means. For Euclid represents two very different things: logical rigour, and geometry. Philosophers emphasise the ideal of rigorous proof, according to which Euclid fulfils the goal set for all science by Aristotle's *Posterior Analytics*, of a set of theorems deduced from self-evident axioms. That is the aspect of Euclid that Matteo Ricci saw as lacking in Chinese mathematics:

> Nothing pleased the Chinese as much as the volume on the Elements of Euclid. This perhaps was due to the fact that no people esteem mathematics as highly as the Chinese, despite their method of teaching, in which they propose all kinds of propositions but without demonstrations. The result of such a system is that anyone is free to exercise his imagination relative to mathematics without offering a definitive proof of anything.[96]

That is, Chinese geometry lacks the discipline of rigorous inference. Modern studies have found that indeed mathematics in traditions that do not descend from the Greeks, though often excellent at developing complicated algorithms, is almost entirely lacking in proof.[97] Non-western mathematics resembles modern computer programming much more than it resembles modern mathematics. But, though the ideal of rigour is exciting, it is not obviously useful for anything outside geometry. For all the talk about axioms and certainty, no other science was put on an axiomatic basis, nor were there even any serious attempts. Not even the sciences closest to geometry became axiomatised: Ptolemy's *Almagest* has no self-evident axioms, and there is hardly a proof in number theory between Diophantus and Fermat, let alone an axiom.

Still less are there any medieval or Renaissance attempts like Spinoza's to develop ethics, for example, *more geometrico.*

On the other hand, Euclid represents geometry, understood as reasoning about spatial magnitudes and shapes. Here, the evidence is clearer for the usefulness of geometrical training. There was a medieval tradition of 'practical geometry' that depends on Roman surveying techniques and pre-dates the influence of the translations of Euclid.[98] Nevertheless, it was already consciously about inference: 'Practical [geometry] is that which is done by certain instruments and by inferring (*coniciendo*) proportionally one [distance] from others'.[99] When Euclid did become available, the practical geometries incorporated some of his ideas. Thus, even those interested only in the practical applications of geometry were encouraged to take some interest in reasoning. The widely-discussed medieval split between theory and practice, while genuine enough in general, is probably at its narrowest in geometry.[100] This is important, if the view is taken that a necessary condition for a scientific revolution is a feedback loop between theoretical science and technological innovation.

A sign of how deeply Euclid penetrated is the non-trivial use made of it by Bartolus, the foremost medieval authority in law, a discipline legendary for its innumeracy. Bartolus wrote a book applying Euclid to the division of lands affected by flooded rivers; it remains one of the few legal books with geometrical diagrams.[101] He also indulges in some simple optical reasoning: 'I saw Titius in the mirror at the barber shop, and recognised him' is not completely certain evidence that I saw Titius, because an image in a mirror is smaller than the thing, and so harder for the eye to discern.[102]

Yet, there are several respects in which Euclid is unsatisfactory as a training in spatial reasoning. For one thing, the proofs are difficult, so difficult that struggling through them can make learning the actual geometry very slow going. Secondly, the geometry of the *Elements* is almost all two-dimensional. It is true that Euclid knew some three-dimensional geometry, and something about how to represent it in two dimensions, as is clear from the diagram of parallelepipeds (fig. 7) from the oldest manuscript.[103]

But three-dimensional geometry occupies only small portions of the text, and they are in the little-read later books. This is a pity, as the difficulties of applying geometry to optics or astronomy or architecture lie mainly in coping with the third dimension. Finally, there is no motion; Euclid does not train the reader in what happens when shapes move, and avoids methods of proof that

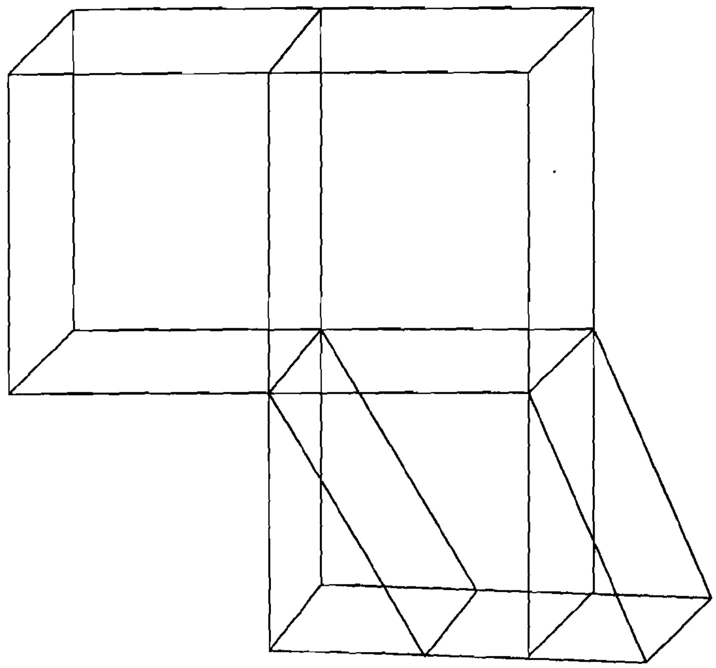

Figure 7. *Euclid XI.31,* from *The 'Heiberg' Manuscript.*

involve, for example, superimposing one figure on another. 'Geometry is occupied with immobile magnitude', Hugh of St Victor says, but adds, 'astronomy with mobile'.[104] Astronomy supplied all the other lacks in Euclid, too.

1.13 Astronomy

The standard introduction to astronomy in the university curriculum from about 1300 to 1600 was one of the best-written textbooks ever, the *Sphere* of Sacrobosco. The *Sphere* was the book on which Copernicus received his training on celestial orbs and their revolutions. It explains the celestial spheres, the zodiac, equinoxes and solstices, eccentrics and epicycles (very briefly) in a way that is always concise, clear and to the point. It expects of its readers a considerable willingness to imagine in three dimensions, but rewards average persistence with genuine understanding. Its strengths are illustrated by the two definitions of a sphere, which open the first chapter:

> A sphere is thus described by Euclid: A sphere is the transit of the circumference of a half-circle upon a fixed diameter until it revolves back to its original position. That is, a sphere is such a round and solid body as is described by the revolution of a semi-circular arc.
>
> By Theodosius a sphere is described thus: A sphere is a solid body contained within a single surface, in the middle of which there is a point from which all straight lines drawn to the circumference are equal, and that point is called the 'center of the sphere'.[105]

A diagram on paper is no use here: the reader must construct one in his imagination. Anyone who has done so and understood that the two definitions are equivalent has learned something substantial about modelling in the imagination.

Some of the later portions of the book are helped by illustrations. The edition of Venice, 1485, prints an eclipse diagram in three colours. Many of the sixteenth-century printed editions had sheets of volvelles to be cut out and pasted in.[106]

The capacity to visualise the universe from different points of view was not restricted to professional astronomers. The round earth itself required some work in the imagination for everyone: Madeville's *Travels* explains that the inhabitants of the southern hemisphere are upside down, to us, but seem to themselves to be the right way up.[107] (As far as is known, no other civilisation was asked to contemplate such a thing of other peoples.) There is something similar in the *Divine Comedy*,[108] a work generally heavy with geometry—its last thirteen lines contain a geometer, a circle, an image, the (faculty of) phantasy, and astronomical revolutions.[109] Dante imagines looking down from the sphere of the fixed stars and seeing the northern hemisphere from Cadiz to Asia.[110] We might think of it as a NASA's-eye view,[111] and see a connection between it and a good deal of later speculation about what is 'out there';[112] in any case, it is exactly the view shown in the map in Ptolemy's *Cosmographia* of 1482, which uses some interesting techniques to make the spherical shape of the earth obvious.[113] Edgerton remarks, 'Only by this kind of three-dimensional mind's-eye imagining could [Columbus] convince himself, as well as others, that it would be possible to reach the East by sailing west'.[114] It would be possible to mount a large thesis concerning the effect on Western thought of the 'what if?'

style of counterfactual imagining, implicating everyone from the Pre-Socratics[115] to science fiction, and no doubt such a thesis would be as defensible as ones of comparable size on religion and the rise of capitalism and the like. 'Here's fine revolution, an we had the trick to see't' (*Hamlet* V.i.89).

Since we are speaking of revolutions, it is pertinent to observe that the use of this astronomical metaphor to describe any major change is a medieval idea.[116]

So, when we speak of the 'heritage of Greek geometry', we should keep in mind not only Euclid, but Parmenides (or his contemporary), who first concluded the earth was round from a leap of geometrical imagination,[117] and Hipparchus, who fitted the numerical data of the Greeks and Babylonians to a geometrical scheme.[118] Astronomy has had a wide sphere of influence.

1.14 Optics and Perspective

Perspective in painting is an even more well-worn topic than geometry, and again we select just those aspects most relevant to spatial inference.

Geometrical optics, including the theory of vision, was, as is well-known, one of the best developed of medieval sciences.[119] The books of Alhazen, Bacon, Grosseteste, Witelo and Pecham are as scientifically sophisticated as anything the medievals produced. Optics include possibly *the* high point of medieval science, Theodoric of Freiberg's explanation of the rainbow (which depends essentially on a diagram).[120]

Books on perspective still sometimes read as if perspective sprang fully formed from the head of Brunelleschi.[121] This is ridiculous from the point of view of geometrical reasoning, and also from the point of view of illusionistic art. The rules of perspective are a marvellous discovery, but they are of the same sort as a number of earlier and simpler ones. (Giotto was 'the inventor and discoverer of many methods which had been buried for about six hundred years', according to Ghiberti.[122] The best one or two surviving antique perspective paintings do seem to justify the claims that the ancients knew some geometrical rules;[123] Giotto should have had little trouble reverse engineering them.) Consider the view of a ceiling with rafters in fig. 8.

Five lines in different directions, all converging, are more than a coincidence; the artist clearly knows something about how to represent three dimensions in two. The artist in question did not have the benefit of

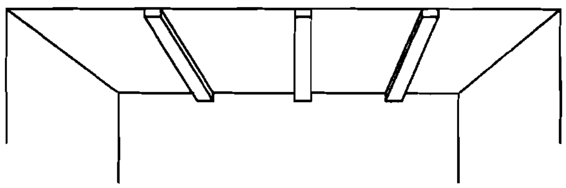

Figure 8. *Duccio's 'Perspective'.*

Brunelleschi's and Alberti's tuition, as he lived a century earlier. The view is from Duccio's *Maesta* of 1308-11;[124] he repeats it a number of times, as does Giotto, who on one ceiling has eleven converging lines, all in slightly different directions.[125] Duccio also has a complicated array of boxes seen from a generic angle, with the angles correct. Giotto draws the circular rims of jars seen from an angle as ellipses.[126] In these cases, the ellipses are rather flat, and could as well be lozenge shapes. This is not the case, however, with the quite wide ellipses Giotto uses to represent circles in his *coretti* in the Arena Chapel, perhaps the first genuine *trompe l'oeil*, at least since antiquity.[127]

The history of ellipses makes an interesting microcosm of the thesis of this paper. Ellipses are, in theory, fully treated in Apollonius' *Conics*, but as that work is almost impossible to read, they had only a marginal role in geometry in the tradition of Euclid's *Elements*.[128] They have little part in practical geometry either, which is based on straight rules and strings, and compasses. But in ancient painting, it was well-known that a shield or wheel seen side-on should be drawn as an ellipse, and theorem 36 of Euclid's *Optics* says that a wheel is seen sometimes as a circle, sometimes as an ellipse.[129] The ellipse belongs more to 'perspective'[130] than to formal geometry until its scientific apotheosis in Kepler's planetary theory. The method of constructing an ellipse with a string stretched between two pins was mentioned by Anthemius of Tralles, the architect of Hagia Sophia, and investigated by Kepler. Kepler opens his *New Astronomy* with a lament that is hard to find suitably prepared readers, as hardly anyone works through Apollonius, so that few can understand diagrams well.[131]

The most obvious way in which Giotto's and Duccio's 'perspective' differs from the real thing, as in the fifteenth-century masters, is that the earlier painters are not prepared to make the lines orthogonal to the picture plane long, with the result that they cannot have their perspective scheme covering the whole painting. They do not attempt real depth for the whole scene: there is, for example, always a blank wall preventing the rafters from continuing into the picture away from the viewer, covering the vanishing point (the 'perspective fig-leaf').

One of the things necessary to make this step—arguably the main one, in view of the effort Alberti and his contemporaries make in explaining it—is a conscious, general method of foreshortening. Lines parallel to the picture plane must recede into the distance, with the distances between equally spaced real things (for example the edges of tiles) diminishing in the picture in geometrical proportion. A method of accomplishing this appears in Pietro Lorenzetti's *Birth of the Virgin* of 1342, which Kemp calls 'the *tour de force* of fourteenth-century perspective'. It uses simply similar triangles, in much the same way as Alberti later describes; the construction lines can be traced incised in the wall in some parts.[132] Some of the painters' methods of geometrical construction on walls are described in Cennini's handbook of the late fourteenth century.[133]

The step from a collection of special tricks to a method of organising a whole picture requires the addition of geometrical theory. The part played by theoretical geometry and optics in the invention of perspective is not as clear as it might be, because neither Brunelleschi's two original perspective panels, nor his method of constructing them, has survived. It has been argued with at least some show of reason that he did not know any rules.[134] But the oldest relevant written evidence, the treatises of Alberti and Ghiberti, are fully based on geometrical theory. Book I of Alberti's *On Painting* is entirely an exposition of geometry, though punctuated by claims that he speaks 'as a painter', not as a mathematician. All that means is that his lines have a finite width, unlike the abstract widthless lines of the mathematicians. In fact, he goes so far as to retain barely relevant theory from Euclid, such as the axion that 'all right angles are equal'.[135] The point is to reason about what to draw in a painting:

> Now, since we have said that the picture is a cross-section of the pyramid we ought to investigate what importance this cross-section has for us. Since we have these knowns, we now have new principles with which to reason about the plane from which we have said the pyramid issues.[136]

What the reader is supposed to reason with is, in effect, an imagined three-dimensional diagram, containing a pyramid of visual rays and a cross-section of it, the plane of the painting. There follows a set of recipes on what lines to draw on the picture plane. Alberti promises that he possesses demonstrations of the recipes from reasoning about the visual pyramid, but omits them 'for the sake of brevity'.[137] The connection of early perspective with optical theory is even clearer in Ghiberti's *Third Commentary*, which is a series of extracts from the optical writings, especially Alhazen.[138] Though Ghiberti did not write the *Commentary* until the 1450s, when he was old, he claims to have studied optics when young and still learning the art of painting.[139]

The various Renaissance treatises on perspective provide the best examples of genuinely applied mathematics of the time. They are highly theoretical, but universally recognised as successful in practice.

An interesting application of optical reasoning in the Scientific Revolution itself is Galileo's reinvention of the telescope. He claims that the Dutch discovered it by accident, but that he, 'incited by the news mentioned above, discovered the same by means of reasoning'; that is, by reasoning as to what combination of convex and concave glasses would give a clear magnified image.[140]

1.15 The Mathematical Revolution

The thematic survey of geometrical reasoning just undertaken has perhaps obscured the broad chronological outlines of the Mathematical Revolution which preceded the Scientific one. Let us briefly review the chronology, emphasising how much was geometry, in the broad sense. To begin with, the ancient inheritance was extremely geometrical: the three most successful ancient sciences (not counting geometry itself) were astronomy, optics and statics, which consisted of great super-structures of geometry anchored in only few and easily acquired observations.[141] For later developments, a table (fig. 9) will provide the most perspicuous representation, as time is linear. Something that

	PURE GEOMETRY	OPTICS	ASTRONOMY	OTHER SPATIAL	NUMERICAL
1100	Latin Euclid, Archimedes			Music with staff Compass, quadrant	
1200	Euclid in curriculum Fibonacci's geometry	*Perspectives* Glass mirrors	Alfonsine tables Sacrobosco's *Sphere*	Villard's sketchbook Cathedral plans Jordanus on weights	Arabic numerals Fibonacci
1300		Theodoric on rainbow Spectacles Giotto's "perspective"	Wallingford Chaucer's *Astrolabe*	Portolan charts Oresme on graphs Merton "intension of forms" Anatomical diagrams Vigevano: machine drawings Compound interest tables	Double-entry Insurance Abacus schools Dice calculations
1400		Rules of perspective		Leonardo's sketches Henry "the Navigator"	
1500	Italian, English Euclids	Telescope	Copernicus Brahe	Dürer's illustrations Vesalius' illustrations Mercator's projection Stevin's "wreath of spheres"	Solution of cubic Decimals Viète's algebra Logarithms

Figure 9. *Table of developments.*

is clearer from the table than it would be from a mass of text is the clustering of applications of mathematics in the fourteenth century. The Scientific Revolution thus inherited a tradition of applying mathematics already two hundred years old. In some sense, it did not inherit much else of use—there was, for example, no comparable tradition of controlled experimentation. A glance at Grant's *Source Book of Medieval Science* shows that almost all medieval and Renaissance science is included in the above table, except for some depressing nonsense on matters like alchemy and herbs.

A snapshot of the state of mathematics after the Mathematical Revolution, but at the very beginning of the Scientific Revolution, as usually calculated, can be found in Billingsley's Euclid, the first English translation, of 1570 (a book otherwise notable for its pop-up figures of 3D geometry[142]). It has a preface by John Dee, 'specifying the chief Mathematicall Sciences, what they are, and wherunto commodious'. He arranges in a tree the two principal sciences, arithmetic and geometry, and some thirty derivative sciences. There are 'vulgar' arithmetic and 'vulgar' geometry, the latter divided into eleven sciences, concerned with surveying, in one, two and three dimensions, on both land and sea. Then there are nineteen miscellaneous sciences, most with a distinctly geometrical aspect. Of these, perspective, astronomy, music, cosmography, astrology, statics and navigation are clear enough, but the remaining eleven have names now unrecognisable. 'Anthropographie' is about the proportions in the human body; 'Zographie' is something like the modern theory of rendering in computer graphics; 'Trochilike' studies circular motions, simple and compound; 'Hydragogie' 'demonstrateth the possible leading of Water by Natures Law, and by artificiall helpe, from any head' and 'Pneumatithmie' 'demonstrateth by close hollow Geometrical figures (Regular and Irregular) the straunge properties (in motion or stay) or the Water, Ayre, Smoke and Fire'. Some of these are plainly more commodious unto nascent capitalism than others, but the total picture is of a suite of mathematical, mostly geometrical, sciences, in common and successful use, with investors queuing.[143] One of the mathematical sciences that certainly did attract serious money was navigation.[144] Military engineering was another steady earner.[145] Renaissance universities, for all the obloquy heaped on them, were also reliable sources of money for research into geometry, and especially astronomy (though not for algebra or non-medical experimental research).[146]

The evidence is, then, that when all due allowance has been made for the religious, hermetic, scholastic, military, astrological, ancient, alchemical and mercantile roots of the Scientific Revolution, the true compost in which those roots struck and grew strong was mathematics, and especially applied geometry.

2. THE VIEW FROM THE INSIDE

'One must consider the affection which is produced in the soul, and in that part of the body which contains the soul—the affection, the lasting state of which we call memory—as a kind of picture': Aristotle[147]

'There are no such things as mental pictures': Ryle[148]

We live at the end of a period which, perhaps more than any other, has hidden the pictorial life of the mind from intellectual view. Philosophy in the midcentury regarded 'sense data' as fictions, arrived at by (bad) inference and suitable for disposal as an undergraduate exercise.[149] It was seriously maintained that all inner representation was propositional.[150] (That is philosophy in the English-speaking tradition, continental philosophy being, if anything, even more word-oriented.) Psychology in the behaviourist decades was not much concerned with any inner life, pictorial or not, and 'imagery' tended to be associated with the Gestalt school, labelled 'unscientific'. Introspection, despite being reproducible with reasonable consistency, was ruled out as a source of experimental data. Frege, Russell and Hilbert, followed by Turing and the computer scientists and Artificial Intelligentsia, imposed on the learned world a view of inference as the manipulation of uninterpreted symbols according to formal rules. No room for pictures there. Even mathematics, once centred on geometry, maintained hardly any formal role for pictures, holding that geometrical intuition is unreliable. Galton was surprised to find even at the end of the nineteenth century that scientists were claiming to think in symbols, not images, and supposed that science had atrophied the imagination.[151] It is true that in the physical sciences there has been a kind of tradition of remarks on the need for mental geometrical intuition, but it is one of those 'traditions' that presents itself as a novelty every time it is reinvented.[152] At about the time Ryle was denying the reality of mental pictures, Einstein was saying:

> The words or the language, as they are written or spoken, do not seem to play any role in my mechanism of thought. The psychical entities which seem to serve as elements in thought are certain signs and more or less clear images which can be 'voluntarily' reproduced and combined ... The above mentioned elements are, in my case, of visual and some of muscular type. Conventional words or other signs have to be sought for laboriously only in a secondary stage, when the mentioned associative play is sufficiently established and can be reproduced at will.[153]

Generally, though, the image was left to intellectual marginals: diagrammatic inference to the engineers, with their slide rules, flow charts and circuit diagrams, and mental images to the Freudians and their dream fantasies.

The result is that the late-millennial intellectual has several layers of defence against accepting the medieval unselfconsciousness about using the imagination as a tool for doing science. First, we doubt the existence of the inner life at all. Or, we regard it as reached only by a chain of inference, not open to view. Or, we regard it as vague, and primarily emotional, not precise and scientific. Or, we presume the 'stream of consciousness', if there is one, is a flow of words, the preserve of psychiatrists and novelists. Or, if we do take the visual side of the imagination seriously, we think of it as something like a muse, 'inspiring' artists and poets in a manner too sublime to analyse. We have every excuse for misunderstanding.

At the leading edge of science, this picture is no longer true. Two developments especially have made the difference: psychological experiments on mental images, and scientific visualisation by computer. The psychological work is especially relevant. Around 1970, Shepard and Metzler found that the time subjects took to decide whether one 3D figure could be rotated to fit in the same space as another was proportional to the angle required, suggesting that the subjects were actually performing the rotation in some kind of mental space.[154] Since then, similar techniques have been used to investigate how subjects imagine themselves in an environment while mentally searching it,[155] and how they construct a mental model from a description of a scene.[156] It is found generally that reasoning about space is done by means of mental models, not via chains of propositions.[157] In view of what was said above about the importance of reasoning in three dimensions, it is significant that 3D imagining is found to be remarkably powerful: one can, for example, mentally scan across an imagined space from *any* viewpoint: the space is mentally encoded in 3D, but can be 'displayed' from any viewpoint in 2D.[158]

The other development that has given images scientific respectability is the very recent availability of computer-intensive visualisation tools, used for such purposes as understanding complex fluid flows.[159] The images are not themselves mental, but of course the point of computer visualisation is to create pictures that lead to understanding, that is, cause suitable mental representations. To some extent, the two themes are connected by research in education, which finds that mental visualisation is a necessary skill for learning in the sciences.[160] Anyone with some sympathy for the idea that ontogeny recapitulates phylogeny will be ready to look for some parallel process in the history of science.

These developments are yet to flow through into general intellectual consciousness, but progress has been sufficient to assert, at least, that Aristotle was more right than Ryle, and that there are no longer any barriers in principle to taking literally what the medievals were saying about mental images.

These studies also encourage us to add some further subtleties to the photo/line-drawing/diagram distinction drawn at the beginning of the paper. In order to *infer* something, whether from a diagram or from propositions, those entities must be represented internally, in the mind, soul or brain. In recent years some understanding has been reached of how this is done: put crudely, there is something like a picture inside; a kind of mental image which one can inspect. The image is, however, much more like a diagram than a photograph, in that it leaves out some things, while emphasising and labelling others.[161] So it is not quite correct to distinguish sharply between an image and a 'model' or metaphor.[162] Because the image is labelled, it contains information about, and can replicate (some of) the structure of its object; and hence support inference about the object. Speaking in an older idiom, Albertus Magnus says that memory is the storehouse not of images alone, but also of the *intentiones* drawn from them by the estimative power, with the image including the *intentio* within itself.[163] (Conversely, of course, from a purely physical point of view, photos, line drawings and diagrams are all just marks on paper. So to distinguish even them one must at least implicitly refer to the internal representation that they cause.)

2.1 The History Of Mental Images: Phantasms, Memory Theatres And Visions

We are now ready to see with new eyes the image-laden, text-poor world of the late medieval at his devotions. It is impossible not to speculate on the richness of the inner imaginal life of those fortunate to have viewed Fra Angelico or Giotto when newly painted. Fortunately, it is not necessary to be content with speculation, since those who did master text (surely a sample biased the wrong way) are eager to tell us in detail all about what it looked like inside.

The ancient legacy of discussion on mental images was moderately rich—richer, certainly, that the available ancient store of actual diagrams, of perspective pictures, or of psychological writings generally. 'There is no thinking without an image'[164] is one of Aristotle's most quoted sayings, and his discussion of imagination posits physical entities like pictures in the sensory apparatus.[165] Talk of 'phantasms', or mental images, is common currency in Stoic and Neoplatonist discussion of perception.[166] If Augustine is not the discoverer of the inner life here to the same extent as he is with autobiography, he certainly talked enthusiastically of inner images. He writes (the portion in italics is quoted in Aquinas' *Summa Theologiae*):

> The bodily sight cannot exist without the spiritual sight, because at the very moment when the bodily sense is touched by a body, there is made in the soul a something which is not this but is like it. If this were not produced, there would be no sense able to perceive those things which lie outside. *For the body does not sense, but the soul through the body, which it uses as a messenger for reproducing within itself what is announced from without.*[167]

He applies the expression 'the mind's eye' (*oculus mentis*) to a kind of intellectual vision.[168] The 'phantasm' occurs in the famous passage of Augustine where he anticipates Descartes' *Cogito, ergo sum*:

> But without any delusive representations of dreams and phantasms, I am most certain that I am ... For if I am deceived, I am.[169]

The western scholastics, following Avicenna, produced a very elaborate, and widely-known, theory of the 'inward wits'. It involved five internal faculties, including the 'imagination', which stored images, and the 'phantasy' (English 'fancy'), which recombined them.[170] There was also a baroque population of entities like the impressed and expressed species in which the various faculties dealt, but the most crucial one was still the 'phantasm'. The startlingly physical view of mental images that the scholastics took is preserved in later discussions of the effect of a mother's imagination on the foetus:

> Now from many instances it is clear that the imagination of the parents has an extraordinary power to modify and inflect the tempering and the formative power. So, if it is extremely strong and intent, sometimes the figure of what is thought, even if quite alien, is induced in the foetus. Thus it sometimes happens that a black child is born of two white parents, because the mother was turning over in her imagination an Ethiopian which she had depicted to herself in the bedchamber ... Given that the imagination can exercise such force, it is difficult to explain it. One opinion is this: The mother apprehends with intent thought, say, the form of an Ethiopian, whose image she imprints on the spirits which are carried back into the passages of the brain. The spirits then act as a vehicle to carry the image to the place of conception, where it

> imbues the material that the foetus is made from with the black colour that later appears. It so modifies the formative power inherent in the semen that what should make the foetus like the parents now makes it like the image, and so what happens is that the foetus degenerates into the form of an Ethiopian.[171]

(Before laughing about quaint views on the efficacy of mental images, one should perhaps recall that imagining walking increases the heart-rate.[172])

There is a dark side to regarding images as real things, which the soul somehow 'has' or 'receives'. It is that one can start brooding over their source, and suspecting their veracity. From Augustine again:

> Whatever we perceive by the body, even when not present to the senses, may be present to the imagination, as when we are asleep or angry: yet we cannot discern by the senses, whether what we perceive be the sensible object, or the deceptive image thereof.[173]

One will be particularly worried if one believes in dark forces whose mission is to deceive, like devils or witches.

> Both in bodily sights and the images of bodies which appear in the spirit, good spirits instruct and bad ones deceive.[174]

The result of these speculations in the fevered imaginations of the witch inquisitors makes an alarming and depressing story, an illuminating case study on the effects to which mistakes in abstract thinking can lead.[175] (Though on the other side the optical writer Witelo wrote a book *On the Nature of Demons*, attributing them to optical mistakes in bad light.[176]) Less harmful, but in principle much the same rationally, were beliefs in the efficacy of the imagination in Renaissance writings on magic and 'fascination'.[177]

After that, the visual imagery (*sic*) in Shakespeare's sonnets reads less as the excesses of the poetical imagination (*sic*) than as sober science:

> Mine eye hath played the painter and hath stell'd
> Thy beauty's form in table of my heart
> My body is the frame wherin 'tis held

And pérspective it is best painter's art.
For through the painter must you see his skill
To find where your true image pictured lies,
Which in my bosom's shop is hanging still *Sonnet 24*

Since I left you mine eye is in my mind
And that which governs me to go about
Doth part his function and is partly blind,
Seems seeing, but effectually is out,
For it no form delivers to the heart
Of bird, of flower, or shape, which it doth latch:
Of his quick object hath the mind no part
Nor his own vision holds what it doth catch. *Sonnet 113*[178]

For some more Shakespearean science:

> A foolish extravagant spirit, full of forms, figures, shapes, objects, ideas, apprehensions, motions, revolutions. These are begat in the ventricle of memory ...[179]

The references here to perspective and the theory of the internal senses are, it will be observed, not just decoration. They concentrate on perception as inference, with the possibility of that inference being faulty.

Conversely, talk about the 'imagination' of poets still meant something literal. Philip Sidney's *Apology for Poetry* explains that poetry is better than both philosophy and history, since the pictures it induces in the imagination have more force than the dry abstractions of philosophy, while still having a generality that the particular facts of history lack. He means by the 'imagination', as usual, a faculty of visualising by recombining images.[180] Sidney occupies a place towards the end of a long tradition of connecting the activity of the poet with the exercise of the (literal) faculty of imagination.[181]

So much for the vivid content of the medieval and Renaissance imagination. What was its purpose? In the first instance, it was for memory. Before memory was exported from the mind to written records, and the art of memory decayed, the geometrical resources of the visual imagination were used as a way of ordering any complicated body of ideas that had to be committed to memory, such as a long speech. The invention of the art was ascribed to Simonides of Ceos, who, leaving a banquet just before the dining hall collapsed, was able to identify the mangled bodies of the diners from the places where they lay; for he found he possessed a mental image of where the diners had been sitting.[182] From the start, a mental image was recognised as a structured entity, in which the relationship between the parts was useful for drawing conclusions. Simonides' idea was taken up in the later antique and medieval 'art of memory' that was the subject of Yates' famous book.[183] Cicero explains:

> The most complete pictures are formed in our minds of the things that have been conveyed to them and imprinted on them by the senses, but the keenest of all our senses is the sense of sight, and consequently perceptions received by the ears or by reflexion can be most easily retained if they are also conveyed to our minds by the mediation of the eyes.[184]

'Keenest' is perhaps not quite right; smells, for example, are 'keen', and one can recognise remarkable numbers of them. But the 'space' of smells does not seem to have a natural structure, whereas it is the geometry of images that makes them so structured, and hence useful for representing the structure of anything else. This is how the actual art works, as explained classically in the *Rhetoric to Herennius*: the orator imagines a building, with rooms, arches, statues. He places in them objects that will stimulate recall of the matter to be remembered. A ram with huge testicles, for example, will suggest testimony. Then while delivering the speech, he mentally visits the places in the correct order, thus recalling the speech. The medievals revived the art, applying it to the many texts that were memorised by all educated people, notably the Psalms. The illuminated capitals and marginal grotesques in medieval books are not just decoration. They are to enhance the visual memory of the page.[185]

Memory was of course closely connected with education, especially training 'by rote'. One of the architects of medieval pedagogy was Hugh of Saint Victor, inventor of possibly the largest diagram of the middle ages. It is his Mystical Ark of Noah, intended to organise the whole of knowledge and allow its recall. None of the surviving manuscripts attempt to picture it, as it is obviously too big to draw, and only fits in the imagination. It has all the patriarchs, popes and so on, also a map of the world, the vices, virtues, seasons and so on and on. It is organised with ladders, wheels and trees. Everything is in it.[186]

By what one may call the *Cutty Sark* phenomenon, the art of memory reached its most perfect form when it was already superseded, after the invention of printing. Father Ricci amazed the Chinese with this piece of Western technology, as with many others, using a vast memory theatre to achieve recall of Chinese characters.[187] Even more remarkable were the vast museum pieces of Giulio Camillo and Robert Fludd. Camillo's description of what he is doing is too overgrown with Hermetic and occult accretions to understand, but it is interesting in that it was actually constructed, in wood (though on what scale is now impossible to tell). It also attracted a brief description by a not entirely sympathetic visitor, who found in the idea of it something still of note:

> The work is of wood, marked with many images, and full of little boxes; there are various orders and grades in it. He gives a place to each individual figure and ornament, and he showed me such a mass of papers that ... He calls this theatre of his by many names, saying now that it is a built or constructed mind or soul, and now that it is a windowed one. He pretends that all things that the human mind can conceive and which we cannot see with the corporeal eye, after being collected together by diligent meditation may be expressed by certain corporeal signs in such a way that the beholder may at once perceive with his eyes everything that is otherwise hidden in the depths of the human mind. And it is because of this corporeal looking that he calls it a theatre.[188]

A mind realised in hardware, with all human knowledge in it, arranged in a natural way, hence content-addressable and immediately accessible! It is a dream yet to be realised by the AI and database communities. The modern equivalent is Lenat's CYC project, which aims to achieve artificial intelligence by having teams of typists enter all commonsense knowledge;[189] so far it is in the same state as Camillo's theatre appeared to his patron, the king of France: money goes in, and promises come back out. But one effect of the CYC project has been to make clear that AI confirms what the medievals presumed: to make sense of experience, one must know (that is, remember) a lot.

The unique aspect of Camillo's plan is that the organisation is topographic, in the most literal possible sense. The reader can no doubt visualise the place in his local library where the books on his favourite subject are kept, and finds unsettling the librarians' periodic rearrangements.

The second use of mental images, perhaps the most widely applied in medieval and Renaissance times, was to encourage meditation during prayer. The majority of mental pictures, like the majority of physical pictures, were painted to assist the soul. The medieval soul is not, according to itself, full of voices, but of forms. If modern introspection reveals neuroses and a stream of words, the medieval tended rather to find sins and visions. If the soul of a medieval was touched by God, it did not experience a voice from God so much as a 'vision'. (Some of the visions are very diagrammatic, too, especially those of Joachim of Fiori and Hildegard of Bingen, regarded as having considerable meaning.[190]) The imagination was made much of by the School of Saint Victor, in the twelfth century.[191] Visions may be for saints, but

> anyone can enter into the interior of his conscience and meditate in his mind's eye on Christ's wound, so that he conforms to Christ's sufferings.[192]

Each of Saint Ignatius of Loyola's *Spiritual Exercises* begins with a 'prelude' along the lines of:

> The second prelude is to form a mental image of the scene and to see in my imagination the road from Nazareth to Bethlehem. I will consider its length and breadth, and whether it is level or winding through valleys and over hills.[193]

Teresa of Avila's Interior Castle is more like a picture gallery than a library; though she is concerned to distinguish the true visions that adorn it from mere works of the imagination.[194]

The typical late medieval, then, was trained to exercise his imagination from his first youth.

2.2. *The Imagination as a Tool of Scientific Visualisation*

It has been argued several times above that various scientific thinkers must have been reasoning spatially in their imagination, particularly in cases like astronomy and perspective that involve heavy use of three dimensions. This ought to be obvious, but the ludicrous misunderstandings possible are illustrated by C.S. Lewis. Lewis would be expected to understand mental images, since, according to his student Ken Tynan, he was usually able to quote from a page, given the bay number in his room, the shelf number, how many books from the left, and the page number. Yet he comments on Albert the Great:

> I do not understand why *boni imaginativi* should tend, as he says they do, to be good at mathematics. Can this mean that paper was too precious to be wasted and you geometrised, as far as possible, with figures merely held before the mind's eye? But I doubt it; there was always sand.[195]

No-one can reason in the sand, since his brain is not there. You might as well say that someone who reads music needs no internal sense of rhythm, on the grounds that the notes are all there on the page. Or that a reader of words does not need to know their meanings, since the meanings are all in the dictionary. *One can only reason with internal representations.*

We now come to the evidence for saying that those who reasoned with the imagination were in general consciously aware of doing so, and saw the imagination as a tool of scientific visualisation.

Plato sometimes uses mental diagrams to represent abstract relations of proportion,[196] and Aristotle has the memory forming a kind of scale model of magnitudes, both spatial and temporal,[197] but generally the earliest writers do not much emphasise the role of the imagination in mathematics. Proclus, however, in the only developed ancient philosophy of mathematics, holds that the imagination is where geometry is done. For the usual Platonist reasons, he thinks physical diagrams are unsuitable: 'the circle in sensible things is inferior in precision, infected with straightness, and falls short of the purity of immaterial circles'. On the other hand, the pure understanding cannot do geometry either, since its concepts are simple, or 'wrapped up', and there is only one of each kind, so that it cannot deal with circles of different sizes. So the understanding projects images 'distinctly and individually on the screen of the imagination', which provides a kind of 'intelligible matter' for them.[198] The passage is an influential one. Kepler is enthusiastic,[199] while A. Piccolomini claims that Proclus' placing of the mathematicals in the imagination explains the certainty of mathematics.[200]

Alhazen remarks that visual rays are imaginary (*lineae imaginabiles* in the Latin);[201] 'lyne ymagined' is also Chaucer's phrase for meridians of longitude and the ecliptic.[202] The point that a diagram is a device to create a three-dimensional model in the imagination is made by a thirteenth-century Islamic writer:

> In drawing ... I have not aimed for completeness. My purpose was to present an arrangement so it can be understood in the whole and in detail. One realizes that there is obscurity in the representation of solid bodies, but in the imagination one can fit one thing to another, view it from any angle, dissect it, and thus assemble it step by step. All the drawings that I have made are simple, so that they give a clear picture.[203]

Hence, the fact that medieval scientific and technological diagrams are sometimes hard to interpret for us does not show that those diagrams failed to support good scientific inference. Living in a post-perspective world, we have it easy.[204]

There may be a case for connecting later medieval reasonings in the imagination with the use in philosophy of cases *secundum imaginationem*, which can only exist through God's absolute power.[205] The work on 'physics' of the 'Merton School', it has been pointed out, is purely 'in the head': it considers only imaginary cases, and confronts real experience only via 'well-known' (that is, remembered) facts such as that a spinning top occupies the same place.[206] Heytesbury says at one point that he is proceeding only *secundum imaginationem*: cases like acceleration to infinity and diminution to zero quantity are not physically possible, but are imaginable and should be considered.[207] It has been maintained that this procedure separates the medieval scientific methodology from the modern, experimental one. It could just as well be maintained that medieval 'physics' is really mathematics. In the modern theory of differential equations, one will certainly want to consider various limiting cases, and for the same reasons as the medievals.

Still, the imaginative and the philosophical methods of reasoning are not entirely compatible. Henry of Ghent, about 1300, calls some opponents

> those of whom the Commentator says that in them the imaginative virtue dominates over the cognitive virtue, and so, he says, they do not believe

> demonstrations unless the imagination accompanies them, for they cannot believe that there is neither plenum nor vacuum nor time outside the world ... mathematical imaginations and what is outside the heaven seem to them infinite ... Therefore such people are melancholy and make the best mathematicians, but the worst metaphysicians, because they cannot extend their understanding beyond site and magnitude.[208]

It will come as no surprise to find Oresme the most explicit on reasoning in the imagination. He thought of his graphs not, in the first instance, as on paper, but in the imagination:

> Every measurable thing except numbers is imagined in the manner of continuous quantity. Therefore, for the mensuration of such a thing, it is necessary that points, lines and surfaces, or their properties, be imagined. For in them [i.e. the geometrical entities], as the Philosopher has it, measure or ratio is initially found, while in other things it is recognized by similarity as they are being referred by the intellect to them [i.e., to geometrical entities]. Although indivisible points, or lines, are non-existent, still it is necessary to feign them mathematically for the measures of things and for the understanding of their ratios. Therefore every intensity which can be acquired successively ought to be imagined by a straight line perpendicularly erected on some point of the space or subject of the intensible thing, e.g., a quality. For whatever ratio is found to exist between intensity and intensity, in relating intensities of the same kind, a similar ratio is found to exist between line and line, and vice versa.[209]

The imagination is better than paper for graphs in higher dimensions:

> By a like imagination the quality of a surface is imagined as a kind of body, whose longitude and latitude is the extent of the surface and whose depth is the intensity of the quality.

So what about the quality of a three-dimensional object? Do we need a fourth dimension to graph its intensity? No,

> because while a flowing point is imagined as causing a line, a line a surface, and a surface a body, it is not necessary, if a body is imagined flowing, that it causes a fourth type of quantity, but only a body.[210]

This can only mean that the flow in the imagination is what represents the fourth dimension. Oresme himself describes the forerunners of his idea, in an effort to excuse himself from the vice of novelty:

> It is sought whether a quality is to be imagined as a surface.
> It is argued on the negative ...
> I respond that the statement is true and could be confirmed by the writers on perspective like Witelo and Lincoln [Grosseteste], who in this manner imagine the intensity of light, and by Aristotle, who in the fourth [book] of the *Physics* imagines time by means of a line, and by the Commentator [Campanus] in the fifth [book] of this [commentary on Euclid's *Elements*], where he holds, in expounding ratios, that everything having the nature of a continuum can be imagined as a line, surface or body.[211]

The originals[212] do use geometrical magnitudes to represent non-geometrical magnitudes, but only the optical writers speak of anything 'imaginary'.

Aristotle had maintained that the form of a work of art, or a building, exists beforehand in the soul of the artist.[213] According to Alberti, architecture does not happen on the building site, or even on paper, but in the imagination:

> It is quite possible to project whole forms in the mind without recourse to the material, by designating and determining a fixed orientation and conjunction for the various lines and angles. Since that is the case, let lineaments be the precise and correct outline, conceived in the mind, made up of lines and angles, and perfected in the learned intellect and imagination.[214]

It is clear that the role of the imagination in both pure and applied mathematical contexts was taken for granted by the time of the Scientific Revolution.

2.3 Galileo's Thought Experiments

It has been a source of embarrassment for many historians of science that Galileo, when he is supposed to be founding modern science by performing experiments, is actually caught doing his experiments in thought. Examples are common; let us take a crucial one in his early work, *On Motion*:

> ... a larger stone does not fall more swiftly than a smaller. Those who are surprised by this conclusion will also be surprised by the fact that a very large piece of wood can float on water, no less than a small piece. For the reasoning is the same. Thus, if we imagine (*si mente conciperemus*) that the water on which a large piece of wood and a small piece of the same wood are afloat, is gradually made successively lighter, so that finally the water becomes lighter than the wood, and both pieces slowly begin to sink, who could every say that the large piece would sink first or more swiftly than the small piece?
>
> ...
>
> Again, if we imagine, for example, a large piece of wax floating on water, and we mix this wax either with sand or some other heavier substances ...
>
> I argue as follows in proving that bodies of the same material but of unequal volume move with the same speed. Suppose there are two bodies of the same material, the larger *a*, and the smaller *b*, and suppose, if it is possible, as asserted by our opponent, that *a* moves more swiftly than *b*. We have, then, two bodies of which one moves more swiftly. Therefore, according to our assumption, the combination of the two bodies will move more slowly than that part which by itself moved more swiftly than the other. If, then, *a* and *b* are combined, the combination will move more slowly than *a* alone. But the combination of *a* and *b* is larger than *a* is alone. Therefore, contrary to the assertion of our opponents, the larger body will move more slowly than the smaller.[215]

Where does the 'combination' of *a* and *b* take place? According to Aquinas, it is the 'phantasy or imagination' that is the organ of 'combining and dividing'.[216] Galileo's 'gradually' and 'successively' in this passage are significant: like Oresme with his 'flowing point', he regards the imagination as a (more or less) continuous medium. By movement through that continuous medium, he can perform his characteristic transformation to a limiting, ideal case, which is still physically meaningful, but inaccessible to real experiment.[217] Where the medievals used imaginary cases, like motion in the void, largely for critical purposes, Galileo uses them constructively, regarding them as simple versions of reality, to which complex real cases approximate.[218]

And is Galileo evasive about the imaginary nature of his experiments? Modest? Repentant? No, he is brazen:

> SIMPLICIO: So you have not made a hundred tests, or even one?
> SALVIATI: Without experiment, I am sure that the effect will happen as I tell you, because it must happen that way.[219]

As Kuhn remarks, 'Surely he did experiments, but he is even more noteworthy as the man who brought the medieval thought-experimental tradition to its highest form'.[220]

Experimenting in the imagination is a lot cheaper than in the lab, of course, and faster, like modern simulations by computer, but like them is exposed to the objection that it will not tell you how the real world is, since in thought, presumably, anything can happen.

This is the nub of the matter. Imagination, used as a tool for scientific reasoning by an expert like Galileo, is *not* infinitely plastic, and hence divorced from reality, but incorporates a lot of structure, parts of which restrict what can happen in other parts. This is why actual experiments on motion on an inclined plane can be relevant to what would happen with vertical motion in a vacuum, and the imagination can mediate between the two. Feynman, a modern physicist who was champion of imagination over mathematical formalism, explains that the point of the scientific imagination is the *constraints* it incorporates:

> The whole question of imagination in science is often misunderstood by people in other disciplines. They overlook the fact that whatever we are *allowed* to imagine in science must be *consistent with everything else we know* ... We can't allow ourselves to seriously imagine things which are obviously in contradiction to the known laws of nature. And so our kind of imagination is quite a difficult game.[221]

The question is, how does the imagination acquire that structure that mimics the world, and which allows it to be used as a bridge between real experiments and what would happen in counterfactual circumstances? One way or another, the imagination must be structured by the flux of ordinary experience. An indication of how it happens can be had by recalling Steven's Wreath of Spheres diagram, which is a thought experiment not unlike Galileo's. The fact that the spheres do not revolve in either direction, but hang in equilibrium, is a deliverance of the imagination, but is not a logical truth. It must, therefore, be a distillation of experience. It follows that the process of using the imagination to learn the results of thought experiments is a kind of reminiscence. That is what Aquinas says: the 'phantasy or imagination' is for the 'retention and conservation' of forms.[222] More to the point, it is what Galileo says. Salviati is encouraging Simplicio to realise that a stone released from a sling moves off tangentially:

> SALVIATI: The unraveling depends upon some data well known and believed by you just as much as by me, but because they do not strike you, I shall cause you to resolve the objection by merely recalling them.
> SIMPLICIO: I have frequently studied your manner of arguing, which gives me the impression that you lean toward Plato's opinion that *nostrum scire sit quoddam reminisci* [our knowing is a kind of reminiscence] ...
> SALVIATI: Well, then, what is its motion?
> SIMPLICIO: Let me think a moment here, for I have not formed a picture of it in my mind.
> SALVIATI: Listen to that, Sagredo; here is the *quoddam reminisci* in action, sure enough. Well, Simplicio, you are thinking a long time.
> SIMPLICIO: So far as I can see, the motion received on leaving the notch can only be along a straight line ...[223]

Two other arguments in the *Dialogue* involve Socratic questioning of Simplicio's 'memory'; they are both purely geometric arguments. In one of them, Simplicio is gradually forced to draw a diagram of the Copernican universe, led by purely geometrical considerations.[224]

Let us separate two questions that may occur on reading Galileo. Can one get a reasonably consistent answer by quizzing people's imagination on, for example, what happens when a body whirled on a string is released? And if so, is the answer the correct one, as Galileo suggests? Modern psychology has investigated and the answers are respectively 'yes' and 'sometimes'. Consistent patterns of expectation about motion are found on eliciting subjects' 'intuitive physics' or 'naive physics' by asking them to imagine what motion would occur in various circumstances. Some subjects give the correct answers, but a high proportion, even those educated in physics, persist in mistakes like expecting curved motion to continue in a curve when released.[225] The authors of these studies claim that all the different medieval errors about the motion of projectiles and circular impetus can be found among present-day college students. This suggests, among other things, that the medievals were doing their physics by the same kind of imaginative reasoning from experience as the psychologists are now studying.

2.4 Epilogue: Philosophy Moves Inside

The story would not be complete without at least a brief mention of the fact that a Philosophical Revolution occurred at the same time as the Scientific one, and that it too was driven by inference from mental images.

Kant claimed to have effected a 'Copernican revolution' in philosophy, meaning a fundamental change in viewpoint. He had in mind his claim that necessities, such as mathematical ones, previously thought to be in the world, were removed, in his philosophy, into the cognitive apparatus. Almost everyone has thought that such a reversal of point of view was indeed a revolution, but that its true Copernicus was Descartes. It is universally agreed that 'modern' philosophy began when the Cartesian *cogito* transferred the central question of philosophy from metaphysics to epistemology. It is the interior viewpoint that Descartes starts from which makes the problem of knowledge of the external world central. The point was argued at length by the modern Thomists, who saw Descartes as having diverted philosophy into several centuries of the 'way of ideas' and idealism by beginning with the certainty of mental images, and asking, 'How do we get out?'[226]

Descartes certainly had a well-trained imagination, and its training was on geometry. In the *Discourse*, he says of his early studies in geometry:

> [Geometry] is so closely tied to the examination of figures that it cannot exercise the intellect without greatly tiring the imagination.[227]

That is the opposite to the naive thought that imagining a few pictures should be easy, even if the logical reasoning is difficult. Not so difficult, though, that he will not recommend hard work with the imagination to others. Rule 14 of *Rules for the Direction of the Mind* is:

> The problem should be re-expressed in terms of the real extension of bodies and should be pictured in our imagination entirely by means of bare figures. Thus it will be perceived much more distinctly by our intellect.[228]

Among the figures suggested is a family tree. It is only with Rule 15 that permission is given for the diagrams to be drawn on paper.

On the question of whether training in mathematics is for the purpose of learning imaginative modelling or logical reasoning, Descartes prefers the latter, but concedes it is possible to do mathematics imaginatively:

> The fact that there are some people who are clever at Mathematics but less successful in subjects like Physics, is not due to any defect in their powers of reasoning, but is the result of their having done Mathematics not by reasoning but by imagining—everything they have accomplished has been by means of imagination. Now, in Physics there is no place for imagination, and this explains their signal lack of success in the subject.[229]

In the *Meditations*, Descartes examines the difference between the imagination and the 'pure understanding':

> When I imagine a triangle, for example, I do not merely understand that it is a figure bounded by three lines, but at the same time I also see the three lines with my mind's eye as if they were present before me; and this is what I call imagining. But if I want to think of a chiliagon, although I understand that it is a figure consisting of a thousand sides just as well as I understand the triangle to be a three-sided figure, I do not in the same way imagine the thousand sides or see them as if they were present before me ... But suppose I am dealing with a pentagon: I can of course understand the figure of a pentagon, just as I can the figure of a chiliagon, without the help of the imagination; but I can also imagine a pentagon, by applying my mind's eye to its five sides and the area contained within them. And in doing this I notice quite clearly that imagination requires a peculiar effort of mind which is not required for understanding ...[230]

Elsewhere, Descartes claims that a heptagon or octagon can only be imagined with difficulty, but that was he, 'who is a fairly imaginative man and has trained his mind in this field for some time', can do it 'reasonably distinctly'. In the same place he emphasises how like imagination perception is, the only difference being that the images are imprinted in one case without and in the other case with external objects.[231] Again, in defining his crucial term 'idea', Descartes distinguishes an idea from an image in the imagination, but the difference is not large:

> Thus it is not only the images depicted in the imagination that I call 'ideas'. Indeed, in so far as these images are in the corporeal imagination, that is, are depicted in some part of the brain, I do not call them 'ideas' at all; I call them 'ideas' only in so far as they give form to the mind itself, when it is directed towards that part of the brain.[232]

It is true that here, by agreeing with the scholastics that the imagination is 'corporeal', Descartes distinguishes some internal pictures from what is really in the 'ego'.[233] Nevertheless, it is clear that Descartes' 'idea' has a spatial content lacking in the modern 'concept', and that this survival of the medieval theory of the imagination is at the bottom of much of what moderns find bizarre in the seventeenth and eighteenth-century 'way of ideas' in philosophy.[234]

And it is no accident that Descartes' 'representative' theory of perception is essentially the same as Aristotle's theory of memory. According to Aristotle, one can regard a memory image, like any picture, either simply as a painted thing or as a likeness.[235] It is in the latter mode that memory is a kind of inference, from image to thing pictured. Indeed, Aristotle emphasises that recollecting is a sort of inference, hence, he thinks, peculiar to humans:

> For when a man is recollecting he infers that he has seen or heard or experienced something of the sort before, and the process is a kind of search.[236]

Aristotle says that memory is inference from internal pictures. Descartes' Copernican revolution in philosophy is the claim that all perception is inference from internal pictures.

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NOTES

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[16] Seneca, *De beneficiis* III.xxviii.2; Pliny, *Natural History* XXXV.ii.6.

[17] M.T. Clanchy, *From Memory to Written Record* (London, 1979) plate XIII.

[18] E.H. Wilkins, 'The trees of the 'genealogia deorum', *Modern Philology* 23 (1925-6) 61-5; Murdoch, *Album of Science*, illustration 41.

[19] Evans, *Medieval Drawings,* plate 71; Murdoch, *Album of Science*, illustrations pp. 37-9.

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