Global and Local

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Discrete versus continuous, simple versus complex, global versus local, linear versus nonlinear, deterministic versus stochastic, analytic versus numerical, constructive versus nonconstructive – those contrasts are among the great organizing themes of mathematics. They are forks in the road of mathematical technique – the concepts along one fork are very different from those along the other, even when they give complementary views on the same phenomena.

It is hard to find a clear and elementary exposition of any one of those contrasts, but perhaps it is the global/local distinction that is worst served by current theory. A beginning graduate student in mathematics is certainly expected to have a sense of the distinction and to be able to talk coherently about "local minima versus global minimum", "a local solution to a d.e. that is not extendable to a global solution", and so on. But there is no article available on the distinction in Wikipedia, the Springer *Encyclopedia of Mathematics*, or Wolfram Mathworld. (Wikipedia and Mathworld do have very brief articles on "local" in the sense of topological spaces.)

This article brings together some mostly familiar examples of the global/local distinction from a range of different areas, as a basis for explaining clearly what the distinction is and why it is central to mathematics.

Local and global behavior of functions

The first example is not the simplest that could be found. It is chosen because it is typical of the kind of theorems that are most commonly thought of as involving the interaction of global and local structure.

It is impossible to build a circular or nearly-circular staircase that goes up all the way round and ends at its starting point. The famous Escher drawings with the structure of "Penrose stairs" [19] which seem to show this kind of thing happening, as in Fig 1, are thus impossible to realise physically.

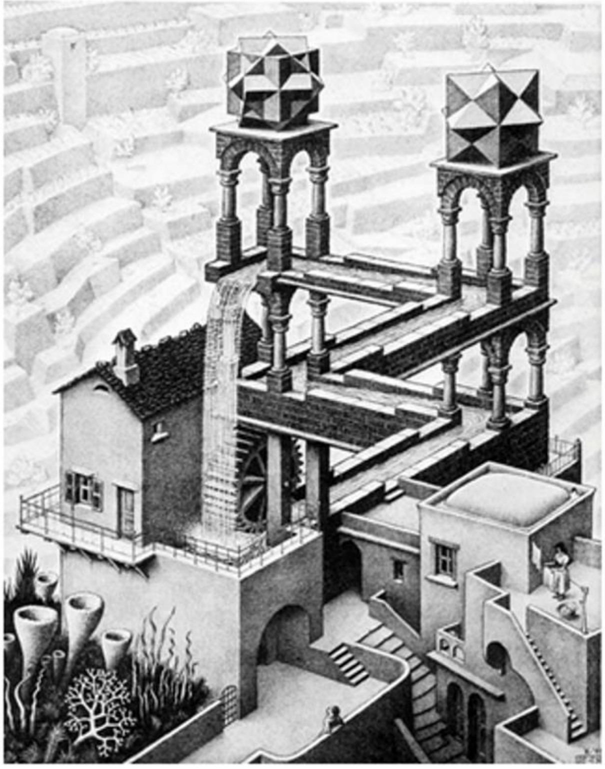

Figure 1: Escher's Waterfall

(http://en.wikipedia.org/wiki/File:Escher_Waterfall.jpg)

The impossibility is not just empirical, since no change in the laws of nature would make such a staircase possible. There is a purely mathematical fact underlying the impossibility, namely, that there exists no continuous function from the (oriented) circle to the real numbers which is increasing all the way round. That theorem involves the global/local distinction in an essential way: locally – at any point on the circle – it is possible to find a neighbourhood and a function from that neighbourhood to the reals which is monotonic increasing; but it is impossible to fit those local choices together to construct a function from the whole oriented circle to the reals which is increasing everywhere.

The example fits into general theory of the extension of continuous functions [23], [13]. There are many more advanced theorems in algebraic topology, differential geometry and related fields, of a similar character to the staircase example. They describe how global structure constrains local structure (or, depending on one's point of view, how local structure gives rise to global). A well-known instance is the "hairy ball theorem", which states that the hair on a ball cannot be combed flat everywhere (formally, every continuous tangent vector field on the sphere vanishes somewhere) [6].

The fact that this is impossible on the sphere while possible on the torus indicates that the theorem is about "relationships between the local differential properties of a space and its topologic structure as a whole" [3] − how the global topology of the space constrains what is possible differentially locally.

Before launching into any more advanced theorems, it should be emphasised that the global/local distinction pervades some much more elementary sections of mathematics.

Local and global in difference and differential equations

Recall how compound interest works. If money is invested in a bank at 2% per month compound interest, the accumulated amount (principal plus interest) after t months, P_t, is related to the amount of the month before, P_{t-1}, by

$$P_t = P_{t-1} + {}^2/_{100}\, P_{t-1}$$

The formula says "each month, add to the accumulated amount 2% of itself to get next month's amount."

That equation expresses the *local* structure, the relation between the accumulated amounts at consecutive months. The bank's computer starts out with the original principal, and goes through step by step using the equation to calculate the accumulated amount after t months. The resulting *global* structure, the general shape of what happens over time, is represented by the familiar rising exponential growth curve. That overall shape is not visible in the local structure: it comes only from *solving* the equation, that is, discovering the global structure implied or induced by the local structure – namely, $P_t = P_0\,(1 + {}^2/_{100})^t$.

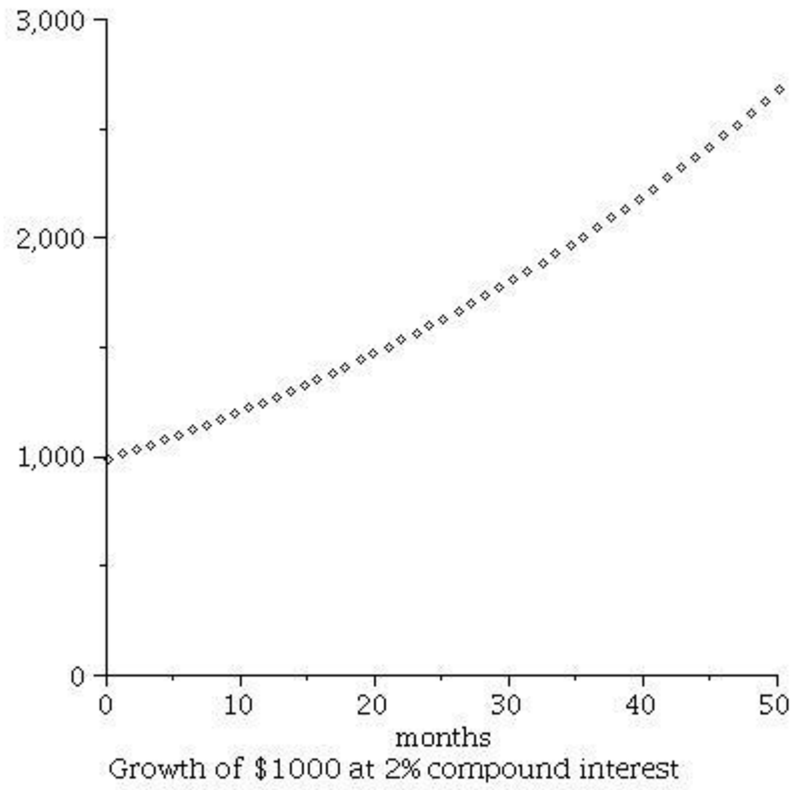

Figure 2 Compound interest

Similar phenomena arise in continuous cases such as the exponential growth equations that are often used to approximately model populations. If a population P grows continuously at an instantaneous rate of, say, 2% a month, then again mathematical reasoning can start with the local structure expressed by the differential equation $\frac{dP}{dt} = 0.02P$, and solve it to extract the global structure, the result being the familiar exponential growth curve $P = P_0e^{0.02t}$. Of course its shape is similar to the compound interest graph except for being continuous – the local versus global duality cut across the discrete/continuous divide.

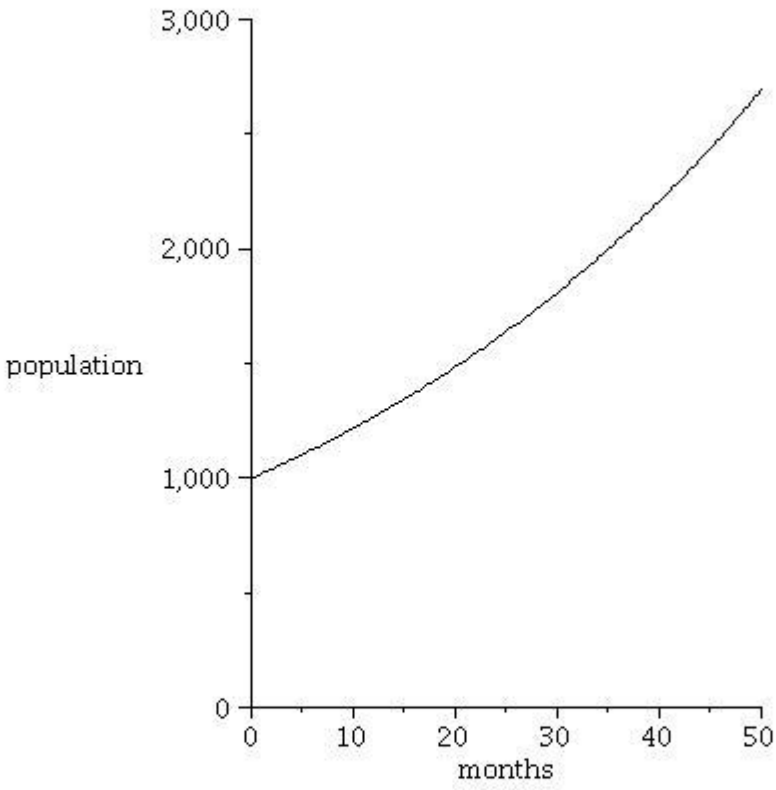

Figure 3 Continuous exponential growth

Again, the exponential shape is not a property of the originally given local structure, the monthly growth rate. It is visible only in the global structure.

So there is fundamentally more to the concept of "solution to a d.e." than there is to "solution to an algebraic equation". A solution to an algebraic equation is just a number that in fact satisfies the equation. But to regard a solution to a d.e. as just a function that in fact satisfies the equation (as pedagogy in pure mathematics often unfortunately does) misses the point of a d.e. A d.e. involves a derivative, a local notion, and asserts some fact about the derivative that holds locally at all points in the space. A solution is a global function, which is the collective result of the d.e. acting at all the points locally; it is how the local solutions fit together. The global solution has properties unlike those of the local solutions.

In general, textbooks on the solution of differential equations emphasise that the fundamental existence and uniqueness theorems are local. Given a point in the space and a d.e. (expressing the flow at that point), there is *some* neighbourhood of the point such that a solution of the d.e. through the point exists (e.g. [12] p. 223). It is a separate and more difficult enterprise to study "global analysis", the theory of "differential equations from a global, or topological point of view", which examines more global and qualitative properties of the solutions on the whole space, such as phase portraits and bifurcations. The geometric picture of a d.e. as a flow on a manifold gives an insight into the global structure arising [21]. It is an example of Hadamard's more general claim of 1921, "almost everywhere, the advance of contemporary mathematical sciences consists of two steps: The local solution *(solution locale)* of problems; the passage from this solution to a solution in the large *(solution d'ensemble*), if this kind of synthesis is possible." ([10] p. 205, discussed in [4] pp. 46-7).

Despite the more or less self-evident nature of the global-local distinction in such contexts, the language of "global" and "local" to describe it is not as old as one might think. The oldest use of "global", in the mathematical sense, known to the (usually omniscient) *Oxford English Dictionary* is in a rather opaque footnote to a 1937 paper on general topology by B. Kaufmann, which says:

> In geometry (and not only in geometry) the local validity or extension of properties given in the large is usually so obvious that one hardly refers to it; for instance, each point of an element (simplex) lies on an arbitrarily small element etc. It seems to me essential to make a clear distinction between properties in a point of local and of integral (or global) origin. [15]

But that is misleading. The detailed historical work of Chorlay [4] shows that the pair "im kleinen" and "im grossen" were used in a similar sense in both German and American mathematics from the 1890s, such as in Osgood's expositions of German developments in analysis. Osgood at one point gives an explicit definition (in German):

> The concept of behaviour of a function *im Kleinen* and *im Grossen* plays an important role in Analysis, and concerns all parts of mathematics (in particular Geometry as well) where a continuous set of elements form the substrate for the configuration to be studied. In the theory of functions, the behaviour of a function *im Kleinen* resp. *in Grossen* means its behaviour in the neighbourhood of a given point a … or a point-set P … resp. in a domain T … the extent of which is set from the start and not determined afterwards to meet the requirements of the given problem ([18] p.12, discussed in [4] p. 20).

Local and global extrema

The other main appearance of the local/global distinction in the more elementary levels of mathematics is in the notion of global and local extrema of functions. A local maximum is a value that exceeds those in *some* neighbourhood of a point, while a global maximum is one that exceeds all other values everywhere. Early instruction in calculus emphasizes that the method of looking for zeros of a function's derivative is adapted to finding local extrema, not global ones. As one clear exposition puts it:

> In order to find the minima of a differentiable function, which is a global property, we examine how the function should behave near such a point and deduce that the point must be critical, i.e. all partial derivatives must vanish. Then, in order to determine whether a critical point is a [local] minimum, a maximum or neither, we apply the second derivative (Hessian) test. Finally, having determined all local minima, we simply compare the values of the function at those points to determine the global minimum. ([14] pp. 169-70)

"Local maximum" is an instance of a more general notion of "local", as described in Wolfram MathWorld's article 'Local': "A mathematical property P holds locally if P is true near every point. In many different areas of mathematics, this notion is very useful. For instance, the sphere, and more generally a manifold, is locally Euclidean. For every point on the sphere, there is a neighborhood which is the same as a piece of Euclidean space." [20] Or to take another example, curvature (of either a curve or a surface) is a local notion: it is "at" a point, that is, definable using only knowledge of behavior in an (indefinitely small) neighbourhood of the point; whereas whether a curve is closed or a surface finite is not a local notion: it depends on what happens to the whole curve or surface.

The concept of global versus local extrema is certainly an easy way in to the global/local distinction. At the same time it quickly leads into themes of global/local interaction, as when the existence of two isolated local maxima of a differentiable function on an open interval implies the existence of a local minimum between them. Or Rolle's Theorem, so basic to the foundations of calculus, in which the local differentiability of a function equal at two distinct points implies that there is a point somewhere between them where the derivative is zero.

Combinatorics

There are many examples in combinatorics of behavior that is possible locally but impossible globally, illustrating that the global/local distinction is not restricted to differential or continuous contexts (or to difference equations that could be regarded as discretizations of the continuous, like compound interest). The “stripped-down” or “bare hands” nature of combinatorics thus exposes the essence of the global/local distinction as lying in the relation of parts and wholes as such. The natural setting of the distinction is not in the world of continuity or differentiability.

Take Euler’s classic example of the bridges of Königsberg. The bridges connected two islands and two riverbanks as shown in the figure 4:

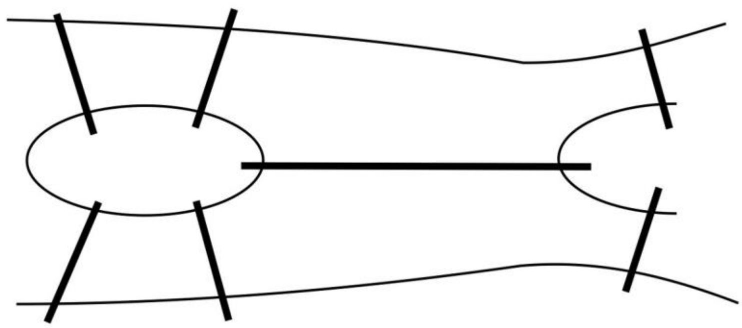

Figure 4 The Bridges of Königsberg

The citizens of Königsberg in the eighteenth century noticed that it was impossible to walk over all the bridges once, without walking over at least one of them twice. In his pioneering work on what we would now call the topology of graphs, Euler proved they were correct [7]. Locally, there is plenty of choice of paths: it is easy to start in any land area and choose a bridge to move to any other land area and then to another one. But it is impossible to fit those local choices together to form a solution to the global problem of finding an “Euler path” – one including all the bridges exactly once.

Or consider the simplest non-trivial example of Ramsey theory. Take six points, with each pair joined by a line. The lines are all colored, in one of two colors (represented by dotted and undotted lines in the figure). Then there must exist a triangle of one color (that is, three points such that all three of the lines joining them have the same color).

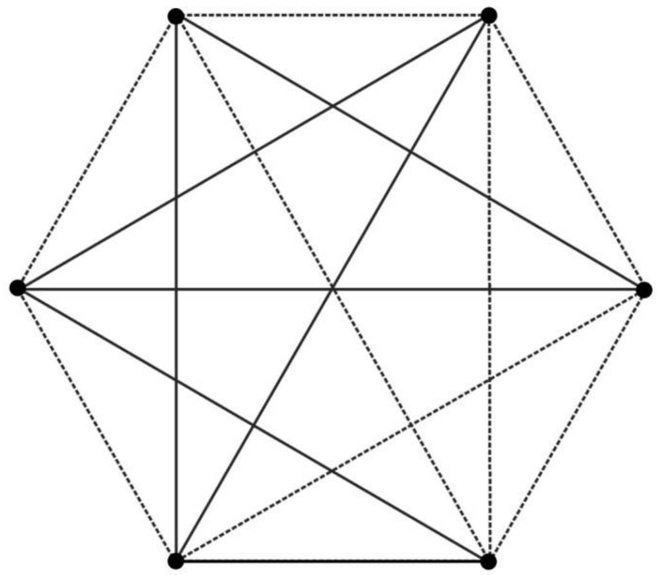

Figure 5 Combinatorics with Six Points

Proof: Take one of the points, and call it O. Then of the five lines from that point to the others, at least three must have the same color, say color A. Consider the three points at the end of those lines. If any two of them are joined by a line of color A, then they and O form an A-color triangle. But if not, then the three points must all be joined by B-color lines, so there is a B-color triangle. So there is always a single-colored triangle.

As in the case of Euler's bridges or the staircase example, what is easy locally is impossible globally. It is easy at any point to choose a line of any color to any other point, but when making all those choices simultaneously, it is impossible to avoid having a same-color triangle configuration somewhere. Similar language could be used naturally of the four-color map theorem.

Even the pigeonhole principle, in its simple way, can be seen as an instance of the same phenomenon – a global obstruction to what is possible locally anywhere. If one tries to place 10 pigeons in 9 pigeonholes, it is easy to start out by placing single pigeons in empty holes, but at the end of the process, no matter what sequence of choices one makes, the global structure forces the assignment of at least two pigeons to at least one pigeonhole.

The pigeonhole principle takes us in the direction of number theory, where the "Hasse principle" concerns the extendability of "local" solutions of Diophantine equations (that is, solutions modulo each prime power) to a "global" solution (a solution over the integers) [2]. The analogy with extendability of continuous functions is not as clear as it might be, and it may be arguable that this is not truly the same notion of global and local. However, the existence of a solution over the integers does imply the existence of a solution for each prime power; and while the integers modulo any number are not exactly a part of the integers but rather a quotient structure, a quotient structure carries part of the information about the full structure. So there is some genuine analogy between the way in which the partial information contained in the modular solutions may or may not fit together into a full solution over the integers, and the usual question as in the original staircase example, whether local functions satisfying a condition fit together into a global function satisfying the condition. But the setting is discrete.

Global and local outside mathematics: physics and economics

Like any good mathematical concept, it is to be expected that the global/local distinction will prove fruitful in many areas outside mathematics proper. The full story would ramify endlessly. We just give a few short examples to indicate the vast range possible.

It is familiar in General Relativity, as is natural because that science is largely an application of differential geometry. It may be that the physicists have a clearer sense of the distinction and the interaction of local and global than most mathematicians do. Hawking and Ellis's classic, *The Large Scale Structure of Space-Time*, begins:

> The view of physics that is most generally accepted at the moment is that one can divide the discussion of the universe into two parts. First, there is the question of the local laws satisfied by the various physical fields. These are usually expressed in the form of differential equations. Secondly, there is the problem of the boundary conditions for these equations, and the global nature of their solutions. This involves thinking about the edge of space-time in some sense. These two parts may not be independent. Indeed it has been held that the local laws are determined by the large scale structure of the universe. This view is generally connected with the name of Mach, and has more recently been developed by Dirac (1938), Sciama (1953), Dicke (1964), Hoyle and Narlikar (1964), and others. We shall adopt a less ambitious approach: we shall take the local physical laws that have been experimentally

> determined, and shall see what these laws imply about the large scale structure of the universe. ([11] p. 1)

It is found, for example, that if the laws observed locally do hold at other places and times, then a global property of the universe is a singularity in the past, the Big Bang.

In economics, Adam Smith's "invisible hand" is a metaphor for the difference between what individual actors intend locally and the global effect of their actions. A buyer or seller "generally, indeed, neither intends to promote the public interest, nor knows how much he is promoting it. … he intends only his own security; and by directing that industry in such a manner as its produce may be of the greatest value, he intends only his own gain, and he is in this, as in many other cases, led by an invisible hand to promote an end which was no part of his intention." ([22] bk. IV ch. 2) The "invisible hand" is not an entity or in any way a cause that acts. All the causes acting are local: the individual decisions of buyers and sellers. The global effect, overall prosperity, is the working out of the sum total of those local effects, in the same way as the solution of a differential equation is the sum of the effects of local actions. General equilibrium economic models have explored how microeconomic behavior involving many free choices results in the global stability (or not) of the economic system. In finance, too, there is a recognised distinction between the risks of individual investments failing in the present climate, and "systemic risk", the risk that there will be a failure across a large part of the financial system ([5]; some mathematical perspectives in [1]).

Examples of the local/global contrast can be found in any science that deals with complex systems, such as psychology [8] and computer science (where the distinction between local and global variables is important to the modularity of programs). But enough has been said to make it clear why the pair of concepts is ubiquitous. And why there are many opportunities for mathematicians, to whose expertise belong questions about global-local duality.

History: Leibniz's best of all possible worlds

The first clear use of the contrast between global and local had nothing to do with mathematics, though it was an idea of one of the great mathematicians, Leibniz. Leibniz was a man with a vast range of intellectual interests. One of them was Christian theory, especially the problem of evil. As most religious people recognize, one of the main difficulties with believing in God is the great amount of evil in the world: if God is good and also all-powerful, how can he allow such horrors? Why does he not make a much better world, if he can?

Leibniz's solution to this difficult puzzle, put forward in his book *Theodicy* (Divine justice) of 1710, is a startling one. God does not create a better world, he says, because there *is* no better world. This is already the best of all possible worlds. By that he does not mean that everything is rosy in the actual world. Quite the contrary, there is a great deal wrong, but any attempt to tinker with it to improve it here or there would make it worse overall. Leibniz writes: "all things are *connected* in each one of the possible worlds: the universe, whatever it may be, is all of one piece, like an ocean: the least movement extends its effect there to any distance whatsoever … Therein God has ordered all things beforehand once for all, having foreseen prayers, good and bad actions, and all the rest; and each thing *as an idea* has contributed, before its existence, to the resolution that has been made upon the existence of all things." ([16], ch 9) He means that from God's-eye point of view, design is global (over time as well as space), while human action and human imagination is local. The point of Leibniz's theory is made in an old joke: An optimist is someone who thinks this is the best of all possible worlds; and a pessimist thinks the same.

So Leibniz's theory relies on the local/global contrast. The world is, he says, easy to improve locally but impossible to improve globally. The limitations of our intellect and imagination make it seem to us easy to suppose this or that thing being made better, without our understanding the necessary cross-connections between things which make it impossible — logically or mathematically impossible — to realise all those improvements at once. As in the staircase example, what is easy to do locally is impossible to do globally – but its impossibility may be hard to see ([9], history in [17]).

Perhaps only a mathematician could have taken that idea seriously.

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