Intertwining metaphysics and mathematics: the development of Georg Cantor's set theory 1871-1887

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Summary

This article discusses Georg Cantor's philosophical defence of the existence of actual infinities in mathematics. The metaphysical background to Cantor's argument is documented, emphasizing the philosophical references in Cantor's main work, *Grundlagen* (1883). Cantor's engagement with the history of philosophy, particularly the writings of Spinoza and Leibniz on infinity, is also documented. Finally, relevant portions of Cantor's correspondence and portions of his *Nachlass* are also used as evidence to document the development of Cantor's philosophy.

Contents

1. The reception and philosophical significance of Cantorian set theory

When Michael Crowe first argued that there can be no revolutions in mathematics, he had in mind the way in which old mathematical theories (e.g. Euclidean geometry) can peacefully co-exist with alternative theories (e.g. non-Euclidean geometries).[1] The cumulative nature of mathematics suggested initially to Crowe that the mathematics was not susceptible to Kuhnian treatment. However, reflection on various episodes in science and mathematics reveal a more complex picture, as Crowe came to acknowledge. Paradigm shifts need not be a matter of total

[1] M. Crowe, 'Ten 'laws' concerning patterns of change in the history of mathematics', *Historia Mathematica* 2: 161-6, 1975. Reprinted in *Revolutions in Mathematics*, edited by D. Gillies, (Oxford, 1992), 15-21.

replacement of one theory by another. Often what happens is a theory becomes subordinated to a more encompassing theory. Rather than throwing out a theory and its existing ontological commitments, the existing scientific theory is incorporated into a more encompassing theory. The result is an ontologically conservative, bloodless revolution. This model of change, well documented in theoretical physics, is applicable to mathematical knowledge as well. Dauben [1992] argues that Cantor's set theory is an example of a mathematical revolution.[2] Cantorian set theory requires a commitment to the existence of actual infinities. There are (according to Cantorian set theory) actually infinite sets of various sizes, each of a certain transfinite cardinality. The 'infinite', once thought to be beyond the scope of mathematics, could be tamed.

Cantorian set theory is revolutionary in many ways: philosophically and mathematically. It throws off the Aristotelian prohibition against actual infinities in mathematics. It provides a new way of conceiving of the structures of mathematics as sets. Set theory is powerful enough to embed hitherto existing mathematical theory within itself, thereby providing a foundation for mathematics. The Cartesian dream of a certain foundation for knowledge seems within reach, if only the axioms and premises of set theory could be known with certainty. Of course we know from subsequent history that the Cartesian dream was shattered. The assumptions of set theory—particularly the assumption that the infinite sets are well-ordered—have not met with universal agreement. Possibly, the best possible way of vindicating these axioms is 'post hoc': they produce powerful, fruitful mathematics and are consistent as far as we know, so they should be assumed to be true.[3]

Cantorian set theory should be strongly differentiated from 'naïve set theory' with its attendant set-theoretical paradoxes. Scholars of Cantor's writings have known for some time that Cantor was aware of the set-theoretic antinomies and took steps to avoid them as early as 1883.[4] That Cantor did not develop a sufficiently formal way of warding off the antinomies is fair enough, but the germ of the idea of how to do so--by assuming the cumulative set theoretical hierarchy-- is present in his work.

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[2] J. Dauben, 'Conceptual revolutions and the history of mathematics: two case studies in the growth of knowledge (1984)', in *Revolutions in Mathematics*, edited by D. Gillies, (Oxford, 1992).

[3] P. Maddy, 'Believing the Axioms I', *The Journal of Symbolic Logic*, Vol. 53, No. 2 (Jun., 1988), 481-511.

[4] G. Moore, ' Towards a History of Cantor's Continuum Problem', in *The History of Modern Mathematics*, edited by D. Rowe, and J. McCleary, vol. I, (Boston, 1989).

Among some professional mathematicians, especially those of a constructivist bent, the view persists even today that Cantorian set theory is a kind of religion. The implication of this view of Cantorian mathematics is that its theorems must be taken on faith without argument. This tone for the reception of Cantor's work was set in the nineteenth century by the caustic remarks apocryphally attributed to Leopold Kronecker: 'I don't know what predominates in Cantor's theory - philosophy or theology, but I am sure that there is no mathematics there.'[5] The view was perpetuated by Poincaré's description of Cantorian set theory as a pathological disease.[6] However, eventually Cantorian set theory went mainstream as far as mathematics is concerned, owing to the staunch advocacy of Hilbert and Russell as well as its axiomatization by Zermelo.[7]

It is probably accurate to say that Cantor's reputation as a philosopher has fared worse than his reputation as a mathematician in the long term. Few contemporary analytic philosophers pay much serious attention to Cantor. Frege's criticism of Cantor, mediated by Dummett's influential advocacy of Frege's work, have undoubtedly played a role in the chilly reception of Cantor's philosophy.[8] As a rule those philosophers with a strong interest in the history of science and mathematics have paid the most positive attention to Cantor. In an exceptional work of careful historical scholarship, Michael Hallett excavates the theological and metaphysical background underlying Cantor's mathematics.[9] Hallett clearly recognizes the influence of Cantor's views on the subsequent development of set theory. However, Hallett is unconvinced as to the philosophical worth of Cantor's metaphysical views. He writes that:

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[5] This apocryphal quotation is given at the top of N. Wildberger, 'Set Theory: Should you believe?', http://web.maths.unsw.edu.au/~norman/views2.htm. Wildberger does not give a source for this Kronecker quotation.

[6] H. Poincaré, *Science and méthode* (Paris, 1908). English version: 'The Future of Mathematics', in *Science and Method* , translated by F. Maitland, (New York, 1952), 25-45.

[7] D. Hilbert, 1926. 'Űber das Unendliche', *Mathematische Annalen* vol. 95: 161-90, 1926. English version: 'On the Infinite', translated by E.Putnam and G. J. Massey in P. Benacerraf and H. Putnam, eds., *Philosophy of Mathematics*, Harvard University Press, 183-201. Bertrand Russell's support for Cantor is evident in B. Russell, *Principles of Mathematics,*(London: Routledge), 1903. The axiomatisation of set theory is carried out in E. Zermelo, 'Untersuchungen über die Grundlagen der Mengenlehre I', *Mathematische Annalen* 65 no.2 June, Berlin: Springer, 1908.

[8] M.A.E. Dummett, *Frege: Philosophy of Mathematics*, (London, 1991).

[9] M. Hallett, *Cantorian Set Theory and Limitation of Size* (Oxford, 1984).

> *In the end*, it is important to divorce the main content of Cantor's ideas… from their theological origins. This is partly because the appeal to God in the way Cantor often does is ultimately no great help. For example, to claim that certain infinite sets or certain infinite numbers exist because it is possible for God to conceive, and therefore to create, them, or that they exist because they are 'ideas in the divine intellect' tells us no more than the bald claim that they exist.[10]

Hallett's study is lauded in a foreword by Michael Dummett. We may wonder whether Dummett would agree with Hallett's assessment of the philosophical worth of Cantor's metaphysics. What is clear in any case is that Dummett would reject Cantor's realist metaphysics. In a series of books starting with *The Elements of Intuitionism*, Michael Dummett argues against realism of the Cantorian sort on the grounds of the epistemic limitations of mathematicians. He urges that the idea of a completed, actual infinity in mathematics is 'incoherent'. Taking the liberal finitist view of Aristotle, Kant, and Brouwer, he argues:

> From an intuitionistic standpoint, the platonistic conception is the result of blatantly transferring, from the finite case to the infinite one, a picture appropriate only to the former. In making this transference, the platonist destroys the whole essence of infinity, which lies in the conception of a structure which is always in growth, precisely because the process of construction is never completed.[11]

Given Dummett's influence on contemporary analytic philosophy, it is not surprising that Cantor's ideas have not been accepted by many contemporary philosophers of mathematics.[12] Ironically, as Kronecker's remarks demonstrate, Cantor was dismissed by some influential mathematicians for being too 'philosophical' in the pejorative sense of the word (i.e. speculative and lacking in rigour). Nonetheless, Cantor's introduction of transfinite arithmetic and set theory constitutes a major philosophical contribution. It provides a coherent theory of actual infinity, and it does so with a level of mathematical detail and rigour not seen before. It thus answers the historical objection that there can be no coherent conception of an actual infinity. Virtually the only thinkers to recognise Cantor's philosophical achievement at the time were a handful of

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[10] Hallett (note 9), 9.

[11] M.A.E. Dummett, *Elements of Intuitionism* (Oxford, 1977), 57.

[12] For example, both A.W. Moore and C. Wright are sympathetic to Dummett and Wittgenstein on the topic of infinite sets. Both of them find serious problems with being a realist about (uncountably) infinite sets. See A.W. Moore, 'A Problem for Intuitionism: The Apparent Possibility of Performing Infinitely Many Tasks in a Finite Time', in *Proceedings of the Aristotelian Society* vol.90, no.1, 17-34. C. Wright, *Rails to Infinity*, (Cambridge, 2001).

neo-scholastics, principally Constantin Gutberlet, author of *Das Unendliche metaphysische und mathematische betrachtet* (1878).[13]

Cantor's reliance on metaphysics in enunciating his set theory does not play well to modern audiences. Cantor's first major presentation of set theory, *Grundlagen einer allgemeine Mannigfaltigkeitslehre* [Foundations of a General Theory of Manifolds] (1883) reads like a defence of a new theory against objections drawn from the history of philosophy.[14] It is self-consciously metaphysical, replete with extensive allusions to Plato, Leibniz, and Spinoza, among many others. With few exceptions (such as Hallett's and Dauben's studies), mathematicians run from the metaphysics and philosophers run from the mathematics.[15] However, even the studies of Hallett and Dauben do not answer all the questions regarding Cantor's engagement with philosophy. Hallett and Dauben emphasize Cantor's engagement with such canonically Christian theologians as Augustine and Constantin Gutberlet, respectively. This article, by contrast, focuses on Cantor's engagement with the philosophy of Spinoza, or more accurately, 'Spinozism' in 19th century German philosophy. For the purposes of this discussion, 'Spinozism' refers to the metaphysics of Spinoza's *Ethics*, especially the first and second parts.[16] One rationale for this focus is that indirectly, Spinoza's metaphysics, through the mediation of German idealists, influenced the philosophical climate in which Cantor worked. Another rationale for the focus is the

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[13] C. Gutberlet, *Das Unendliche: metaphysisch und mathematisch betrachtet*, (Mainz, G Faber, 1878).

[14] G. Cantor, 'Grundlagen einer allgemeine Mannigfaltigkeitslehre' in *Georg Cantor Gesammelte Abhandlungen*, edited by Ernst Zermelo, Berlin: Springer-Verlag ,1932 (Hildesheim: Georg Olms, 1962), 165-209. English version: 'Foundations of a General Theory of Manifolds: A Mathematico-Philosophical Investigation into the Theory of the Infinite', translated by William Ewald in W. Ewald, ed., *From Kant to Hilbert*, vol. II, (Oxford: Clarendon Press, 1996), 878-920.

[15] J. Dauben, *Georg Cantor: His Mathematics and Philosophy of the Infinite*. (Cambridge, 1979).

[16] In this context, the essential doctrine of Spinoza's metaphysics is that there is but one absolutely infinite substance, which Spinoza calls 'God or nature' ('Deus sive natura'). According to Spinoza, this unique substance has two principal attributes: intellect and extension. The usual interpretation of these attributes is that they comprise an infinite divine intellect and infinite physical universe. Finally, Spinoza also holds that all things follow of necessity from the essence of the one substance. This constellation of views was identified as atheist and pantheist in the *Pantheismusstreit* . For discussion of *Pantheismusstreit,* see F. Beiser, *Fate of Reason* (Cambridge, Massachusetts, 1987).

parallelism between the logical difficulties confronting rationalist metaphysics and set theory.[17]

2. Changes in Cantor's early metaphysics: a broad sketch

We may divide the development of Cantor's philosophical thought into two periods: an early period running from 1872 until 1883, and a later period stretching from 1886 until 1895. 1884 marks the occasion of Cantor's first hospitalization for serious depression.[18] Soon thereafter Cantor began associating with neo-scholastic theologians willing to defend the reality of the actual infinite, such as Constantin Gutberlet. By 1886, he corresponded with Cardinal Johannes Franzelin concerning the theological status of his ideas.

Prior to 1886 Cantor adopted a metaphysical system that may be described as both neo-Platonist and Spinozist in inspiration. The key principle of this metaphysical system is that God produces everything possible, i.e. everything 'that can fall under the scope of an infinite intellect'. In a letter to Eberhard Illigens of 21 May 1886, for example, Cantor argues in effect that, given God's omnipotence, every possible (consistent internally) can exist and become real:

> 'If I have recognised the inner consistency of a concept which points to a being, then the idea of God's omnipotence impels me to think of the being expressed by the concept as in some way actually realizable.[19]

On this basis, we may characterise Cantor's early metaphysics as subscribing to a version of the principle of plenitude: everything possible is actual.[20]

The principle of plenitude, uncritically applied, gives rise to paradoxes. Suppose one allows that all possibilities are actual. (Let every possibility be described

[17] After the research for this article was carried out (starting in 1998), another article discussing Cantor's engagement with Spinozism appeared: J. Ferreiros, 'The Motives Behind Cantor's Set Theory—Physical, Biological, and Philosophical Questions', *Science in Context* 17 (1/2), 2004, 49-83. My article focuses more on Spinoza, whereas Ferreiros's article very helpfully focuses on Spinoza as interpreted through the lens of 19th century German philosophy. Ferreiros discusses the influence of Romantic *Naturphilosophie* and cites the influence of Leibniz's monadology on Cantor's attempt to apply point-set theory to nature.

[18] J. Dauben (note 15), 280.

[19] Cantor, 1886b, quoted in Hallett (note 9), 20.

[20] The modern equivalent of this position is David Lewis's modal realism in *The Plurality of Worlds* (Oxford: Basil Blackwell, 1986). It is no accident either, that Lewis's metaphysics is threatened by a Cantorian problem: How can all the possible worlds be actual? Assume the number of actual worlds is n. Then we can create 2 to the n more possible worlds, which must be actual, and so *ad infinitum*. We can never assume the universe is finished and complete. Lewis tries to answer this type of objection in his chapter 2.

by some well formed formula in (first order) logic.) What, then, prevents one from gathering those possibilities together in a set (obtained by conjoining all the formulas)? If there is no restriction against doing so, the result is an 'inconsistent multiplicity', since for contingent propositions, contradictories are both possible. Clearly some restriction is needed. Cantor's version of restricted plenitude is his domain principle: every variable with a potentially infinite domain presupposes the existence of an actually infinite domain over which it can vary.[21] Once again if the domain principle were applied in an unrestricted fashion, to the whole of the set-theoretic universe, it would lead to the paradox known as "the set of all sets" (also known as 'Cantor's Paradox'). For suppose that the entire set-theoretic universe V were a set. Then, like every set, V must have a power-set, P(V). It can be shown that a contradiction ensues. The cardinality of P(V) must be larger than (V) as power-sets are always larger than the sets of which they are power-sets. Yet V is supposed to be all-inclusive and so P(V) must be a subset of V. Consequently, V must be at least as large as P(V), which contradicts the claim that P(V) is larger than V. To dissolve the paradox, one must refrain from treating V as a set. V should be viewed a potentially infinite class, rather than as an actually infinite set. In other words, the *domains* in question in Cantor's 'domain principle' should already be assumed to be 'consistent multiplicities', suitable for sethood.

It is entirely possible that the change in Cantor's metaphysics precisely reflects his discovery of the set-theoretic antinomies as early as 1883.[22] This discovery alone, quite apart from the potentially unorthodox religious implications of his work, would have been enough to influence Cantor to change his views. Cantor's work provides an opportunity to study the relation of metaphysics to mathematics: is mathematics autonomous (free of metaphysical influence) as Cantor at one point claimed, or does Cantor's own practice suggest that the mathematics is inextricably bound up with metaphysics?

Cantor's set theory and transfinite number theory seem to derive logically and directly from his work in mathematics, specifically his work on the uniqueness of

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[21] Hallett (note 9), 25. The original source of the principle is G. Cantor, 'Mitteilungen zur Lehre vom Transfiniten'(1886), in *Georg Cantor Gesammelte Abhandlungen* edited by Ernst Zermelo, Berlin: Springer-Verlag ,1932 (Hildesheim: Georg Olms, 1962), 410-411.

[22] As discussed below, the evidence is largely contained in Cantor's second footnote to his (1883) (as cited in note 14).

trigonometric series to represent functions. However, once Cantor provides the essential concepts of set theory (the infinite point sets), one starts to see metaphysical pictures or analogies guiding Cantor's mathematical work. Is the role of such metaphysical pictures confined to the polemical defence of set theory, or is it in fact a substantial part of Cantorian set theory? We know that modern set theory [ZF] can be (at least superficially) shorn of the metaphysics. Ideally, we want to know whether Cantor's creative process of developing set theory would have been possible without his preoccupation with metaphysics. We do not answer this question directly, as it involves counterfactual history and may well be under-determined by the evidence. Instead, we present the evidence for the strong impact of metaphysics on Cantor's mathematical thought.

3. Cantor as a student of Spinoza's philosophy

As a graduate student in mathematics, Cantor studied the first book of Spinoza's major work *Ethica ordine geometrico demonstrate* ('Ethics'), first published in 1677, carefully. An unpublished notebook dated 1871-2 from the archives at the Staatsbibliothek at the University of Göttingen contains Cantor's notes in Latin commenting on Part I ('De Deo') of Spinoza's *Ethics*.[23] The notebook dates from the same time that Cantor was working on his uniqueness proof [*Eindeutigkeitsbeweis*] for the representation of functions using trigonometric series. This work eventually led to his view that there are actually infinite point-sets, since he proved the uniqueness of the representation holds even when an infinite number of points are excepted from a continuous curve.

There is evidence from the report on his *Habilitationsschrift* oral examination of 24 November, 1868 that Cantor conceived of his mathematics as a refinement of Spinoza's project of treating of metaphysics using a geometrical method.[24] The report comments:

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[23] Cantor's short set of Latin notes on Book I of Spinoza's *Ethics* have not, to my knowledge, been published anywhere. The seven pages follow the Latin text of the definitions of part I of Spinoza's *Ethics* verbatim on the lefthand side of the notebook and contain Cantor's close summary (with paraphrases of propositions) of Spinoza's philosophy on the righthand side. It has been suggested by J. Ferreiros (note 17), p.61 that Cantor's notes on Spinoza may reflect the influence of Adolf Trendelenberg, whose lectures he would have heard in Berlin. Trendelenberg is listed in Cantor's 'Vita' in Cantor (note 14), 31.

[24] H. Bandmann, *Die Unendlichkeit des Seins: Cantors Unendlichkeitsidee und ihre metaphysische Wurzeln*, (Frankfurt am Main, 1992).

> The philosophical work of the candidate has been to answer question: What does Spinoza understand in his *Ethics* by the geometrical method and how is his application of it to be judged?...Besides an overall knowledge of the history of philosophy, he showed a unique acquaintance with Spinoza's *Ethics*.[25]

Unfortunately the text of Cantor's lecture is lost to us; we have only the report of his examiners. On the face of it, it is curious that Cantor should be interested in Spinoza's metaphysics and geometrical method. After all, the main thesis of Part I of Spinoza's Ethics is that there is *only one* absolutely infinite substance, 'God or Nature' (Deus sive Natura). However, the main theme of Cantor's *Grundlagen* is that there are *multiple actual infinities*, because there is a realm of an actual, but increasable infinite known as the transfinite. To be sure, Cantor retains the traditional theological idea that *absolute* infinity pertains to God alone (of which, he says, the sequence of transfinite ordinal numbers are a symbol). However, Cantor disagrees with the broadly Aristotelian philosophical tradition that, on one widespread interpretation, assumes that all infinities, apart from absolute infinity, are potential rather than actual.[26]

To be sure, the standard interpretation of Spinoza as recognizing only one truly infinite thing (substance) may be challenged. Spinoza appears to recognise actual infinities that are non-absolute in the form of the attributes of substance, which are each 'infinite in kind'. A thing is 'infinite in its kind' just in case it is not limited or bounded by another thing of the same kind. Spinoza holds that physical extension is an attribute of substance and is both infinite and eternal.[27] This might be interpreted to mean that space is actually infinite.[28] However, even if Spinoza did recognize actual infinities, he certainly rejected the conception of *infinite numbers*. The reason for this position is that for Spinoza, the true infinite is indeterminate, unique, measureless, unsurpassable and so on. Cantor, of course, rejects the idea that one can have infinite collections or series, without also having infinite numbers. He says 'Infinite number

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[25] The report is quoted in W. Purkert and H.J. Illgauds, *Georg Cantor 1845-1918*, (Basel, 1987), 183-185 (my translation).

[26] For a challenge to the conventional interpretation of Aristotle as a liberal finitist, see P. Massie, 'The Actual Infinite as a Day at the Games', *Review of Metaphysics*, March 2007, vol. 60 no.3, 573-596.

[27] B. Spinoza, Ethica I definition 2. Translation: *Ethics, Treatise on the Emendation of the Intellect and Selected Letters*, translated by S. Shirley, Indianapolis: Hackett, 1981.

[28] A.W. Moore, *The Infinite* (London, 1990), 78.

and set are indissolubly bound up with each other; if we give up one of them, we no longer have the right to the other'.[29] Cantor's essential innovation was to suggest that an infinity can be mathematically determinate (as a number or set) while still retaining its character as truly infinite.[30] He insists, of course, that the actual infinite is the true infinite. This rehabilitation of the notion of the actual infinite is something Cantor shares with philosophers such as Spinoza and Leibniz.

In the *Grundlagen* of 1883 there are repeated allusions to Spinoza's metaphysics.[31] In particular, Cantor distinguishes between the immanent and transient reality of ideas. The immanent reality of an idea consists in its internal coherence and consistency of an idea. In endnote 5 to the *Grundlagen*, Cantor quotes Spinoza's definition of an adequate idea in the *Ethics*, and says that the definition captures what he means by the immanent reality of an idea.[32] Finally, in a remark that Hallett describes as 'mysterious', Cantor says that the correspondence of immanent and transient reality has its ground in 'the unity of the All, to which we ourselves belong'. The 'All' should be interpreted as Spinoza's all inclusive infinite substance, with its dual attributes of thought and extension.

Cantor's references to Spinoza may simply reflect Spinoza's prominence in 19th century German culture. Goethe, Schelling, and Hegel embraced pantheism, bringing about a reversal of the rejection of Spinoza's 'atheist' metaphysics during the *Pantheismusstreit* a century earlier.[33] Recently, José Ferreiros has argued that Cantor's references to Spinoza and Leibniz suggest 'an image of Cantor the philosopher as a Romantic thinker in the grand style of the professors whose lectures he heard at Berlin and Göttingen'.[34] In particular, Ferreiros establishes the influence of Trendelenburg's philosophy lectures on Cantor's philosophical outlook in his early period. Cantor's notes on Spinoza's *Ethics* from the winter semester of 1871 confirm Cantor's interest in Spinoza's philosophy as a result of Trendelenburg's lectures on Spinoza. Moreover, Cantor's interest in Spinoza's metaphysics persists in his

[29] G. Cantor (1886) (note 21), 394.

[30] However, finitists would argue that Cantor's transfinite numbers are *too determinate, too similar to finite numbers*, to be truly infinite Finitists agree with Aristotle that the proper conception of infinity is that of something that is endless and essentially incomplete and indeterminate.

[31] Cantor (1883) (note 14), 175.

[32] Cantor (1883) (note 14), 206.

[33] F. Beiser, *Fate of Reason* (Cambridge, Massachusetts, 1987), 45.

[34] J. Ferreiros, (note 17), 61.

philosophical writings, re-appearing in his *Grundlagen* of 1883. Cantor appears to have had more than a casual interest in Spinoza's metaphysics.

To be sure, Cantor's early acceptance of something like Spinoza's views on substance is coupled with a strong rejection of Spinoza's antiquated views on infinite number. Spinoza's views on the infinite in mathematics are set out in his correspondence, especially his letter to Lodewijk Meyer dated 20 April 1663. Spinoza was primarily concerned to defend the view, advanced in the *Ethics* (Book I, proposition 15) that substance is indivisible and infinite. Specifically, he wanted to resist the suggestion that, because substance has extension or matter, it can be divided infinitely. Descartes had claimed 'the number of particles into which matter is divided is in fact indefinite, although it is beyond our power to grasp them all'.[35] Spinoza maintains the distinction between 'what is called infinite because it has no limits' and 'that whose parts we cannot explain or equate with any number, though we know its maximum or minimum'.[36] The later kind of infinite Spinoza calls 'indefinite'. As an example of the indefinite, Spinoza considers the space that exists between two non-concentric circles, one contained within the other, with diameters AD (the diameter of larger circle) and BC (the diameter of the smaller circle embedded in the larger circle) respectively. Spinoza claims that contradictions result from trying to attach a number to the number of divisions we can make in each curve. He appears puzzled by the fact that one circle clearly has a greater circumference than the other, and yet it seems impossible to say that the path travelled by a particle on one circular path is more 'infinitely' divisible (greater in parts) than the path travelled by particle on the other circular path. Spinoza does not argue clearly for this conclusion in the letter to Meyer, but does explain his reasoning more clearly in his letter to Tschirnhaus in reply to a request for clarification:

> As to what I stated in my letter regarding the infinite, that it is not from the multitude of their parts that the infinity is inferred, this is evident from the fact that if infinity were inferred from the multitude of parts, it would be impossible for us to conceive a greater multitude of parts, but this multitude of parts ought to be greater than any given number. This is untrue, for in the total space between the two non-concentric circles, we conceive twice as many parts as in half that

[35] Descartes, *Principles of Philosophy*, Part II: 34.

[36] Letter 12, 'BDS to the very learned and expert Lodewijk Meyer, Doctor of Medicine and Philosophy' in E. Curley, ed. and trans., *The Collected Works of Spinoza*, Princeton: Princeton University, 1985, p. 200.

> space, and yet the number of parts in both the half-space and the whole space exceeds any assignable number.[37]

The same concern with infinite quantities greater or lesser than one another appears in condensed form in the letter to Meyer: 'Lastly, there are things called infinite, or if you prefer indefinite, because they cannot be accurately expressed by any number, while yet being conceivable as greater or less.'[38] Faced with the puzzle of different sizes of infinity (in tension with the view that the infinite is a maximum), Spinoza avoids equating sets that appear to be of different measures, by taking refuge in their uncountable, anumerical nature.

Cantor refers to Spinoza's letter to Meyer in the *Grundlagen*, which he describes as 'highly important' and 'rich in content' and promises to provide later 'a detailed and thorough discussion' of Spinoza's letter as well as other works containing arguments against infinite numbers by Spinoza, Descartes, Leibniz, Locke, Hobbes, and Berkeley.[39] In the absence of Cantor's promised discussion of the letter to Meyer, we can only conjecture based on knowledge of Cantor's point-set topology how he would have solved Spinoza's problem of the two circles. Cantor would have equated the number of points in the two areas, just as he would have equated the number of points in each circumference of the circles, because a one to one correspondence between the points in one circle and the points in the other circle could be established. This is obvious in the case of the two circumferences, because the radius of the larger can sweep through every point on the circumference of the smaller circle and on its own circumference. This is in fact a simple case of the more general result that any continuous space (of any dimension) is equipollent with, i.e., has the same cardinality as, the linear (one-dimensional) continuum, a surprising result in topology known as 'the invariance of dimension'.[40] Cantor proved a limited version of the invariance of dimension in his early paper [1874]. In his famous 1877 letter to Dedekind, Cantor showed that there was a one-to-one mapping between the points on a line segment of unit length [0,1] and the points on a square with sides of unit length [0,1] x [0, 1]. We

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[37] B. Spinoza, Letter 81, to Tschirnhaus, in trans. S. Shirley, *Ethics, Treatise on the Emendation of the Intellect and Selected Letters*, (Indianapolis, 1981), 291-2.

[38] B. Spinoza, Letter 12, 'BDS to the very learned and expert Lodewijk Meyer, Doctor of Medicine and Philosophy' in *The Collected Works of Spinoza*, edited and translated by E. Curley, (Princeton, 1985), 200.

[39] Cantor (note 14), (Zermelo, ed.), 175; English version (Ewald), 890.

[40] D.M. Johnson, 'The Problem of the Invariance of Dimension in the Growth of Modern Topology, Part I', *Archives for the History of Exact Sciences* 20 (2), 1979, 91-188.

may conjecture that one reason Spinoza's letter to Meyer would have interested Cantor so much would have been its perceived relevance to fundamental questions in basic topology.

3. Avoiding the antinomies

It can be argued that Cantor was motivated by a desire to avoid the antinomies attending the equation of possibility and actuality. For if all possibilities had to be realised, then, since each number is possible, there would be nothing to stop one from concluding that a universal set, a collection consisting of all the numbers, actually exists and like each of the numbers, exists necessarily. Yet Cantor knew that such a collection would be a 'inconsistent multipicity'. For it would thereby have to have a cardinal number itself, which would contradict the hypothesis that it contained all numbers.

Cantor informs us in his letter to Hilbert of 15 November 1899 that his knowledge of the set-theoretic antinomy was 'clearly though intentionally hidden in my 1883 Grundlagen, specially in the end notes'.[41] The passage Cantor alludes to does indeed show an understanding of the problem:

> I have no doubt that, as we pursue this path [of generating numbers] even further, we shall never reach a boundary that cannot be crossed; but that we shall also never achieve even an approximate conception of the absolute. The absolute can only be acknowledged [anerkannt] but never known [erkannt]—and not even approximately known. For just as in the number class (I) every finite number, however great, always has the same power of the finite numbers greater than it, so every supra-finite number, however great, of any of the higher number-classes (II) or (III), etc., is followed by an aggregate of numbers and number-classes whose power is not in the slightest reduced compared to the entire absolutely infinite aggregate of numbers, starting with 1. As Albrecht von Haller says of eternity: 'I attain to the enormous number, but you, O eternity, lie always ahead of me.' *The absolutely infinite sequence of numbers thus seems to me to be an appropriate symbol of the absolute*; in contrast, the infinity of the first number-class (I), which has hitherto sufficed, because I consider it to be a graspable idea (not a representation [Vorstellung]) seems to me to dwindle into nothingness by comparison.
>
> It also seems to me remarkable that to each of the number classes (and hence to each of the powers) there corresponds a

[41] Walter Purkert, "Cantor's Views on the Foundations of Mathematics" in *The History of Modern Mathematics* I, edited by D. Rowe and J. McCleary, (Poughkeepsie, New York, 1989), 49-65.

> completely determinate member of the absolutely infinite aggregate of numbers—and in such a way that for every supra-finite number γ there is a power that is the γth; so the different powers also form an absolutely infinite sequence. This is all the more peculiar in that the number γ which gives the order of a power (in case the number γ has an immediate predecessor) stands to the number of the number-class that has the power in a relationship of size whose smallness mocks all description; and all the more so, the greater we take γ to be. (Cantor, *Grundlagen*, Ewald translation pp. 916-7 ; emphasis mine).

In this paragraph Cantor shows that he has a metaphysical picture that prevents the antinomy from arising at all. Cantor compares the collection of all number-classes and numbers they contain to the 'absolute infinity' of God. According to the traditional conception of God as an *ens realissimum,* God is the most perfect and real being, and possesses all attributes or perfections. This conception of God is evident in Spinoza's *Ethics*, where God is defined as 'an absolutely infinite being'.[42] However, given such an infinity of attributes, it supposedly follows that God cannot be adequately conceived by a finite mind.[43] In the same way, the collection of absolutely all the numbers is beyond adequate comprehension by a finite mind. In the quotation above Cantor considers such a number sequence as 'an appropriate symbol of the absolute', whereas elsewhere (in his 'Mitteilungen' (1886)) he entertains Augustine's view that all the numbers exist as ideas in the divine mind.[44] In any case, the sequence of all the numbers, like the notion of all of God's mathematical ideas, is not an object of (human) mathematical thought. As Hallett suggests, for Cantor, the collection of everything mathematizable, which is 'absolutely infinite', is of a radically different type from ordinary mathematical objects, including the transfinite numbers. It is in this context that we should consider the qualification in Cantor's motto: "Omnia seu finita seu infinita *definita* sunt et excepto Deo ab intellectu

[42] B. Spinoza (note 24), Book I, proposition 8, scholium 1.
[43] To be sure, it is fallacious to claim that, *because* a mind is finite, therefore it cannot conceive of infinitely many items. Cantor identifies this fallacy in the *Grundlagen* §5, [4] (G. Cantor (note 14),176; Ewald translation, 891). For discussion of how the fallacy appears in arguments for finitism by empiricists, see A.Newstead and J. Franklin, 'What Finite Minds Can Know: Epistemic Limitations and Our Knowledge of Infinity', forthcoming.
[44] G. Cantor (note 21), 401-403, footnote 3. For discussion of aspects of Augustine's views in Cantor, see M. Hallett (note 9), 35-7.

determinari possunt" (All things, whether finite or infinite are definite and, with the exception of God, can be determined by the intellect).[45] God, or the absolute infinite, cannot be determined by the intellect: that is, it cannot be rendered an object of mathematical knowledge.

Cantor's recognition of the problematic nature of the ideal of omniscience and the notion of an absolutely infinite intellect is lacking in Spinoza. One reason for this may be that for Spinoza the proposition that the divine intellect is an "inconsistent multiplicity" can never arise. The nature of Spinoza's substance, although it can be conceived in infinitely many ways, is simple. As Leibniz once complained, nowhere does Spinoza argue for the compatibility of God or substance's infinitely many attributes.[46] Spinoza does not do so because he insists that each 'attribute must be conceived in itself' (Ethics, Book I, definition 10), and such things of different natures can have nothing to do with one another, much less conflict with one another.[47] At a deeper level, the question does not arise for Spinoza because he not consider modes essential to the divine substance. In particular, modes cannot be used to individuate substance (Ethics, Book I, proposition 1, proposition 5, demonstration), so that substances that possess different modes from another are not, in reality, distinct. Consequently the idea of contradictory modes (e.g. being extended and not-extended at the same time and place) all pertaining to the same infinite substance, or forming the basis for a dissolution of substance into a plurality of substances, cannot arise for Spinoza. It is important to realise that for Cantor such a problem can arise, because he considered numbers to be 'ideas in the divine intellect' and yet was aware that although all numbers were possible in themselves, the very notion of collecting all numbers into a single unity, the divine intellect, was riddled with contradiction.

Of course, Cantor still lacked a precisely formulated mathematical rationale for blocking the antinomy. All he had was a metaphysical picture that prevented the

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[45]G. Cantor (note 14), 176; Ewald, vol. II, 891.
[46] G. Leibniz, 'Comments on Spinoza's Philosophy (c. 1707), in *Philosophical Essays,* edited and translated by R. Ariew and D. Garber, (Indianapolis, 1989), 272-281, especially 276.
[47] B. Spinoza (note 24), Book I, definition 10, 36.

antinomy from arising. Yet he did not yet abandon the equation of possibility and actuality. Why? The reason is that he needed as loose a criterion for the reality of ideas as logical possibility to prove that his transfinite numbers were as real as finite numbers. Whereas initially he did think that his transfinite numbers were mere ‘definite defined infinity symbols’, by the time of the *Grundlagen* (1883), he was convinced that they had as much right to be considered ‘concrete numbers of real significance’ as the finite numbers.[48] In light of the support Cantor’s early metaphysical picture, and its strong existential assumptions, provides for the reality of his transfinite numbers, it seems plausible to suppose that Cantor had some non-mathematical reasons for changing his metaphysics. The evidence does in fact point to one such possible motive: the desire to avoid heresy.

4. The controversy over pantheism

Spinoza is traditionally regarded as a pantheist, because he identifies God with the whole of nature, and its immanent cause (*Ethics* I p18). Cantor is clearly not a pantheist in this sense, because he did not reject the idea of a transcendent God who stands outside of his creation. Nonetheless, as we have seen, at the time of the *Grundlagen*, he shared with Spinoza the principle that the immanent reality suffices for the transient reality of an idea. This principle amounts to the collapse of the possible and the actual. The coincidence of the possible and the actual is strictly independent of pantheism. However, once the divine essence is conceived of as the realm of all possibilities, the coincidence of possibility and actuality follows from the pantheist (and neo-Platonist) doctrine of the necessary emanation of things from the divine essence. So pantheism, necessitarianism, and the modal principle of plenitude (‘everything possible is actual’) are a closely allied set of doctrines.

Pantheism was both fashionable and heretical in mid-19th century Germany. In 1861, Pope Pius IX felt the need to issue a formal prohibition against pantheism, a

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[48] G.Cantor (note 14), *Grundlagen*, §1, 166; Ewald, vol. II, 883: ‘The infinite real integers, which I shall define later (and which I discovered many years ago, without becoming clearly aware that they are concrete numbers of real significance)…’.

sure sign that the doctrine was popular as well as threatening. Given the moral and intellectual support that Cantor received from Catholic neo-Scholastic theologians (such as Constantin Gutberlet) in the face of heavy criticism from mathematicians, it is likely that Cantor would have been motivated to avoid (at least the appearance of) heresy. [49] Cantor was aware that his opinions contradicted those of Thomas Aquinas, whose philosophy was fast becoming the official philosophy of the Church. Some theologians feared his transfinite numbers might undermine the doctrine of creation by making a return to the notion of an eternal world possible.[50] As Cantor came to realise in the early months of 1886 through his correspondence with Cardinal Johannes Franzelin, by far the biggest challenge orthodoxy posed against his belief in the transfinite was to show that it did not lead to pantheism.

Cantor sent the Cardinal an essay he had written in 1885, later published in the *Zeitschrift für Philosophie*, vol.88, as "Über die verschiedenen Standpunkte in bezug auf das aktuelle Unendliche" (Concerning Various Perspectives on the Infinite). He asked the Cardinal to submit the essay to examination for any possible errors, especially those which might pose a danger to religion. In this essay, Cantor claims to be the first to affirm the reality of the infinite in the abstract as well as in the concrete. The infinite in the concrete [or transfinite] he identified with Spinoza's 'natura naturata', the created universe; the infinite in the abstract with transfinite order types and cardinals; and the infinite in God with 'natura naturans', the creator of the universe.

In this letter, Cantor claims that pantheism results from the failure to distinguish between the two forms of the actual infinite, the transfinite and the

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[49] Dauben (note 15), 145. Cantor's honeymoon with Catholicism appears to have soured in his final years. In his later (unpublished) writings such as his letter of 9 May 1894, he speaks of "the offensive character that the holy church has for all heretics and the proverbial intolerance of catholicism". (An unpublished note of 9 May, 1894. From an analysis of a Rosicrucian emblem. Abteilung für die Handschriften under seltene Ausdruck, University of Göttingen, Cantor's Nachlass, "Philosophical Notes", Cod. Ms. G. Cantor 29.)

[50] See for example Cantor's correspondence with the Protestant theologian, Heman, at the end of his first letter book in the Nachlass, 16 (as cited in note 49).

absolute. Only if one held that God's absolute infinity were fully realised in nature would one truly be a pantheist:

> Another common confusion is seen with the two forms of the actual infinite, in which namely the transfinite is mixed up with the absolute, whole these concepts are strongly distinguished, insofar as the former is infinite, but can still be added to, while the later is essentially such that it cannot be added to, and therefore is not be determinable by mathematical thought; we encounter this failure, for example, in pantheism, and it constitutes the Achilles' heel of Spinoza's *Ethics*, of which in fact F.H. Jacobi remarked, that it cannot be refuted on reasonable grounds.[51]

Despite Cantor's attempts to differentiate his position from pantheism, the Cardinal seems to have recognised the Spinozist, necessitarian consequences of Cantor's doctrine of the identity of possibility and actuality. In his response to Cantor, he warns him of the pantheist nature of his argument, and its necessitarian consequences:

> ....I confess, however, that in my opinion, what the author calls "transfinitum in natura naturatum", cannot be defended, and in a certain sense, which the author appears not to have given, would contain the error of pantheism.[52]

Cantor's reply to Franzelin illustrates simultaneously his desire to avoid pantheism and the appeal it held for him. In each attempt where Cantor technically disassociated his philosophy from pantheism, he reveals his sympathy with it. In his letter to Franzelin of January 22nd, 1886, he attempts again to distinguish his position from pantheism by drawing attention to his distinction between the transfinite and the absolute. He tries to explain the special sense he gives to Spinoza's expressions, *natura naturata* and *natura naturans*:

> I use the following expressions "natura naturans" and "natura naturata" in my short essay 'Concerning Various Perspectives on the Infinite' in the same sense, which the Thomists have given them, so that the former expresses God as the creator and preserver of substances produced

[51] Cantor (1886) (note 21), 375, translation mine.
[52] H. Meschowski, and W. Nilson, eds., 1991. *Georg Cantor. Briefe*. Berlin: Springer, 253.

> from Him out of nothing, while the later designates the world created by Him. Accordingly, I distinguish between an "Infinitum aeternum sive Absolutum" , that refers to God and his attributes and an "Infinitum creatum sive Transfinitum" , that above all testifies to where in natura creata an actual infinite must be acknowledged, as for example, according to my firm conviction, in the actual infinite number of created individuals , in the entire universe and also on our earth, and in all probability, in every smallest part of extended space itself, with which I completely agree with Leibniz. (Letter to Foucher, part II of works, p.I. pag.243).[53]

Unlike pantheists, Cantor does not think that God's infinity is exhausted in nature. However, one is hardly reassured by his enthusiastic endorsement of Leibniz's theory of monads, which is arguably just a consistent version of pantheism, overlaid with a few concessions to orthodox religion.[54] One finds traces, too, of his allegiance to pantheism, in his use of a kind of principle of plenitude:

> One proof [of the reality of the Transfinite] proceeds from the concept of God and infers from the greatest perfection of God's essence the possibility of the creation of a transfinite order, from his supreme goodness to the necessity that there should actually follow a Transfinite.[55]

Yet the letter ends with an explicit repudiation of pantheism:

> ...no system leads further away from my chief convictions than pantheism, when I foresee its consequence of materialism, with which I have absolutely no association .[56]

The Cardinal's response to Cantor's attempt to distance his position from pantheism was more vigorous and drawn out than his initial remarks on Cantor's essay. Thus it must have been with some consternation that Cantor read:

> ..On the assumption that your actual Transfinitum contains no contradiction in itself , your conclusion of the <u>possibility of the creation</u> of a Transfinitum from the concept of God's omnipotence is almost right. Only to my regret, you go further and end "out of his absolute goodness [Allgüte] and splendour [Herrlichkeit] follows <u>of</u>

[53] Meschkowski-Nilson, eds., (note 52), 254, translation mine.
[54] For the argument that Leibniz's metaphysics is Neo-Platonist and pantheistic, see Stephen A. Erickson, "Leibniz on Essence, Existence and Creation," *Review of Metaphysics* 18 (1965), 476-87.
[55] Meschkowski-Nilson, eds., (note 52), 255, translation mine.
[56] Meschkowski-Nilson, eds., (note 52), 255, translation mine.

> necessity the actual creation of the Transfinitum."Even though God is in himself absolute infinite goodness and splendour, qualities which cannot wax nor wane, the necessity of a creation , would always be like a contradiction of the freedom of creation--an even as necessary perfection as all his other perfections, or better, God's infinite perfection is (according to our necessary distinction) just as much freedom, as omnipotence, wisdom, justice, etc.
>
> From your conclusion of the necessity of the creation of the Transfinitum, you might have to go even further. Your actual transfinite is increasable [Vermehrbares]; only if God's infinite goodness and splendour actually requires the creation of the Transfinitum with necessity, so that it follows from almost the very ground of the infinitude of his goodness and splendour, the necessity of increase until it is not increasable anymore, which contradicts your own concept of the Transfinite. In other words: Whoever appeals to the necessity of creation out of the infinite goodness and splendour of God, must claim that everything creatable is actually created from eternity, and that there are no possibilities before God's eyes, that the infinite being could call forth [rufen] in the realm of being [ins Dasein].
>
> In my opinion the necessity of creation will be very hindersome to you in your so commendable fight against pantheism, and at the least, weakens the power of your proof. I have dwelled on this point for so long, because I fervently wish that your great sagacity [Scharfsinn] will free itself from so fatal an error, which many other great minds have lapsed into taking for orthodox [rechtglaubig]. [57]

Cantor's next letter to Franzelin, dated 29 January 1886, contains a significant retreat from his previous position. He no longer tries to maintain the content of his philosophy while arguing that it separability from pantheism. Instead he modifies that system to meet the demands of orthodoxy.[58] Cantor now claims in his letter that the creation of the finite as well as the transfinite realm has a merely 'subjective necessity for us' that follows from our contemplation of the divine nature:

> ...it was not [upon further consideration] my opinion, to speak of an objective, metaphysical necessity of the creative act, which would have subjugated God's absolute freedom, but rather I wanted to point to a subjective necessity for us, that from God's absolute goodness and splendour there must result an actual creation, not merely of a finite

[57]G. Cantor (note 21),386 and Nachlass (note 49), Notebook Cod. Ms. 16, 41.

[58]At the end of his work *Das Unendliche* (1878), Cantor's friend, Constantin Gutberlet observes that it is characteristic of pantheism to postulate the necessity of the existence of the actual order of nature. Gutberlet was a student of Cardinal Franzelin, and a comparison of his book with Franzelin's letter shows that there was a consensus among neo-Scholastic theologians that dogma of God's free creation of the world was the only orthodox option.

> order (not following from a part of God), but also of a transfinite order.[59]

Cantor's notion of 'purely subjective necessity' could be a contortion designed to bend his views into orthodox shape. But it is clear that he cannot have it both ways. Either his transfinite numbers are necessary, and God does not create freely, or the numbers are contingent beings and God creates freely. In the end, Cantor's desire to protect what he saw as 'religious truth' won out over his desire to find a secure philosophical foundation for his transfinite numbers. In his later letters[60], he is careful to emphasize God's freedom, and cites Franzelin's final approval of his revised system repeatedly. In his letter to Ignatius Jeiler of 8 June 1888, Cantor goes so far as to say that he thinks his letter to the Cardinal of 22nd January 1886 will actually prove fatal to pantheism and positivism.

5. Persecution, orthodoxy, and sincerity

How sincere was Cantor about the modification of his doctrine? A unpublished letter of 6 February 1887 to his friend and fellow believer in the actual infinite, the theologian Constantin Gutberlet, suggests Cantor was ambivalent. He continued to affirm the correctness of the Cardinal's opinion and his modified doctrine, but nonetheless characterised the Cardinal's criticism as a polemic:

> Now however follows a polemic against such a creation [of the transfinite] as the necessary consequence of God's goodness and splendour, in which the Cardinal is thoroughly correct.[61]

Nor can their be any doubt of Cantor's respect for the Cardinal, whom he praises profusely upon notifying Gutberlet of his former teacher's death (11 December 1887).

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[59] Meschkowski-Nilson, eds., (note 52), 258, translation mine.

[60] See, e.g. his letters to Pater Jeiler in Meschkowski, 'Auf den Briefbüchern Georg Cantors', *Archive for the History of the Exact Sciences* 2, (1965), 503-19, and his letter to Pater Esser in Meschkowski -Nilson (1991) (note 49). The letter to Jeiler of 8 June 1888 is unpublished, but exists in a typescript for the doctoral thesis of Bernhard Jaegar, "Das actuale and das absolute Unendliche. Zum Dialog zwischen Georg Cantor und Theologen seiner Zeit". Eberhards Karls Univ. Tübingen 1992.

[61] The letter is mentioned in Jaeger (note 59), 50. The whole letter exists in Cantor's *Nachlass*, G. Cantor (note 49), Cod. Ms. G. Cantor 16, 94.

And yet belief in an actual infinity realized apart from God has always been dangerous. Giordano Bruno was burned at the stake for believing in an infinity of worlds. His successor, Benedictus Spinoza was reviled as an atheist for two centuries after his work, his reputation following the opinion of his early critic, Pierre Bayle, in his influential *Dictionnaire historique et critique* (1697). With the birth of German romanticism, and the revelation that Lessing and Jacobi admired his work, Spinoza's reputation was rehabilitated. A Spinoza Renaissance flourished in Germany, and Spinoza's monism influenced the philosophies of Hegel, Schelling, and Goethe. It would not be surprising if this atmosphere contributed to Cantor's initial positive reception of Spinoza. But like Spinoza, Cantor's belief in an infinity apart from a transcendent personal deity flew in the face of orthodox religion. Mindful of the lessons of history, Cantor choose to walk a narrow path between orthodoxy and insincerity.

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