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Aristotle and Modern Mathematical Theories of the Continuum*

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1 Introduction: Mathematical and Physical Continua

The mathematical structure of the continuum, in the guise of the domain of continuous, differentiable functions, has proved immensely useful in the study of nature. However, we have learned to be sceptical of any claim to the effect that our current favourite mathematical theory necessarily describes the actual structure of the physical universe. The continuous manifold of space-time may be no more than a helpful idealisation, when in fact space-time has a minutely grainy or quantised structure. Nonetheless, the question of whether the classical continuum is an accurate representation of the structure of space-time is a separate question from the one which we have to answer today. We are interested in the mathematical concept of the continuum itself.

In saying that we may develop a mathematical theory of the continuum regardless of whether such a continuum is actually to be found in the universe, we are relying on the premise that there is such a thing as *pure mathematics*, a body of knowledge whose evidential basis rests on something other than observation of the physical world. Consider, for example, the proposition that the tangent to a point P on a circle's circumference is perpendicular to the radius connecting P and the centre of the circle. As a proposition of pure mathematics, this proposition is true independently of whether there really are entities in the physical world that meet the mathematical definition of a circle, namely, that of a figure all of whose points are equidistant from a given point. Similarly, the proposition that the continuum is infinitely divisible is true as a proposition about the purely mathematical continuum regardless of whether there are continua perfectly instantiated in the physical world.

This way of dividing mathematics into pure and applied, with its concomitant separation of the subject matter of mathematics from physics, is quite foreign to Aristotle's way of thinking about mathematics.[1] In his view, the mathematician and the physicist, are both interested in the properties of natural bodies (τα φυσικα σώματα), but they differ in their emphasis. The mathematician studies the properties of natural bodies, which include their surfaces and volumes, lines, and points (*Phys*. 193b 23-5, Waterfield).[2] The mathematician is not interested in 'the properties of natural bodies as the properties of natural bodies,' as the physicist is (*Phys.* 193b33-4, Waterfield). Instead, the mathematician is interested in the properties of natural bodies that are 'separable in thought from the world of change' (χωριστα γαρ τη νοήσει κινήσεώς εστι) (*Phys*. 193b33, Waterfield). But, Aristotle assures us, the procedure of separating these properties in thought from the world of change does not make any difference or result in any falsehood (*Phys*.193a36). Aristotle's philosophy of mathematics is rightly classified as *abstractionist*, because abstraction (separation in thought) is the process whereby Aristotle's mathematician comes up with his subject matter, having started with bodies and their properties in the natural world.[3]

An *apparent* implication of Aristotle's description of mathematical properties at *Physics* 193b34-5 as those that are separable in thought from motion is that the mathematician cannot study properties pertaining to the motion of bodies. But motion is continuous, and continuity would seem to be an inextricably mathematical notion insofar as it involves the idea of the infinite divisibility of magnitude. So the implication cannot be what Aristotle intended. Rather, then, it must be the case that both the physicist and mathematician study continuity, the one as it pertains to the actual movements of particular bodies, and the other insofar as the general property can be abstracted from the trajectories of moving bodies. The continuity of a trajectory in space can be considered independently of the body that traces out that trajectory, since any other body of the same size moving continuously along the same path would exemplify the same mathematical property.[4] A further level of abstraction occurs if we take the body to be a point-mass, and disregard the width of its path entirely. Moreover, when studying the continuum, it really doesn't matter if you take the continuous, regular motion of one body or another: the mathematical property of continuity is the same everywhere and always. That is why Aristotle is fond of saying that mathematical items are not at all like the notion of snub-nosedness, since unlike the latter notion, they *can* be conceived of apart from the particular material substances in which they happen to inhere (e.g. *Phys.* 194a6; *Meta*. 1025b30-1026a1).

An advantage of reading Aristotle as an abstractionist is that it encourages us to look to Aristotle's remarks on physical continua in order to recover his mathematical theory of continuity. An investigation respecting this principle of interpretation has been given by Michael White in his excellent article, "On Continuity: Aristotle versus Topology."[5] I think he has been quite persuasive in showing that Aristotle has a 'proto-topological' conception of the continuum that overlaps to a considerable extent with the classical modern account. In this essay, I develop this claim in further detail with special reference to Georg Cantor (1845-1918), the creator of set theory, transfinite numbers, and point set topology. The rationale for undertaking such a comparison stems largely from Cantor's numerous references to Aristotle, which suggest that, *in Cantor's view*, Aristotle's denial of the actual infinite posed the single greatest obstacle towards the acceptance of his transfinite number theory.[6] In fact the relation between Aristotelianism and modern Cantorian mathematics is more complicated, and this is particularly evident in the case of the continuum.

There are significant dangers in comparing the views of thinkers from such different time periods, and so in section II, as a guard against anachronism, I sketch some of the major differences between the state of mathematics, particularly in the conception of real quantities, in ancient and modern times. In section III, to further motivate the comparison between Aristotle and Cantor, I present some evidence, primarily from Cantor's letters, which reveals Cantor's attitude towards Aristotle's theory of the continuum. In section IV, I argue that of the three features that Cantor uses to define the continuum (density, connectedness and closure), Aristotle makes wide use of first two in his general theory of the continuum, and includes the third when discussing time and change. In Section V, I give a reconstruction of Aristotle's argument against constituting the continuum out of points, claiming that this conclusion can be derived from intuitions about the impossibility of giving a rule for ordering the densely arrayed points of a continuum. Although this 'reconstruction' is broadly Aristotelian in spirit, I do not claim, however, that it represents what Aristotle actually thought about this matter: it is a reconception of the Aristotelian argument from a modern (and unabashedly classical realist) perspective. Finally, I leave it to the reader to judge whether such agreement as can be found between such opposed thinkers as Cantor and Aristotle lends some support to a crucial premise from Aristotle's philosophy of mathematics: the claim that mathematical structures begin as abstractions from the behaviour and properties of physical objects.

2 An Apology to Historians

The demands of good history and creative philosophical thinking are not always in harmony. While some philosophers are happy to suppose that there is a logical space of eternal philosophical problems, and delight in tracing the development of 'a unit idea' through successive periods, many historians and historically sensitive philosophers would reject such a method as anachronistic.[7] As a guard against such anachronism, before proceeding to a comparison of the similarities between their views in section III and IV, I shall outline a few of the significant differences between Aristotle's and Cantor's views of mathematics. I am aware, however, of not being able to do justice to the intricacies of such a discussion.

The first major obstacle to carrying out a comparison between Aristotle and Cantor is that Aristotle does not have a concept of a real number *per se*, a concept central to the modern arithmetical account of the continuum. It is easy to overestimate this obstacle in light of the fact that the extension of Aristotle's concept of number (αριθμος) is unambiguously restricted to the domain of whole numbers. Aristotle speaks of number as a multitude of indivisibles (πληθος αδιαιρετων) (*Meta*. 1085b22), and as several ones (να πλείω) (*Phys.* 207b5-9). It is clear that, as the one is indivisible (τ ν στιν διαίρετον) (Phys. 207b6) these two descriptions are co-extensive. It is clear, moreover, from *Categories* 4b25ff. that Aristotle views the division between number and magnitude to map neatly onto an exclusive division between the discrete and the continuous. Number is said to be discrete, because its parts do not share any boundary, whereas sharing a boundary is one of Aristotle's criteria for the continuity of one item with another (see also *Phys*. 227a7ff.). Lines are continuous, in contrast, because each point may be regarded as a boundary shared between segments. Given this conception of number as discrete, and given the notion that the parts of number are 'ones,' it would be natural to conclude that Aristotle thinks of numbers as whole numbers that are composed of a certain number of ones. The interpretation of these 'ones' is a matter of some dispute. Aristotle's remarks at *Physics* 224a2ff, however, suggest very much that he conceives of numbers as concrete collections, since he claims that although ten sheep and ten dogs are equinumerous, these collections do not constitute the very same (number) ten. If this view is representative, then Aristotelian ones must be concrete individuals of a certain kind. Assuming the kind of individuals that may serves as 'ones' exclude individual lengths or magnitudes, then it follows that any collection of 'ones' must be a whole number (or positive natural number). So fractions as well as irrational quantities are not numbers for Aristotle. Confirmation that irrationals are not numbers for Aristotle is found in his remark that "the relation of that which exceeds to that which is exceeded is numerically quite indefinite; for number is always commensurable; and number is not said of the non-commensurable" (*Meta.* 1021a5, Ross).

Such remarks aside, Aristotle is perfectly aware that there are magnitudes which cannot be expressed as the ratio of whole numbers, these are λογοι μέγεθοι, or incommensurable magnitudes (συμμετροί μέγεθοι) which he mentions numerous times in various works, (see especially *Meta*. 983a13-20; *An. Pr*. 41a29), and which he would have been aware of from discussions at the Academy.[8] Almost without exception, Aristotle gives as an example of an incommensurable magnitude, the length d of a square's diagonal, which is incommensurable with the length l of one of its sides precisely in the sense that there is no whole number n such that n times l is exactly equal to m times d.

The reader should not be given the impression that the irrationals were somehow intractable entities for the ancient Greeks. In Book 5 of Euclid's *Elements*, there is a completely general theory of proportions between magnitudes, applicable to incommensurable and commensurable magnitudes alike. Eudoxus of Cnidos (ca. 408 BC -ca. 347 BC) is credited with developing this theory of proportion, and I shall follow the general practice by referring to it as the Eudoxean theory.[9] The core of the Eudoxean theory is found in definitions 4 and 5. According to definition 4, 'Magnitudes are said to have a ratio towards one another which are capable, when multiplied of exceeding each other.'[10] So a and b have a ratio just in case there are whole numbers n, m such that $an > b$ and $bm > a$.

Since n and m are finite, this definition rules out the possibility that a or b might be infinitesimal or infinite, since multiplying either type of quantity by a finite number does not increase its size. However, the definition does not exclude the possibility that a or b might be incommensurables. Definition 5 gives the condition for when magnitudes are said to be in the same ratio.[11] In modern notation, definition 5 states that for any magnitudes a, b, c, and d, the two ratios a:b and c:d are equal if for any whole numbers n and m, one of the following cases obtains:

(1) if $na < mb$, then $nc < md$

(2) if $na=mb$, then $nc=md$,

(3) if $na > mb$, then $nc > md$.

In case (2), $a/b = m/n$, and since m and n are whole numbers, a/b is commensurable. Cases (1) and (3) provide for the situations in which $a/b < m/n$ and $a/b > m/n$. If m and n range over all the whole numbers, this means that a/b is incommensurable, that is, there are no whole numbers m and n such that $a/b=m/n$.

Many modern authors have been struck by the resemblance between Eudoxus' theory and the notion of a Dedekind cut. A Dedekind cut partitions the rational numbers into two (mutually exclusive, non-empty) classes A and B, such that every member of A precedes every member of B, and such that the union of A and B is equal to the set of all rational numbers. Dedekind showed that, in the case where A has no greatest member and B has no least member, there will be a gap in the rational numbers. An irrational number may therefore be posited to correspond to this gap. Eudoxus' theory is similar, except that instead of partitioning rational numbers into two classes, he partitions ratios of magnitudes into three classes, with the second class encompassing the case where the ratio is commensurable. If we let A stand for the ratio of any given quantities a:b, and let m and n be whole numbers as before, then we have three classes of quantities, corresponding to the cases where $A < m/n$, $A=m/n$, and $A> m/n$. In order to make this move, however, we have to treat the ratio a:b as a single item, and it is therefore natural (for us) to conceptualise it as a number. It is not clear, however, whether Euclid (and so far as we can tell Eudoxus) would have considered such ratios as numbers, however, since the theory of proportions is stated twice, once for magnitudes (Book 5, definition 5) and once for numbers (Book 7, definition 20). But one might say that the ratios are implicitly treated like (real) numbers insofar as many quasi-arithmetical operations are performed on them (e.g. inversion, composition, separation, and conversion, Book 5, definitions, 13-16).[12]

Commentators disagree about significance of the Eudoxean theory of ratios in answering the question of whether the ancient Greeks can be said to have anything like the modern theory of real numbers. Heath shows the equivalence of claims of sameness of Eudoxean ratio with sameness of real numbers as defined by Dedekind cuts.[13] Bostock (1979) shows that if we introduce arithmetical operations on Eudoxean ratios, we can derive the same results in geometry that we would now state using real numbers. Finally, Stein (1995) shows that there is a one to one mapping of Eudoxean ratios onto the set of positive real numbers as defined by Dedekind cuts. In light of these results it seems undeniable that the the Eudoxean theory of ratios can act as a surrogate for the theory of real numbers, but it does not follow, in my opinion, that the ancient Greeks had the theory of real numbers. The reason why is broadly Fregean: the fact that one theory T is co-extensive with some other theory T,' does not suffice to show that T and T' are conceptually equivalent. Indeed, the Eudoxean theory and the modern theory could not possibly be conceptually the same, since it is clear that statements involving magnitudes have a different sense from statements involving numbers. Statements about lengths of a square do not have the same sense as statements about real numbers. Statements stated in terms of ratios of lengths and their ratios have a different sense from statements stated in terms of real numbers. The multiplication of two lengths would have immediately called to mind in the Greek geometer a rectangle with a certain area, whereas the multiplication of two real numbers (for us) simply results in another real number.[14]

Since Eudoxus spent time at Plato's Academy, it would not be implausible to suppose that his ideas reached Aristotle through the teachings of the Academy.[15] In a remarkable passage in the *Posterior Analytics*, Aristotle says that whereas the demonstration that proportionals alternate used to be carried out separately with respect to magnitudes and with respect to number, 'now it is proved universally' (*An. Post*. 74a18-25). Aristotle seems on the cusp of recognising that the Eudoxean theory of ratios might apply to numbers as well as magnitudes, but ultimately must disappoint modern readers in insisting on a strict separation between the subject matter of geometry and arithmetic. He fleetingly entertains the proposition that magnitudes might be numbers (*An. Post*. 75b5), but only in the context of a contrary to fact hypothetical. He uses the possibility that magnitudes might be numbers to rebut the idea that one could give a purely arithmetical proof of a geometric proposition. It is clear from the proceeding remarks *(An. Pos.t* 75a38), that Aristotle does not think one can give a purely arithmetical proof of a geometrical proposition. Such a proof would violate his principle of the homogeneity of explanation: the premises of a proof, its axioms, and its conclusion, must refer only to things which belong to the same genus. So, if these remarks are right, then Aristotle is sceptical for methodological reasons about the very possibility of arithmetising the theory of continuous quantities. A different attitude towards the arithmetisation of mathematics, then, constitutes a significant difference between Aristotle and Cantor.

Cantor's definition of the irrationals as equivalence sequents of convergent sequences of rational numbers (in Cantor (1872)) was part of a push to arithmetise analysis, a goal which he shared with his contemporaries Dedekind and Weierstrass.[16] The goal behind arithmetisation was to increase the rigour of mathematics by eliminating, if possible, the reliance on geometrical intuition. Geometrical intuition had shown itself to be fallible and even misleading; it carried with it prejudices that prevented mathematicians from seeing certain possibilities, such as: the existence of continuous but nowhere differentiable functions, the equinumerosity of the points on a line with those in a plane, and the possibility of alternative, non-Euclidean geometries.[17] Not surprisingly, then, both Dedekind and Cantor reversed the priority of geometry and arithmetic, preferring to account for the continuity of a line in terms of properties of the real numbers corresponding to the points on the line. Such a perspective transformed the notion of the continuity of a line from an empirical postulate into a stipulation about a mathematical space.

Cantor went so far as to proclaim the hypothesis of the continuity of space to be none other than an *arbitrary*, conventional assumption of a complete one to one correspondence between the three dimensional purely arithmetical continuum consisting of the ordered triples (x,y,z) of real numbers and the space of the world of appearances (*Erscheinungswelt*).[18] He offered a mathematical argument for the independence of the mathematical structure of the continuum from that of space which consisted of a proof that continuous motion is possible through a discontinuous (not everywhere connected) space. He did this by showing that, even if a countably infinite dense subset M is removed from a continuous space A of two or more dimensions, it is possible to draw a continuous path connecting any two points in A, without crossing over any gaps in A.[19]

The argument by Cantor to the effect that there can be no direct inference from the appearance of motion to the structure of the underlying space would seem to undermine Aristotle's claim that "Magnitude, time, and movement are all liable to the same reasoning. Either they all consist of indivisible components and are divisible into indivisibles, or none of them does" (*Phys*. 232a23ff.). Since a continuum is "that which is divisible into parts which are always further divisible" (*Phys*. 232b25), Aristotle is claiming that either space, time, and motion are all continuous or none of them are. Following Fred Miller's terminology, we may call this claim 'the isomorphism thesis,' since it asserts that the structures of space, time, and motion are isomorphic.[20] In Cantor's example, we have continuous motion in a not-everywhere continuous space, so that not all spaces and motions are isomorphic. Nonetheless, it remains true for Cantor as for Aristotle, that any segment of a continuum is isomorphic to any other. This claim is true in a technical sense for Cantor, since it is possible to map, in an order preserving fashion, any segment of the reals onto any other segment. For Aristotle, the claim is true in the non-technical sense that any part of a continuum is a continuum. A further philosophical difference is that Cantor does not take claims about the mathematical structure of the continuum to follow from the apprehension of the physically extended world. That is why Cantor could view the thesis that space is continuous as arbitrary. But a philosopher sensitive to contemporary physics could not possibly view the claim as arbitrary. As noted at the outset, our best physical theories *do* assume that space and time are continuous.[21]

In departing from Aristotle's view that mathematical structures are merely abstracted aspects of physical processes and things, Cantor freed himself from the constraint that the mathematical structure of space must mirror that of the space in which we live and move. As Cantor was well aware, Aristotle drew his conclusions about the nature of the (mathematical) continuum from a consideration of physical continua.[22] This imposed a considerable constraint on the freedom of the mathematician. If the abstractionist interpretation of Aristotle's philosophy of mathematics is correct, then for Aristotle there could be no such thing as a mathematical continuum with peculiar *mathematical* features not already found in the physical continuum. The isomorphism thesis, then, would follow from Aristotle's philosophy of mathematics. Although it would also be consistent with the isomorphism thesis to claim that space, time, and motion are all discontinuous, Aristotle does not do so, because he takes the continuity of motion as a *datum* of perception.[23] Despite the wealth of evidence suggesting that Aristotle was concerned with the physical continuum, it would be a mistake, given either interpretation of Aristotle's philosophy of mathematics, to suppose that he is concerned with the physical continuum to the exclusion of the mathematical.[24] The exhaustive opposition between the physical and the mathematical simply has no place in Aristotle's anti-Platonist philosophy of mathematics.

3 Cantor as a Critic of Aristotle

Most of what Cantor has to say about philosophers who reject his actually infinite numbers is negative, and in this respect, Aristotle is no exception. In the *Grundlagen*, Cantor locates Aristotle at the head of a tradition against the actual infinite based on what he regards as a mistaken assumption, the assumption that only finite numbers can be counted. In contrast, Cantor thinks that by providing a hierarchy of transfinite numbers ordered according to size, he has shown that there are infinities that can be counted in the extended sense of coming in a certain order in a sequence.[25] Even as a metaphysical argument, Cantor does not find Aristotle's rejection of the actual, determinate infinite compelling:

> The reasons of the Stagirite prove nothing, however, other than that the arguments which the ancient natural philosophers put forward for the necessary existence of a α πειρον φωρισμένον [determinate infinite] are not compelling; he does not prove, the impossibility of an existing α πειρον φωρισμένον; in other words, he does not prove, that the latter concept, when one conceives of it as a *Transfinitum*, is self-contradictory, and it would be difficult for him, or more correctly said, impossible for him to have done so.[26]

Given his fundamental disagreement with Aristotle on the issue of whether there can be actually infinite magnitudes, it is a curious fact that Cantor was sympathetic to certain features of Aristotle's theory of the continuum. In his letter to Paul Tannery of 5 October 1888, Cantor wrote:

> You are right to point out that, I so to speak, renew the Pythagorean view, insofar as I teach that the geometrical continuum is a real compound of separate points, geometrical individuals, just as a forest is composed out of trees, but because the Pythagoreans understood the continuum as a sum of points, [a view] which is powerless against the demonstrations [*Beweise*] of Zeno of Elea, I take the continuum to be a point set (ensemble of points) of a more definite, precisely specified nature.
>
> My grasp of the geometrical (and temporal) continuum is one which harmoniously combines the advantages of the Aristotelian view with what is true in the Pythagorean way of understanding [*Auffasungsweise*], so that there will be no Zeno waiting for me who will demonstrate any kind

> of contradiction whatsoever in my most well-considered [*wohlerwogenen*] concept of the continuum.[27]

Aside from the historically interesting fact that Cantor accepted Tannery's now discredited claim that the target of Zeno's paradoxes was atomism, this passage also suggests that Cantor credits the Aristotelian view as having certain advantages which allow it to elude Zeno's paradoxes, advantages that he wishes to preserve.[28] The features of Aristotle's account that I think Cantor admired were Aristotle's emphasis on the connectedness of each segment of a continuum with the next, and his insight that there is a line between any two points, so that no that no point is immediately next to another point. We shall see in more detail where the agreement between Aristotle and Cantor lies in the next section.

4 Criteria for Continuity

Cantor and modern topologists consider the continuum to be constituted out of points, a claim which Aristotle tirelessly combats at *Phys*ics 231a21-37 and *De Generatione et Corruptione* 316a1-317b30. In the first argument for this claim, Aristotle argues on the basis of his definitions of contiguity, succession, and contact, that no point can be contiguous to another, and so not continuous. In the second argument, Aristotle argues that nothing which has magnitude can be composed of points which lack magnitude. Cantor thought that despite his point-set ontology, he could still preserve the basic Aristotelian insight into the connectedness of the continuum. As we shall see in this section and the next, Cantor would agree with the first argument to the extent that he also thinks that it is not obvious that the points on a continuum cannot be ordered consecutively. As White has argued, Cantor's response to the second argument is to reject the assumption altogether and substitute in its place a conception of continuity as an emergent, non-additive property. In this way, a point set can have magnitude even though none of its points do.[29]

In modern mathematics, the continuum is defined as any collection of points (or a point-set) possessing a certain structure, a structure that is exemplified by, e.g. the collection of real numbers, or the collection of points on a line. According to Cantor's famous definition, a continuum is a point set that is both perfect (dense in itself and closed) and well-connected. He considers these predicates to jointly constitute the necessary and sufficient conditions for being a continuum.[30]

Those not familiar with topology, as well as those unacqainted with Cantor's writings, might want to know Cantor's definitions. For both Cantor and modern topologists, a *limit point* x of a set S is a point around which in any neighbourhood of x, no matter how small, there are points other than x which belong to S. A neighbourhood around a point p is defined as the set of all points with some distance less than some arbitrary real number ε from point p. In one dimension, the neighbourhood of p is the interval [p-ε, p+ε], while in two dimensions, it is the set of points in the circular disk with centre p and with radius ε, and in three dimensions, the set of points in the sphere with centre p and radius ε. An open set is a set all of whose points are interior points, where p is an interior point of S just in case there is some $\varepsilon > 0$ such that if the distance between p and p' is less than ε, then p' is also in S.

A set S is everywhere dense (*überall dicht*) in an interval (a, b) if every sub-interval of (a, b), no matter how small, contains points of S. Cantor calls the set of limit points of a set S its derived set, which is denoted by S.' A set S is dense in itself (*in sich dicht*) if every member of it is a limit point, i.e. S ⊆S.' A set S is said to be closed (*abgeschlossen*) if all of its limit points are members of it, S' ⊆ S. Cantor defines a perfect (*perfekt*) set as one which is both dense in itself and closed. Finally, Cantor defines a set T to be connected (*zusammenhängend*) if for any two members of the set t and t,' and for any arbitrarily small number ε, there is always a finite number of points t, t1, …tn,t' of T such that the distances t-t1, t1-t2, t2-t3 … tn-t' are all less than ε. This definition differs from the now standard definition of connectedness, according to which a set is connected just in case it cannot be represented as the union of two non-empty non-overlapping sets.

Cantor considers previous attempts to define the continuum to be lacking in their failure to include one or another of his criteria.[31] This criticism applies to Aristotle's attempt, too, since Aristotle emphasises the connectedness of the continuum without considering its perfection. However, judged according to Cantor's criteria, Aristotle does not fare poorly, anticipating in a clear fashion two out of the three properties Cantor deems essential to continua. Aristotle has the concepts of density and connectedness, at least in a non-technical sense. It is less clear whether Aristotle has the concept of closure. As we shall see, he seems to differentiate between open and closed intervals of time, but cannot be said to have the concept of closure in a general sense, nor to have the concept of a closed interval of space.

Density

The thesis that the points of the continuum are densely packed is implicit in Aristotle's claim that there is a line between any two points and a stretch of time between any two 'nows' (Phys. 231b6). For each line is indefinitely divisible into points, so if there is a line between two points, there are indefinitely many points between the two points. Aristotle defines two things X and Y to be successive (φεξ ς) just in case X comes after Y, where X and Y are of the same type, and there is nothing of the same type as X and Y between X and Y (*Phys.*226b35-227a9). It follows that entities, which are densely arranged, e.g. points, cannot be successive to one another. It may be objected that it cannot be said that there are points between any two points for Aristotle, since points only come into (actual) existence for Aristotle when a division is made between two line segments. However, such an objection must explain what Aristotle says at *Phys.* 231b9: there is always a line between two points. Of course, for Aristotle, a line is not a collection of points, so strictly speaking the claim at 231b9 does not imply the points *themselves* are densely packed. Nonetheless, there is a potential infinity of points between any two points, since one can always make a bisection of a line and take the midpoint. This potential infinity of points is dense in the requisite sense.

The kind of density Aristotle recognises in the continuum corresponds to Cantor's notion of a point set being *überall dicht* (everywhere dense) in an interval. This is not quite the same as the density of a perfect set, which Cantor says is *insichdicht* (dense in itself), a density which a point set possesses just in case every point in the set is a limit point. The predicate '*insichdicht'* pertains to the set, while the predicate '*überall dicht'* denotes a property of the set relative to the space in which it is embedded. Although related the two concepts are clearly not co-extensive. [32]

Connectedness

An essential aspect of any theory of the continuum is the claim that its parts are connected. Intuitively, a set is connected if it contains no gaps. Recall that Cantor defined a point set T to be connected, if whenever for any two points t and t,' and for any ε, no matter how small, there is always a finite number points t1, t2, … tk of T, such that each of the successive distances t-t1, t1-t2, t2-t3, …tk-t' is less than ε.[33] So a connected set in this sense can always be divided up into a number of segments, the end-points of which all belong to the set and can never be very far apart from each other.

To understand why a continuum must be connected, consider the pathological case of Cantor's disconnected semi-continuum, also known as the Cantor set. The Cantor set is formed by removing successive middle thirds from an interval, starting with [0,1]. The first few steps in generating the set are pictured below:

stage 1: 0____1

stage 2: 0__________________1/3   2/3 ____________1

stage 3: 0_____1/9   2/9 ___1/3   2/3 ___7/9 8/9 ___1

The procedure is 'performed' *ad infinitum* (a countably infinite number of times!). The Cantor set is the union of all the points that remain when this operation is 'done.' A moment's reflection will show that, since we always remove the middle thirds, the points remaining in the Cantor set will include points like 1/3, 2/3, 1/9, 2/9, 7/9, 8/9, 1/27, 2/27 and so on. So in ternary notation, the points in the Cantor set will include only those points expressed by 0.n1n2n3… where the n's are 2's or 0's, but never 1's. (A ternary fraction is an expression of form 0.n1n2n3… where n1 represents the number of thirds, n2 the number of ninths, n3 the number of twenty-sevenths, and in general nk the number of $(1/3^k)$ths.) Cantor gave the equation for the series of all the points in the Cantor set as: $C = c1/3 + c2/9 + \ldots cn/3^n + \ldots$, where the co-efficients c_i can take on the value of 0 or 2, and n ranges from 0 to ∞.[34]

It is easy to see pictorially that the Cantor set will contain isolated islands of points. But we can also show arithmetically that the Cantor set doesn't contain any line segments at all, because the sum of the lengths of all of the removed intervals (1/3+ 2/9+4/27+…) equals 1. The measure of the Cantor set, like that of any collection of disconnected points, is 0. How then can it be that there is still something remaining? The continuum is uncountably infinite, but in removing segments, we have also removed uncountably many points. Yet the Cantor set, like the continuum [0,1], is uncountably infinite. If we wrote an expression in ternary notation for each point in the Cantor set, such as.222000222, we could show that for each expression, there is a unique decimal expression of a number in the interval [0,1].[35] So the Cantor set is equipotent with the set of points in the continuum [0,1].

So clearly uncountable magnitude does not suffice for being a continuum. Nor can being perfect be enough, since the Cantor set is perfect. Every point in C is a limit point of C, since around any point in C there are other points belonging to C. Yet the Cantor set fails to be a continuum because it is not connected and not everywhere dense in the interval [0,1]. It is not connected, obviously, because there are huge gaps between consecutive segments in the interval, between 1/3 and 2/3 for example. For the same reason it is not everywhere dense in the interval.

The fractal pioneer Benoit Mandelbrot called the Cantor set 'the dust set' in his book *The Fractal Geometry of Nature.*[36] It is interesting to note that Aristotle, too, imagines a magnitude that is divided through and through (ad infinitum) to resemble saw dust:

> But suppose that as the body is being divided, something like sawdust [πρισμα] is produced, and that in this sense a body comes away from the magnitude, even then the same argument applies…[that] it is absurd that magnitude should consist of things which are not magnitudes (Joachim, GC 316b1-5).

Aristotle even goes some way towards explaining why a dust set is not a continuum. For he realises that there can be no contact between two points, since the points being indivisible, lack extremities: "And every contact is always a contact of two somethings, i.e. there is always something besides the contact or division or the point (GC 316a6)."[37] So it would not be wholly inaccurate to say that Aristotle's insight is shared by Cantor and instituted in the latter's requirement that a continuum be connected.

According to Aristotle the parts of a continuum must be such that their extremities are not only touching but identical:

> …things are called continuous (συνεχ)when the touching limits of each become one and the same and are, as the word implies, contained in each other (συνέχηται): continuity is impossible if the extremeties are two. This definition makes it plain that continuity belongs to things that naturally in virtue of their mutual contact (σύναψιν) form a unity (*Phys*., 227a10ff.).

Whereas for connectedness Cantor requires that we be able to divide up the continuum into a finite number of segments such that the distance between adjacent segments approaches 0, Aristotle requires that the distance between two 'consecutive' parts of a continuum be zero. A line with its midpoint removed could not be continous for Aristotle, but it does count as connected for Cantor. However, it is not a continuum for Cantor, since it is not closed and dense in itself.

To be sure, Aristotle has already established that consecutive parts of the continuum cannot be points, because points are dense in the continuum. Aristotle's requirement creates a problem for the point-set concept of the continuum. Since points are indivisible and of zero magnitude, they have no extremities and therefore it is not possible for them to be distinct if their extremities are one. Hence, the attempt to string together points in such a way that their limits are one will have the absurd consequence that all of the points will coincide with one another, and a single extensionless point will be left. This sort of consideration may underlie Aristotle's claim that any attempt to construct a continuum out of points will amount to nothing (GC 316a30-5). The lesson to be learned from the Cantor set, and from Aristotle's 'sawdust' is that it is not enough to have a collection of uncountably many points to form a continuum. In addition, the set must have a certain structure: it must be connected, dense and closed.

Closure

So far we have seen that Aristotle's criteria for continuity include both density and connectedness. This leaves closure, the other aspect of perfection. Without this concept, we cannot distinguish between the rational numbers which are dense, but not closed, and the reals, which are closed. Does Aristotle have the topological concept of an open as opposed to closed interval? Some Aristotelian scholars, such as Michael White, think that Aristotle lacks this distinction and that the lack of this distinction prevents him from offering a satisfactory (i.e. mathematically acceptable) account of the continuum. Other Aristotelian scholars, notably David Bostock, argue that Aristotle does distinguish between open and closed intervals, at least sometimes. It seems to me that the preponderance of evidence supports Bostock's conclusion.

First, the evidence in favour of the hypothesis that Aristotle lacks the distinction between closed and open intervals. White cites Aristotle's explanation of why time, though potentially divisible into infinitely many nows, can nonetheless elapse at Physics 263a23ff. A key feature of the explanation is that we must not think of a movement as divided into infinitely many actual points, rather than as potentially divisible *ad infinitum*. According to Aristotle,

> … anyone who divides a continuous line into two halves is treating the single point at which the division occurs as two points, because he is making it both a starting point and an ending point; and counting out halves is no different from dividing into halves. But to make these divisions is to destroy the continuity of movement as well as the line, because continuous movement is movement over a continuum, and although there are infinitely many halves in any continuum, these are potential, not actual. Any actual division puts an end to continuous movement and creates a standstill (Phys. 263a23ff., Waterfield, emphasis mine).

According to White (1992), the explanation for why Aristotle thinks it is obvious that dividing a line into two segments involves treating one point as two, is that he cannot imagine that the point could be assigned to one interval only, with the other interval being left open. Since Aristotle tends to think in terms of magnitude, White argues, there would be no reason for him to distinguish between [0,1) and [0,1]. As Aristotle knew, a point does not have magnitude, so subtracting a point from an interval does not alter its magnitude.[38] In fact Aristotle makes the stronger claim elsewhere that any mere (disconnected) collection of points cannot add up to a positive magnitude (GC 316a30.).

Now for the evidence against the hypothesis. The evidence in favour of the hypothesis is drawn entirely from Aristotle's remarks on movement and spatial extension it seems. But Aristotle's remarks on temporal extension in the context of his discussion of change may suggest that he does take into account open intervals.[39] For Aristotle thinks that there is no first moment of change, but there is a time by which the change is completed. So if the stretch of time it takes for a change to occur is conceived of as an interval, it will be open on one end and closed on the other. In his article, 'Aristotle on Continuity in Physics VI,' David Bostock notes that

> One might claim that Aristotle himself may be said to recognize the existence of such things as half-open intervals at Phys. VIII.8 263b9-264a6, where he perceptively remarks that an interval without its end-point is no shorter than the interval with its end-point (264a4-6).

Following up this clue does indeed suggest that Aristotle distinguishes between open and closed intervals in time. Aristotle discusses the case of an object D changing from white to non-white during the time interval ACD. During the entire interval of time from A to C, the object is white, and during the interval from C to D it is not white. In order to avoid the paradox that D is both white and non-white at C, Aristotle says

> The solution is not to grant that it is white for the whole stretch of time, but to say that it is white for the whole stretch of time except the final now, namely C, which is already part of the later stretch of time. (*Phys*. 263b20).

Here we have the half open interval [A, C). Moreover,

> ..it is clear that if time A as a whole is the time during which it was becoming white, the time in which it was becoming white and became white [A,C] is no greater than all the time in which it was becoming white [A,C). (*Phys.* 264a5).

So here Aristotle recognises that the magnitude of the half-open interval [A, C) is no greater than that of the closed interval [A, C].[40]

But Aristotle appears to only talk of open-intervals with reference to stretches of time, not stretches of space. This introduces an asymmetry into the account of space and time. So we should not take Miller's isomorphism thesis to imply that intervals of space and time are exactly alike. The isomorphism thesis will still be valid insofar as both space and time are continua for Aristotle, but it will no longer be true that stretches of space and time have the same topological structure.

Well-Ordering and the Continuum

In an eminently sensible passage, Aristotle argues that a continuum cannot be constituted out of points, since a continuum has magnitude, whereas points (as ultimate indivisibles) lack magnitude. Aristotle's analysis of the problem is worth quoting in full:

> For, since no point is contiguous to another point, magnitudes are divisible through and through in one sense, and yet not in another. When, however, it is admitted that a magnitude is divisible through and through in one sense, it is thought that there is a point not only *anywhere*, but also *everywhere,* in it: hence it follows that the magnitude must be divided away into nothing. For there is a point everywhere within it, so that it consists either of contacts or points. But it is only *in one sense* that magnitude is divisible through and through; viz. insofar as there is one point anywhere within it and all of its points are everywhere within it if you take them singly. But there are not more points than one *anywhere* within it, for the points are not consecutive; hence, it is not divisible through and through. For if it were, then,….if it be divisible at its centre, it will be divisible also at a contiguous point. But it is not so divisible; for position is not contiguous to position, nor point to point (GC 317a2-10, Joachim translation).

In this section, I will suggest a novel and somewhat speculative reconstruction of this argument. I should note, before doing so, however, that Aristotle's argument is rejected by Cantor, who does think that some collections of points can be continua. A major difference between the Aristotelian and Cantorian conceptions here, is that, for Cantor the measure of a set may be non-additive, 'emergent property,' whereas for Aristotle the sum of the whole must equal the sum of each of its parts.[41] I suggest that the technical reason why Aristotle could not have conceived of a continuum as being constituted out of points was that he lacked the sophisticated apparatus of point set theory. Aristotle was, of course, philosophically opposed to the idea of an infinite number. But given his respect for the use of mathematics in laying the conceptual foundations of scientific theory, we cannot help but wonder what Aristotle would have made of transfinite numbers. What follows is just such an imaginative exercise.

For Aristotle, all numbers are determinate and countable. Consequently the idea of an uncountably infinite number would have struck him as an oxymoron. However, since Cantor, modern mathematicians take a different view of countability. Countability is only a feature of one portion of the universe of numbers, the natural numbers. The more basic fact is that sets that are comparable with regard to cardinality can be well-ordered. A set A can be well ordered if it is possible to define a relation R on the set A such that R constitutes a linear ordering of the set A and for every member of a non-empty subset of A, there is an R minimal element, a first element to bear R to something else. Cantor drew attention to the well-ordering principle in his *Grundlagen*, calling it 'a basic law of thought.'[42]

Is there any sense in which we can say that Aristotle recognised that the continuum was uncountably infinite? To say that the continuum is *uncountably infinite* for Aristotle (or a modern Aristotelian) would be to allude to the fact that it is always possible to make an additional division in a continuum. Any point can be numbered, but there are countlessly many potential points on the continuum. At no time will all of the points be numbered. For modern mathematicians, to call a set 'uncountable' is just to say that its members cannot be put in one to one correspondence with the natural numbers. If we consider the act of counting to be like that of making a division, then the two notions of countability might seem analogous. However, an Aristotelian notion of uncountability is meant to connote the indeterminate number of points that lie ready to be made in a continuum. The mathematical notion of uncountability carries with it no such connotation of indeterminacy. Cantor did recognise a kind of uncountability *in the extended sense* of not being well-orderable. Counting depends on the possibility of arranging things in order. So it might be said that there is an analogy between uncountability in the extended sense and uncountability in the Aristotelian sense of being indeterminately, or potentially, infinite. Such an analogy proves useful in interpreting Aristotle's argument for the impossibility of dividing a continuum through and through at GC 316a24-34.

A good basic analysis of the passage is given by William Charlton in 'Aristotle's Potential Infinites.'[43] Charlton's analysis is that Aristotle is drawing a distinction of scope in a universal modal claim. Thus Aristotle is affirming the proposition (1) For all x, it's possible to divide body b at point x, while denying the proposition (2) It's possible that a body is divided at every point x for all x. This interpretation makes good sense of Aristotle's claim that 'it is only in one sense that the magnitude is divisible through and through, viz. only insofar as there is one point anywhere within it.' It also explains why each point can be actualised individually, even though the infinite collection cannot be actualised at once, *en masse.*

However, this interpretation neglects the other aspect of Aristotle's explanation, i.e. his mention of the non-contiguity of points in the continuum, and subsequent inference that this prevents the continuum from being divided through and through. At 317a9-10, Aristotle recognises the need for an ordering procedure for counting or dividing things. We know from Phys. 262a21 that a point on a line is not considered to be actual for Aristotle unless an object comes to rest there, dividing the path of its journey into two. We might also suppose that counting is another way to actualise something. This hypothesis yields the desired result that there can be no actually infinite number for Aristotle, since then the infinite would be countable, and what is more, actual. In the case of the continuum where no point is immediately next to another given point, there is no constructive means for ordering the points. Hence, the points are uncountable in the extended sense and there is thus no way in which the process of division can deliver the totality of points produced by each possible division, rendering them uncountable in the sense we defined for Aristotle.

There is, of course, a non-constructive means of imposing order on the continuum: by applying the Axiom of Choice, we can arbitrarily select a first element from every non-empty subset of the continuum.[44] But the Axiom is unjustified from the constructive point of view, because it does not give a *rule* for selecting the chosen elements. Moreover, it is especially disturbing to a finitist, because it licenses the inference that we can perform these infinitely many choices simultaneously.

For Aristotle, it is non-sensical to conjure up an actually infinite set out of nothing, out of a mere collection of points. In this respect, Aristotle is closer to Brouwer than Cantor. The continuum as a whole exists, but the termini of intervals within it, the points, are only realised in time as the thinker or mathematician performs a certain act, such as counting or dividing intervals into 'before' and 'after.' There are affinities here between Aristotle's insistence that there must be a soul to count the passage of time (Phys. 223a21-28), and Brouwer's notion that the structure of the continuum is given in the intuition of time. Since time is potentially infinite, it forever grows in the direction of the future, and so there will never be a time at which all of its moments exist simultaneously. Thus far Aristotle and Brouwer are in agreement that a potential infinity is as Brouwer would say, 'denumerably unfinished' or incomplete. Despite the frequency with which the comparison between Aristotelian and intuitionistic theories of the continuum is casually proposed, it is seldom carried out in detail, perhaps because there is a significant disanalogy between Aristotle's approach, with its coupling of classical logic, abstractionism, and finitism about the universe, and Brouwer's approach, with its epistemic and proof-theoretic reasons for rejecting infinite totalities. Moreover, although a good deal of mathematics of the continuum can be reconstructed using intuitionism, intuitionistic mathematics is plainly not adequate to capture all of contemporary mathematical practice, a goal which was still feasible for Aristotelian mathematics (see Phys. 207b27-34). There is thus a sense in which Aristotle's philosophy of mathematics is more successfully conservative of mathematical practice. For such reasons, a comparison with Brouwer is especially difficult.

Cantor, in contrast to Aristotle, proceeded from the conception that the continuum and all of its uncountably many points already exist in actuality prior to the activity of the human mathematician. This conception legitimates many of Cantor's non-constructivist claims. Yet despite their different metaphysics, there is a substantial area of agreement between Cantor and Aristotle as to the topological nature of the continuum. Cantor upholds Aristotle's insight insofar as he too insists that a mere collection of points, no matter how many, cannot in themselves constitute a continuum. The collection must have a specific structure: it must be connected, dense, and closed. Of these three criteria, Aristotle already singles out two as essential to continua, and shows some understanding of the third (at least for the case of time-intervals). What might account for the remarkable similarity in these theories of the continuum? I believe, even though Cantor would reject such an explanation, that the similarity is best explained by taking a leaf out of Aristotle's philosophy of mathematics. Cantor and Aristotle share a common intuition of the structure of the continuum, an intuition that has its source in the experience of the physical world.

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[*] I am grateful to David Bostock and Gianni Rigamonto for their comments.

[1] I do not say that the division between pure and practical mathematics was *unfamiliar* to Aristotle, since it appears in Plato's distinction between arithmetic and logistic. Roughly speaking, the distinction between arithmetic and logistic is taken to be the distinction between pure number theory and the practical application of arithmetic in counting. As Klein has argued, however, it is difficult to map the distinction between arithmetic and logistic directly onto the distinction between, say, pure arithmetic and applied arithmetic. Plato at times seems to have called for a theoretical logistical science which would study the presuppositions about numbers and their relations presupposed by the practice of counting. See Jacob Klein, *Greek Mathematical Thought and the Origin of Algebra*, translated by E. Brann, New York: Dover Publications, 1992.

[2] Aristotle *Physics* 2. 2. 193b22-5. I have consulted several translations, which are abbreviated in the main text as follows:

Waterfield: Aristotle, *Physics*, translated by R.Waterfield, (New York: Oxford University Press, 1996).

Ross: Aristotle's Physics, translated by W.D. Ross, (Oxford: Clarendon Press, 1936).

Hardin and Gaye: Aristotle, "Physics." Translated by R. P. Hardin and R.K. Gaye. In *The Complete Works of Aristotle, The Revised Oxford Translation*, ed. J. Barnes, (Princeton: Princeton University Press, 1984).

[3] For some of the difficulties associated with abstractionism, see Frege's *Grundlagen der Arithmetik*, 1884. An adequate reply to Frege's criticisms could not be attempted here. I should note also that this interpretation is somewhat controversial. On Mueller's interpretation, Aristotle's mathematician is concerned with intelligible matter and geometrical properties are not possessed by material objects. See Ian Mueller. "Aristotle on Geometrical Objects," *Archiv für Geschichte der Philosophie* 52 (1970): 156-71. I have followed Lear's interpretation in the main in insisting that natural bodies do indeed possess geometrical properties for Aristotle, given in Jonathan Lear. "Aristotle's Philosophy of Mathematics," *The Philosophical Review* 111 (1982): 161-192. However, it seems to me that the difference between these two interpretations does not affect the general point I am making, namely that mathematical structures are derived, via abstraction, from physical structures, and thus must in some sense mirror physical structures. The notion of intelligible matter comes in only when the mathematical item is treated as an object of thought. And of course, it is clear from the remark that mathematicals are 'separable in thought' as well as from the remarks on intelligible matter in the *Metaphysics*, that mathematicals do exist as objects of thought.

[4] For a treatment of Aristotle's mathematical physics, see Edward Hussey, "Aristotle's Mathematical Physics: A Reconstruction." In *Aristotle's Physics*, edited by Lindsay Judson, Oxford: Clarendon Press, 1991. For a less sympathetic treatment, see Carl Boyer, *The History of the Calculus and its Conceptual Development*, New York: Dover Press, 1959.

[5] See Michael White, "On Continuity: Aristotle versus Topology," *History and Philosophy of Logic* 9 (1988):1-12. White says, 'It is, I think, rather amazing to what degree the "proto-topological intuitions" concerning continuity enunciated by Aristotle have been mathematically incorporated into contemporary topology" (op cit, p. 2). While I am largely in agreement with White, I do differ in my treatment of density and closure, crediting Aristotle with more of a grasp of these notions than White does. Moreover, I also differ in stressing along with Bostock (1991) that there are two distinct strands in Aristotle's account of the continuum at *Phys*. 231a21ff. and Phys. 227a6-16, whereas White (1988) focuses entirely on the account of the continuum as a natural unity at *Phys*. 227a6-a16.

[6] See Section III below, 'Cantor as Critic of Aristotle' for some references.

[7] The notion of a 'unit idea', of course, is taken from: Arthur Lovejoy, *The Great Chain of Being*, (Cambridge: Harvard University Press, 1936).

[8] A comprehensive list of Aristotle's references to incommensurables is given by D.H. Fowler, *The Mathematics of Plato's Academy*, (Oxford: Clarendon Press, 1987): 295. Some of the more important references are: *An.Pr*. 41a23-40 and 50a37-8 (which suggests a reductio proof for the incommensurability of the diagonal based on the fact that odd and even numbers are mutually exclusive classes, cf. Euclid's *Elements* X, prop. 117); *Phys*. 221b23-5, 222a3-7 ; *De Cael.* 281a6-7, b5-6, b12-14; *Meta*. 1019b24-7 (which says that the commensurability of a diagonal is not a possibility realised at any time); *GA* 742b27-8 (which states that the incommensurability of the diagonal is an eternal truth susceptible of proof); *Meta.* 983a12-21 (the incommensurability of a diagonal as a source of wonder), *Meta*. 1053a17-8 (no common unit measures the diagonal and side of a square).

[9] According to Heath, the theory is attributed to Eudoxus by the author of an anonymous scholium to Book 5, possibly Proclus. See Euclid. *The Thirteen Books of the Elements*. Translated and with an introduction by Sir Thomas Heath, 3 vols. (New York: Dover Publications, 1956), Book 5, vol. 2: 112. There is some disagreement about the dates of Eudoxus. The *Dictionary of Scientific Biography* gives his dates as ca 400 BC-347 BC. See G.L. Huxley, 'Euoxus of Cnidos." In *The Dictionary of Scientific Biography*, vol. 4, (New York: Charles Scribner's Sons, 1971) : 465-7.

[10] Euclid. *Elements*. Heath translation, vol 2: 114.

[11] According to Elements 5. definition 6, magnitudes which are in the same ratio are said to be proportional. However, such a fine distinction does not matter for our purposes.

12 Here as in general in this discussion I have benefitted from the insights in Howard Stein, “Eudoxus and Dedekind.” *In Frege’s Philosophy of Mathematics*, Cambridge, Massachusetts: Harvard University Press, 1995. Stein discusses the performance of algebraic operations on ratios in section IV of his essay, see especially, p. 348.

13 Euclid, op. cit., vol 2: 125.

14 For a discussion, see David Bostock, *Logic and Arithmetic*, 2 vols. (Oxford: Clarendon Press, 1979). Bostock (1979) rightly notes, “Though it makes perfectly good sense to multiply two real numbers together, the product being again a real number, there is no evident sense to the suggestion that two lines may be ‘multiplied together’ to yield a plane figure as a result “ (p. 237).

15 Actually the relation may be more direct than this cautious statement suggests. Dating is a difficult matter, but Knorr suggests that Eudoxus entered Plato’s Academy roughly at the same time as Aristotle, during 367 BC. See Wilbur Knorr, *The Evolution of the Euclidean Elements*, (Dordrecht: D. Reidel, 1975): 285.

16 Cantor,Georg. “über die Ausdehnung eines Satzes aus der Theorie der trigonometrischen Reihen,*” Mathematischen Annalen* 5, (1872): 123-132. Reprinted in his *Gesammelte Abhandlungen*., edited by Ernst Zermelo, Berlin, 1932: 92-101.

17 See Hans Hahn, “The Crisis in Intuition.” In *Empiricism, Logic and Mathematics*, edited by Brian McGuinness, (Dordrecht: D.Reidel, 1980): 73-102.

18 Georg Cantor, *Gesammelte Abhandlungen*, 156.

19 This example was brought to my attention by White, *The Continuous and Discrete*,1992: 186, fn.74. See also Joseph Dauben *Georg Cantor: His Mathematics and Philosophy of the Infinite* (Cambridge, Massachuetts: Harvard University Press, 1979), 85-86. The original example is found in Georg Cantor, “*Ueber unendliche lineare Punktmannigfaltigkeiten*, nr. 1-3,” in *Gesammelte Abhandlungen*, 139-156.

20 Fred Miller, “Aristotle Against the Atomists,” in *Infinity and Continuity in Ancient and Medieval Thought*, ed. N. Kretzmann, (Ithaca: Cornell University Press, 1982), 102. Aristotle of course does not have the formal logical concept of isomorphism, with its trappings of model theory, but a lack of the formal theory need not imply a lack of the intuition of ‘sameness of structure’. Indeed, without such ‘intuitions’, theories could never get started.

21 A very entertaining discussion of this point can be found in W. Newton-Smith, *The Structure of Time*, (London: Routledge, Kegan and Paul, 1980): 118-124.

22 Cantor, “Letter to Benno Kerry,” in *Briefe Georg Cantors*, ed. Herbert Meschkowski and Wilfrid Nilson, (Berlin: Springer Verlag, 1991).

23 See Bostock’s commentary to Phys. 232a23-233b32, in Aristotle, *Physics*, trans. Waterfield, (Oxford: Oxford University Press, 1996), 274-275.

24 See, e.g., Aristotle’s argument for the finitude of magnitude, Phys.208a14; his acceptance of only convergent infinite series, 206b3-12; the analogy between a point and a ‘now’ in *De Caelo* 300a6-20; his choice of examples in discussing the continuum Phys. 227a16; his discussion of the impossibility of division everywhere with respect to a physical body (σ˜μα).

25 Cantor, *Gesammelte Abhandlungen*, 174; Ewald, *From Kant to Hilbert*, 889.

26 Cantor, *Gesammelte Abhandlungen*, p. 396. My translation.

The original reads:

Die Gründe des Stagiriten beweisen aber nichts anderes, als daß die Argumente, welche die alten Naturphilosophen für die *notwendige* Existenz eines [πειρον φωρισμένον] vorgebracht haben, nicht zwingend sind; er beweist aber nicht die Unmöglichkeit eines existierenden [πειρον φωρισμένον]; mit anderen Worten, er beweist nicht, das letzterer Begriff, wenn man ihn als *Transfinitum* faßt, ein widersprechender sei, und es würde ihm solches auch schwer oder richtiger gesagt unmöglich gewesen sein.

27 Cantor, “Briefe an Tannery,” *Briefe Georg Cantors*, ed. Meschkowski and Nilson, 1991, 323-5. My translation.

28 Tannery’s hypothesis, advanced in his *Pour l’Histoire de la Science Hellene* (1887), was that the Pythagoreans were atomists and that Zeno’s paradoxes were designed to refute their atomism. This hypothesis has been discredited by contemporary scholars, e.g. Gregory Vlastos, “Review of J.E. Raven Pythagoreans and Eleatics,” *Gnomon* 25, 1953: 29-35, reprinted in *Studies in Greek Philosophy*, ed. D. Graham, vol. 1, (Princeton; Princeton University Press, 1995) ; David Furley (*Two Studies in Greek Atomism*, Princeton University Press, 1967, c.3, 47ff.).

29 Michael White, *The Continuous and Discrete*, 185.

30 Georg Cantor, *Gesammelte Abhandlungen*, 194.

31 Cantor, *ibid.*,194.

32 Georg Cantor, *Gesammelte Abhandlungen*, 229. Cantor notes that a point set can be dense in itself without being everydense in an interval. As an example, consider the set A= [0,1) (2,3], the set of real numbers between 0 and 3 inclusive, omitting those between 1 and 2. A is not dense in [0,3] but dense in itself. Conversely, if some points of a set B lie outside a space H, B can be everywhere dense in H, without being dense in itself. However, the two concepts

seem to be related in the following fashion: if a set P is in a domain H entirely and P is everywhere dense in H, then P is dense in itself.

33 Cantor, Georg. *Gesammelte Abhandlungen*, 194.

34 Georg Cantor, *Gesammelte Abhandlungen*, 207, fn. 11.

35 Of course, we would need to adopt the convention that.59999… was to be written as.6, and so on, in order to guarantee the unique representation of each point.

36 Benoit Mandelbrot, *The Fractal Geometry of Nature*, (New York: W.H. Freeman Co., 1983).

The Cantor set is an early instance of a fractal, because it has a fractional dimension. The Haussdorf dimension of the Cantor set is given by log 2/log 3=.63093. See Witold Hurewicz, and Henry Wallman, *Dimension Theory*, (Princeton: Princeton University Press, 1948): 107.

37 Note that here Aristotle says that a point is not something, but a division made between two things, e.g. halves of a line segment.

38 This suggests a sorites paradox: how many points must be substracted from an interval before it is empty? Before it is no longer continous? Modern mathematics gives strangely precise answers: uncountably many points.

39 I am indebted to David Bostock for this observation.

40 I owe this insight to Bostock's commentary in the Waterfield translation of the Physics.

41 I am indebted again to White (1991) for this observation, who in turn also notes some remarks by Skyrms. See White, *The Continuous and the Discrete*, 8.

42 Cantor, *Gesammelte Abhandlungen*, 169:

"Der Begriff der *wohlgeordneten* Menge weist sich als fundamental für die ganze Mannigfaltigkeitslehre aus. Daß es immer möglich ist, jede *wohldefinierte* Menge in die Form einer wohlgeordneten Menge zu bringen, auf dieses, wie mir scheint, grundlegende und folgenreiche, durch seine Allgemeingültigkeit besonders merkwürdige Denkgesetz werde ich in einer späteren Abhandlungen zurückkommen. Hier beschränke ich mich auf den Nachweis, wie aus dem Begriffe der wohlgeordneten Menge die Grundoperationen für die ganzen, sei es endlichen oder bestimmt-unendlichen Zahlen, in der einfachsten Weise sich ergeben und wie die Gesetze derselben aus der unmittelbaren inneren Anschauung mit apodiktischer Gewißheit erschlossen werden." Cantor never made good on his promise.

43 William Charlton, "Aristotle's Potential Infinites," in *Aristotle's Physics: A Collection of Essays,* ed. by L. Judson (Oxford: Clarendon Press, 1991), 129-149.

44 The Axiom of Choice is known to be equivalent to Cantor's principle that all sets can be well-ordered.