SLIM CYCLOTOMIC q-SCHUR ALGEBRAS

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ABSTRACT. We construct a new basis for a slim cyclotomic q-Schur algebra $\mathcal{S}_{\mathbf{m}}(n,r)$ via symmetric polynomials in Jucys–Murphy operators of the cyclotomic Hecke algebra $\mathcal{H}_{\mathbf{m}}(r)$. We show that this basis, labelled by matrices, is not the double coset basis when $\mathcal{H}_{\mathbf{m}}(r)$ is the Hecke algebra of a Coxeter group, but coincides with the double coset basis for the corresponding group algebra, the Hecke algebra at $q=1$. As further applications, we then discuss the cyclotomic Schur–Weyl duality at the integral level. This also includes a category equivalence and a classification of simple objects.

1. INTRODUCTION

Cyclotomic q-Schur algebras were first introduced by Dipper, James and Mathas (DJM) [20] as a natural generalisation of Dipper and James' q-Schur algebra [19] associated with the symmetric groups $\mathfrak{S}_r$ to a similar algebra associated with the wreath products $\mathbb{Z}_m \wr \mathfrak{S}_r$ of the cyclic group $\mathbb{Z}_m$ with $\mathfrak{S}_r$. See also [29] for the q-Schur$^{2\mathrm{B}}$ (or q-Schur^2) algebra associated with the type B Weyl group $\mathbb{Z}_2 \wr \mathfrak{S}_r$ and [37, 49, 13] for the affine q-Schur algebra associated with the affine symmetric group $\mathbb{Z} \wr \mathfrak{S}_r$. Like the q-Schur algebras, the cyclotomic ones are quasi-hereditary with a cellular basis; see [20, (6.12), (6.18)].

As an endomorphism algebra of the direct sum of certain cyclic modules over the cyclotomic Hecke algebra [4], the construction of a cyclotomic q-Schur algebra involves *all* m-fold multipartitions of r, which index all irreducible characters of the finite group $\mathbb{Z}_m \wr \mathfrak{S}_r$. However, when the "cyclotomic Schur–Weyl duality" is under consideration, these algebras seem a bit too "fat". Thus, as suggested in [42], a certain centraliser algebra, involving only partitions of r, has been considered to play such a role. We will call these centraliser algebras *slim cyclotomic q-Schur algebras* in this paper.

A further study on slim cyclotomic q-Schur algebras is motivated by the following recent works: First, the investigation on affine q-Schur algebras and affine Schur–Weyl duality has made significant progress. This includes a full generalisation to the affine case [13, 24] of the original work [8] of Beilinson–Lusztig–MacPherson (BLM) and a classification of irreducible representations of affine q-Schur algebras. These works should have applications to slim cyclotomic q-Schur algebras. Second, there has been also a breakthrough in generalising the BLM work to types other than A series by Bao–Kujawa–Li–Wang [7] for type B/C; see also [33] for type D and [34] for

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affine type C. The q-Schur algebra $\mathbf{S}^\imath$ defined in [7, §6.1] is used in linking the Hecke algebra of type C with a certain quantum symmetric pair, where a coideal subalgebra is constructed to play the role required in the Schur–Weyl duality in this case. The algebra $\mathbf{S}^\imath$ is the $m = 2$ slim cyclotomic q-Schur algebra. This algebra is also called the hyperoctahedral Schur algebra in [37] or the q-Schur$^{1\mathrm{B}}$ algebra in [30].

As a centraliser algebra of DJM's cyclotomic q-Schur algebra, a slim cyclotomic q-Schur algebra inherits a cellular basis that does not provide enough information for the representations of the algebra. However, the most natural basis, analogous to the well-known orbital basis for the endomorphism algebra of a permutation module (see [46]), is not available if the complex reflection groups $\mathbb{Z}_m \wr \mathfrak{S}_r$ is not a Coxeter group. In this paper, we will reveal a new integral basis for a slim cyclotomic q-Schur algebra. This basis is different from the usual double coset basis for the Hecke endomorphism algebra associated with a Coxeter group. However, it still reduces to a double coset basis for the group algebra. So we may regard our new basis as a new quantisation of the double coset basis, generalising the natural basis for the endomorphism algebra of a permutation module considered in [46] to this cyclotomic world.

It is interesting to point out that the definition of this new basis involves symmetric polynomials in Jucys–Murphy operators and can be lifted to affine q-Schur algebras. In this way, we obtain an algebra epimorphism from an affine q-Schur algebra to a slim cyclotomic q-Schur algebra of the same degree. Thus, an epimorphism from the integral quantum loop algebra of $\mathfrak{gl}_n$ to a slim cyclotomic q-Schur algebra can be constructed. Thus, we may put slim cyclotomic q-Schur algebras in the context of a possible cyclotomic Schur–Weyl duality. In fact, we propose a candidate for a possible "finite type" quantum subalgebra which can be used to replace $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$ in the cyclotomic Schur–Weyl duality. We will also establish a Morita equivalence in the cyclotomic case and present a classification of irreducible modules for a slim cyclotomic q-Schur algebra. It should be noted that the Schur–Weyl duality for cyclotomic Hecke algebras has been investigated by many authors; see, for example, [6, 45, 38]. However, the tensor spaces considered in these works are different from those used in the current paper.

In a forthcoming paper [16], we will determine which slim cyclotomic q-Schur algebras are quasi-hereditary and further investigate the representations of such an algebra at roots of unity.

We organize the paper as follows. In Section 2, we give a brief background on cyclotomic Hecke algebras and slim cyclotomic q-Schur algebras. In Section 3, we will see how the symmetric polynomials in Jucys–Murphy operators are used in the construction of an integral basis for $\mathcal{S}_{\mathbf{m}}(1, r)$, which is a centraliser algebra of $\mathcal{S}_{\mathbf{m}}(n, r)$ associated with the partition (r), and prove that it is commutative. In Section 4, we generalise the construction to the entire algebra $\mathcal{S}_{\mathbf{m}}(n, r)$ by employing Mak's double coset description in term of matrices [43]. In Section 5, we will make a comparison of our new basis with the usual double coset basis for the Coxeter group, $\mathfrak{S}_{2,r}$ of type B. Sections 6 and 7 are devoted to the cyclotomic Schur–Weyl duality. Partial double centraliser property is established through affine q-Schur algebras and quantum loop algebra of $\mathfrak{gl}_n$. A certain category equivalence has also been lifted to the cyclotomic case. Finally, as a further application of our discovery, we classify all simple objects for the slim cyclotomic q-Schur algebras as an application of a similar classification for affine q-Schur algebras. The two appendices cover some technical proofs and a generalisation of Lemma 3.2 to the affine case.

2. Ariki–Koike algebras and slim cyclotomic q-Schur algebras

For any nonnegative integer $m \geqslant 0$, let $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$. We always identify $\mathbb{Z}_m$ with the subset $\mathbb{Z}_m = \{0, 1, \ldots, m-1\}$ of $\mathbb{Z}$ if $m \geqslant 1$ (and, of course, $\mathbb{Z}_0 = \mathbb{Z}$). Thus,

$$\mathbb{Z}_m^r = \{\mathbf{a} = (a_i) \in \mathbb{Z}^r \mid a_i \in \mathbb{Z}_m, \ \forall\, 1 \leqslant i \leqslant r\}$$

is a subset of $\mathbb{Z}^r$.

Let $\mathfrak{S}_r = \mathfrak{S}_{\{1,2,\ldots,r\}}$ be the symmetric group on r letters with the basic transpositions $s_i = (i, i+1)$, $1 \leqslant i \leqslant r-1$, as generators, and let $\mathfrak{S}_{m,r} = \mathbb{Z}_m \wr \mathfrak{S}_r$ be the wreath product of the cyclic group $\mathbb{Z}_m$ and $\mathfrak{S}_r$. Then $\mathfrak{S}_{0,r} = \mathfrak{S}_{\vartriangle,r}$ is the affine symmetry group, $\mathfrak{S}_{1,r} = \mathfrak{S}_r$, $\mathfrak{S}_{2,r}$ is the Weyl group of type B, and $\mathfrak{S}_{m,r}$ for $m \geqslant 3$ are the complex reflection groups of type $G(m,1,r)$.

From now onwards, $\mathscr{R}$ always denotes a commutative ring (with 1) containing invertible element q.

For $m \geqslant 1$ and an m-tuple

$$\mathbf{m} = (u_1, \ldots, u_m) \in \mathscr{R}^m,$$

the *cyclotomic Hecke algebra* (or the Ariki–Koike algebra) over $\mathscr{R}$

$$\mathcal{H}_{\mathbf{m}}(r) = \mathcal{H}_{\mathbf{m}}(r)_{\mathscr{R}},$$

associated with $\mathfrak{S}_{m,r}$ (and parameters $q, \mathbf{m}$), is by definition the $\mathscr{R}$-algebra generated by T_i, L_j $(1 \leqslant i \leqslant r-1, 1 \leqslant j \leqslant r)$ with defining relations

(CH1) $(T_i + 1)(T_i - q) = 0$,
(CH2) $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$, $T_iT_j = T_jT_i$ $(|i-j| > 1)$,
(CH3) $L_iL_j = L_jL_i$, $(L_1 - u_1)\cdots(L_1 - u_m) = 0$,
(CH4) $T_iL_iT_i = qL_{i+1}$, $L_jT_i = T_iL_j$ $(j \neq i, i+1)$.

For $m = 0$, the affine Hecke algebra

$$\mathcal{H}_{\vartriangle}(r) = \mathcal{H}_{\vartriangle}(r)_{\mathscr{R}},$$

associated with $\mathfrak{S}_{\vartriangle,r}$ (and parameter q), is presented by the generators T_i, $X_j^{\pm}$ for $1 \leqslant i \leqslant r-1$ and $1 \leqslant j \leqslant r$ and relations (CH1), (CH2) together with

(CH3′) $X_iX_j = X_jX_i$, $X_iX_i^{-1} = 1 = X_i^{-1}X_i$,
(CH4′) $T_iX_iT_i = qX_{i+1}$, $X_jT_i = T_iX_j$ $(j \neq i, i+1)$.

In both cases, the subalgebra generated by $T_1, \ldots, T_{r-1}$ is the Hecke algebra $\mathcal{H}(r) = \mathcal{H}(r)_{\mathscr{R}}$ of $\mathfrak{S}_r$. If *all $u_i \in \mathscr{R}$ are invertible*, then the L_i are invertible in $\mathcal{H}_{\mathbf{m}}(r)$. Thus, in this case, there is a natural surjective homomorphism[1]

$$\epsilon_{\mathbf{m}} : \mathcal{H}_{\vartriangle}(r) \longrightarrow \mathcal{H}_{\mathbf{m}}(r),\ T_i \longmapsto T_i,\ X_j \longmapsto L_j, \tag{2.0.1}$$

which gives an algebra isomorphism

$$\mathcal{H}_{\vartriangle}(r)/\langle (X_1 - u_1)\cdots(X_1 - u_m)\rangle \cong \mathcal{H}_{\mathbf{m}}(r).$$

Thus, the algebra $\mathcal{H}_{\mathbf{m}}(r)$ is a cyclotomic quotient of the affine Hecke algebra $\mathcal{H}_{\vartriangle}(r)$ when all the parameters u_i are invertible.

[1] When $m = 1$, $\epsilon_{\mathbf{m}}$ is known as the evaluation map.

The algebra $\mathcal{H}_{\mathbf{m}}(r)$ admits an anti-automorphism

$$\tau : \mathcal{H}_{\mathbf{m}}(r) \longrightarrow \mathcal{H}_{\mathbf{m}}(r), \quad T_i \longmapsto T_i, L_j \longmapsto L_j. \tag{2.0.2}$$

Since q is invertible in $\mathscr{R}$, all the T_i for $1 \leqslant i \leqslant r-1$ are invertible with

$$T_i^{-1} = q^{-1}(T_i - (q-1)).$$

By multiplying T_i^{-1} and T_i on both sides of $T_iL_iT_i = qL_{i+1}$, respectively, we get the following two equalities

$$L_iT_i = T_iL_{i+1} + (1-q)L_{i+1}, \quad L_{i+1}T_i = T_iL_i + (q-1)L_{i+1}. \tag{2.0.3}$$

For each $\mathbf{a} = (a_1, \ldots, a_r) \in \mathbb{N}^r$, write

$$L^{\mathbf{a}} = L_1^{a_1} \cdots L_r^{a_r} \in \mathcal{H}_{\mathbf{m}}(r), \quad X^{\mathbf{a}} = X_1^{a_1} \cdots X_r^{a_r} \in \mathcal{H}_{\vartriangle}(r).$$

Then, by an inductive argument, we have for each $1 \leqslant i \leqslant r-1$ and $\mathbf{a} \in \mathbb{N}^r$,

$$L^{\mathbf{a}}T_i = T_iL^{\mathbf{a}s_i} + (1-q)\frac{L_{i+1}(L^{\mathbf{a}} - L^{\mathbf{a}s_i})}{L_i - L_{i+1}}, \tag{2.0.4}$$

where $\mathbf{a}s_i$ is given by permuting the i-th and $(i+1)$-th entries of $\mathbf{a}$. The affine version of the formula (with $L^{\mathbf{a}}$ replaced by $X^{\mathbf{a}}$) is well known; see, e.g., [13, (3.3.0.2)].

For each $w \in \mathfrak{S}_r$ and $\mathbf{a} \in \mathbb{Z}^r$, define the place permutation

$$\mathbf{a}w = (a_{w(1)}, a_{w(2)}, \cdots, a_{w(r)}). \tag{2.0.5}$$

For each $1 \leqslant i \leqslant r-1$, set

$$\alpha_i = (0, \ldots, 1, -1, 0, \ldots, 0) \in \mathbb{Z}^r$$

with 1 in the i-th position. Thus, $\mathbf{a}s_i = \mathbf{a} + (a_{i+1} - a_i)\alpha_i$. Applying (2.0.4) gives the following lemma.

Lemma 2.1. *Suppose that $1 \leqslant i \leqslant r-1$ and $\mathbf{a} \in \mathbb{N}^r$. Then in $\mathcal{H}_{\mathbf{m}}(r)$*

$$L^{\mathbf{a}}T_i = \begin{cases} T_iL^{\mathbf{a}} = T_iL^{\mathbf{a}s_i}, & \text{if } a_i = a_{i+1}; \\ T_iL^{\mathbf{a}s_i} + (q-1)\sum_{t=1}^{a_{i+1}-a_i} L^{\mathbf{a}s_i - t\alpha_i}, & \text{if } a_i < a_{i+1}; \\ T_iL^{\mathbf{a}s_i} + (1-q)\sum_{t=0}^{a_i-a_{i+1}-1} L^{\mathbf{a}s_i + t\alpha_i} & \text{if } a_i > a_{i+1}. \end{cases}$$

A similar formula holds in $\mathcal{H}_{\vartriangle}(r)$.

The following result is taken from [4, Thm 3.10].

Lemma 2.2. *The algebra $\mathcal{H}_{\mathbf{m}}(r)$ is a free $\mathscr{R}$-module with bases*

$$\{T_wL^{\mathbf{a}} \mid w \in \mathfrak{S}_r, \mathbf{a} \in \mathbb{Z}_m^r\} \quad \text{and} \quad \{L^{\mathbf{a}}T_w \mid w \in \mathfrak{S}_r, \mathbf{a} \in \mathbb{Z}_m^r\}.$$

Replacing $\mathbb{Z}_m^r$ by $\mathbb{Z}^r$ and $L^{\mathbf{a}}$ by $X^{\mathbf{a}}$ yields bases for $\mathcal{H}_{\vartriangle}(r)$.

A sequence $\lambda = (\lambda_1, \ldots, \lambda_n)$ of non-negative integers is called a composition (resp., partition) of r, if $|\lambda| = \sum_i \lambda_i = r$ (resp., if, in addition, $\lambda_1 \geqslant \lambda_2 \geqslant \ldots$). For a composition λ of r, let $\mathfrak{S}_\lambda$ be the corresponding standard Young (parabolic) subgroup of $\mathfrak{S}_r$. Let $\mathscr{D}_\lambda$ be the set of minimal length right coset representatives of $\mathfrak{S}_\lambda$ in $\mathfrak{S}_r$. Then $\mathscr{D}_\lambda^{-1} = \{d^{-1} \mid d \in \mathscr{D}_\lambda\}$ is the set of minimal length left coset representatives of $\mathfrak{S}_\lambda$ in $\mathfrak{S}_r$.

Set $x_\lambda = \sum_{w \in \mathfrak{S}_\lambda} T_w \in \mathcal{H}(r) \in \mathcal{H}_{\mathbf{m}}(r)$. Then (see, e.g., [14, Lem. 7.32])

$$x_\lambda T_i = T_i x_\lambda = q x_\lambda, \ \forall\, i \in J_\lambda, \tag{2.2.1}$$

where

$$J_\lambda = \big\{1 \leqslant i \leqslant r-1 \mid i \neq \sum_{j=1}^{s} \lambda_j; \forall\, 1 \leqslant s < n\big\}. \tag{2.2.2}$$

We may use a similar condition to characterise the "permutation modules" $x_\lambda \mathcal{H}_{\mathbf{m}}(r)$ and $\mathcal{H}_{\mathbf{m}}(r)x_\lambda$.

Lemma 2.3. *Let $\lambda \in \Lambda(n,r)$. We have*

$$x_\lambda \mathcal{H}_{\mathbf{m}}(r) = \{h \in \mathcal{H}_{\mathbf{m}}(r) \mid T_i h = qh,\ \forall i \in J_\lambda\},$$
$$\mathcal{H}_{\mathbf{m}}(r) x_\lambda = \{h \in \mathcal{H}_{\mathbf{m}}(r) \mid hT_i = qh,\ \forall i \in J_\lambda\}.$$

The two modules are $\mathscr{R}$-free with respective bases $\{x_\lambda T_d L^{\mathbf{a}} \mid d \in \mathscr{D}_\lambda,\ \mathbf{a} \in \mathbb{Z}_m^r\}$ and $\{L^{\mathbf{a}} T_d x_\lambda \mid d \in \mathscr{D}_\lambda^{-1},\ \mathbf{a} \in \mathbb{Z}_m^r\}$. Moreover, we have an $\mathscr{R}$-module isomorphism

$$\mathrm{Hom}_{\mathcal{H}_{\mathbf{m}}(r)}(x_\mu \mathcal{H}_{\mathbf{m}}(r), x_\lambda \mathcal{H}_{\mathbf{m}}(r)) \cong x_\lambda \mathcal{H}_{\mathbf{m}}(r) \cap \mathcal{H}_{\mathbf{m}}(r) x_\mu.$$

Proof. By Lemma 2.2, $\{T_w L^{\mathbf{a}} \mid w \in \mathfrak{S}_r, \mathbf{a} \in \mathbb{Z}_m^r\}$ is an $\mathscr{R}$-basis of $\mathcal{H}_{\mathbf{m}}(r)$. Suppose $h = \sum_{w,\mathbf{a}} c_{w,\mathbf{a}} T_w L^{\mathbf{a}} \in \mathcal{H}_{\mathbf{m}}(r)$ and $T_i h = qh,\ \forall i \in J_\lambda$. Equating gives $T_i \sum_w c_{w,\mathbf{a}} T_w = q \sum_w c_{w,\mathbf{a}} T_w$ for all $i \in J_\lambda, \mathbf{a} \in \mathbb{Z}_m^r$. Now our assertion follows from the fact that $x_\lambda \mathcal{H}(r) = \{h \in \mathcal{H}(r) \mid T_i h = qh,\ \forall i \in J_\lambda\}$ (see [18] or [14, Lem. 7.33]). The basis assertion follows from the fact that $\{x_\lambda T_d \mid d \in \mathscr{D}_\lambda\}$ is an $\mathscr{R}$-basis of $x_\lambda \mathcal{H}(r)$. Finally, if $f \in \mathrm{Hom}_{\mathcal{H}_{\mathbf{m}}(r)}(x_\mu \mathcal{H}_{\mathbf{m}}(r), x_\lambda \mathcal{H}_{\mathbf{m}}(r))$, then $f(x_\mu)T_i = qf(x_\mu)$ for all $i \in J_\mu$. By the first assertion, $f(x_\mu) \in \mathcal{H}_{\mathbf{m}}(r)x_\mu$ and so $f(x_\mu) = h_f x_\mu$ for some $h_f \in \mathcal{H}_{\mathbf{m}}(r)$. Hence, the right module homomorphism f is completely defined by the left multiplication by h_f. □

We now consider the set of all compositions of r into n parts:

$$\Lambda(n,r) = \big\{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{N}^n \mid \sum_{1 \leqslant i \leqslant n} \lambda_i = r\big\}.$$

For $\bullet \in \{\mathbf{m}, \vartriangle\}$, let $\mathcal{T}_\bullet(n,r) = \oplus_{\lambda \in \Lambda(n,r)} x_\lambda \mathcal{H}_\bullet(r)_{\mathscr{R}}$. The endomorphism algebra

$$\mathcal{S}_{\mathbf{m}}(n,r) = \mathcal{S}_{\mathbf{m}}(n,r)_{\mathscr{R}} = \mathrm{End}_{\mathcal{H}_{\mathbf{m}}(r)}(\mathcal{T}_{\mathbf{m}}(n,r))$$
$$\text{resp.,}\quad \mathcal{S}_\vartriangle(n,r) = \mathcal{S}_\vartriangle(n,r)_{\mathscr{R}} = \mathrm{End}_{\mathcal{H}_\vartriangle(r)}(\mathcal{T}_\vartriangle(n,r))$$

is called the *slim cyclotomic q-Schur algebra*, resp., *the affine q-Schur algebra*.

Note that $\mathcal{S}_{\mathbf{m}}(n,r)$ is indeed a centraliser algebra of the cyclotomic q-Schur algebra introduced in [20], which is called the q-Schur$^{2\mathrm{B}}$ algebra in [29] when $\mathbf{m} = (-1, q_0)$. Note also that the algebra $\mathcal{S}_{\mathbf{m}}(n,r)$ has been studied in [42]. See also [37, 29] and more recently in [7, §5.1] for the $m = 2$ case.

For each $\mu \in \Lambda(n,r)$, let $\mathfrak{l}_\mu$ denote the following composition of maps

$$\mathfrak{l}_\mu : \bigoplus_{\lambda \in \Lambda(n,r)} x_\lambda \mathcal{H}_{\mathbf{m}}(r) \xrightarrow{\pi} x_\mu \mathcal{H}_{\mathbf{m}}(r) \xrightarrow{\mathrm{id}} x_\mu \mathcal{H}_{\mathbf{m}}(r) \xrightarrow{\iota} \bigoplus_{\lambda \in \Lambda(n,r)} x_\lambda \mathcal{H}_{\mathbf{m}}(r) \tag{2.3.1}$$

where π, ι are the canonical projection and inclusion, respectively. This is an idempotent $\mathfrak{l}_\mu^2 = \mathfrak{l}_\mu$ which defines a *centraliser algebra* of $\mathcal{S}_{\mathbf{m}}(n,r)$:

$$\mathfrak{l}_\mu \mathcal{S}_{\mathbf{m}}(n,r) \mathfrak{l}_\mu \cong \mathrm{End}_{\mathcal{H}_{\mathbf{m}}(r)}(x_\mu \mathcal{H}_{\mathbf{m}}(r)). \tag{2.3.2}$$

The next two sections are devoted to constructing explicitly an $\mathscr{R}$-basis of $\mathcal{S}_{\mathbf{m}}(n,r)$.

3. The commutative algebra $\mathcal{S}_{\mathbf{m}}(1,r)$

In this section we deal with the special composition $\lambda=(r,0,\dots,0)\in\Lambda(n,r)$. Our aim is to construct an $\mathscr{R}$-basis of $x_\lambda\mathcal{H}_{\mathbf{m}}(r)\cap\mathcal{H}_{\mathbf{m}}(r)x_\lambda$ and to prove that the centraliser algebra $\mathfrak{l}_\lambda\mathcal{S}_{\mathbf{m}}(n,r)\mathfrak{l}_\lambda\cong\mathcal{S}_{\mathbf{m}}(1,r)$ is commutative. For simplicity, we write

$$x_{(r)}:=x_\lambda=\sum_{w\in\mathfrak{S}_r}T_w\ \text{ and }\ \mathfrak{l}_{(r)}:=\mathfrak{l}_\lambda\in\mathcal{S}_{\mathbf{m}}(n,r).$$

First of all, we have by (2.2.1) that

$$T_ix_{(r)}=x_{(r)}T_i=qx_{(r)},\ \forall\,1\leqslant i\leqslant r-1.$$

This together with Lemma 2.2 implies the following.

Lemma 3.1. (1) *The $\mathcal{H}_{\mathbf{m}}(r)$-modules $x_{(r)}\mathcal{H}_{\mathbf{m}}(r)$ and $\mathcal{H}_{\mathbf{m}}(r)x_{(r)}$ are $\mathscr{R}$-free with, respectively, bases $\{x_{(r)}L^{\mathbf{a}}\mid\ \mathbf{a}\in\mathbb{Z}_m^r\}$ and $\{L^{\mathbf{a}}x_{(r)}\mid\ \mathbf{a}\in\mathbb{Z}_m^r\}$.*

(2) *Likewise, the $\mathcal{H}_\vartriangle(r)$-modules $x_{(r)}\mathcal{H}_\vartriangle(r)$ and $\mathcal{H}_\vartriangle(r)x_{(r)}$ are $\mathscr{R}$-free with, respectively, bases $\{x_{(r)}X^{\mathbf{a}}\mid\ \mathbf{a}\in\mathbb{Z}^r\}$ and $\{X^{\mathbf{a}}x_{(r)}\mid\ \mathbf{a}\in\mathbb{Z}^r\}$.*

The following result provides a description of elements in $x_{(r)}\mathcal{H}_{\mathbf{m}}(r)\cap\mathcal{H}_{\mathbf{m}}(r)x_{(r)}$.

Lemma 3.2. *Suppose that*

$$z=\sum_{\mathbf{b}\in\mathbb{Z}_m^r}c_{\mathbf{b}}x_{(r)}L^{\mathbf{b}}\in x_{(r)}\mathcal{H}_{\mathbf{m}}(r)\cap\mathcal{H}_{\mathbf{m}}(r)x_{(r)},\quad\text{where } c_{\mathbf{b}}\in\mathscr{R}.$$

Then $c_{\mathbf{b}}=c_{\mathbf{b}w}$ for all $\mathbf{b}\in\mathbb{Z}_m^r$ and $w\in\mathfrak{S}_r$.

Proof. It suffices to prove that $c_{\mathbf{a}}=c_{\mathbf{a}s_i}$ for each fixed $\mathbf{a}\in\mathbb{Z}_m^r$ and $1\leqslant i<r$. Write $\mathbf{a}=(a_1,\dots,a_r)$. If $a_i=a_{i+1}$, then $\mathbf{a}=\mathbf{a}s_i$ and hence, $c_{\mathbf{a}}=c_{\mathbf{a}s_i}$, as desired. Now let $a_i\neq a_{i+1}$. We may suppose $a_i<a_{i+1}$ (Otherwise, we replace $\mathbf{a}$ by $\mathbf{a}s_i$).

By Lemma 2.1, we obtain that

$$\begin{aligned}zT_i&=\sum_{\mathbf{b}\in\mathbb{Z}_m^r}c_{\mathbf{b}}x_{(r)}L^{\mathbf{b}}T_i\\&=\sum_{\mathbf{b}\in\mathbb{Z}_m^r}c_{\mathbf{b}}x_{(r)}T_iL^{\mathbf{b}s_i}+\sum_{\substack{\mathbf{b}\in\mathbb{Z}_m^r\\ b_i<b_{i+1}}}(q-1)c_{\mathbf{b}}x_{(r)}\sum_{t=1}^{b_{i+1}-b_i}L^{\mathbf{b}s_i-t\alpha_i}\\&\quad+\sum_{\substack{\mathbf{b}\in\mathbb{Z}_m^r\\ b_i>b_{i+1}}}(1-q)c_{\mathbf{b}}x_{(r)}\sum_{t=0}^{b_i-b_{i+1}-1}L^{\mathbf{b}s_i+t\alpha_i}.\end{aligned}$$

Since $z\in\mathcal{H}_{\mathbf{m}}(r)x_{(r)}$ and $x_{(r)}T_i=qx_{(r)}$, it follows that $qz=zT_i$ which gives rise to the equality

$$\begin{aligned}\sum_{\mathbf{b}\in\mathbb{Z}_m^r}qc_{\mathbf{b}}x_{(r)}L^{\mathbf{b}}=\sum_{\mathbf{b}\in\mathbb{Z}_m^r}qc_{\mathbf{b}}x_{(r)}L^{\mathbf{b}s_i}&+\sum_{\substack{\mathbf{b}\in\mathbb{Z}_m^r\\ b_i<b_{i+1}}}(q-1)c_{\mathbf{b}}x_{(r)}\sum_{t=1}^{b_{i+1}-b_i}L^{\mathbf{b}s_i-t\alpha_i}\\&+\sum_{\substack{\mathbf{b}\in\mathbb{Z}_m^r\\ b_i>b_{i+1}}}(1-q)c_{\mathbf{b}}x_{(r)}\sum_{t=0}^{b_i-b_{i+1}-1}L^{\mathbf{b}s_i+t\alpha_i}.\end{aligned}$$

For the fixed $\mathbf{a} \in \mathbb{Z}_m^r$, comparing the coefficients of $x_{(r)}L^{\mathbf{a}}$ on both sides implies that

$$qc_{\mathbf{a}} = qc_{\mathbf{a}s_i} + c' + c'', \tag{3.2.1}$$

where c' (resp., c'') denotes the coefficient of $x_{(r)}L^{\mathbf{a}}$ in the second (resp., third) sum of the right hand side. In the following we calculate c' and c''.

Consider those $\mathbf{b} = (b_1, \dots, b_r) \in \mathbb{Z}_m^r$ with $b_i < b_{i+1}$ such that

$$\mathbf{b}s_i - t\alpha_i = \mathbf{a} \text{ for some } 1 \leqslant t \leqslant b_{i+1} - b_i.$$

With $\mathbf{a}$ fixed, it is equivalent to determine the precise range of t for which $\mathbf{b} = \mathbf{a}s_i - t\alpha_i$. Since $b_i = a_{i+1} - t$ and $b_{i+1} = a_i + t$, it follows that $t \leqslant b_{i+1} - b_i = a_i - a_{i+1} + 2t$. Hence,

$$a_{i+1} - a_i \leqslant t.$$

On the other hand, since $0 \leqslant b_i = a_{i+1} - t \leqslant m - 1$ and $0 \leqslant b_{i+1} = a_i + t \leqslant m - 1$, it follows that $t \leqslant a_{i+1}$ and $t \leqslant m - 1 - a_i$. Consequently,

$$a_{i+1} - a_i \leqslant t \leqslant \min\{a_{i+1}, m - 1 - a_i\}.$$

Since $\mathbf{a}s_i - t\alpha_i = \mathbf{a} - (t - (a_{i+1} - a_i))\alpha_i$, by substituting t for $t - (a_{i+1} - a_i)$, we obtain that $\mathbf{b} = \mathbf{a} - t\alpha_i$ with

$$0 \leqslant t \leqslant \min\{a_i, m - 1 - a_{i+1}\}.$$

Similarly, those $\mathbf{b} = (b_1, \dots, b_r) \in \mathbb{Z}_m^r$ with $b_i > b_{i+1}$ such that

$$\mathbf{b}s_i + t\alpha_i = \mathbf{a} \text{ for some } 0 \leqslant t \leqslant b_i - b_{i+1} - 1$$

are simply $\mathbf{b} = \mathbf{a}s_i + t\alpha_i$ for all

$$0 \leqslant t \leqslant \min\{a_i, m - 1 - a_{i+1}\}.$$

We treat the following two cases.

Case 1: $a_i \leqslant m - 1 - a_{i+1}$. Then

$$c' = (q-1)\sum_{t=0}^{a_i} c_{\mathbf{a}-t\alpha_i} \quad \text{and} \quad c'' = (1-q)\sum_{t=0}^{a_i} c_{\mathbf{a}s_i+t\alpha_i}.$$

Hence, we obtain from (3.2.1) that

$$\begin{aligned} qc_{\mathbf{a}} &= qc_{\mathbf{a}s_i} + (q-1)\sum_{t=0}^{a_i} c_{\mathbf{a}-t\alpha_i} + (1-q)\sum_{t=0}^{a_i} c_{\mathbf{a}s_i+t\alpha_i} \\ &= qc_{\mathbf{a}s_i} + (q-1)\big(c_{\mathbf{a}} + \sum_{t=1}^{a_i} c_{\mathbf{a}-t\alpha_i}\big) + (1-q)\big(c_{\mathbf{a}s_i} + \sum_{t=1}^{a_i} c_{\mathbf{a}s_i+t\alpha_i}\big). \end{aligned}$$

This together with the fact that $\mathbf{a}s_i + t\alpha_i = (\mathbf{a} - t\alpha_i)s_i$ implies that

$$c_{\mathbf{a}} - c_{\mathbf{a}s_i} = \sum_{t=1}^{a_i}(q-1)\big(c_{\mathbf{a}-t\alpha_i} - c_{(\mathbf{a}-t\alpha_i)s_i}\big).$$

If $a_i = 0$, then $c_{\mathbf{a}} = c_{\mathbf{a}s_i}$. Let $1 \leqslant k \leqslant m-1$ and suppose $c_{\mathbf{a}} = c_{\mathbf{a}s_i}$ in case $a_i < k$. Now consider the case $a_i = k$. Since for each $1 \leqslant t \leqslant a_i$, $\mathbf{a} - t\alpha_i =: \mathbf{b}^{(t)} = (b_1^{(t)}, \dots, b_r^{(t)})$ satisfies $b_i^{(t)} = a_i - t < a_i = k$,

$$b_i^{(t)} = a_i - t < a_{i+1} + t = b_{i+1}^{(t)} \text{ and } b_i^{(t)} = a_i - t \leqslant m - 1 - a_{i+1} - t = m - 1 - b_{i+1}^{(t)},$$

we have $c_{\mathbf{a}-t\alpha_i} = c_{(\mathbf{a}-t\alpha_i)s_i}$, $\forall\, 1 \leqslant t \leqslant a_i$. Hence, $c_{\mathbf{a}} = c_{\mathbf{a}s_i}$.

Case 2: $a_i > m-1-a_{i+1}$. In this case,

$$c' = (q-1)\sum_{t=0}^{m-1-a_{i+1}} c_{\mathbf{a}-t\alpha_i} \quad\text{and}\quad c'' = (1-q)\sum_{t=0}^{m-1-a_{i+1}} c_{\mathbf{a}s_i+t\alpha_i}.$$

From (3.2.1) it follows that

$$qc_{\mathbf{a}} = qc_{\mathbf{a}s_i} + (q-1)\sum_{t=0}^{m-1-a_{i+1}} c_{\mathbf{a}-t\alpha_i} + (1-q)\sum_{t=0}^{m-1-a_{i+1}} c_{\mathbf{a}s_i+t\alpha_i}.$$

Using an argument similar to that in Case 1, we obtain that

$$c_{\mathbf{a}} - c_{\mathbf{a}s_i} = \sum_{t=1}^{m-1-a_{i+1}} (q-1)\big(c_{\mathbf{a}-t\alpha_i} - c_{(\mathbf{a}-t\alpha_i)s_i}\big).$$

Finally, proceeding by induction on $m-1-a_{i+1}$ gives the equality $c_{\mathbf{a}} = c_{\mathbf{a}s_i}$. □

Remark 3.3. In Lemma B.1 in the appendix, we will modify the proof to get the affine version of the lemma.

Let $\mathscr{R}[L_1,\ldots,L_r]_m$ be the free $\mathscr{R}$-submodule of $\mathcal{H}_{\mathbf{m}}(r)$ with basis $\{L^{\mathbf{a}} \mid \mathbf{a}\in\mathbb{Z}_m^r\}$. Then the symmetric group $\mathfrak{S}_r$ acts on $\mathscr{R}[L_1,\ldots,L_r]_m$ defined by

$$w\cdot L^{\mathbf{a}} = L^{\mathbf{a}w^{-1}},\ \forall\, w\in\mathfrak{S}_r,\ \mathbf{a}\in\mathbb{Z}_m^r.$$

By $\mathscr{R}[L_1,\ldots,L_r]_m^{\mathfrak{S}_r}$ we denote the set of fixed elements in $\mathscr{R}[L_1,\ldots,L_r]_m$.

Consider the elementary symmetric polynomials $\sigma_1,\ldots,\sigma_r$ in $L_1,\ldots,L_r$, i. e.,

$$\sigma_i = \sigma_i(L_1,\ldots,L_r) = \sum_{1\leqslant j_1<\cdots<j_i\leqslant r} L_{j_1}\cdots L_{j_i},\ \forall\, 1\leqslant i\leqslant r.$$

It is clear that $\sigma_1,\ldots,\sigma_r\in\mathscr{R}[L_1,\ldots,L_r]_m^{\mathfrak{S}_r}$. Set

$$\Lambda(r,{<}m) = \bigcup_{k=1}^{m-1}\Lambda(r,k) = \big\{\mathbf{a}=(a_1,\ldots,a_r)\in\mathbb{N}^r \mid \sum_{i=1}^r a_i\leqslant m-1\big\}. \tag{3.3.1}$$

Then for each $\mathbf{a}=(a_1,\ldots,a_r)\in\Lambda(r,{<}m)$, the monomial

$$\sigma^{\mathbf{a}} = \sigma^{\mathbf{a}}(L_1,\ldots,L_r) = \sigma_1^{a_1}\cdots\sigma_r^{a_r}$$

lies in $\mathscr{R}[L_1,\ldots,L_r]_m^{\mathfrak{S}_r}$. Indeed, we have the following result.

Lemma 3.4. *The set of fixed elements $\mathscr{R}[L_1,\ldots,L_r]_m^{\mathfrak{S}_r}$ is a free $\mathscr{R}$-module with basis*

$$\{\sigma^{\mathbf{a}} \mid \mathbf{a}\in\Lambda(r,{<}m)\}.$$

Moreover, for each $1\leqslant i\leqslant r-1$ and $f\in\mathscr{R}[L_1,\ldots,L_r]_m^{\mathfrak{S}_r}$,

$$T_i f = f T_i.$$

Proof. By a standard method on symmetric polynomials (see [40, I, §2.13]), each $f\in\mathscr{R}[L_1,\ldots,L_r]_m^{\mathfrak{S}_r}$ can be written as an $\mathscr{R}$-linear combination of monomials

$$\sigma_1^{t_1-t_2}\sigma_2^{t_2-t_3}\cdots\sigma_{r-1}^{t_{r-1}-t_r}\sigma_r^{t_r},$$

where $t_1,\ldots,t_r\in\mathbb{N}$ satisfy

$$m-1\geqslant t_1\geqslant t_2\geqslant\cdots\geqslant t_r\geqslant 0.$$

Therefore, $\mathscr{R}[L_1,\dots,L_r]_m^{\mathfrak{S}_r}$ is spanned by the set $\{\sigma^{\mathbf{a}} \mid \mathbf{a}\in\Lambda(r,{<}m)\}$.

For each $\mathbf{a}\in\Lambda(r,{<}m)$, define $\widehat{\mathbf{a}}=(\widehat{a}_1,\dots,\widehat{a}_r)\in\mathbb{Z}_m^r$ by setting

$$\widehat{a}_i=\sum_{i\leqslant j\leqslant r} a_j,\ \forall\, 1\leqslant i\leqslant r.$$

Then the leading term of $\sigma^{\mathbf{a}}$ as a polynomial in $L_1,\dots,L_r$ is $L^{\widehat{\mathbf{a}}}$ and, moreover, for $\mathbf{a},\mathbf{b}\in\Lambda(r,{<}m)$,

$$\mathbf{a}=\mathbf{b}\Longleftrightarrow \widehat{\mathbf{a}}=\widehat{\mathbf{b}}.$$

Consequently, the linear independence of the set $\{\sigma^{\mathbf{a}}\mid \mathbf{a}\in\Lambda(r,{<}m)\}$ follows from that of $\{L^{\mathbf{a}}\mid \mathbf{a}\in\mathbb{Z}_m^r\}$.

Let $1\leqslant i,j\leqslant r$ with $i\neq r$. By summing up both sides of the formula (2.0.4) over $L^{\mathbf{a}}=L_{k_1}\dots L_{k_j}$ with $1\leqslant k_1<\cdots<k_j\leqslant r$, we obtain that $T_i\sigma_j=\sigma_jT_i$. Thus, the second assertion follows. □

Proposition 3.5. *As an $\mathscr{R}$-module, $x_{(r)}\mathcal{H}_{\mathbf{m}}(r)\cap\mathcal{H}_{\mathbf{m}}(r)x_{(r)}$ is free with basis*

$$\mathcal{B}_{r,r}=\mathcal{B}_{r,r}^{(m)}=\{x_{(r)}\sigma^{\mathbf{a}}=\sigma^{\mathbf{a}}x_{(r)}\mid \mathbf{a}\in\Lambda(r,{<}m)\}.$$

Proof. By Lemma 3.2, each element in $x_{(r)}\mathcal{H}_{\mathbf{m}}(r)\cap\mathcal{H}_{\mathbf{m}}(r)x_{(r)}$ has the form $x_{(r)}f$ for some $f\in\mathscr{R}[L_1,\dots,L_r]_m^{\mathfrak{S}_r}$. This together with Lemma 3.4 implies that $x_{(r)}\mathcal{H}_{\mathbf{m}}(r)\cap\mathcal{H}_{\mathbf{m}}(r)x_{(r)}$ is spanned by the set $\mathcal{B}_{r,r}$. Finally, the linear independence of $\mathcal{B}_{r,r}$ follows from Lemma 3.4 and the fact that $\{T_wL^{\mathbf{a}}\mid w\in\mathfrak{S}_r,\mathbf{a}\in\mathbb{Z}_m^r\}$ is an $\mathscr{R}$-basis of $\mathcal{H}_{\mathbf{m}}(r)$. □

Combining the results above gives the following corollary.

Corollary 3.6. *The centraliser algebra $\mathcal{S}_{\mathbf{m}}(1,r)=\mathfrak{l}_{(r)}\mathcal{S}_{\mathbf{m}}(n,r)\mathfrak{l}_{(r)}$ is a commutative $\mathscr{R}$-algebra.*

Proof. By (2.3.2), there is an algebra isomorphism

$$\mathfrak{l}_{(r)}\mathcal{S}_{\mathbf{m}}(n,r)\mathfrak{l}_{(r)}\cong \mathrm{End}_{\mathcal{H}_{\mathbf{m}}(r)}(x_{(r)}\mathcal{H}_{\mathbf{m}}(r))=\mathcal{S}_{\mathbf{m}}(1,r).$$

The latter as an $\mathscr{R}$-module is isomorphic to $x_{(r)}\mathcal{H}_{\mathbf{m}}(r)\cap\mathcal{H}_{\mathbf{m}}(r)x_{(r)}$, by Lemma 2.3. For each $\mathbf{a}\in\Lambda(r,{<}m)$, if we define a right $\mathcal{H}_{\mathbf{m}}(r)$-module homomorphism

$$\Phi_{\mathbf{a}}:\ x_{(r)}\mathcal{H}_{\mathbf{m}}(r)\longrightarrow x_{(r)}\mathcal{H}_{\mathbf{m}}(r),\quad x_{(r)}h\longmapsto\sigma^{\mathbf{a}}x_{(r)}h,\ \forall h\in\mathcal{H}_{\mathbf{m}}(r),$$

then the proposition above implies that $\{\Phi_{\mathbf{a}}\mid \mathbf{a}\in\Lambda(r,{<}m)\}$ forms an $\mathscr{R}$-basis for $\mathrm{End}_{\mathcal{H}_{\mathbf{m}}(r)}(x_{(r)}\mathcal{H}_{\mathbf{m}}(r))$. Finally, it follows from Lemma 3.4 that $\Phi_{\mathbf{a}}\circ\Phi_{\mathbf{b}}=\Phi_{\mathbf{b}}\circ\Phi_{\mathbf{a}}$ for all $\mathbf{a},\mathbf{b}\in\Lambda(r,{<}m)$. Hence, $\mathcal{S}_{\mathbf{m}}(1,r)$ is commutative, as desired. □

For $t\in\mathbb{N}$ and $\nu=(\nu_1,\dots,\nu_t)\in\Lambda(t,r)$, let

$$\begin{aligned}\Lambda(\nu,{<}m)&=\Lambda(\nu_1,{<}m)\times\cdots\times\Lambda(\nu_t,{<}m)\\&=\{(\mathbf{a}^{(1)},\dots,\mathbf{a}^{(t)})\mid \mathbf{a}^{(i)}\in\Lambda(\nu_i,{<}m),\ \forall\,1\leqslant i\leqslant t\},\end{aligned}\tag{3.6.1}$$

where $\Lambda(\nu_i,<m)=\{\mathbf{b}\in\mathbb{N}^{\nu_i}\mid |\mathbf{b}|<m\}$ as defined in (3.3.1).

Definition 3.7. For each $\nu=(\nu_1,\dots,\nu_t)\in\Lambda(t,r)$ and each $1\leqslant i\leqslant t$, let $\sigma_1^{(i)},\dots,\sigma_{\nu_i}^{(i)}$ denote the elementary symmetric polynomials in

$$L_{\nu_1+\cdots+\nu_{i-1}+1},\dots,L_{\nu_1+\cdots+\nu_{i-1}+\nu_i}.$$

For every $\mathbf{a}(\nu) = (\mathbf{a}^{(1)}, \dots, \mathbf{a}^{(t)}) \in \Lambda(\nu, <m)$ with $\mathbf{a}^{(i)} = (a_1^{(i)}, \dots, a_{\nu_i}^{(i)})$, define

$$\sigma^{\mathbf{a}(\nu)} := (\sigma_1^{(1)})^{a_1^{(1)}} (\sigma_2^{(1)})^{a_2^{(1)}} \cdots (\sigma_{\nu_1}^{(1)})^{a_{\nu_1}^{(1)}} \cdots\cdots (\sigma_1^{(t)})^{a_1^{(t)}} (\sigma_2^{(t)})^{a_2^{(t)}} \cdots (\sigma_{\nu_t}^{(t)})^{a_{\nu_t}^{(t)}}.$$

For example, let $m = 3$, $t = 4$ and $r = 7$, and take $\nu = (1, 2, 1, 3)$. Then, for $\mathbf{a}(\nu) = ((1), (1, 1), (1), (1, 0, 1))$, $\mathbf{b}(\nu) = ((1), (1, 0), (0), (0, 2, 0)) \in \Lambda(\nu, < 3)$, we have

$$\sigma^{\mathbf{a}(\nu)} = L_1 \cdot \big((L_2 + L_3)L_2L_3\big) \cdot L_4 \cdot \big((L_5 + L_6 + L_7)L_5L_6L_7\big)$$

and

$$\sigma^{\mathbf{b}(\nu)} = L_1 \cdot (L_2 + L_3) \cdot (L_5L_6 + L_5L_7 + L_6L_7)^2.$$

Let $\nu = (\nu_1, \dots, \nu_t) \in \Lambda(t, r)$ for some $t \in \mathbb{N}$. The action of $\mathfrak{S}_r$ on $\mathscr{R}[L_1, \dots, L_r]_m$ restricts to an action of its subgroup $\mathfrak{S}_\nu$. The following result will be needed in the next section. Recall the set J_λ defined in (2.2.2).

Proposition 3.8. (1) *The set of fixed elements* $\mathscr{R}[L_1, \dots, L_r]_m^{\mathfrak{S}_\nu}$ *is a free* $\mathscr{R}$*-module with basis* $\{\sigma^{\mathbf{a}(\nu)} \mid \mathbf{a}(\nu) \in \Lambda(\nu, <m)\}$. *Moreover, for* $i \in J_\nu$ *and* $f \in \mathscr{R}[L_1, \dots, L_r]_m^{\mathfrak{S}_\nu}$, *we have* $T_i f = f T_i$ *in* $\mathcal{H}_{\mathbf{m}}(r)$.

(2) *Suppose that* $z \in x_\nu \mathcal{H}_{\mathbf{m}}(r) \cap \mathcal{H}_{\mathbf{m}}(r) x_\nu$ *has the "pure" form*

$$z = \sum_{\mathbf{a} \in \mathbb{Z}_m^r} g_{\mathbf{a}} x_\nu L^{\mathbf{a}} = \sum_{\mathbf{b} \in \mathbb{Z}_m^r} h_{\mathbf{b}} L^{\mathbf{b}} x_\nu, \quad \text{where } g_{\mathbf{a}},\ h_{\mathbf{b}} \in \mathscr{R}.$$

Then z *is an* $\mathscr{R}$*-linear combination of* $x_\nu \sigma^{\mathbf{a}(\nu)}$ *with* $\mathbf{a}(\nu) \in \Lambda(\nu, <m)$.

Proof. (1) This statement follows from an argument similar to that in the proof of Proposition 3.5 together with an induction on t.

(2) Clearly, for each $i \in J_\nu$ (i. e., $1 \leqslant i \leqslant r-1$ with $i \neq \sum_{j=1}^s \nu_j$, $\forall\, 1 \leqslant s \leqslant t-1$), we have

$$T_i x_\nu = x_\nu T_i = q x_\nu \ \text{ and } \ z T_i = q z.$$

By applying similar arguments to those in the proof of Lemma 3.2, we obtain that

$$g_{\mathbf{a}} = g_{\mathbf{a}s_i},\ \forall\, i \in J_\nu.$$

Thus, $z = x_\nu f$ with $f \in \mathscr{R}[L_1, \dots, L_r]_m^{\mathfrak{S}_\nu}$. The statement then follows from (1). □

Remark 3.9. By Lemma B.1 and a similar argument, the $\mathscr{R}$-module $x_{(r)}\mathcal{H}_\vartriangle(r) \cap \mathcal{H}_\vartriangle(r)x_{(r)}$ contains the linearly independent set $\{x_{(r)}\sigma^{\mathbf{a}}(X_1, \cdots, X_r) \mid \mathbf{a} \in \mathbb{N}^r\}$. Hence, there is an affine version of Proposition 3.8.

4. A new integral basis for $\mathcal{S}_{\mathbf{m}}(n, r)$

In this section we describe an $\mathscr{R}$-basis for $x_\lambda \mathcal{H}_{\mathbf{m}}(r) \cap \mathcal{H}_{\mathbf{m}}(r) x_\mu$ ($\lambda, \mu \in \Lambda(n, r)$), all of which give rise to a new basis for $\mathcal{S}_{\mathbf{m}}(n, r)$. We keep all the notations in previous sections.

For two compositions λ and μ of r, let $\mathscr{D}_{\lambda,\mu} = \mathscr{D}_\lambda \cap \mathscr{D}_\mu^{-1}$. Then $\mathscr{D}_{\lambda,\mu}$ is the set of minimal length $\mathfrak{S}_\lambda$-$\mathfrak{S}_\mu$ double coset representatives in $\mathfrak{S}_r$.

Lemma 4.1. ([10], [18, Lem. 1.6]) *Let* λ, μ *be two compositions of* r *and* $d \in \mathscr{D}_{\lambda,\mu}$.

(1) *There exist compositions* $\nu(d) = \lambda d \cap \mu, \nu(d^{-1}) = \lambda \cap \mu d^{-1}$ *of* r *such that*

$$\mathfrak{S}_{\nu(d)} = d^{-1}\mathfrak{S}_\lambda d \cap \mathfrak{S}_\mu, \quad \mathfrak{S}_{\nu(d^{-1})} = \mathfrak{S}_\lambda \cap d\mathfrak{S}_\mu d^{-1}.$$

(2) *Each element* $w \in \mathfrak{S}_\lambda d\mathfrak{S}_\mu$ *can be uniquely written as* $w = udv = u'dv'$ *with* $u \in \mathfrak{S}_\lambda$, $v \in \mathscr{D}_{\nu(d)} \cap \mathfrak{S}_\mu$, $u' \in \mathscr{D}^{-1}_{\nu(d^{-1})} \cap \mathfrak{S}_\lambda$ *and* $v' \in \mathfrak{S}_\mu$, *which satisfy* $\ell(w) = \ell(u) + \ell(d) + \ell(v) = \ell(u') + \ell(d) + \ell(v')$. *(Here* ℓ *is the length function.)*

In particular, we have

$$x_\lambda T_d = \Big(\sum_{u \in \mathscr{D}^{-1}_{\nu(d^{-1})} \cap \mathfrak{S}_\lambda} T_u\Big) T_d x_{\nu(d)} \quad \textit{and} \quad T_d x_\mu = T_d x_{\nu(d)} \Big(\sum_{v \in \mathscr{D}_{\nu(d)} \cap \mathfrak{S}_\mu} T_v\Big). \tag{4.1.1}$$

For $\lambda, \mu \in \Lambda(n, r)$, $d \in \mathscr{D}_{\lambda,\mu}$, the double coset $\mathfrak{S}_\lambda d \mathfrak{S}_\mu$ defines an $n \times n$ matrix

$$\theta(\lambda, d, \mu) = \big(|R_i^\lambda \cap d(R_j^\mu)|\big), \tag{4.1.2}$$

where, for $\nu = (\nu_1, \nu_2, \ldots, \nu_n) \in \Lambda(n, r)$,

$$R_i^\nu = \{\widetilde{\nu}_{i-1} + 1,\ \widetilde{\nu}_{i-1} + 2, \ldots, \widetilde{\nu}_{i-1} + \nu_i\} \text{ with } \widetilde{\nu}_0 = 0,\ \widetilde{\nu}_k = \sum_{j=1}^k \nu_j, \tag{4.1.3}$$

see, e.g., [14, Thm 4.15]. Let

$$\Theta(n, r) = \big\{\theta(\lambda, d, \mu) \mid \lambda, \mu \in \Lambda(n, r),\ d \in \mathscr{D}_{\lambda,\mu}\big\} \ \text{ and } \ \Theta(n) = \bigcup_{r \geqslant 0} \Theta(n, r).$$

Then $\Theta(n)$ is the set $M_n(\mathbb{N})$ of $n \times n$ matrices with nonnegative integer coefficients. For each $A = (a_{ij}) \in \Theta(n)$, define its row and column vectors by

$$\mathrm{ro}(A) = \Big(\sum_{j=1}^n a_{1j}, \ldots, \sum_{j=1}^n a_{nj}\Big),\ \mathrm{co}(A) = \Big(\sum_{i=1}^n a_{i1}, \ldots, \sum_{i=1}^n a_{in}\Big) \in \mathbb{N}^n.$$

Thus, if $A = \theta(\lambda, d, \mu)$ and $\nu(d)$ as defined in Lemma 4.1(1), then $\lambda = \mathrm{ro}(A)$, $\mu = \mathrm{co}(A)$, and

$$\nu(d) = \nu(A) =: (a_{11}, \ldots, a_{n1}, a_{12}, \ldots, a_{n2}, \ldots, a_{1n}, \ldots, a_{nn}) \in \Lambda(n^2, r). \tag{4.1.4}$$

We also write $d = d_A$. Hence, the inverse map of θ has the form

$$\begin{aligned} \theta^{-1} : \Theta(n, r) &\longrightarrow \{(\lambda, d, \mu) \mid \lambda, \mu \in \Lambda(n, r),\ d \in \mathscr{D}_{\lambda,\mu}\}, \\ A &\longmapsto (\mathrm{ro}(A),\ d_A,\ \mathrm{co}(A)), \end{aligned} \tag{4.1.5}$$

and the subset $\Theta(n, r)_{\lambda,\mu} = \{A \in \Theta(n, r) \mid \mathrm{ro}(A) = \lambda, \mathrm{co}(A) = \mu\}$ identifies the double cosets in $\mathfrak{S}_\lambda \backslash \mathfrak{S}_r / \mathfrak{S}_\mu$.

C. Mak [43, Thm 4.1.2] has generalised this double coset correspondence from $\mathfrak{S}_r$ to $\mathfrak{S}_{m,r}$ ($m \geqslant 1$).

The group $\mathfrak{S}_{m,r} = \mathbb{Z}_m \wr \mathfrak{S}_r$ admits a presentation with generators $s_0, s_1, \ldots, s_{r-1}$ and relations:

$$\begin{aligned} &s_0^m = 1,\ s_i^2 = 1\ (1 \leqslant i \leqslant r-1),\ s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0, \\ &s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}\ (1 \leqslant i < r-1),\ s_i s_j = s_j s_i\ (|i-j| > 1). \end{aligned} \tag{4.1.6}$$

There is an alternative description of $\mathfrak{S}_{m,r}$, following [43]. Let ξ be a primitive m-th root of unity. Set $C_m = \{1, \xi, \ldots, \xi^{m-1}\}$ and $R = \{1, 2, \ldots, r\}$. Consider the set

$$C_m R = \{\xi^a i \mid 0 \leqslant a \leqslant m-1,\ 1 \leqslant i \leqslant r\}.$$

Then $\mathfrak{S}_{m,r}$ is identified with the group of all permutations w of C_mR satisfying

$$w(\xi^a i) = \xi^a w(i),\ \forall\, 1 \leqslant i \leqslant r.$$

In other words, each element w in $\mathfrak{S}_{m,r}$ is uniquely determined by the sequence $(w(1), \ldots, w(r))$. In particular, s_0 corresponds to the sequence $(\xi, 2, \ldots, r)$ and for $1 \leqslant i \leqslant r-1$, s_i corresponds to the sequence $(1, 2, \ldots, i-1, i+1, i, i+2, \ldots, r)$

Let $\Theta_m(n, r)$ be the set of $n \times nm$-matrices with nonnegative integer entries sum to r. We will write elements in $\Theta_m(n, r)$ as an $n \times n$ array with m-tuples of nonnegative integers as entries. More precisely,

$$\Theta_m(n, r) = \big\{\mathbb{A} = (\mathbf{a}_{ij})_{n\times n} \mid \mathbf{a}_{ij} \in \mathbb{N}^m, 1 \leqslant i, j \leqslant n; \sum_{i,j=1}^{n} |\mathbf{a}_{ij}| = r\big\}. \tag{4.1.7}$$

Then each element $\mathbb{A} \in \Theta_m(n, r)$ gives a matrix $|\mathbb{A}| := (|\mathbf{a}_{ij}|)_{n\times n} \in \Theta(n, r)$. Thus, we obtain a matrix-valued map

$$\mathfrak{f} = |\ \ | : \Theta_m(n, r) \longrightarrow \Theta(n, r), \quad (\mathbf{a}_{ij}) \longmapsto (|\mathbf{a}_{ij}|).$$

For $\lambda, \mu \in \Lambda(n, r)$, let

$$\Theta_m(n, r)_{\lambda,\mu} = \{\mathbb{A} \in \Theta_m(n, r) \mid \mathrm{ro}(|\mathbb{A}|) = \lambda,\ \mathrm{co}(|\mathbb{A}|) = \mu\}.$$

The following can be seen from [43, Thm 4.1.2] and (4.1.5).

Lemma 4.2. *Let $\lambda, \mu \in \Lambda(n, r)$.*

(1) *If $A = \theta(\lambda, d, \mu)$ with $d \in \mathscr{D}_{\lambda,\mu}$, then the fibre of $\mathfrak{f}$ at A is*

$$\mathfrak{f}^{-1}(A) = \Theta_m(n, r)^d_{\lambda,\mu} := \{\mathbb{A} \in \Theta_m(n, r)_{\lambda,\mu} \mid d_{|\mathbb{A}|} = d\}.$$

(2) *There is a bijection $\theta_{\lambda,\mu} = \theta^{(m)}_{\lambda,\mu} :\ \mathfrak{S}_\lambda\backslash\mathfrak{S}_{m,r}/\mathfrak{S}_\mu \to \Theta_m(n, r)_{\lambda,\mu}$ sending a double coset $\mathfrak{S}_\lambda w \mathfrak{S}_\mu$ to $\mathbb{A} = (\mathbf{a}_{ij})_{n\times n}$, where $\mathbf{a}_{ij} = (a^{(1)}_{ij}, \ldots, a^{(m)}_{ij})$ is defined by $a^{(t)}_{ij} = |\xi^t R^\lambda_i \cap w(R^\mu_j)|$ for $1 \leqslant t \leqslant m$.*

For each $\mathbb{A} \in \Theta_m(n, r)_{\lambda,\mu}$, define a composition, as in (4.1.4), associated with the parabolic subgroup $\mathfrak{S}^{d_{|\mathbb{A}|}}_{\mathrm{ro}(|\mathbb{A}|)} \cap \mathfrak{S}_{\mathrm{co}(|\mathbb{A}|)}$:

$$\nu(|\mathbb{A}|) = \nu(d_{|\mathbb{A}|}) = (|\mathbf{a}_{11}|, \ldots, |\mathbf{a}_{n1}|, |\mathbf{a}_{12}|, \ldots, |\mathbf{a}_{n2}|, \ldots, |\mathbf{a}_{1n}|, \ldots, |\mathbf{a}_{nn}|) \in \Lambda(n^2, r), \tag{4.2.1}$$

We also define a composition

$$\nu(\mathbb{A}) = (\mathbf{a}_{11}, \ldots, \mathbf{a}_{n1}, \mathbf{a}_{12}, \ldots, \mathbf{a}_{n2}, \ldots, \mathbf{a}_{1n}, \ldots, \mathbf{a}_{nn}) \in \Lambda(mn^2, r). \tag{4.2.2}$$

Then we have the associated Young subgroup $\mathfrak{S}_{\nu(\mathbb{A})}$ of $\mathfrak{S}_r$. The following lemma generalizes [18, Lem. 1.6] (for type A) and can be deduced from [43, Lem. 4.3.1(d)]. However, we provide a proof here for completeness and later use.

Let

$$\mathcal{I} = [1, n]^2, \quad \mathcal{J} = \mathcal{J}(\mathbb{A}) = \{(i, j) \in \mathcal{I} \mid 1 \leqslant i, j \leqslant n, a^{(m)}_{ij} < |\mathbf{a}_{ij}|\}. \tag{4.2.3}$$

We order $\mathcal{I}$ by setting

$$(i, j) \preceq (i', j') \text{ if } j < j' \text{ or } i \leqslant i' \text{ whenever } j = j'. \tag{4.2.4}$$

For $1 \leqslant i, j \leqslant n$, $1 \leqslant k \leqslant m$, let

$$a(i,j,k) = \widetilde{a}_{ij} + \sum_{1\leqslant x\leqslant k-1} a_{ij}^{(x)}, \quad \text{where } \widetilde{a}_{ij} = \sum_{1\leqslant i'\leqslant n,\, l<j} |\mathbf{a}_{i'l}| + \sum_{1\leqslant i'\leqslant i-1} |\mathbf{a}_{i'j}| \tag{4.2.5}$$

is the partial sum of the sequence in (4.2.1) up to $|\mathbf{a}_{ij}|$, but not include $|\mathbf{a}_{ij}|$.

For any $(i,j) \in \mathcal{I}$ and $1 \leqslant k \leqslant m$, let

$$\begin{aligned} &t_1 = s_0, t_i = s_{i-1}\cdots s_2 s_1 t_1 s_1 s_2 \cdots s_{i-1}, \quad 2 \leqslant i \leqslant r; \\ &t_{ij}^{(k)} = \begin{cases} t_{a+1}t_{a+2}\cdots t_{a+a_{ij}^{(k)}}, & \text{if } (i,j)\in\mathcal{J}, k\leqslant m-1, a_{ij}^{(k)}\geqslant 1; \\ 1, & \text{otherwise,} \end{cases} \end{aligned} \tag{4.2.6}$$

where $a = a(i,j,k)$.

Lemma 4.3. *For $\lambda, \mu \in \Lambda(n,r)$ and $w \in \mathfrak{S}_{m,r}$, if $\mathbb{A} = (\mathbf{a}_{ij}) = \theta_{\lambda,\mu}(\mathfrak{S}_\lambda w \mathfrak{S}_\mu) \in \Theta_m(n,r)_{\lambda,\mu}$, then $w^{-1}\mathfrak{S}_\lambda w \cap \mathfrak{S}_\mu = \mathfrak{S}_{\nu(\mathbb{A})}$. Moreover, we may choose*

$$w = d_{|\mathbb{A}|}\cdot \prod_{(i,j)\in\mathcal{J}} \big((t_{ij}^{(1)})^1 (t_{ij}^{(2)})^2 \cdots (t_{ij}^{(m-1)})^{m-1}\big) \tag{4.3.1}$$

(noting $(t_{ij}^{(m)})^m = 1$).

Proof. Write $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\mu = (\mu_1, \ldots, \mu_n)$. For $1 \leqslant i \leqslant n$, let $\mathfrak{S}_\lambda^{(i)}$ be the subgroup of $\mathfrak{S}_\lambda$ generated by $s_{\widetilde{\lambda}_{i-1}+1}, \ldots, s_{\widetilde{\lambda}_{i-1}+\lambda_i-1}$ ($\widetilde{\lambda}_{i-1} = \sum_{1\leqslant j\leqslant i-1}\lambda_j$ and $\lambda_0 = 0$ by convention). Similarly, we obtain the subgroups $\mathfrak{S}_\mu^{(i)}$ of $\mathfrak{S}_\mu$. Then

$$\mathfrak{S}_\lambda = \mathfrak{S}_\lambda^{(1)} \times \cdots \times \mathfrak{S}_\lambda^{(n)} \ \text{ and } \ \mathfrak{S}_\mu = \mathfrak{S}_\mu^{(1)} \times \cdots \times \mathfrak{S}_\mu^{(n)}.$$

This implies that $w^{-1}\mathfrak{S}_\lambda w = (w^{-1}\mathfrak{S}_\lambda^{(1)} w) \times \cdots \times (w^{-1}\mathfrak{S}_\lambda^{(n)} w)$. Hence,

$$\begin{aligned} w^{-1}\mathfrak{S}_\lambda w \cap \mathfrak{S}_\mu &= (w^{-1}\mathfrak{S}_\lambda w \cap \mathfrak{S}_\mu^{(1)}) \times \cdots \times (w^{-1}\mathfrak{S}_\lambda w \cap \mathfrak{S}_\mu^{(n)}) \\ &= \big(\prod_{i=1}^n w^{-1}\mathfrak{S}_\lambda^{(i)} w \cap \mathfrak{S}_\mu^{(1)}\big) \times \cdots \times \big(\prod_{i=1}^n w^{-1}\mathfrak{S}_\lambda^{(i)} w \cap \mathfrak{S}_\mu^{(n)}\big). \end{aligned} \tag{4.3.2}$$

By the definition, $\mathfrak{S}_\lambda^{(i)}$ is the subgroup of $\mathfrak{S}_{m,r}$ consisting of all the elements which fix the set $C_m(R \setminus R_i^\lambda)$, and map R_i^λ onto R_i^λ (Thus, they map $\xi^t R_i^\lambda$ onto $\xi^t R_i^\lambda$ for all $1 \leqslant t \leqslant m$). Consequently, $w^{-1}\mathfrak{S}_\lambda^{(i)} w$ is a subgroup of $\mathfrak{S}_{m,r}$ consisting of all the elements which fix the set $C_m w^{-1}(R \backslash R_i^\lambda)$ and map $w^{-1}(R_i^\lambda)$ onto $w^{-1}(R_i^\lambda)$ (They also map $\xi^t w^{-1}(R_i^\lambda)$ onto $\xi^t w^{-1}(R_i^\lambda)$ for $1 \leqslant t \leqslant m$).

Therefore, for $1 \leqslant i, j \leqslant n$, we can write

$$w^{-1}\mathfrak{S}_\lambda^{(i)} w \cap \mathfrak{S}_\mu^{(j)} = \prod_{t=1}^m \mathfrak{S}_{ij}^{(t)},$$

where $\mathfrak{S}_{ij}^{(t)}$ is the subgroup formed by all elements in $\mathfrak{S}_{m,r}$ which permute elements in $\xi^t w^{-1}(R_i^\lambda) \cap R_j^\mu$ and fix other elements in $C_m w^{-1}(R_i^\lambda) \cap R_j^\mu$.

By Lemma 4.2(2), for $1 \leqslant i, j \leqslant n$ and $1 \leqslant k \leqslant m$,

$$|\xi^k w^{-1}(R_i^\lambda) \cap R_j^\mu| = |\xi^k R_i^\lambda \cap w(R_j^\mu)| = a_{ij}^{(k)}.$$

Therefore, $\mathfrak{S}_{ij}^{(k)}$ is the subgroup of $\mathfrak{S}_{m,r}$ generated by $s_{a+1}, s_{a+2}, \ldots, s_{a+a_{ij}^{(k)}-1}$, where $a = a(i,j,k)$ as defined in (4.2.5). This together with (4.3.2) implies that $w^{-1}\mathfrak{S}_\lambda w \cap \mathfrak{S}_\mu = \mathfrak{S}_{\nu(\mathbb{A})}$.

It remains to prove the last assertion. Assume now w is given as in (4.3.1). For each element $u \in \mathfrak{S}_{m,r}$, we write $u = (u(1), \ldots, u(r))$. Then, for $(i,j) \in \mathscr{J}, 1 \leqslant k \leqslant m$ and $a = a(i,j,k)$,

$$(t_{ij}^{(k)})^k = (1, 2, \ldots, a, \xi^k(a+1), \xi^k(a+2), \ldots, \xi^k(a+a_{ij}^{(k)}), a+a_{ij}^{(k)}+1, a+a_{ij}^{(k)}+2, \ldots, r).$$

If we put $\beta_{ij}^{(k)} = (\xi^k(a+1), \xi^k(a+2), \ldots, \xi^k(a+a_{ij}^{(k)}))$, then (after concatenation)

$$d_{|\mathbb{A}|}^{-1} w := y = (\beta_{11}^{(1)}, \ldots, \beta_{1,1}^{(m)}, \beta_{21}^{(1)}, \ldots, \beta_{2,1}^{(m)}, \ldots \beta_{n-1,n}^{(1)}, \ldots, \beta_{n-1,n}^{(m)}, \beta_{n,n}^{(1)}, \ldots, \beta_{nn}^{(m)}).$$

Since $|\mathbf{a}_{ij}| = |d_{|\mathbb{A}|}^{-1} R_i^\lambda \cap R_j^\mu|$, it follows that

$$a_{ij}^{(k)} = |\xi^k(d_{|\mathbb{A}|}^{-1} R_i^\lambda) \cap y(R_j^\mu)| = |\xi^k R_i^\lambda \cap w R_j^\mu|,$$

for all $1 \leqslant i,j \leqslant n$ and $1 \leqslant k \leqslant m$. Hence, $\theta_{\lambda\mu}(\mathfrak{S}_\lambda w \mathfrak{S}_\mu) = \mathbb{A}$. □

We need more notations. For $k \in \mathbb{N}$ and $\lambda = (\lambda_1, \ldots, \lambda_m) \in \Lambda(m,k)$, let

$$b_i = \sharp\{t \in \{1, \ldots, m-1\} \mid \lambda_1 + \cdots + \lambda_t \geqslant i\} \quad (i = 1, 2, \ldots, k).$$

Then $m-1 \geqslant b_1 \geqslant b_2 \geqslant \cdots \geqslant b_k$. In other words, with the notation in (4.1.3), $(b_1, \ldots, b_k)$ is the partition dual to $(\widetilde{\lambda}_{m-1}, \ldots, \widetilde{\lambda}_2, \widetilde{\lambda}_1)$. Finally, define

$$\ddot{\lambda} = (b_1 - b_2, \ldots, b_{k-1} - b_k, b_k), \tag{4.3.3}$$

which clearly lies in $\Lambda(k, <m)$. Indeed, it is easy to check that the correspondence $\lambda \mapsto \ddot{\lambda}$ induces a bijection

$$g_{m,k} : \Lambda(m,k) \longrightarrow \Lambda(k, <m). \tag{4.3.4}$$

For example, let $m = 3$, $k = 6$ and take $\lambda = (2,3,1) \in \Lambda(3,6)$. Then

$$(b_1, b_2, b_3, b_4, b_5, b_6) = (2,2,1,1,1,0) \text{ and } \ddot{\lambda} = (0,1,0,0,1,0).$$

Let $\mathbb{A} = (\mathbf{a}_{ij})_{n\times n} \in \Theta_m(n,r)$ with $\mathbf{a}_{ij} = (a_{ij}^{(1)}, \ldots, a_{ij}^{(m)})$ for $1 \leqslant i,j \leqslant n$. By the construction above, each $\mathbf{a}_{ij}$ as a composition in $\Lambda(m, |\mathbf{a}_{ij}|)$ gives rise to a vector $\ddot{\mathbf{a}}_{ij} \in \Lambda(|\mathbf{a}_{ij}|, <m)$ and a "matrix" $\ddot{\mathbb{A}} = (\ddot{\mathbf{a}}_{ij})$. Thus, by juxtaposition, $\mathbb{A}$ defines a composition

$$\mathbf{a}(\ddot{\mathbb{A}}) := (\ddot{\mathbf{a}}_{11}, \ldots, \ddot{\mathbf{a}}_{n1}, \ddot{\mathbf{a}}_{12}, \ldots, \ddot{\mathbf{a}}_{n2}, \ldots, \ddot{\mathbf{a}}_{1n}, \ldots, \ddot{\mathbf{a}}_{nn}) \in \Lambda(\nu(|\mathbb{A}|), <m),$$

which[2] further defines, in the notation of Definition 3.7, an element

$$\sigma^{\ddot{\mathbb{A}}} := \sigma^{\mathbf{a}(\ddot{\mathbb{A}})} \in \mathscr{R}[L_1, \ldots, L_r]_m^{\mathfrak{S}_{\nu(|\mathbb{A}|)}}. \tag{4.3.5}$$

Lemma 4.4. *Suppose $\lambda, \mu \in \Lambda(n,r)$, $d \in \mathscr{D}_{\lambda,\mu}$ and let $\nu = \lambda d \cap \mu$. Then for each $\mathbf{a}(\nu) \in \Lambda(\nu, <m)$, there is a unique $\mathbb{A} \in \Theta_m(n,r)^d_{\lambda,\mu}$ such that $\sigma^{\mathbf{a}(\nu)} = \sigma^{\ddot{\mathbb{A}}}$.*

[2]Unlike the compositions $\nu(|\mathbb{A}|), \nu(\mathbb{A})$ defined in (4.2.1),(4.2.2), $\mathbf{a}(\ddot{\mathbb{A}})$ is not a composition of r.

Proof. Let $A = (a_{ij}) = \theta(\lambda, d, \mu) \in \Theta(n, r)$, i. e., $(\mathrm{ro}(A),\, d_A,\, \mathrm{co}(A)) = (\lambda, d, \mu)$. Then

$$\nu = \lambda d \cap \mu = \nu(A) = (a_{11}, \ldots, a_{n1}, a_{12}, \ldots, a_{n2}, \ldots, a_{1n}, \ldots, a_{nn}) \in \Lambda(n^2, r).$$

Applying the bijections $g_{m,a_{ij}}$ defined as in (4.3.4) induces a bijection

$$g_A : \Theta_m(n, r)^d_{\lambda,\mu} \longrightarrow \Lambda(\nu(A), {<}m),\ \mathbb{A} \longmapsto \mathbf{a}(\ddot{\mathbb{A}}),$$

noting that $|\mathbb{A}| = A$. Hence, $\mathbb{A} = g_A^{-1}\big(\mathbf{a}(\nu)\big)$. □

The lemma above together with Proposition 3.8(1) gives the following result.

Proposition 4.5. *For $\lambda, \mu \in \Lambda(n, r)$ and $d \in \mathscr{D}_{\lambda,\mu}$, the set*

$$\{\sigma^{\ddot{\mathbb{A}}} \mid \mathbb{A} \in \Theta_m(n, r)^d_{\lambda,\mu}\}$$

is linearly independent. Moreover, for each $\mathbb{A} \in \Theta_m(n, r)$,

$$x_{\nu(d_{|\mathbb{A}|})}\sigma^{\ddot{\mathbb{A}}} = \sigma^{\ddot{\mathbb{A}}}x_{\nu(d_{|\mathbb{A}|})}. \tag{4.5.1}$$

Example 4.6. Let $n = 2$, $m = 3$, and $r = 11$. Choose

$$\mathbb{A} = \begin{pmatrix} (1, 1, 1) & (1, 0, 2) \\ (1, 1, 0) & (1, 2, 0) \end{pmatrix} \in \Theta_3(2, 11).$$

Then $\mathbf{a}_{11} = (1, 1, 1)$, $\mathbf{a}_{21} = (1, 1, 0)$, $\mathbf{a}_{12} = (1, 0, 2)$, $\mathbf{a}_{22} = (1, 2, 0)$, and

$$|\mathbb{A}| = \begin{pmatrix} 3 & 3 \\ 2 & 3 \end{pmatrix}.$$

By the definition above, we obtain

$$\begin{aligned} &\ddot{\mathbf{a}}_{11} = (1, 1, 0),\ \ddot{\mathbf{a}}_{21} = (1, 1),\ \ddot{\mathbf{a}}_{12} = (1, 0, 0),\ \ddot{\mathbf{a}}_{22} = (1, 0, 1), \ \text{ and} \\ &\sigma^{\ddot{\mathbb{A}}} = \sigma^{\ddot{\mathbf{a}}_{11}}\sigma^{\ddot{\mathbf{a}}_{21}}\sigma^{\ddot{\mathbf{a}}_{12}}\sigma^{\ddot{\mathbf{a}}_{22}} \\ &\quad = (L_1 + L_2 + L_3)(L_1L_2 + L_2L_3 + L_1L_3)(L_4 + L_5)(L_4L_5) \\ &\quad\quad (L_6 + L_7 + L_8)(L_9 + L_{10} + L_{11})L_9L_{10}L_{11}. \end{aligned}$$

Moreover, $\nu(d_{|\mathbb{A}|}) = (3, 2, 3, 3)$, and the corresponding Young subgroup $\mathfrak{S}_{\nu(d_{|\mathbb{A}|})}$ is generated by $s_1, s_2, s_4, s_6, s_7, s_9, s_{10}$, where s_i $(1 \leqslant i \leqslant 10)$ are the generators of $\mathfrak{S}_{11}$.

Using the element $\sigma^{\ddot{\mathbb{A}}}$ associated with $\mathbb{A} \in \Theta_m(n, r)$ in (4.3.5), we further define with $\lambda = \mathrm{ro}(|\mathbb{A}|)$ and $\mu = \mathrm{co}(|\mathbb{A}|)$,

$$\mathfrak{b}_{\mathbb{A}} = x_\lambda T_{d_{|\mathbb{A}|}}\sigma^{\ddot{\mathbb{A}}} \sum_{w \in \mathscr{D}_{\nu(d_{|\mathbb{A}|})} \cap \mathfrak{S}_\mu} T_w \in x_\lambda \mathcal{H}_{\mathbf{m}}(r). \tag{4.6.1}$$

Theorem 4.7. *For $\lambda, \mu \in \Lambda(n, r)$, the $\mathscr{R}$-module $x_\lambda\mathcal{H}_{\mathbf{m}}(r) \cap \mathcal{H}_{\mathbf{m}}(r)x_\mu$ is free and the set*

$$\mathcal{B}_{\lambda,\mu} = \mathcal{B}^{(m)}_{\lambda,\mu} = \big\{\mathfrak{b}_{\mathbb{A}} \mid \mathbb{A} \in \Theta_m(n, r)_{\lambda,\mu}\big\}$$

forms a basis.

The proof of the theorem is somewhat standard and will be given in Appendix A; compare [31].

We remark that the assertion that $\mathscr{R}$-module $x_\lambda\mathcal{H}_{\mathbf{m}}(r) \cap \mathcal{H}_{\mathbf{m}}(r)x_\mu$ is $\mathscr{R}$-free is not new. In [20, (6.3)], a basis that can give rise to a cellular basis for the cyclotomic q-Schur algebra is constructed. However, the basis $\mathcal{B}_{\lambda,\mu}$ is new. We will show in §5 that it is a q-analogue of the usual double coset basis for the endomorphism algebra of a permutation module [46].

For given $\mathbb{A} \in \Theta_m(n,r)$, define

$$\Phi_{\mathbb{A}} \in \mathcal{S}_{\mathbf{m}}(n,r) = \mathrm{End}_{\mathcal{H}_{\mathbf{m}}(r)}\big(\bigoplus_{\lambda\in\Lambda(n,r)} x_\lambda \mathcal{H}_{\mathbf{m}}(r)\big)$$

taking $x_\mu h \mapsto \delta_{\mu,\mathrm{co}(|\mathbb{A}|)}\mathfrak{b}_{\mathbb{A}} h$ for all $\mu \in \Lambda(n,r)$ and $h \in \mathcal{H}_{\mathbf{m}}(r)$.

Now we can state the main result of this section.

Theorem 4.8. *The slim cyclotomic q-Schur algebra $\mathcal{S}_{\mathbf{m}}(n,r)$ is a free $\mathscr{R}$-module with basis $\{\Phi_{\mathbb{A}} \mid \mathbb{A} \in \Theta_m(n,r)\}$ and rank $\binom{mn^2+r-1}{r}$.*

Proof. Since $\mathcal{S}_{\mathbf{m}}(n,r) = \bigoplus_{\lambda,\mu\in\Lambda(n,r)} \mathrm{Hom}_{\mathcal{H}_{\mathbf{m}}(r)}(x_\mu\mathcal{H}_{\mathbf{m}}(r), x_\lambda\mathcal{H}_{\mathbf{m}}(r))$, the first assertion follows from Lemma 2.3 and Theorem 4.7. The second assertion is clear as $|\Theta_m(n,r)| = \binom{mn^2+r-1}{r}$. □

Remark 4.9. (1) It would be interesting to generalise the construction to obtain a similar basis for the affine q-Schur algebra $\mathcal{S}_\vartriangle(n,r)$. However, by simply replacing L_j's by X_j's in the definition, we obtain a linearly independent set $\{\Phi^\vartriangle_{\mathbb{A}} \mid \mathbb{A} \in \Theta_m(n,r)\}$ for $\mathcal{S}_\vartriangle(n,r)$. See Remarks 3.3 and 3.9.

(2) We also observe that the basis in Theorem 4.8 cannot be naturally extended to a basis for the cyclotomic q-Schur algebra discussed in [20].

View $\Theta(n,r)$ as a subset of $\Theta_m(n,r)$ via the following map:

$$\iota^{(m)} : \Theta(n,r) \longrightarrow \Theta_m(n,r), \quad A \longmapsto A^{(m)}, \tag{4.9.1}$$

where if $A = (a_{ij}) \in \Theta(n,r)$, then $A^{(m)} = (\mathbf{a}_{ij}) \in \Theta_m(n,r)$ with all $\mathbf{a}_{ij} = (0,\ldots,0,a_{ij}) \in \mathbb{N}^m$. Then by the definition, $\sigma^{\ddot{A}^{(m)}} = 1$ and, hence,

$$\Phi_{A^{(m)}}(x_\mu h) = \delta_{\mu,\mathrm{co}(A)} x_\lambda T_{d_A} \sum_{v\in\mathscr{D}_{\nu(d_A)}\cap\mathfrak{S}_\mu} T_v h, \ \forall\, h \in \mathcal{H}_{\mathbf{m}}(r).$$

Note that the same rule with $h \in \mathcal{H}(r)$ defines an element ϕ_A in the q-Schur algebra $\mathcal{S}_q(n,r) := \mathrm{End}_{\mathcal{H}(r)}(\oplus_{\lambda\in\Lambda(n,r)} x_\lambda\mathcal{H}(r))$. Thus, we have immediately the following:

Proposition 4.10. *The q-Schur algebra $\mathcal{S}_q(n,r)$ can be embedded into $\mathcal{S}_{\mathbf{m}}(n,r)$ via the map $\phi_A \mapsto \Phi_{A^{(m)}}$ for $A \in \Theta(n,r)$.*

5. Comparison with the double coset basis of type B/C

Recall the presentation of $\mathfrak{S}_{m,r}$ in (4.1.6). If $m = 2$, then

$$W := \mathfrak{S}_{2,r}$$

is a Coxeter (or Weyl) group of type B with Coxeter generators $s_0, s_1, \ldots, s_{r-1}$. In this section we mainly deal with the cyclotomic Hecke algebra $\mathcal{H}_{\mathbf{m}}(r)$ associated with W. We will also choose

$$\mathbf{m} = \mathbf{2} = (-1, q_0)$$

with q_0 invertible in $\mathscr{R}$ (q also invertible in $\mathscr{R}$ as before) so that $\mathcal{H}_{\mathbf{2}}(r)$ is isomorphic to the Hecke algebra of type B defined in [29, Sect. 3] by sending T_i to T_{s_i} and L_j

to $q^{1-j}T_{s_{j-1}}\cdots T_{s_1}T_{s_0}T_{s_1}\cdots T_{s_{j-1}}$. We will identify the two. Thus, $T_i = T_{s_i}$, $L_1 = T_0$, and $\mathcal{H}_{\boldsymbol{2}}(r)$ has a basis $\{T_w\}_{w\in W}$, where $T_w = T_{i_1}T_{i_2}\cdots T_{i_l}$ for any reduced expression $w = s_{i_1}s_{i_2}\cdots s_{i_l}$. The associated slim cyclotomic q-Schur algebra is known as the *q-Schur$^{\text{1B}}$ algebra*.[3] See Remark 5.10 for connections to other similar algebras.

For any subset D of W, define

$$T_D = \sum_{w\in D} T_w \in \mathcal{H}_{\boldsymbol{2}}(r), \quad \underline{D} = \sum_{w\in D} w \in \mathscr{R}W.$$

For arbitrary $\lambda,\mu \in \Lambda(n,r)$, we shall make a comparison between the integral basis $\mathcal{B}^{(2)}_{\lambda,\mu}$ for $x_\lambda\mathcal{H}_{\boldsymbol{2}}(r)\cap\mathcal{H}_{\boldsymbol{2}}(r)x_\mu$ given in Theorem 4.7 and the double coset basis $\{T_D \mid D \in \mathfrak{S}_\lambda\backslash W/\mathfrak{S}_\mu\}$ given in [29, Prop. 4.2.5] (and their counterparts in the non-quantum case).

Following [29, Sect. 2], for each composition λ of r, let $\widetilde{\mathscr{D}}_\lambda$ be the set of shortest right coset representatives of the Young subgroup $\mathfrak{S}_\lambda$ in W. Set $\widetilde{\mathscr{D}}_\lambda^{-1} = \{d^{-1} \mid d \in \widetilde{\mathscr{D}}_\lambda\}$. Then for two compositions λ,μ of r, $\widetilde{\mathscr{D}}_{\lambda,\mu} = \widetilde{\mathscr{D}}_\lambda \cap \widetilde{\mathscr{D}}_\mu^{-1}$ is the set of shortest $\mathfrak{S}_\lambda$-$\mathfrak{S}_\mu$ double coset representatives in W.

We will prove the following theorem in this section.

Theorem 5.1. *For $\lambda,\mu\in\Lambda(n,r)$, let $\mathcal{B}^{(2)}_{\lambda,\mu}$ be the basis for the free $\mathscr{R}$-module $x_\lambda\mathcal{H}_{\boldsymbol{2}}(r)\cap\mathcal{H}_{\boldsymbol{2}}(r)x_\mu$ as given in Theorem 4.7.*

(1) *For $\lambda=\mu=(r)=(r,0,\ldots,0)\in\Lambda(n,r)$, write $\widetilde{\mathscr{D}}_{r,r} := \widetilde{\mathscr{D}}_{\lambda,\lambda}$. Then*

$$\mathcal{B}^{(2)}_{r,r} = \{T_{\mathfrak{S}_r d\mathfrak{S}_r} \mid d \in \widetilde{\mathscr{D}}_{r,r}\}.$$

(2) *If $\mathcal{H}_{\boldsymbol{2}}(r)$ is the group algebra of W over $\mathscr{R}$ (i. e., if $q=q_0=1$), then*

$$\mathcal{B}^{(2)}_{\lambda,\mu} = \big\{\underline{\mathfrak{S}_\lambda d\mathfrak{S}_\mu} \mid d\in\widetilde{\mathscr{D}}_{\lambda,\mu}\big\}.$$

Remark 5.2. By this result, we see the basis constructed in Theorem 4.8 for the (non-quantum) Schur$^{\text{1B}}$ algebra coincides with the standard basis for the endomorphism algebra of a permutation module [46]. Thus, both the basis in Theorem 4.8 and the basis given in [29, Thm 4.2.6] for the q-Schur$^{\text{1B}}$ algebra are quantum analogues of the classical double coset basis for the Schur$^{\text{1B}}$ algebra.

Thus, the basis constructed in Theorem 4.8 can be regarded as a generalisation of the double coset basis to the Hecke algebra of a complex reflection group.

We proceed to prove the theorem.

Let $d_0 = 1$ and, for $1\leqslant i\leqslant r$, let

$$d_i = \tau_1\cdot\tau_2\cdots\tau_i, \qquad \text{where} \quad \tau_j = s_{j-1}s_{j-2}\ldots s_1s_0. \tag{5.2.1}$$

Lemma 5.3. *We have $\widetilde{\mathscr{D}}_{r,r} = \{d_i \mid 0\leqslant i\leqslant r\}$ and $\ell(d_i) = i(i+1)/2$.*

Proof. This is clear since the double coset $\mathfrak{S}_r d_i\mathfrak{S}_r$ is the subset of $\mathfrak{S}_{2,r}$ consisting of elements satisfying the property that s_0 occurs exactly i times in every reduced expression. $\square$

[3] The (nonslim) cyclotomic q-Schur algebra in this case is called the q-Schur$^{\text{2B}}$ (or q-Schur^2) algebra; see [29].

Lemma 5.4. *For $0 \leqslant i \leqslant r$, we have*

$$\mathfrak{S}_{\nu(d_i)} := d_i^{-1}\mathfrak{S}_r d_i \cap \mathfrak{S}_r = \begin{cases} \mathfrak{S}_r, & \text{if } i = 0; \\ \mathfrak{S}_{\{1,2,\ldots,i\}} \times \mathfrak{S}_{\{i+1,i+2,\ldots,r\}}, & \text{if } i \geqslant 1. \end{cases}$$

Proof. We first observe that $\mathfrak{S}_{\nu(d_1)}$ is the subgroup of $\mathfrak{S}_r$ generated by $\{s_2, s_3, \ldots, s_{r-1}\}$. If $2 \leqslant i \leqslant r$, then, for $1 \leqslant j \leqslant r-1$,

$$d_i^{-1}s_j d_i = \begin{cases} s_{i-j}, & \text{if } j \leqslant i-1; \\ s_0 s_1 \ldots s_{i-1}s_i s_{i-1}\ldots s_1 s_0, & \text{if } j = i; \\ s_j, & \text{if } j \geqslant i+1. \end{cases} \tag{5.4.1}$$

Therefore, $\mathfrak{S}_{\nu(d_i)}$ is generated by $\{s_1, s_2, \ldots, s_{i-1}, s_{i+1}, \ldots s_{r-1}\}$, as desired. □

We now compute $\mathscr{D}_{\nu(d_i)} = \widetilde{\mathscr{D}}_{\nu(d_i)} \cap \mathfrak{S}_r$. For $2 \leqslant i \leqslant r$, let

$$\mathcal{E}_i = \{\mathbf{j} = (j_1, \ldots, j_i) \in \mathbb{N}^i \mid 1 \leqslant j_1 < \ldots < j_i \leqslant r\}.$$

Then, for each $\mathbf{j} \in \mathcal{E}_i$, we have $j_k \geqslant k$ for all $1 \leqslant k \leqslant i$.

Lemma 5.5. *We have $\mathscr{D}_{\nu(d_0)} = \{1\}$, $\mathscr{D}_{\nu(d_1)} = \{1, s_1, s_1s_2, \ldots, s_1s_2\ldots s_{r-1}\}$, and, for $2 \leqslant i \leqslant r$,*

$$\mathscr{D}_{\nu(d_i)} = \{v_{\mathbf{j}} = (s_i \ldots s_{j_i-1})(s_{i-1}\ldots s_{j_{i-1}-1})\ldots(s_1\ldots s_{j_1-1}) \mid \mathbf{j} = (j_1, \ldots, j_i) \in \mathcal{E}_i\}.$$

(If $j_k = k$, set $s_k \ldots s_{j_k-1} = 1$ by convention.)

Proof. Let $v_{\mathbf{j}}^{-1} = (s_{j_1-1}\cdots s_1)(s_{j_2-1}\cdots s_2)\cdots(s_{j_i-1}\cdots s_i)$ act on the ordered sequence $1, \ldots, r$. Then we get an ordered sequence such that the first i numbers are $j_1, \ldots, j_i$ and the remaining ones are in increasing order if and only if the number of inversions of the sequence is $\sum_{1\leqslant k\leqslant i}(j_k - 1)$, which equals the length of $v_{\mathbf{j}}^{-1}$. Hence, $v_{\mathbf{j}}^{-1}$, as well as $v_{\mathbf{j}}$, is reduced. The lemma then follows from the equalities $|\mathcal{E}_i| = |\mathscr{D}_{\nu(d_i)}| = \frac{r!}{i!(r-i)!}$ and the fact that $v_{\mathbf{j}} \neq v_{\mathbf{j}'}$ whenever $\mathbf{j} \neq \mathbf{j}'$. □

Thus, the double coset basis of $x_{(r)}\mathcal{H}_{\mathbf{2}}(r) \cap \mathcal{H}_{\mathbf{2}}(r)x_{(r)}$ consists of the following elements

$$x_{(r)}, \quad x_{(r)}T_0(T_1 + T_1T_2 + \cdots + T_1T_2\cdots T_{r-1}), \quad x_{(r)}T_{d_i}\sum_{\mathbf{j}\in\mathcal{E}_i} T_{v_{\mathbf{j}}}, \quad 2 \leqslant i \leqslant r.$$

By taking $\mathbf{m} = \mathbf{2} = (-1, q_0)$, the basis for $x_{(r)}\mathcal{H}_{\mathbf{2}}(r) \cap \mathcal{H}_{\mathbf{2}}(r)x_{(r)}$ in Proposition 3.5 becomes

$$\mathcal{B}_{r,r}^{(2)} = \{x_{(r)}\sigma_i \mid 0 \leqslant i \leqslant r\} \quad (\text{where } \sigma_0 = 1).$$

Proof of Theorem 5.1(1). It is obvious for $i = 0$ as $x_{(r)} = T_{\mathfrak{S}_r}$. Since $x_{(r)}T_i = qx_{(r)}$ for $1 \leqslant i \leqslant r-1$ and $L_i = q^{-(i-1)}T_{i-1}T_{i-2}\cdots T_1T_0T_1\cdots T_{i-1}$, it follows from Lemmas 5.5 and 4.1(2) that

$$\begin{aligned} x_{(r)}\sigma_1 &= x_{(r)}(L_1 + L_2 + \ldots + L_r) \\ &= x_{(r)}T_0(1 + T_1 + T_1T_2 + \ldots + T_1T_2\ldots T_{r-1}) = x_{(r)}T_0T_{\widetilde{\mathscr{D}}_{\nu(d_1)}\cap\mathfrak{S}_r} = T_{\mathfrak{S}_r d_1 \mathfrak{S}_r}. \end{aligned}$$

We now assume $2 \leqslant i \leqslant r$. We first claim that for $\mathbf{j} = (j_1, \ldots, j_i) \in \mathcal{E}_i$,

$$x_{(r)}T_{d_i}T_{v_{\mathbf{j}}} = x_{(r)}L_{\mathbf{j}} \quad \text{where} \quad L_{\mathbf{j}} = L_{j_1}L_{j_2}\ldots L_{j_i}.$$

Indeed, by repeatedly applying (5.4.1) or a well-known fact, we have

$$t_1t_2\cdots t_i = s_1(s_2s_1)\cdots(s_{i-1}\cdots s_1)d_i = w_{0,i}d_i,$$

where $w_{0,i}$ is the longest element of $\mathfrak{S}_{\{1,2,\dots,i\}}$, and $t_1,\dots,t_i$ are defined in (4.2.6). Thus, $x_{(r)}T_{d_i} = x_{(r)}L_1L_2\cdots L_i$. By (2.0.3), for $2 \leqslant i \leqslant r-1$,

$$\begin{aligned} x_{(r)}L_1\dots L_{i-1}L_iT_i &= x_{(r)}L_1L_2\dots L_{i-1}(L_iT_i)\\ &= x_{(r)}L_1L_2\dots L_{i-1}\big(T_iL_{i+1}+(1-q)L_{i+1}\big)\\ &= x_{(r)}\big(T_iL_1L_2\dots L_{i-1}L_{i+1}+(1-q)L_1L_2\dots L_{i-1}L_{i+1}\big)\\ &= x_{(r)}\big(qL_1L_2\dots L_{i-1}L_{i+1}+(1-q)L_1L_2\dots L_{i-1}L_{i+1}\big)\\ &= x_{(r)}L_1L_2\dots L_{i-1}L_{i+1}. \end{aligned}$$

Hence, inductively, we obtain

$$x_{(r)}L_1L_2\dots L_i(T_iT_{i+1}\dots T_{j_i-1}) = x_{(r)}L_1\dots L_{i-1}L_{j_i},\ \forall\, 2\leqslant i<j_i\leqslant r-1.$$

Since $T_{v_{\mathbf{j}}} = (T_iT_{i+1}\dots T_{j_i-1})(T_{i-1}T_i\dots T_{j_{i-1}-1})\dots(T_1T_2\dots T_{j_1-1})$, repeatedly applying the formula gives

$$x_{(r)}T_{d_i}T_{v_{\mathbf{j}}} = x_{(r)}L_{j_1}L_{j_2}\dots L_{j_i} \tag{5.5.1}$$

proving the claim. Now, by Lemmas 4.1(2) and 5.5, and the claim,

$$T_{\mathfrak{S}_rd_i\mathfrak{S}_r} = x_{(r)}T_{d_i}T_{\widetilde{\mathscr{D}}_{\nu(d_i)}\cap\mathfrak{S}_r} = \sum_{\mathbf{j}\in\mathcal{E}_i} x_{(r)}T_{d_i}T_{v_{\mathbf{j}}} = x_{(r)}\sum_{\mathbf{j}\in\mathcal{E}_i} L_{\mathbf{j}} = x_{(r)}\sigma_i, \tag{5.5.2}$$

as desired. □

For $a,b\in\mathbb{N}$ with $b\geqslant 1, a+b\leqslant r$, define the subgroups $W_b^a, \mathfrak{S}_b^a$ of $W=\mathfrak{S}_{2,r}$:

$$\begin{aligned} W_b^a &= \langle s_{a+1}, s_{a+2},\dots,s_{a+b-1},\ t_{a+1}\rangle,\\ \mathfrak{S}_b^a &= \langle s_{a+1}, s_{a+2},\dots,s_{a+b-1}\rangle, \end{aligned} \tag{5.5.3}$$

where $t_{a+1} = s_as_{a-1}\cdots s_1s_0s_1\cdots s_{a-1}s_a$. Note that W_b^a is not a parabolic subgroup of W. However, since $t_{a+1}^2 = 1$ and $t_{a+1}s_{a+1}t_{a+1}s_{a+1} = s_{a+1}t_{a+1}s_{a+1}t_{a+1}$, there is a group embedding

$$\iota:\mathfrak{S}_{2,b}\longrightarrow W,\ \ s_0\longmapsto t_{a+1}, s_i\longmapsto s_{a+i}\ (1\leqslant i\leqslant b-1), \tag{5.5.4}$$

which induces a group isomorphism $\mathfrak{S}_{2,b}\cong W_b^a$.

Note that $\mathfrak{S}_b^a$ is a parabolic subgroup of W and $\mathfrak{S}_b^a = \mathfrak{S}_{\mu_{a,b}}$, the Young subgroup associated with $\mu_{a,b} = (1^a, b, 1^{r-a-b})$. Set

$$\widetilde{\mathscr{D}}_{b,b}^a = \widetilde{\mathscr{D}}_{\mu_{a,b},\mu_{a,b}}\cap W_b^a,$$

the set of shortest double coset representatives of $\mathfrak{S}_b^a$ in W_b^a. By (5.5.4) and Lemma 5.3, we have the following result.

Lemma 5.6. *For $a,b\in\mathbb{N}$ with $b\geqslant 1, a+b\leqslant r$, we have*

$$\widetilde{\mathscr{D}}_{b,b}^a = \{d_{a,b}^{(0)}=1,\ \ d_{a,b}^{(i)} = \tau_1'\tau_2'\tau_3'\dots\tau_i' \mid 1\leqslant i\leqslant b\},$$

where $\tau_t' = s_{a+t-1}s_{a+t-2}\dots s_{a+1}t_{a+1}$ for $2\leqslant t\leqslant b$ and $\tau_1' = t_{a+1}$.

Proof. By Lemma 5.3, the elements 1, $\tau'_1\tau'_2\cdots\tau'_i$ $(1\leqslant i\leqslant b)$ form a complete set of $(\mathfrak{S}_b^a,\mathfrak{S}_b^a)$-double coset representatives in W_b^a. Since, for every s_{a+i} $(1\leqslant i<b)$, the product $s_{a+i}\tau'_1\tau'_2\cdots\tau'_i$ has length increased, we conclude that

$$\widetilde{\mathscr{D}}^a_{b,b}=\{1,\ \tau'_1\tau'_2\cdots\tau'_i\mid 1\leqslant i\leqslant b\}.$$

□

If there is no confusion, we often drop the subscripts a,b in $d^{(i)}_{a,b}$. For $0\leqslant i\leqslant b$, let $\mathfrak{S}^a_{b,\nu(d^{(i)})}=d^{(i)^{-1}}\mathfrak{S}^a_b d^{(i)}\cap\mathfrak{S}^a_b$ and let $\mathscr{D}^a_{b,\nu(d^{(i)})}$ be the set of shortest right coset representatives of $\mathfrak{S}^a_{b,\nu(d^{(i)})}$ in $\mathfrak{S}^a_b$.

Set $x^a_b=\sum_{w\in\mathfrak{S}^a_b}T_w$. For $1\leqslant i\leqslant b$, let $\sigma^a_{b,i}$ be the i-th elementary symmetric polynomial in $L_{a+1},\dots,L_{a+b}$ and set $\sigma^a_{b,0}=1$. By a similar argument at the end of the proof of Lemma 3.4, we have

$$\sigma^a_{b,i}T_j=T_j\sigma^a_{b,i},\quad\forall\,0\leqslant i\leqslant b,\ a+1\leqslant j<a+b-1.\tag{5.6.1}$$

The following lemma is an analogue to (5.5.2).

Lemma 5.7. *Let $a,b\in\mathbb{N}$ with $b\geqslant1,a+b\leqslant r$. Then for each $0\leqslant i\leqslant b$,*

$$x^a_b\sigma^a_{b,i}=q^{-ai}T_{\mathfrak{S}^a_b d^{(i)}\mathfrak{S}^a_b}=q^{-ai}x^a_bT_{d^{(i)}}\Big(\sum_{w\in\mathscr{D}^a_{b,\nu(d^{(i)})}}T_w\Big).$$

Proof. Let $\mathcal{H}(\mathfrak{S}_{2,b})$ be the Hecke algebra associated with the Coxeter group with basis $T'_w,w\in\mathfrak{S}_{2,b}$. The map ι in (5.5.4) induces an $\mathscr{R}$-module embedding

$$\widetilde{\iota}:\mathcal{H}(\mathfrak{S}_{2,b})\longrightarrow\mathcal{H}_{\boldsymbol{2}}(r),T'_w\longmapsto T_{\iota(w)}.$$

This map satisfies the property: if $w=s_{i_1}\cdots s_{i_l}$ is a reduced expression, then $\widetilde{\iota}(T'_w)=T_{\iota(w)}=T_{\iota(s_{i_1})}\cdots T_{\iota(s_{i_l})}=\widetilde{\iota}(T'_{s_{i_1}})\cdots\widetilde{\iota}(T'_{s_{i_l}})$, as $\iota(s_{i_1})\cdots\iota(s_{i_l})$ is reduced. Since $\widetilde{\iota}(L'_i)=q^aL_{a+i}$, applying $\widetilde{\iota}$ to the formula in (5.5.2) for $\mathcal{H}(\mathfrak{S}_{2,b})$ yields the required one. □

Consider a basis element $\mathfrak{b}_{\mathbb{A}}\in\mathcal{B}^{(2)}_{\lambda\mu}$ with $\mathbb{A}=(\mathbf{a}_{ij})\in\Theta_2(n,r)_{\lambda,\mu}$, where $\mathbf{a}_{ij}=(a^{(1)}_{ij},a^{(2)}_{ij})$. Then $\mathcal{J}=\{(i,j)\in\mathcal{I}\mid a^{(1)}_{ij}\neq0\}$; see (4.2.3). Note that $\mathfrak{S}_{\nu(\mathbb{A})}\subseteq\mathfrak{S}_{\nu(|\mathbb{A}|)}\subseteq\mathfrak{S}_{\mathrm{co}(|\mathbb{A}|)}\subseteq\mathfrak{S}_r$. Thus,

$$\mathscr{D}_{\nu(\mathbb{A})}\cap\mathfrak{S}_\mu=(\mathscr{D}_{\nu(\mathbb{A})}\cap\mathfrak{S}_{\nu(|\mathbb{A}|)})(\mathscr{D}_{\nu(|\mathbb{A}|)}\cap\mathfrak{S}_\mu).\tag{5.7.1}$$

Every number $\widetilde{a}_{ij}$ defined in (4.2.5) defines a subgroup $W^{\widetilde{a}_{ij}}_{|\mathbf{a}_{ij}|}$ of W (by convention, $W^{\widetilde{a}_{ij}}_{|\mathbf{a}_{ij}|}=1$ if $|\mathbf{a}_{ij}|=0$), as well as the subgroup $\mathfrak{S}^{\widetilde{a}_{ij}}_{|\mathbf{a}_{ij}|}$ of $W^{\widetilde{a}_{ij}}_{|\mathbf{a}_{ij}|}$. By applying Lemma 5.6 to the case $a=\widetilde{a}_{ij}$ and $b=|\mathbf{a}_{ij}|$, we obtain an element

$$d_{ij}:=d^{(a^{(1)}_{ij})}_{\widetilde{a}_{ij},|\mathbf{a}_{ij}|}\in\widetilde{\mathscr{D}}^{\widetilde{a}_{ij}}_{|\mathbf{a}_{ij}|,|\mathbf{a}_{ij}|}.$$

Proposition 5.8. *For arbitrary $\lambda,\mu\in\Lambda(n,r)$, $\mathbb{A}\in\Theta_m(n,r)_{\lambda\mu}$ and $\mathcal{J}=\mathcal{J}(\mathbb{A})$, we have*

$$\mathfrak{b}_{\mathbb{A}}=\Big(\prod_{(i,j)\in\mathcal{J}}q^{-\widetilde{a}_{ij}a^{(1)}_{ij}}\Big)x_\lambda T_{d_{|\mathbb{A}|}}\prod^{(\preceq)}_{(i,j)\in\mathcal{J}}T_{d_{ij}}\sum_{w\in\mathscr{D}_{\nu(\mathbb{A})}\cap\mathfrak{S}_\mu}T_w,$$

where the product of $T_{d_{ij}}$ is taken over the order $\preceq$ given in (4.2.4).

Proof. Let $d = d_{|\mathbb{A}|}$. Then $\nu(d) = \nu(|\mathbb{A}|)$ satisfies $\mathfrak{S}_{\nu(|\mathbb{A}|)} = d^{-1}\mathfrak{S}_\lambda d \cap \mathfrak{S}_\mu$. By (4.1.1) and (4.6.1),

$$\mathfrak{b}_{\mathbb{A}} = x_\lambda T_d \sigma^{\ddot{\mathbb{A}}} \sum_{v\in\mathscr{D}_{\nu(d)}\cap\mathfrak{S}_\mu} T_v = \sum_{u\in\mathscr{D}^{-1}_{\nu(d^{-1})}\cap\mathfrak{S}_\lambda} T_u T_d x_{\nu(d)} \sigma^{\ddot{\mathbb{A}}} \sum_{v\in\mathscr{D}_{\nu(d)}\cap\mathfrak{S}_\mu} T_v. \tag{5.8.1}$$

Then, by the definition,

$$x_{\nu(d)} = \prod_{i,j=1}^{n} x^{\widetilde{a}_{ij}}_{|\mathbf{a}_{ij}|}.$$

Moreover, the terms $x^{\widetilde{a}_{ij}}_{|\mathbf{a}_{ij}|}$ commute with each other. Also, by definition,

$$x_{\nu(d)}\sigma^{\ddot{\mathbb{A}}} = \prod_{i,j=1}^{n} x^{\widetilde{a}_{ij}}_{|\mathbf{a}_{ij}|}\sigma^{\ddot{\mathbf{a}}_{ij}}. \tag{5.8.2}$$

Since $\ddot{\mathbf{a}}_{ij} \in \Lambda(|\mathbf{a}_{ij}|, {<}2)$ is computed by the partition dual to $(a^{(1)}_{ij})$, it follows that

$$\ddot{\mathbf{a}}_{ij} = \begin{cases} (0,\dots,0,1,0,\dots,0), & \text{if } a^{(1)}_{ij} \neq 0;\\ (0,\dots,0), & \text{otherwise,}\end{cases}$$

where the 1 is at the $a^{(1)}_{ij}$-th position.

Note that, if $(i,j) \in \mathcal{J}$, then $\sigma^{\ddot{\mathbf{a}}_{ij}} = \sigma^{\widetilde{a}_{ij}}_{|\mathbf{a}_{ij}|,a^{(1)}_{ij}}$ is the $a^{(1)}_{ij}$-th elementary symmetric polynomial in $L_{\widetilde{a}_{ij}+1}, L_{\widetilde{a}_{ij}+2}, \dots, L_{\widetilde{a}_{ij}+|\mathbf{a}_{ij}|}$. We set $\sigma^{\ddot{\mathbf{a}}_{ij}} = 1$ for $(i,j) \notin \mathcal{J}$. By Lemma 5.7, for $(i,j) \in \mathcal{J}$,

$$x^{\widetilde{a}_{ij}}_{|\mathbf{a}_{ij}|}\sigma^{\ddot{\mathbf{a}}_{ij}} = x^{\widetilde{a}_{ij}}_{|\mathbf{a}_{ij}|}\sigma^{\widetilde{a}_{ij}}_{|\mathbf{a}_{ij}|,a^{(1)}_{ij}} = q^{-\widetilde{a}_{ij}a^{(1)}_{ij}} \cdot x^{\widetilde{a}_{ij}}_{|\mathbf{a}_{ij}|} T_{d_{ij}} \sum_{x\in\mathscr{D}_{ij}} T_x,$$

where $\mathscr{D}_{ij} = \mathscr{D}^{\widetilde{a}_{ij}}_{|\mathbf{a}_{ij}|,\nu(d_{ij})}$ is the set of shortest right coset representatives of

$$\mathfrak{S}_{\nu(d_{ij})} = \mathfrak{S}_{(1^{\widetilde{a}_{ij}},\mathbf{a}_{ij},1,\dots)} = d^{-1}_{ij}\mathfrak{S}^{\widetilde{a}_{ij}}_{|\mathbf{a}_{ij}|} d_{ij} \cap \mathfrak{S}^{\widetilde{a}_{ij}}_{|\mathbf{a}_{ij}|} \quad \text{in} \quad \mathfrak{S}^{\widetilde{a}_{ij}}_{|\mathbf{a}_{ij}|}.$$

By substituting into (5.8.2), we obtain that

$$\begin{aligned} x_{\nu(d)}\sigma^{\ddot{\mathbb{A}}} &= \prod_{(i,j)\in\mathcal{I}}^{(\preceq)} \Big(q^{-\widetilde{a}_{ij}a^{(1)}_{ij}} \cdot x^{\widetilde{a}_{ij}}_{|\mathbf{a}_{ij}|} T_{d_{ij}} \big(\sum_{x\in\mathscr{D}_{ij}} T_x \big) \Big) \\ &= \Big(\prod_{(i,j)\in\mathcal{J}} q^{-\widetilde{a}_{ij}a^{(1)}_{ij}} \Big) x_{\nu(d)} \prod_{(i,j)\in\mathcal{J}}^{(\preceq)} T_{d_{ij}} \prod_{(i,j)\in\mathcal{J}}^{(\preceq)} \big(\sum_{x\in\mathscr{D}_{ij}} T_x \big), \end{aligned}$$

where $(\preceq)$ indicates the products are taken over the order $\preceq$ defined in (4.2.4). The second equality is seen from the fact that if $(k,l) \preceq (i,j)$, then $T_{d_{kl}}\big(\sum_{x\in\mathscr{D}_{kl}} T_x\big)$

commutes with $x^{\widetilde{a}_{ij}}_{|\mathbf{a}_{ij}|}$, and $\sum_{x\in\mathscr{D}_{kl}} T_x$ with $T_{d_{ij}}$. Then (5.8.1) becomes

$$\begin{aligned}\mathfrak{b}_{\mathbb{A}} &= \sum_{u\in\mathscr{D}^{-1}_{\nu(d^{-1})}\cap\mathfrak{S}_\lambda} T_uT_dx_{\nu(d)}\sigma^{\ddot{\mathbb{A}}}\sum_{v\in\mathscr{D}_{\nu(d)}\cap\mathfrak{S}_\mu} T_v\\ &= \sum_{u\in\mathscr{D}^{-1}_{\nu(d^{-1})}\cap\mathfrak{S}_\lambda} T_uT_dx_{\nu(d)}\prod_{i,j=1}^n q^{-\widetilde{a}_{ij}a^{(1)}_{ij}}T_{d_{ij}}\prod_{i,j=1}^n\big(\sum_{x\in\mathscr{D}_{ij}}T_x\big)\sum_{v\in\mathscr{D}_{\nu(d)}\cap\mathfrak{S}_\mu} T_v \\ &= \Big(\prod_{i,j=1}^n q^{-\widetilde{a}_{ij}a^{(1)}_{ij}}\Big)x_\lambda T_d\prod_{i,j=1}^n T_{d_{ij}}\prod_{i,j=1}^n\big(\sum_{x\in\mathscr{D}_{ij}}T_x\big)\sum_{v\in\mathscr{D}_{\nu(d)}\cap\mathfrak{S}_\mu} T_v.\end{aligned}\tag{5.8.3}$$

By Lemma 4.3,

$$\prod_{i,j=1}^n\big(\sum_{x\in\mathscr{D}_{ij}}T_x\big)=\sum_{w\in\mathscr{D}_{\nu(\mathbb{A})}\cap\mathfrak{S}_{\nu(|\mathbb{A}|)}}T_w,$$

it follows by (5.7.1) that

$$\prod_{i,j=1}^n\big(\sum_{x\in\mathscr{D}_{\nu(d_{ij})}}T_x\big)\sum_{v\in\mathscr{D}_{\nu(d)}\cap\mathfrak{S}_\mu}T_v=\sum_{w\in\mathscr{D}_{\nu(\mathbb{A})}\cap\mathfrak{S}_{\nu(|\mathbb{A}|)}}T_w\sum_{v\in\mathscr{D}_{\nu(|\mathbb{A}|)}\cap\mathfrak{S}_\mu}T_v=\sum_{w\in\mathscr{D}_{\nu(\mathbb{A})}\cap\mathfrak{S}_\mu}T_w.$$

Substituting into (5.8.3) gives the desired formula. □

Proof of Theorem 5.1(2). Let w be the element given in (4.3.1) and assume that $d_{\mathbb{A}}$ is the shortest representative of $\mathfrak{S}_\lambda w\mathfrak{S}_\mu$. We claim that if $\ddot{d}=d\cdot\prod^{(\preceq)}_{(i,j)\in\mathcal{J}}d_{ij}$, then $\mathfrak{S}_\lambda\ddot{d}\mathfrak{S}_\mu=\mathfrak{S}_\lambda w\mathfrak{S}_\mu$. Indeed, for $a=\widetilde{a}_{ij}$, $b=|a_{ij}|$ and $c=a^{(1)}_{ij}>0$, we have $d_{ij}=1=t_{ij}$ if $c=0$, $d_{ij}=t_{a+1}=t_{ij}$ if $c=1$, and, for $2\leqslant c\leqslant b$

$$\begin{aligned}d_{ij}&=t_{a+1}\cdot(s_{a+1}t_{a+1})\cdot(s_{a+2}s_{a+1}t_{a+1})\cdot\cdots\cdot(s_{a+c-1}s_{a+c-2}\cdots s_{a+1}t_{a+1})\\ &=t_{a+1}\cdot(t_{a+2}s_{a+1})\cdot(t_{a+3}s_{a+2}s_{a+1})\cdot\cdots\cdot(t_{a+c}s_{a+c-1}s_{a+c-2}\cdots s_{a+1})\\ &=t_{ij}\cdot s_{a+1}\cdot(s_{a+2}s_{a+1})\cdot(s_{a+3}s_{a+2}s_{a+1})\cdots(s_{a+c-1}s_{a+c-2}\cdots s_{a+1}),\end{aligned}$$

(thus, $d_{ij}d_{kl}=d_{kl}d_{ij}$) proving the claim. By Proposition 5.8, we have in the group algebra $\mathfrak{b}_{\mathbb{A}}=\underline{\mathfrak{S}_\lambda}\cdot\ddot{d}\cdot\underline{\mathscr{D}_{\nu(\mathbb{A})}\cap\mathfrak{S}_\mu}=\underline{\mathfrak{S}_\lambda}\cdot d_{\mathbb{A}}\cdot\underline{\mathscr{D}_{\nu(\mathbb{A})}\cap\mathfrak{S}_\mu}=\underline{\mathfrak{S}_\lambda d_{\mathbb{A}}\mathfrak{S}_\mu}$.
□

Example 5.9. Let $n=2$ and $r=3$. For the matrix

$$\mathbb{A}=\begin{pmatrix}(0,0)&(1,0)\\(1,1)&(0,0)\end{pmatrix}\in\Theta_2(2,3)\quad\text{with }\ |\mathbb{A}|=\begin{pmatrix}0&1\\2&0\end{pmatrix},$$

we have $\lambda=\mathrm{ro}(|\mathbb{A}|)=(1,2)$, $\mu=\mathrm{co}(|\mathbb{A}|)=(2,1)$, and $\nu(|\mathbb{A}|)=(0,2,1,0)$. Then $\mathfrak{S}_\lambda=\{1,s_2\}$, $\mathfrak{S}_\mu=\{1,s_1\}$, $d_{|\mathbb{A}|}=s_1s_2$, and

$$\ddot{d}=d_{|\mathbb{A}|}d_{21}d_{12}=s_1s_2t_1t_3=s_0s_1s_0s_2.$$

But

$$T_{|\mathbb{A}|}T_{t_1}T_{t_3}=q^2T_{0102}+q(q-1)T_{10102}+(q-1)T_{1201012},$$

where we write $T_w = T_{i_1\cdots i_l}$ whenever $w = s_{i_1}\cdots s_{i_l}$ is reduced. Note that $d_{\mathbb{A}} = s_0s_1s_0s_2$ with $\nu(d_{\mathbb{A}}) = \nu(\mathbb{A}) = (0,0,1,1,1,0,0,0)$. By Proposition 5.8,

$$\begin{aligned}\mathfrak{b}_{\mathbb{A}} &= q^{-2}x_\lambda T_{|\mathbb{A}|}T_{t_1}T_{t_3}x_\mu\\ &= x_\lambda T_{0102}x_\mu + q^{-1}(q-1)x_\lambda T_{10102}x_\mu + q^{-1}(q-1)x_\lambda T_{120102}x_\mu\\ &= T_{\mathfrak{S}_\lambda d_{\mathbb{A}}\mathfrak{S}_\mu} + q^{-1}(q-1)T_{\mathfrak{S}_\lambda d'\mathfrak{S}_\mu} + q^{-1}(q-1)T_{\mathfrak{S}_\lambda d''\mathfrak{S}_\mu},\end{aligned}$$

where $d' = s_1s_0s_1s_0s_2$ and $d'' = s_1s_2s_0s_1s_0s_2$.

Remark 5.10. The slim cyclotomic q-Schur algebra $\mathcal{S}_{\mathbf{2}}(n,r)$ is isomorphic to the hyperoctahedral Schur algebras (defined over $\mathscr{R} = \mathbb{Z}[q,q^{-1},Q,Q^{-1}]$ in indeterminates q,Q); see [37, Defs 3.2.1, 4.3.3] and [37, Lem 4.3.4]. Note that our labelling $0,1,2,\ldots,r-1$ of the type B/C Dynkin diagram corresponds to their labelling $r,\ldots,2,1$. The isomorphism can be induced from this correspondence. By [7, Rem. 6.3], the type C Schur algebra $\mathbf{S}^\imath$ geometrically defined in [7, §6.1] is isomorphic to the hyperoctahedral Schur algebra. Hence, $\mathcal{S}_{\mathbf{2}}(n,r)$ is isomorphic to $\mathbf{S}^\imath$.

6. The cyclotomic Schur–Weyl duality: double centraliser property

Let $C = (c_{ij})_{n\times n}$ be the Cartan matrix of affine type A; see, e. g., [13, (1.2.3.1)].

Definition 6.1. Let $\mathbf{U}_v(\widehat{\mathfrak{sl}}_n)$ be the quantum enveloping algebra over the field $\mathbb{Q}(v)$ of rational functions in indeterminate v associated with the Cartan matrix C. Then it can be presented by the generators $E_i,\ F_i,\ K_i,\ K_i^{-1}$ $(1\leqslant i\leqslant n)$ and the relations (QGL1)–(QGL5) in [13, Thm 2.3.1].[4] Define the quantum affine $\mathfrak{gl}_n$ by

$$\mathbf{U}_v(\widehat{\mathfrak{gl}}_n) = \mathbf{U}_v(\widehat{\mathfrak{sl}}_n)\otimes\mathbb{Q}(v)[\mathsf{c}_1^+,\mathsf{c}_1^-,\mathsf{c}_2^+,\mathsf{c}_2^-,\ldots,\mathsf{c}_s^+,\mathsf{c}_s^-,\ldots],$$

where $\mathsf{c}_s^+,\mathsf{c}_s^-$ $(s\in\mathbb{Z}^+)$ are commuting indeterminates.

The algebra $\mathbf{U}_v(\widehat{\mathfrak{gl}}_n)$ is also a Hopf algebra with the structure maps Δ, ε, and σ acting on the $\widehat{\mathfrak{sl}}_n$ part as usual plus the following extra action on the $\mathsf{c}_s^\pm$:

$$\Delta(\mathsf{c}_s^\pm) = \mathsf{c}_s^\pm\otimes 1 + 1\otimes\mathsf{c}_s^\pm,\quad \varepsilon(\mathsf{c}_s^\pm) = 0,\quad \sigma(\mathsf{c}_s^\pm) = -\mathsf{c}_s^\pm.$$

This Hopf algebra is also isomorphic to the quantum loop algebra of $\mathfrak{gl}_n$, known as the Drinfeld new realisation in [22]; see [13, Thm 2.5.3].

The quantum group $\mathbf{U}_v(\widehat{\mathfrak{gl}}_n)$ has a second construction in terms of a double Ringel–Hall algebra. This new construction is crucial in the study of the (enhanced) affine quantum Schur–Weyl duality as presented in [13]. In this section, we investigate new applications of this theory to its cyclotomic counterpart. We first briefly review the definition of the double Ringel–Hall algebra associated with a cyclic quiver.

Let $\triangle = \triangle(n)$ $(n\geqslant 2)$ be the cyclic quiver with vertex set $I = \mathbb{Z}/n\mathbb{Z}$ and arrow set $\{i\longrightarrow i+1\}_{i\in I}$. Let $\mathfrak{H}_\triangle(n) = \mathrm{span}\{u_A \mid A\in\Theta_\triangle^+(n)\}$ be the Ringel–Hall algebra over $\mathcal{Z} := \mathbb{Z}[v,v^{-1}]$ associated with $\triangle(n)$ and let $\boldsymbol{\mathfrak{H}}_\triangle(n) = \mathfrak{H}_\triangle(n)\otimes_{\mathcal{Z}}\mathbb{Q}(v)$. There are extended Ringel–Hall algebras over $\mathbb{Q}(v)$

$$\boldsymbol{\mathfrak{H}}_\triangle(n)^{\geqslant 0} = \boldsymbol{\mathfrak{H}}_\triangle(n)\otimes\mathbb{Q}(v)[K_1^{\pm1},\ldots,K_n^{\pm1}]$$

[4]The quantum enveloping algebra $\mathbf{U}_v(\widehat{\mathfrak{sl}}_n)$ here is called the extended quantum affine $\mathfrak{sl}_n$ defined in [13, p. 43]. The non-extended version is defined in [13, §1.3].

and, similarly, $\boldsymbol{\mathfrak{H}}_\triangle(n)^{\leqslant 0}$ which possess the Green–Xiao Hopf algebra structure. By considering a certain skew–Hopf pairing $\psi : \boldsymbol{\mathfrak{H}}_\triangle(n)^{\geqslant 0} \times \boldsymbol{\mathfrak{H}}_\triangle(n)^{\leqslant 0} \to \mathbb{Q}(v)$, one defines the Drinfeld double $\widehat{\boldsymbol{\mathfrak{D}}_\triangle(n)}$ as the quotient algebra of the free product $\boldsymbol{\mathfrak{H}}_\triangle(n)^{\geqslant 0} * \boldsymbol{\mathfrak{H}}_\triangle(n)^{\leqslant 0}$ by an ideal defined by ψ; see [13, §2.1]. The *double Ringel–Hall algebra* is a reduced version of $\widehat{\boldsymbol{\mathfrak{D}}_\triangle(n)}$:

$$\boldsymbol{\mathfrak{D}}_\triangle(n) = \widehat{\boldsymbol{\mathfrak{D}}_\triangle(n)}/\mathscr{I} \cong \boldsymbol{\mathfrak{H}}_\triangle^+(n) \otimes \mathbb{Q}(v)[K_1^{\pm 1}, \dots, K_n^{\pm 1}] \otimes \boldsymbol{\mathfrak{H}}_\triangle^-(n)$$

where $\mathscr{I}$ denotes the ideal generated by $1 \otimes K_\alpha - K_\alpha \otimes 1$ for all $\alpha \in \mathbb{Z}I$ and $\mathfrak{H}_\triangle^+(n) = \mathfrak{H}_\triangle(n)$, $\mathfrak{H}_\triangle^-(n) = \mathfrak{H}_\triangle(n)^{\mathrm{op}}$.

Each vertex $i \in I$ of $\triangle(n)$ defines a simple representation which in turn defines generators u_i^+ and u_i^-. There are so-called Schiffmann–Hubery generators z_s^+ and z_s^- for $\boldsymbol{\mathfrak{D}}_\triangle(n)$ constructed in [13, §2.3]. By [13, Thm 2.3.1], there is a Hopf algebra isomorphism from $\mathbf{U}_v(\widehat{\mathfrak{gl}}_n)$ to $\boldsymbol{\mathfrak{D}}_\triangle(n)$ given by

$$E_i \longmapsto u_i^+,\ F_i \longmapsto u_i^-,\ K_i^{\pm} \longmapsto K_i^{\pm},\ \mathsf{c}_s^{\pm} \longmapsto \mathsf{z}_s^{\pm}.$$

We identify the two algebras under this isomorphism.

Let Ω be a free $\mathcal{Z}$-module with basis $\{\omega_i \mid i \in \mathbb{Z}\}$ and let $\boldsymbol{\Omega} = \Omega \otimes_{\mathcal{Z}} \mathbb{Q}(v)$. Using the action given in [13, (3.5.0.1)], $\boldsymbol{\Omega}$ becomes a left $\boldsymbol{\mathfrak{D}}_\triangle(n)$-module. Hence, the Hopf algebra structure on $\boldsymbol{\mathfrak{D}}_\triangle(n)$ induces a $\boldsymbol{\mathfrak{D}}_\triangle(n)$-module structure on $\boldsymbol{\Omega}^{\otimes r}$. On the other hand, $\Omega^{\otimes r}$ has a right $\mathcal{H}_\triangle(r)_{\mathcal{Z}}$-module structure; see [13, (3.3.0.4)], commuting with the left $\boldsymbol{\mathfrak{D}}_\triangle(n)$-module structure. Thus, we obtain a $\mathbb{Q}(v)$-algebra homomorphism

$$\xi_r : \boldsymbol{\mathfrak{D}}_\triangle(n) \longrightarrow \mathrm{End}_{\mathcal{H}_\triangle(r)_{\mathbb{Q}(v)}}(\boldsymbol{\Omega}^{\otimes r}) = \mathcal{S}_\triangle(n,r)_{\mathbb{Q}(v)}. \tag{6.1.1}$$

Let $\mathfrak{D}_\triangle(n)^0_{\mathcal{Z}}$ denote the $\mathcal{Z}$-subalgebra of $\boldsymbol{\mathfrak{D}}_\triangle(n)$ generated by $K_i^{\pm 1}$ and $\left[{K_i;0 \atop t}\right]$ for $i \in I$ and $t > 0$ and put $\mathfrak{D}_\triangle(n)^{\pm}_{\mathcal{Z}} = \mathfrak{H}_\triangle(n)^{\pm}$; see [13, §2.4] for the details.

Theorem 6.2. *Maintain the notations introduced above.*

(1) *The $\mathcal{Z}$-submodule*

$$\mathfrak{D}_\triangle(n)_{\mathcal{Z}} := \mathfrak{H}_\triangle(n)^+ \mathfrak{D}_\triangle(n)^0 \mathfrak{H}_\triangle(n)^- \cong \mathfrak{H}_\triangle(n)^+ \otimes \mathfrak{D}_\triangle(n)^0 \otimes \mathfrak{H}_\triangle(n)^-$$

is a Hopf $\mathcal{Z}$-subalgebra of $\boldsymbol{\mathfrak{D}}_\triangle(n)$.

(2) *There is a $\mathcal{Z}$-algebra isomorphism $\mathcal{S}_\triangle(n,r)_{\mathcal{Z}} \cong End_{\mathcal{H}_\triangle(r)_{\mathcal{Z}}}(\Omega^{\otimes r})$.*

(3) *The restriction of ξ_r to $\mathfrak{D}_\triangle(n)_{\mathcal{Z}}$ induces a $\mathcal{Z}$-algebra epimorphism*

$$\xi_r : \mathfrak{D}_\triangle(n)_{\mathcal{Z}} \longrightarrow End_{\mathcal{H}_\triangle(r)_{\mathcal{Z}}}(\Omega^{\otimes r}).$$

(4) *The map ξ_r induces an $\mathscr{R}$-algebra epimorphism*

$$\xi_{r,\mathscr{R}} : \mathfrak{D}_\triangle(n)_{\mathcal{Z}} \otimes \mathscr{R} \longrightarrow \mathcal{S}_\triangle(n,r)_{\mathscr{R}}.$$

Note that (1) is done in [25]; (2) is given in [13, Prop. 3.3.1]; (3) can be derived from [13, Thm 3.7.7] and the proof of [13, Thm 3.8.1], and (4) follows from (2). Note also that an explicit action of the generators of $\mathfrak{D}_\triangle(n)_{\mathcal{Z}}$ on Ω can be found in [13, Prop. 3.5.3].

We now extend the algebra homomorphism to slim cyclotomic q-Schur algebras. Recall the canonical epimorphism $\epsilon_{\mathbf{m}} : \mathcal{H}_\triangle(r) \to \mathcal{H}_{\mathbf{m}}(r)$ given in (2.0.1) under the assumption that all the $u_i \in \mathscr{R}$ in $\mathbf{m}$ are invertible. Regarding $\mathcal{H}_{\mathbf{m}}(r)$ as an $\mathcal{H}_{\vartriangle}(r)$-module via this map, we obtain the canonical right $\mathcal{H}_{\mathbf{m}}(r)$-module isomorphism

$$\mathcal{T}_{\vartriangle}(n,r)\otimes_{\mathcal{H}_{\vartriangle}(r)}\mathcal{H}_{\mathbf{m}}(r)=\bigoplus_{\lambda\in\Lambda(n,r)}x_\lambda\mathcal{H}_{\vartriangle}(r)\otimes_{\mathcal{H}_{\vartriangle}(r)}\mathcal{H}_{\mathbf{m}}(r)\cong\bigoplus_{\lambda\in\Lambda(n,r)}x_\lambda\mathcal{H}_{\mathbf{m}}(r)=\mathcal{T}_{\mathbf{m}}(n,r).$$

Thus, we obtain an $\mathscr{R}$-algebra homomorphism

$$\widetilde{\epsilon}_{\mathbf{m}}:\mathcal{S}_{\vartriangle}(n,r)\longrightarrow\mathcal{S}_{\mathbf{m}}(n,r),\ f\longmapsto f\otimes\mathrm{id}.$$

Proposition 6.3. *Suppose that all the $u_i \in \mathscr{R}$ are invertible. The $\mathscr{R}$-algebra homomorphism $\widetilde{\epsilon}_{\mathbf{m}}$ is surjective. In particular, the map $\xi_{r,\mathscr{R}}$ extends to the $\mathscr{R}$-algebra epimorphism*

$$\widetilde{\epsilon}_{\mathbf{m}}\circ\xi_{r,\mathscr{R}}:\mathfrak{D}_{\vartriangle}(n)_{\mathcal{Z}}\otimes\mathscr{R}\longrightarrow\mathcal{S}_{\mathbf{m}}(n,r)_{\mathscr{R}}.$$

Proof. Consider the linearly independent elements $\Phi^{\vartriangle}_{\mathbb{A}}$, $\mathbb{A}\in\Theta_m(n,r)$, for $\mathcal{S}_{\vartriangle}(n,r)$, which are defined similarly to the basis elements $\Phi_{\mathbb{A}}$ with all L_i replaced by X_i; see Remarks 3.3, 3.9 and 4.9. By definition, we see easily that $\widetilde{\epsilon}_{\mathbf{m}}(\Phi^{\vartriangle}_{\mathbb{A}})=\Phi_{\mathbb{A}}$. In other words, the images of these elements form a basis for $\mathcal{S}_{\mathbf{m}}(n,r)$. Hence, $\widetilde{\epsilon}_{\mathbf{m}}$ is surjective. □

Remarks 6.4. (1) The surjectivity of $\widetilde{\epsilon}_{\mathbf{m}}$ was given in [42, Prop. 5.7(a)].

(2) For any $t\geqslant 0$, let $\mathfrak{D}_{\vartriangle}(n)^{(t)}=\mathbf{U}_v(\widehat{\mathfrak{sl}}_n)\otimes\mathbb{Q}(v)[\mathsf{c}_1^+,\mathsf{c}_1^-,\mathsf{c}_2^+,\mathsf{c}_2^-,\dots,\mathsf{c}_t^+,\mathsf{c}_t^-]$. This is a Hopf subalgebra of $\mathfrak{D}_{\vartriangle}(n)$. In fact, by [13, Thm 3.8.1(2)], if t satisfies $r=tn+t_0$, where $0\leqslant t_0<n$, then the restriction of the map ξ_r in (6.1.1) gives an epimorphism

$$\xi_r^{(t)}:\mathfrak{D}_{\vartriangle}(n)^{(t)}\longrightarrow\mathcal{S}_{\vartriangle}(n,r)_{\mathbb{Q}(v)}.$$

This together with $\widetilde{\epsilon}_{\mathbf{m}}$ induces an epimorphism

$$\xi_{r,\mathbf{m}}^{(t)}:\mathfrak{D}_{\vartriangle}(n)^{(t)}\longrightarrow\mathcal{S}_{\mathbf{m}}(n,r)_{\mathbb{Q}(v)}.$$

It would be interesting to conjecture that there exists a t, independent of r, such that $\xi_{r,\mathbf{m}}^{(t)}$ is surjective for all $r\geqslant 0$.

Note that, if $\mathbf{U}_v(\mathfrak{gl}_n)$ denotes the subalgebra generated by $E_i,F_i,K_j^{\pm1}$ for $1\leqslant i,j\leqslant n, i\neq n$, then $\mathfrak{D}_{\vartriangle}(n)^{(t)}$ contains the subalgebra

$$\mathbf{U}_v^{(t)}(\mathfrak{gl}_n):=\mathbf{U}_v(\mathfrak{gl}_n)\otimes\mathbb{Q}(v)[\mathsf{c}_1^+,\mathsf{c}_1^-,\mathsf{c}_2^+,\mathsf{c}_2^-,\dots,\mathsf{c}_t^+,\mathsf{c}_t^-].$$

If the conjecture were true and the restriction of $\xi_{r,\mathbf{m}}^{(t)}$ to $\mathbf{U}_v^{(t)}(\mathfrak{gl}_n)$ remained surjective, then the new object $\mathbf{U}_v^{(t)}(\mathfrak{gl}_n)$ would be called the *cyclotomic quantum* $\mathfrak{gl}_n$.

(3) We will prove in [16] that if $\mathcal{H}_{\mathbf{m}}(r)$ has a semisimple bottom in the sense of [27], then the algebra homomorphism

$$\xi_{r,\mathbf{m}}^{\vee}:\mathcal{H}_{\mathbf{m}}(r)\longrightarrow\mathrm{End}_{\mathcal{S}_{\mathbf{m}}(n,r)}(\Omega^{\otimes r})$$

is surjective. So it is natural to conjecture that the cyclotomic double centraliser property holds in general.

7. The cyclotomic Schur–Weyl duality: Morita equivalence

As part of the Schur–Weyl duality, it is well-known that, for any field $\mathscr{K}$, the q-Schur algebra $\mathcal{S}_q(n,r)_{\mathscr{K}}$ is Morita equivalent to the Hecke algebra $\mathcal{H}_q(r)_{\mathscr{K}}$ if $n \geqslant r$ and q is not a root of the Poincaré polynomial of $\mathfrak{S}_r$. We now establish a similar result for the cyclotomic counterpart.

As in previous sections, $\mathscr{R}$ is a commutative ring and $q \in \mathscr{R}$ is invertible. Recall the notations $\mathcal{H}_{\mathbf{m}}(r) = \mathcal{H}_{\mathbf{m}}(r)_{\mathscr{R}}$ and $\mathcal{S}_{\mathbf{m}}(n,r) = \mathcal{S}_{\mathbf{m}}(n,r)_{\mathscr{R}}$ and from Proposition 4.10 that $\mathcal{S}_q(n,r)$ is (isomorphic to) a subalgebra of $\mathcal{S}_{\mathbf{m}}(n,r)$.

Every $\lambda = (\lambda_1,\dots,\lambda_n) \in \Lambda(n,r)$ gives a diagonal matrix $D_\lambda = \mathrm{diag}(\lambda)$ which in turn defines a matrix $D_\lambda^{(m)} \in \Theta_m(n,r)$ via the embedding $\iota^{(m)}$ in (4.9.1). Then $\Phi_{D_\lambda^{(m)}} = \mathfrak{l}_\lambda$ is the idempotent map defined in (2.3.1). Note that all $\mathfrak{l}_\lambda$ live in the q-Schur subalgebra $\mathcal{S}_q(n,r)$ of $\mathcal{S}_{\mathbf{m}}(n,r)$; see Proposition 4.10. In particular, $\mathfrak{l}_\lambda^2 = \mathfrak{l}_\lambda$, $\mathfrak{l}_\lambda\mathfrak{l}_\mu = \delta_{\lambda,\mu}\mathfrak{l}_\lambda$, for $\lambda,\mu \in \Lambda(n,r)$ and

$$\sum_{\lambda\in\Lambda(n,r)} \mathfrak{l}_\lambda = 1. \tag{7.0.1}$$

Moreover, like the q-Schur algebra case, we have, for each $\mathbb{A} \in \Theta_m(n,r)$, the following relations in $\mathcal{S}_{\mathbf{m}}(n,r)$

$$\Phi_{\mathbb{A}}\mathfrak{l}_\lambda = \begin{cases} \Phi_{\mathbb{A}}, & \text{if } \lambda = \mathrm{co}(|\mathbb{A}|); \\ 0, & \text{otherwise} \end{cases} \qquad \text{and} \quad \mathfrak{l}_\lambda\Phi_{\mathbb{A}} = \begin{cases} \Phi_{\mathbb{A}}, & \text{if } \lambda = \mathrm{ro}(|\mathbb{A}|); \\ 0, & \text{otherwise .} \end{cases}$$

In the rest of the section, we assume $n \geqslant r$ and let $\omega = (\underbrace{1,\dots,1}_{r},0,\dots,0) \in \Lambda(n,r)$.

Then $\mathcal{H}_{\mathbf{m}}(r)$ is a centraliser algebra of $\mathcal{S}_{\mathbf{m}}(n,r)$ and (2.3.2) becomes

$$\mathfrak{l}_\omega\mathcal{S}_{\mathbf{m}}(n,r)\mathfrak{l}_\omega \cong \mathcal{H}_{\mathbf{m}}(r).$$

We now consider some particular basis elements given in Theorem 4.8. For $\lambda,\mu \in \Lambda(n,r)$, let $\Phi_{\lambda,\mu} = \Phi_{A^{(m)}} \in \mathcal{S}_{\mathbf{m}}(n,r)$, where $A = \theta(\lambda,1,\mu)$ as defined in (4.1.2). In other words,

$$A = (|R_i^\lambda \cap R_j^\mu|)_{1\leqslant i,j\leqslant n}$$

where R_i^λ, R_j^μ are defined in (4.1.3). Thus, by definition,

$$\Phi_{\lambda,\mu}(x_\nu h) = \delta_{\nu,\mu} x_\lambda \sum_{w\in\mathscr{D}_{\lambda\cap\mu}\cap\mathfrak{S}_\mu} T_w h, \quad \forall h \in \mathcal{H}_{\mathbf{m}}(r), \tag{7.0.2}$$

where $\mathfrak{S}_{\lambda\cap\mu} = \mathfrak{S}_\lambda \cap \mathfrak{S}_\mu$. In particular, if $\lambda = \mu = \nu$, then

$$\Phi_{\lambda,\lambda}(x_\lambda) = x_\lambda = \mathfrak{l}_\lambda(x_\lambda) \text{ or } \Phi_{\lambda,\lambda} = \mathfrak{l}_\lambda. \tag{7.0.3}$$

For each $\lambda = (\lambda_1,\dots,\lambda_n) \in \Lambda(n,r)$, let

$$P_\lambda(q) = P_{\mathfrak{S}_\lambda}(q) = \sum_{w\in\mathfrak{S}_\lambda} q^{\ell(w)}.$$

Then $x_\lambda^2 = P_\lambda(q)x_\lambda$ for $\lambda \in \Lambda(n,r)$. Note that every $P_\lambda(q)$ is a factor of $P_{\mathfrak{S}_r}(q)$.

Proposition 7.1. *Keep the notations above and suppose that $n \geqslant r$. If $P_{\mathfrak{S}_r}(q)$ is invertible in $\mathscr{R}$, then $\mathcal{S}_{\mathbf{m}}(n,r)$ and $\mathcal{H}_{\mathbf{m}}(r)$ are Morita equivalent.*

Proof. We apply the following general fact: for a ring A together with an idempotent $e \in A$, A and eAe are Morita equivalent if and only if $AeA = A$; see [9, Cor. 1.10].

Since $\mathfrak{l}_\omega \mathcal{S}_{\mathbf{m}}(n,r)\mathfrak{l}_\omega \cong \mathcal{H}_{\mathbf{m}}(r)$, it suffices to prove that

$$\mathcal{S}_{\mathbf{m}}(n,r)\mathfrak{l}_\omega \mathcal{S}_{\mathbf{m}}(n,r) = \mathcal{S}_{\mathbf{m}}(n,r). \tag{7.1.1}$$

By (7.0.1), we only need to show $\mathfrak{l}_\lambda = \Phi_{\lambda,\lambda} \in S_{\mathbf{m}}(n,r)\mathfrak{l}_\omega \mathcal{S}_{\mathbf{m}}(n,r)$ for each $\lambda \in \Lambda(n,r)$. Since $\mathfrak{S}_\omega = \{1\}$ and $x_\omega = 1$, by (7.0.2), we have

$$\begin{aligned}\Phi_{\lambda,\omega}\Phi_{\omega,\lambda}(x_\lambda) &= \Phi_{\lambda,\omega}(x_\omega \sum_{w\in\mathfrak{S}_\lambda} T_w) = \Phi_{\lambda,\omega}(x_\omega \cdot x_\lambda)\\ &= x_\lambda \cdot x_\lambda = x_\lambda^2 = P_\lambda(q)x_\lambda = P_\lambda(q)\Phi_{\lambda,\lambda}(x_\lambda).\end{aligned}$$

This implies that

$$\Phi_{\lambda,\omega}\Phi_{\omega,\lambda} = P_\lambda(q)\Phi_{\lambda,\lambda}. \tag{7.1.2}$$

Hence, by the hypothesis on $P_\lambda(q)$, we obtain that

$$\mathfrak{l}_\lambda = \Phi_{\lambda,\lambda} = P_\lambda(q)^{-1}\Phi_{\lambda,\omega}\Phi_{\omega,\lambda} \in \mathcal{S}_{\mathbf{m}}(n,r)\mathfrak{l}_\omega \mathcal{S}_{\mathbf{m}}(n,r).$$

This finishes the proof. □

Remarks 7.2. (1) The proposition above is a (slim) cyclotomic analogue of [17, Appendix, Lem. 1.4] which states that if $n \geqslant r$ and $P_{\mathfrak{S}_r}(q)$ is invertible in $\mathscr{R}$, then $\mathcal{S}_\vartriangle(n,r)_{\mathscr{R}}$ is Morita equivalent to $\mathcal{H}_\vartriangle(r)_{\mathscr{R}}$. Note that this fact is proved through a category equivalence in [13, Thm 4.1.3] under the assumption that q is not a root of unity.

(2) For the cyclotomic q-Schur algebra, this Morita equivalence does not seem to be established in the literature. It is easy to see that the proof above does not work in this case, since, for the idempotent $\Phi_{\lambda,\lambda}$ associated with a multipartition $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(m)})$ of r, formula (7.1.2) can not be established whenever $|\lambda^{(m)}| < r$.

Let $\mathcal{H}_{\mathbf{m}}(r)$-Mod (resp., $\mathcal{S}_{\mathbf{m}}(n,r)$-Mod) be the category of left $\mathcal{H}_{\mathbf{m}}(r)$-modules (resp., $\mathcal{S}_{\mathbf{m}}(n,r)$-modules). By Proposition 7.1, we immediately have the following.

Corollary 7.3. *Let $\mathscr{R}$ be a commutative ring in which $P_{\mathfrak{S}_r}(q)$ is invertible and assume $n \geqslant r$. Then the categories $\mathcal{H}_{\mathbf{m}}(r)$-Mod and $\mathcal{S}_{\mathbf{m}}(n,r)$-Mod are equivalent.*

By a standard argument for Morita equivalence, the category equivalence is given by the following two functors

$$\begin{aligned}\mathcal{F} &: \mathcal{H}\text{-Mod} \longrightarrow \mathcal{S}\text{-Mod}, \quad N \longmapsto \mathcal{S}e \otimes_{\mathcal{H}} N,\\ \mathcal{G} &: \mathcal{S}\text{-Mod} \longrightarrow \mathcal{H}\text{-Mod}, \quad M \longmapsto e\mathcal{S} \otimes_{\mathcal{S}} M,\end{aligned} \tag{7.3.1}$$

where $e = \mathfrak{l}_\omega$, $\mathcal{S} = \mathcal{S}_{\mathbf{m}}(n,r)$ $(n \geqslant r)$, and $\mathcal{H} = \mathcal{H}_{\mathbf{m}}(r) = e\mathcal{S}e$.

Let $\mathscr{K}$ be a field and let $\mathcal{H}_{\mathbf{m}}(r)_{\mathscr{K}}$-mod (resp., $\mathcal{S}_{\mathbf{m}}(n,r)_{\mathscr{K}}$-mod) be the category of finite dimensional left $\mathcal{H}_{\mathbf{m}}(r)_{\mathscr{K}}$-modules (resp., $\mathcal{S}_{\mathbf{m}}(n,r)_{\mathscr{K}}$-modules). Then the functors above induce a category equivalence between $\mathcal{H}_{\mathbf{m}}(r)_{\mathscr{K}}$-mod and $\mathcal{S}_{\mathbf{m}}(n,r)_{\mathscr{K}}$-mod. In the next section, we will describe the simple objects in $\mathcal{S}_{\mathbf{m}}(n,r)_{\mathscr{K}}$-mod.

8. Irreducible objects in $\mathcal{S}_{\mathbf{m}}(n,r)_{\mathscr{K}}$-mod

Throughout this section, let $\mathscr{K}$ be a field and let $0 \neq q \in \mathscr{K}$. We now use the Morita equivalence in the previous section and the classification of irreducible $\mathcal{H}_{\mathbf{m}}(r)_{\mathscr{K}}$-modules to obtain a classification of irreducible $\mathcal{S}_{\mathbf{m}}(n,r)_{\mathscr{K}}$-modules.

The classification of irreducible $\mathcal{H}_{\mathbf{m}}(r)_{\mathscr{K}}$-modules is well known. In [35, 20], it is done by using a cellular basis, while more precise labelling in terms of Kleshchev multipartitions is given in [5, 3]. We now have a brief review.

Recall that a partition λ of r is a sequence of non-negative integers $\lambda_1 \geqslant \lambda_2 \geqslant \cdots$ such that $|\lambda| = \sum_i \lambda_i = r$. By definition, an m-fold multipartition of r (or simply, an m-multipartition) is a sequence of m partitions $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(m)})$ such that $\sum_{1\leqslant i\leqslant m} |\lambda^{(i)}| = r$. We use $\mathcal{P}_m(r)$ to denote the set of m-multipartitions of r.

The dominance order on $\mathcal{P}_m(r)$ is a partial order $\trianglerighteq$ defined by setting, for $\lambda, \mu \in \mathcal{P}_m(r)$, $\lambda \trianglerighteq \mu$ if, for all j, k,

$$\sum_{1\leqslant l\leqslant k-1} |\lambda^{(l)}| + \sum_{1\leqslant i\leqslant j} \lambda_i^{(k)} \geqslant \sum_{1\leqslant l\leqslant k-1} |\mu^{(l)}| + \sum_{1\leqslant i\leqslant j} \mu_i^{(k)}.$$

To each $\lambda \in \mathcal{P}_m(r)$ we can associate a left $\mathcal{H}_{\mathbf{m}}(r)_{\mathscr{K}}$-module $S^\lambda = \mathcal{H}_{\mathbf{m}}(r)_{\mathscr{K}} \mathsf{z}_\lambda$, where $\mathsf{z}_\lambda = \tau(z_\lambda) \in \mathcal{H}_{\mathbf{m}}(r)_{\mathscr{K}}$ with τ, z_λ defined in (2.0.2) and [28, (2.1),(2.9)] (cf. [20, Def. 3.28]). These modules are called Specht modules. Each Specht module is naturally equipped with a bilinear form. Set $D^\lambda = S^\lambda/\mathrm{rad} S^\lambda$, where $\mathrm{rad} S^\lambda$ is the radical of the bilinear form. By the cellular algebra theory, those nonzero D^λ form a complete set of irreducible $\mathcal{H}_{\mathbf{m}}(r)_{\mathscr{K}}$-modules.

By determining Kashiwara's crystal graphs, Ariki and Mathas gave in [5] another classification of irreducible $\mathcal{H}_{\mathbf{m}}(r)_{\mathscr{K}}$-modules in terms of Kleshchev multipartitions. Let $\mathcal{KP}_m(r) \subseteq \mathcal{P}_m(r)$ denote the subset of Kleshchev multipartitions[5] of r. We now summarise these results in the following theorem.

Theorem 8.1. *Suppose that $\mathscr{K}$ is a field with $q, u_1, \ldots, u_m$ all nonzero.*

1. ([20, Thm 3.30]) *The set $\{D^\lambda \mid \lambda \in \mathcal{P}_m(r), D^\lambda \neq 0\}$ forms a complete set of non-isomorphic irreducible left $\mathcal{H}_{\mathbf{m}}(r)_{\mathscr{K}}$-modules. Further, these modules are absolutely irreducible.*
2. ([5, Thm C], [3]) *$D^\lambda \neq 0$ if and only if $\lambda \in \mathcal{KP}_m(r)$.*
3. *Let λ, μ be m-multipartitions of r with $\mu \in \mathcal{KP}_m(r)$. If $[S^\lambda : D^\mu] \neq 0$, then $\lambda \trianglerighteq \mu$. In particular, $[S^\lambda : D^\lambda] = 1$ for all $\lambda \in \mathcal{KP}_m(r)$.*

The map τ defined in (2.0.2) can be extended to an anti-automorphism of $\mathcal{S}_{\mathbf{m}}(n,r)$ (see [20]). We may turn a module over these algebras to its opposite side module. For example, we may twist the $(\mathcal{S}_{\mathbf{m}}(n,r), \mathcal{H}_{\mathbf{m}}(r))$-bimodule structure on $\mathcal{T}_{\mathbf{m}}(n,r)$ into the $(\mathcal{H}_{\mathbf{m}}(r), \mathcal{S}_{\mathbf{m}}(n,r))$-bimodule $\mathcal{T}_{\mathbf{m}}(n,r)^\tau$ with the action $h * x * s = \tau(s)x\tau(h)$ for all $h \in \mathcal{H}_{\mathbf{m}}(r), x \in \mathcal{T}_{\mathbf{m}}(n,r), s \in \mathcal{S}_{\mathbf{m}}(n,r)$. Note also that there is an obvious left $\mathcal{H}_{\mathbf{m}}(r)$-module isomorphism

$$\mathcal{T}_{\mathbf{m}}(n,r)^\tau \cong \bigoplus_{\lambda\in\Lambda(n,r)} \mathcal{H}_{\mathbf{m}}(r) x_\lambda. \tag{8.1.1}$$

[5] See [5] for the definition. If q is not a root of unity, and the parameters $u_1, \ldots, u_m$ are powers of q, see, e.g., [32, 3.5.4] for a combinatorial definition.

For $\mu \in \mathcal{KP}_m(r)$, let
$$L(\mu) = \mathcal{T}_{\mathbf{m}}(n,r) \otimes_{\mathcal{H}_{\mathbf{m}}(r)_{\mathscr{K}}} D^\mu.$$
By Proposition 7.1 and Theorem 8.1, we have the following Corollary.

Corollary 8.2. *Suppose $P_{\mathfrak{S}_r}(q) \neq 0$ in $\mathscr{K}$. If $n \geqslant r$, then $\{L(\mu) \mid \mu \in \mathcal{KP}_m(r)\}$ is a complete set of non-isomorphic irreducible $\mathcal{S}_{\mathbf{m}}(n,r)$-modules. Moreover, we have an $\mathcal{S}_{\mathbf{m}}(n,r)_{\mathscr{K}}$-module isomorphism*
$$L(\mu) \cong \operatorname{Hom}_{\mathcal{H}_{\mathbf{m}}(r)_{\mathscr{K}}}(\mathfrak{l}_\omega \mathcal{S}_{\mathbf{m}}(n,r)_{\mathscr{K}}, D^\mu)$$

Proof. We only need to prove the second assertion. Let $\mathcal{S} = \mathcal{S}_{\mathbf{m}}(n,r)_{\mathscr{K}}$. Then, for $e = \mathfrak{l}_\omega$, $e\mathcal{S}e \cong \mathcal{H}_{\mathbf{m}}(r)_{\mathscr{K}}$. Since $\mathcal{S}$ is Morita equivalent to $e\mathcal{S}e$, the multiplication map $\mathcal{S}e \otimes_{e\mathcal{S}e} e\mathcal{S} \to \mathcal{S}$ is an $(\mathcal{S},\mathcal{S})$-bimodule isomorphism (see, e. g., [9, Prop. 1.9(2)]). Then the following (left) $\mathcal{S}$-module[6] isomorphisms hold:
$$\begin{aligned} L(\mu) &\cong \operatorname{Hom}_{\mathcal{S}}(\mathcal{S}, L(\mu)) \cong \operatorname{Hom}_{\mathcal{S}}(\mathcal{S}e \otimes_{e\mathcal{S}e} e\mathcal{S}, L(\mu)) \\ &\cong \operatorname{Hom}_{e\mathcal{S}e}(e\mathcal{S}, \operatorname{Hom}_{\mathcal{S}}(\mathcal{S}e, L(\mu)) \cong \operatorname{Hom}_{e\mathcal{S}e}(e\mathcal{S}, D^\mu), \end{aligned} \tag{8.2.1}$$
since $\operatorname{Hom}_{\mathcal{S}}(\mathcal{S}e, L(\mu)) \cong eL(\mu) \cong D^\mu$. □

Recall from [20] for the notion of semi-standard λ-tableaux of type μ.

Lemma 8.3. *For an m-multipartition $\lambda = (\lambda^{(1)}, \dots, \lambda^{(m)})$ and a composition $\mu \in \Lambda(n,r)$, if the set $\mathcal{T}^{ss}(\lambda, {}^\circ\mu)$ of semistandard λ-tableaux of type ${}^\circ\mu := ((0), \dots, (0), \mu)$ is not empty, then $\lambda^{(1)} + \cdots + \lambda^{(m)} \trianglerighteq \mu$.*

Proof. If $S = (S^{(1)}, \dots, S^{(m)}) \in \mathcal{T}^{ss}(\lambda, {}^\circ\mu)$, then each $S^{(i)}$ is a semistandard tableau of type, say $\mu^{(i)}$. Thus, $\lambda^{(i)} \trianglerighteq \mu^{(i)}$ for all i. Since $\mu^{(1)} + \cdots + \mu^{(m)} = \mu$, it follows that $\lambda^{(1)} + \cdots + \lambda^{(m)} \trianglerighteq \mu$. □

For a partition λ, we use $l(\lambda)$ to denote the length of λ. Set
$$\mathcal{KP}_m(n,r) = \{(\lambda^{(1)}, \dots, \lambda^{(m)}) \in \mathcal{KP}_m(r) \mid l(\lambda^{(i)}) \leqslant n, 1 \leqslant i \leqslant m\}.$$

Lemma 8.4. *Assume now that $q \in \mathscr{K}$ is not a root of unity. Suppose that $n < r$ and $\mu \in \mathcal{KP}_m(r)$. Then $\mathcal{T}_{\mathbf{m}}(n,r) \otimes_{\mathcal{H}_{\mathbf{m}}(r)_{\mathscr{K}}} D^\mu \neq 0$ if and only if $\mu \in \mathcal{KP}_m(n,r)$.*

Proof. Assume $N \geqslant r > n$ and consider the idempotent $f = \sum_{\lambda \in \Lambda(n,r)} \mathfrak{l}_{\widetilde{\lambda}} \in \mathcal{S} = \mathcal{S}_{\mathbf{m}}(N,r)_{\mathscr{K}}$, where $\widetilde{\lambda} = (\lambda_1, \dots, \lambda_n, 0, \dots, 0) \in \Lambda(N,r)$. Then $\mathcal{S}_{\mathbf{m}}(n,r)_{\mathscr{K}} \cong f\mathcal{S}f$. By (8.2.1), we have
$$L(\mu) \cong fL(\widetilde{\mu}) \cong \operatorname{Hom}_{e\mathcal{S}e}(e\mathcal{S}f, D^\mu) \cong \operatorname{Hom}_{\mathcal{H}_{\mathbf{m}}(r)_{\mathscr{K}}}(\mathcal{T}_{\mathbf{m}}(n,r)^\tau, D^\mu). \tag{8.4.1}$$
Here we have used the $(\mathcal{H}_{\mathbf{m}}(r)_{\mathscr{K}}, \mathcal{S}_{\mathbf{m}}(n,r)_{\mathscr{K}})$-bimodule isomorphism $e\mathcal{S}f \cong \mathcal{T}_{\mathbf{m}}(n,r)^\tau$. By [20, Cor. 4.15], for each $\lambda \in \Lambda(n,r)$, there is a Specht module filtration of $\mathcal{H}_{\mathbf{m}}(r)_{\mathscr{K}} x_\lambda$ such that $[\mathcal{H}_{\mathbf{m}}(r)_{\mathscr{K}} x_\lambda : S^\nu]$ equals the number of semistandard ν-tableaux of type ${}^\circ\lambda$, where $\nu \in \mathcal{P}_m(r)$. By Lemma 8.3, if S^ν occurs in the filtration of $\mathcal{H}_{\mathbf{m}}(r)_{\mathscr{K}} x_\lambda$, then $\sum_{1 \leqslant i \leqslant m} \nu^{(i)} \trianglerighteq \lambda$ and, consequently, $\nu \in \mathcal{KP}_m(n,r)$.

Thus, if $\mathcal{T}_{\mathbf{m}}(n,r) \otimes_{\mathcal{H}_{\mathbf{m}}(r)_{\mathscr{K}}} D^\mu \neq 0$, by (8.4.1) and (8.1.1), D^μ is a composition factor of S^ν for some $\nu \in \mathcal{KP}_m(n,r)$. Then, by Theorem 8.1(3), $\nu \trianglerighteq \mu$. Since $\nu \in \mathcal{KP}_m(n,r)$, we have $\mu \in \mathcal{KP}_m(n,r)$.

[6] See, e.g., [40, II,Prop. 3.5] for various left module structures on the Hom-space.

Conversely, suppose that $\mu = (\mu^{(1)}, \ldots, \mu^{(m)}) \in \mathcal{KP}_m(n,r)$. Then the irreducible $\mathcal{H}_{\mathbf{m}}(r)_{\mathscr{K}}$-module $D^\mu = \mathrm{hd}(S^\mu)$ by Theorem 8.1. Inflate D^μ to an irreducible $\mathcal{H}_\vartriangle(r)_{\mathscr{K}}$-module via (2.0.1) and, by [32, Lem. 4.1.1], there exists a multisegment $\mathbf{s} = \mathbf{s}^{\mathsf{c}}_{\mu,\mathbf{m}} = (\mathsf{s}_1, \ldots, \mathsf{s}_t)$ consisting of column residual segments of μ such that the simple $\mathcal{H}_\vartriangle(r)_{\mathscr{K}}$-module $V_{\mathbf{s}}$ associated with $\mathbf{s}$ is isomorphic to D^μ. Since $\mu \in \mathcal{KP}_m(n,r)$, each of the s_i's has length at most n. Now, by [12, Thm 6.6], we obtain a simple $\mathcal{S}_\vartriangle(n,r)_{\mathscr{K}}$-module $\mathcal{T}_\vartriangle(n,r) \otimes_{\mathcal{H}_\vartriangle(r)_{\mathscr{K}}} V_{\mathbf{s}}$. Hence,

$$\begin{aligned} 0 \neq \mathcal{T}_\vartriangle(n,r) \otimes_{\mathcal{H}_\vartriangle(r)_{\mathscr{K}}} V_{\mathbf{s}} &\cong \mathcal{T}_\vartriangle(n,r) \otimes_{\mathcal{H}_\vartriangle(r)_{\mathscr{K}}} \mathcal{H}_{\mathbf{m}}(r)_{\mathscr{K}} \otimes_{\mathcal{H}_{\mathbf{m}}(r)_{\mathscr{K}}} V_{\mathbf{s}} \\ &\cong \mathcal{T}_{\mathbf{m}}(n,r) \otimes_{\mathcal{H}_{\mathbf{m}}(r)_{\mathscr{K}}} D^\mu, \end{aligned}$$

as desired. □

Now we are ready to prove the main result of this section.

Theorem 8.5. *Assume $q \in \mathscr{K}$ is not a root of unity. For arbitrary positive integers n, r, the following set*

$$\{\mathcal{T}_{\mathbf{m}}(n,r) \otimes_{\mathcal{H}_{\mathbf{m}}(r)_{\mathscr{K}}} D^\mu \mid \mu \in \mathcal{KP}_m(n,r)\}$$

forms a complete set of non-isomorphic irreducible $\mathcal{S}_{\mathbf{m}}(n,r)_{\mathscr{K}}$-modules.

Proof. If $n \geqslant r$, then $\mathcal{KP}_m(n,r) = \mathcal{KP}_m(r)$ and the result can be seen in Corollary 8.2.

Now suppose now $n < r \leqslant N$. Then there exists an idempotent $f \in \mathcal{S}_{\mathbf{m}}(N,r)_{\mathscr{K}}$ such that

$$\{f(\mathcal{T}_{\mathbf{m}}(N,r) \otimes_{\mathcal{H}_{\mathbf{m}}(r)_{\mathscr{K}}} D^\mu) \mid \mu \in \mathcal{KP}_m(N,r) = \mathcal{KP}_m(r)\} \setminus \{0\} \tag{8.5.1}$$

forms a complete set of non-isomorphic irreducible $\mathcal{S}_{\mathbf{m}}(n,r)$-modules. Since

$$f(\mathcal{T}_{\mathbf{m}}(N,r) \otimes_{\mathcal{H}_{\mathbf{m}}(r)_{\mathscr{K}}} D^\mu) \cong \mathcal{T}_{\mathbf{m}}(n,r) \otimes_{\mathcal{H}_{\mathbf{m}}(r)_{\mathscr{K}}} D^\mu,$$

which is nonzero if and only if $\mu \in \mathcal{KP}_m(n,r)$ by Lemma 8.4. Hence, our assertion follows. □

Remark 8.6. The hypothesis that q is not a root of unity in Theorem 8.5 is stronger than the hypothesis that $P_{\mathfrak{S}_r}(q) \neq 0$ in $\mathscr{K}$. This is required in order to use [12, Thm 6.6]. There should be a direct proof under the latter hypothesis.

We also note that when q is not a root of unity the algebra $\mathcal{H}_{\mathbf{m}}(r)_{\mathscr{K}}$, and hence $\mathcal{S}_{\mathbf{m}}(n,r)_{\mathscr{K}}$, can still be non-semisimple.

Appendix A. Proof of Theorem 4.7

Let $\leqslant$ be the Bruhat order on $\mathfrak{S}_r$, that is, $w' \leqslant w$ if w' is a subexpression of some reduced expression of w. For any $y, w \in \mathfrak{S}_r$, there is an element $y * w \in \mathfrak{S}_r$ such that $\ell(y * w) \leqslant \ell(y) + \ell(w)$ and

$$T_y T_w = \sum_{z \leqslant y * w} f_{y,w,z} T_z, \tag{A.0.1}$$

where $f^{y,w}_z \in \mathscr{R}$; see [14, Prop. 4.30]. Also, by Lemma 2.1 and an induction on the length $\ell(w)$ of w, we have taht, for $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^r_m$ and $w, y \in \mathfrak{S}_r$, there exist $f^{\mathbf{a},w}_{y,\mathbf{b}} \in \mathscr{R}$ such that

$$L^{\mathbf{a}} T_w = T_w L^{\mathbf{a}w} + \sum_{y < w, \mathbf{b} \in \mathbb{Z}^r_m} f^{\mathbf{a},w}_{y,\mathbf{b}} T_y L^{\mathbf{b}}. \tag{A.0.2}$$

Lemma A.1. *The set*

$$\mathcal{X} = \bigcup_{\lambda,\mu\in\Lambda(n,r)} \{T_u T_d L^{\mathbf{a}} T_v \mid u \in \mathfrak{S}_\lambda, d \in \mathscr{D}_{\lambda,\mu}, \mathbf{a} \in \mathbb{Z}_m^r, v \in \mathscr{D}_{\nu(d)} \cap \mathfrak{S}_\mu\}$$

forms an $\mathscr{R}$-basis of $\mathcal{H}_{\mathbf{m}}(r)$.

Proof. Recall form Lemma 2.2 that $\mathcal{H}_{\mathbf{m}}(r)$ is a free $\mathscr{R}$-module with basis

$$\{T_w L^{\mathbf{a}} \mid w \in \mathfrak{S}_r, \mathbf{a} \in \mathbb{Z}_m^r\},$$

which, by Lemma 4.1(2), has the same cardinality as $\mathcal{X}$. Thus, it suffices to prove that each basis element $T_w L^{\mathbf{a}}$ of $\mathcal{H}_{\mathbf{m}}(r)$ is an $\mathscr{R}$-linear combination of elements in $\mathcal{X}$. We proceed by induction on the length $\ell(w)$ of w.

Take arbitrary $w \in \mathfrak{S}_r$ and $\mathbf{a} \in \mathbb{Z}_m^r$. If $\ell(w) = 0$, then $T_w L^{\mathbf{a}} = L^{\mathbf{a}} \in \mathcal{X}$. Suppose now $\ell(w) \geqslant 1$. By Lemma 4.1, there are uniquely determined elements $u \in \mathfrak{S}_\lambda$, $d \in \mathscr{D}_{\lambda,\mu}$ and $v \in \mathscr{D}_{\nu(d)} \cap \mathfrak{S}_\mu$ such that $w = udv$ and $\ell(w) = \ell(u) + \ell(d) + \ell(v)$. Applying (A.0.2) and (A.0.1) gives the equalities

$$\begin{aligned}
T_w L^{\mathbf{a}} &= T_u T_d (T_v L^{\mathbf{a}}) \\
&= T_u T_d (L^{\mathbf{a}v^{-1}} T_v + \sum_{y<v,\, \mathbf{b}\in\mathbb{Z}_m^r} f_{y,\mathbf{b}}^{\mathbf{a},v}\, T_y L^{\mathbf{b}}) \\
&= T_u T_d L^{\mathbf{a}v^{-1}} T_v + \sum_{y<v,\, \mathbf{b}\in\mathbb{Z}_m^r} f_{y,\mathbf{b}}^{\mathbf{a},v}\, T_{ud} T_y L^{\mathbf{b}} \\
&= T_u T_d L^{\mathbf{a}v^{-1}} T_v + \sum_{y<v,\, \mathbf{b}\in\mathbb{Z}_m^r} f_{y,\mathbf{b}}^{\mathbf{a},v} \sum_{z\leqslant(ud)*y,\, \mathbf{b}\in\mathbb{Z}_m^r} f_{ud,y,z} T_z L^{\mathbf{b}}.
\end{aligned}$$

Since $y < v$, we obtain that

$$\ell(z) \leqslant \ell(u) + \ell(d) + \ell(y) < \ell(u) + \ell(d) + \ell(v) = \ell(w).$$

By induction, $T_w L^{\mathbf{a}}$ is an $\mathscr{R}$-linear combination of elements in $\mathcal{X}$. □

Proof of Theorem 4.7. Take $\mathbb{A} \in \Theta_m(n,r)_{\lambda,\mu}$ and write $d = d_{|\mathbb{A}|}$ for notational simplicity. By (4.5.1), $x_{\nu(d)}\sigma^{\ddot{\mathbb{A}}} = \sigma^{\ddot{\mathbb{A}}} x_{\nu(d)}$. This together with (4.1.1) implies that

$$\begin{aligned}
\mathfrak{b}_{\mathbb{A}} &= x_\lambda T_d \sigma^{\ddot{\mathbb{A}}} \sum_{v\in\mathscr{D}_{\nu(d)}\cap\mathfrak{S}_\mu} T_v \\
&= \sum_{u\in\mathscr{D}^{-1}_{\nu(d^{-1})}\cap\mathfrak{S}_\lambda} T_u T_d x_{\nu(d)} \sigma^{\ddot{\mathbb{A}}} \sum_{v\in\mathscr{D}_{\nu(d)}\cap\mathfrak{S}_\mu} T_v \\
&= \sum_{u\in\mathscr{D}^{-1}_{\nu(d^{-1})}\cap\mathfrak{S}_\lambda} T_u T_d \sigma^{\ddot{\mathbb{A}}} \big(x_{\nu(d)} \sum_{v\in\mathscr{D}_{\nu(d)}\cap\mathfrak{S}_\mu} T_v\big) \\
&= \sum_{u\in\mathscr{D}^{-1}_{\nu(d^{-1})}\cap\mathfrak{S}_\lambda} T_u T_d \sigma^{\ddot{\mathbb{A}}} x_\mu \in \mathcal{H}_{\mathbf{m}}(r) x_\mu.
\end{aligned}$$

Hence, $\mathfrak{b}_{\mathbb{A}} \in x_\lambda \mathcal{H}_{\mathbf{m}}(r) \cap \mathcal{H}_{\mathbf{m}}(r) x_\mu$.

Linear Independence of $\mathcal{B}_{\lambda,\mu}$. Suppose that

$$\sum_{\mathbb{A}\in\Theta_m(n,r)_{\lambda,\mu}} \gamma_{\mathbb{A}}\mathfrak{b}_{\mathbb{A}} = 0, \text{ where } \gamma_{\mathbb{A}} \in \mathscr{R}.$$

By the definition, each $\mathfrak{b}_{\mathbb{A}}$ is an $\mathscr{R}$-linear combination of elements of the form $T_u T_{d_{|\mathbb{A}|}} L^{\mathbf{a}} T_v$ for $u \in \mathfrak{S}_\lambda$, $\mathbf{a} \in \mathbb{Z}_m^r$, and $v \in \mathscr{D}_{\nu(d_{|\mathbb{A}|})} \cap \mathfrak{S}_\mu$. Since, by Lemma 4.2(1),

$$\Theta_m(n,r)_{\lambda,\mu} = \bigcup_{d\in\mathscr{D}_{\lambda,\mu}} \Theta_m(n,r)^d_{\lambda,\mu},$$

it follows from Lemma A.1 that for each fixed $d \in \mathscr{D}_{\lambda,\mu}$,

$$\sum_{\mathbb{A}\in\Theta_m(n,r)^d_{\lambda,\mu}} \gamma_{\mathbb{A}}\mathfrak{b}_{\mathbb{A}} = 0. \tag{A.1.1}$$

We now fix such $d \in \mathscr{D}_{\lambda,\mu}$. Let d_0 be the (unique) longest element in $\mathscr{D}_{\nu(d)} \cap \mathfrak{S}_\mu$ such that $w^0_\mu = w^0_{\nu(d)} d_0$, where w^0_μ (resp., $w^0_{\nu(d)}$) denotes the longest element of $\mathfrak{S}_\mu$ (resp., $\mathfrak{S}_{\nu(d)}$). If $\mathbb{A} \in \Theta_m(n,r)^d_{\lambda,\mu}$, then, by (A.0.2) and (A.0.1),

$$\begin{aligned}\mathfrak{b}_{\mathbb{A}} &= T_{w^0_\lambda} T_d\, \sigma^{\ddot{\mathbb{A}}} T_{d_0} + \sum_{\substack{u\in\mathfrak{S}_\lambda,\, v\in\mathscr{D}_{\nu(d)}\cap\mathfrak{S}_\mu \\ u<w^0_\lambda \text{ or } v<d_0}} T_u T_d\, \sigma^{\ddot{\mathbb{A}}} T_v \\ &= T_{w^0_\lambda} T_d\, \sigma^{\ddot{\mathbb{A}}} T_{d_0} + \mathfrak{f}_{\mathbb{A}},\end{aligned}$$

where $\mathfrak{f}_{\mathbb{A}}$ is an $\mathscr{R}$-linear combination of the elements $T_u T_d\, L^{\mathbf{a}} T_v$ with $\mathbf{a} \in \mathbb{Z}_m^r$, $u \in \mathfrak{S}_\lambda$ and $v \in \mathscr{D}_{\nu(d)} \cap \mathfrak{S}_\mu$ satisfying $\ell(udv) < \ell(w^0_\lambda d d_0) = \ell(w^0_\lambda) + \ell(d) + \ell(d_0)$. Substituting $\mathfrak{b}_{\mathbb{A}}$ in (A.1.1) gives that

$$\sum_{\mathbb{A}\in\Theta_m(n,r)^d_{\lambda,\mu}} \gamma_{\mathbb{A}} T_{w^0_\lambda} T_d\, \sigma^{\ddot{\mathbb{A}}} T_{d_0} + \sum_{\mathbb{A}\in\Theta_m(n,r)^d_{\lambda,\mu}} \gamma_{\mathbb{A}}\mathfrak{f}_{\mathbb{A}} = 0.$$

By Lemma A.1 again, we obtain that

$$\sum_{\mathbb{A}\in\Theta_m(n,r)^d_{\lambda,\mu}} \gamma_{\mathbb{A}} T_{w^0_\lambda} T_d\, \sigma^{\ddot{\mathbb{A}}} T_{d_0} = 0.$$

Since all T_w are invertible in $\mathcal{H}_{\mathbf{m}}(r)$, the above equality gives that

$$\sum_{\mathbb{A}\in\ddot{\Theta}_m(n,r)^d_{\lambda,\mu}} \gamma_{\mathbb{A}}\, \sigma^{\ddot{\mathbb{A}}} = 0.$$

Applying Proposition 4.5 forces that $\gamma_{\mathbb{A}} = 0$ for each $\mathbb{A} \in \Theta_m(n,r)^d_{\lambda,\mu}$. We conclude that $\mathcal{B}_{\lambda,\mu}$ is linearly independent.

Proof of the Spanning Condition. We now prove that $\mathcal{B}_{\lambda,\mu}$ spans $x_\lambda \mathcal{H}_{\mathbf{m}}(r) \cap \mathcal{H}_{\mathbf{m}}(r) x_\mu$. Take an arbitrary element $z \in x_\lambda \mathcal{H}_{\mathbf{m}}(r) \cap \mathcal{H}_{\mathbf{m}}(r) x_\mu$. By Lemma A.1, z can be written as

$$z = \sum_{d,\mathbf{a},v} \gamma_{(d,\mathbf{a},v)} x_\lambda T_d L^{\mathbf{a}} T_v,$$

where $\gamma_{(d,\mathbf{a},v)} \in \mathscr{R}$, and the sum is taken over all $d \in \mathscr{D}_{\lambda,\mu}$, $\mathbf{a} \in \mathbb{Z}_m^r$, and $v \in \mathscr{D}_{\nu(d)} \cap \mathfrak{S}_\mu$. By applying (4.1.1),

$$\begin{aligned} z &= \sum_{d \in \mathscr{D}_{\lambda,\mu}} \Big(\sum_{u \in \mathscr{D}^{-1}_{\nu(d^{-1})} \cap \mathfrak{S}_\lambda} T_u \Big) T_d \sum_{\mathbf{a},v} \gamma_{(d,\mathbf{a},v)} x_{\nu(d)} L^{\mathbf{a}} T_v \\ &= \sum_{d \in \mathscr{D}_{\lambda,\mu}} \Big(\sum_{u \in \mathscr{D}^{-1}_{\nu(d^{-1})} \cap \mathfrak{S}_\lambda} T_u \Big) T_d z_d, \end{aligned}$$

where

$$z_d = \sum_{\mathbf{a} \in \mathbb{Z}_m^r, v \in \mathscr{D}_{\nu(d)} \cap \mathfrak{S}_\mu} \gamma_{(d,\mathbf{a},v)} x_{\nu(d)} L^{\mathbf{a}} T_v. \tag{A.1.2}$$

We claim that $z_d \in \mathcal{H}_{\mathbf{m}}(r)x_\mu$ for all $d \in \mathscr{D}_{\lambda,\mu}$. Indeed, since $z \in \mathcal{H}_{\mathbf{m}}(r)x_\mu$, we have by Lemma 2.3 that $zT_k = qz$ for $s_k \in J_\mu$. Thus,

$$\sum_{d \in \mathscr{D}_{\lambda,\mu}} \sum_{u \in \mathscr{D}^{-1}_{\nu(d^{-1})} \cap \mathfrak{S}_\lambda} T_u T_d (z_d T_k - q z_d) = 0.$$

Further, by (A.0.2), we have for each fixed $d \in \mathscr{D}_{\lambda,\mu}$,

$$z_d = \sum_{\mathbf{a},v} \gamma_{(d,\mathbf{a},v)} x_{\nu(d)} L^{\mathbf{a}} T_v = \sum_{\mathbf{a},v} \gamma_{(d,\mathbf{a},v)} x_{\nu(d)} \big(T_v L^{\mathbf{a}v} + \sum_{y<v, \mathbf{b} \in \mathbb{Z}_m^r} f^{\mathbf{a},v}_{y,\mathbf{b}} T_y L^{\mathbf{b}} \big).$$

Since $\mathfrak{S}_{\nu(d)} \subseteq \mathfrak{S}_\mu$, z_d can be written as an $\mathscr{R}$-linear combination of $T_w L^{\mathbf{a}}$ with $w \in \mathfrak{S}_\mu$ and $\mathbf{a} \in \mathbb{Z}_m^r$. Thus, for each $s_k \in J_\mu$, $z_d T_k$ can also be written as an $\mathscr{R}$-linear combination of $T_w L^{\mathbf{a}}$ with $w \in \mathfrak{S}_\mu$ and $\mathbf{a} \in \mathbb{Z}_m^r$. By Lemmas 4.1(2) and 2.2, the elements $T_u T_d T_w L^{\mathbf{a}}$ for $u \in \mathscr{D}^{-1}_{\nu(d^{-1})} \cap \mathfrak{S}_\lambda$, $w \in \mathfrak{S}_\mu$ and $\mathbf{a} \in \mathbb{Z}_m^r$ are linearly independent. This forces that $z_d T_k - q z_d = 0$ for all $s_k \in J_\mu$. By Lemma 2.3, $z_d \in \mathcal{H}_{\mathbf{m}}(r)x_\mu$, proving the claim.

Let $d \in \mathscr{D}_{\lambda,\mu}$. If $w \in \mathfrak{S}_{\nu(d)}$ and $w' < w$ (the Bruhat order of $\mathfrak{S}_r$), then $w' \in \mathfrak{S}_{\nu(d)}$. By applying (A.0.2), we obtain that for each $d \in \mathscr{D}_{\lambda,\mu}$ and $v \in \mathscr{D}_{\nu(d)} \cap \mathfrak{S}_\mu$,

$$\sum_{\mathbf{a} \in \mathbb{Z}_m^r} \gamma_{(d,\mathbf{a},v)} x_{\nu(d)} L^{\mathbf{a}} = \sum_{\mathbf{b} \in \mathbb{Z}_m^r, w \in \mathfrak{S}_{\nu(d)}} h_{(\mathbf{b},w)} L^{\mathbf{b}} T_w, \tag{A.1.3}$$

where $h_{(\mathbf{b},w)} \in \mathscr{R}$. Thus, we obtain that

$$z_d = \sum_{\mathbf{b} \in \mathbb{Z}_m^r,\, w \in \mathfrak{S}_{\nu(d)}} h_{(\mathbf{b},w)} L^{\mathbf{b}} T_w \big(\sum_{v \in \mathscr{D}_{\nu(d)} \cap \mathfrak{S}_\mu} T_v \big).$$

Further, for each $w \in \mathfrak{S}_{\nu(d)}$ and $v \in \mathscr{D}_{\nu(d)} \cap \mathfrak{S}_\mu$, we have $T_w T_v = T_{wv}$ with $wv \in \mathfrak{S}_\mu$. Then by the fact that $z_d \in \mathcal{H}_{\mathbf{m}}(r)x_\mu$ and Lemma 2.3, z_d has the form

$$z_d = \sum_{\mathbf{c} \in \mathbb{Z}_m^r} f_{(d,\mathbf{c})} L^{\mathbf{c}} x_\mu$$

for some $f_{(d,\mathbf{c})} \in \mathscr{R}$. By further applying (4.1.1), we obtain that

$$z_d = \sum_{\mathbf{c} \in \mathbb{Z}_m^r} f_{(d,\mathbf{c})} L^{\mathbf{c}} x_\mu = \sum_{\mathbf{c} \in \mathbb{Z}_m^r} f_{(d,\mathbf{c})} L^{\mathbf{c}} x_{\nu(d)} \sum_{v \in \mathscr{D}_{\nu(d)} \cap \mathfrak{S}_\mu} T_v.$$

In other words,

$$\sum_{\mathbf{a}\in\mathbb{Z}_m^r, v\in\mathscr{D}_{\nu(d)}\cap\mathfrak{S}_\mu}\gamma_{(d,\mathbf{a},v)}x_{\nu(d)}L^{\mathbf{a}}T_v=\sum_{\mathbf{c}\in\mathbb{Z}_m^r}f_{(d,\mathbf{c})}L^{\mathbf{c}}x_{\nu(d)}\sum_{v\in\mathscr{D}_{\nu(d)}\cap\mathfrak{S}_\mu}T_v,$$

that is,

$$\sum_{v\in\mathscr{D}_{\nu(d)}\cap\mathfrak{S}_\mu}\Big(\sum_{\mathbf{a}\in\mathbb{Z}_m^r}\gamma_{(d,\mathbf{a},v)}x_{\nu(d)}L^{\mathbf{a}}-\sum_{\mathbf{c}\in\mathbb{Z}_m^r}f_{(d,\mathbf{c})}L^{\mathbf{c}}x_{\nu(d)}\Big)T_v=0.$$

By (A.0.2), each term $\sum_{\mathbf{a}\in\mathbb{Z}_m^r}\gamma_{(d,\mathbf{a},v)}x_{\nu(d)}L^{\mathbf{a}}-\sum_{\mathbf{c}\in\mathbb{Z}_m^r}f_{(d,\mathbf{c})}L^{\mathbf{c}}x_{\nu(d)}$ can be written as an $\mathscr{R}$-linear combination of $L^\alpha T_w$ with $w\in\mathfrak{S}_{\nu(d)}$ and $\alpha\in\mathbb{Z}_m^r$. By Lemma 2.2, the elements $L^\alpha T_wT_v$ for $\alpha\in\mathbb{Z}_m^r$, $w\in\mathfrak{S}_{\nu(d)}$ and $v\in\mathscr{D}_{\nu(d)}\cap\mathfrak{S}_\mu$ are linearly independent. It follows that for each $v\in\mathscr{D}_{\nu(d)}\cap\mathfrak{S}_\mu$,

$$\sum_{\mathbf{a}\in\mathbb{Z}_m^r}\gamma_{(d,\mathbf{a},v)}x_{\nu(d)}L^{\mathbf{a}}-\sum_{\mathbf{c}\in\mathbb{Z}_m^r}f_{(d,\mathbf{c})}L^{\mathbf{c}}x_{\nu(d)}=0,$$

and thus,

$$\sum_{\mathbf{a}\in\mathbb{Z}_m^r}\gamma_{(d,\mathbf{a},v)}x_{\nu(d)}L^{\mathbf{a}}=\sum_{\mathbf{c}\in\mathbb{Z}_m^r}f_{(d,\mathbf{c})}L^{\mathbf{c}}x_{\nu(d)}\in x_{\nu(d)}\mathcal{H}_{\mathbf{m}}(r)\cap\mathcal{H}_{\mathbf{m}}(r)x_{\nu(d)}.$$

Then by Proposition 3.8(2), $\sum_{\mathbf{a}}\gamma_{(d,\mathbf{a},v)}x_{\nu(d)}L^{\mathbf{a}}$ is an $\mathscr{R}$-linear combination of elements in $\{x_{\nu(d)}\sigma^{\mathbf{e}}\mid\mathbf{e}\in\Gamma(m,\nu(d))\}$. Applying Lemma 4.4 implies that $\sum_{\mathbf{a}}\gamma_{(d,\mathbf{a},v)}x_{\nu(d)}L^{\mathbf{a}}$ is an $\mathscr{R}$-linear combination of elements in $\{x_{\nu(d)}\sigma^{\ddot{\mathbb{A}}}\mid\mathbb{A}\in\Theta_m(n,r)^d_{\lambda,\mu}\}$.

We conclude that

$$\begin{aligned}
z&=\sum_{d\in\mathscr{D}_{\lambda,\mu}}\Big(\sum_{u\in\mathscr{D}^{-1}_{\nu(d^{-1})}\cap\mathfrak{S}_\lambda}T_u\Big)T_d\sum_{\mathbf{a},v}\gamma_{(d,\mathbf{a},v)}x_{\nu(d)}L^{\mathbf{a}}T_v\\
&=\sum_{d\in\mathscr{D}_{\lambda,\mu}}\Big(\sum_{u\in\mathscr{D}^{-1}_{\nu(d^{-1})}\cap\mathfrak{S}_\lambda}T_u\Big)T_d\sum_{\mathbf{e},v}f_{(d,\mathbf{e})}x_{\nu(d)}\sigma^{\mathbf{e}}T_v\\
&=\sum_{d\in\mathscr{D}_{\lambda,\mu}}\sum_{\mathbb{A}\in\Theta_m(n,r)^d_{\lambda,\mu}}f_{(d,\mathbb{A})}x_\lambda T_d\sigma^{\ddot{\mathbb{A}}}\sum_{v\in\mathscr{D}_{\nu(d)}\cap\mathfrak{S}_\mu}T_v\\
&=\sum_{d\in\mathscr{D}_{\lambda,\mu}}\sum_{\mathbb{A}\in\Theta_m(n,r)^d_{\lambda,\mu}}f_{(d,\mathbb{A})}\mathbf{b}_{\mathbb{A}}\in\sum_{\mathbb{A}\in\Theta_m(n,r)_{\lambda,\mu}}\mathscr{R}\mathfrak{b}_{\mathbb{A}}.
\end{aligned}$$

This finishes the proof. □

Appendix B. The affine version of Lemma 3.2

Lemma B.1. *Suppose that*

$$z=\sum_{\mathbf{b}\in\mathbb{Z}^r}c_{\mathbf{b}}x_{(r)}X^{\mathbf{b}}\in x_{(r)}\mathcal{H}_\vartriangle(r)_{\mathscr{R}}\cap\mathcal{H}_\vartriangle(r)_{\mathscr{R}}x_{(r)},\quad\textit{where } c_{\mathbf{b}}\in\mathscr{R}.$$

Then $c_{\mathbf{b}}=c_{\mathbf{b}w}$ *for all* $\mathbf{b}\in\mathbb{Z}^r$ *and* $w\in\mathfrak{S}_r$.

Proof. For each $\mathbf{a}=(a_1,\dots,a_r)\in\mathbb{Z}^r$, set

$$\|\mathbf{a}\|=\max\{|a_i|\mid 1\leqslant i\leqslant r\}.$$

It is clear that $\|\mathbf{a}\| = \|\mathbf{a}w\|$ for each $w \in \mathfrak{S}_r$. Since only finitely many $c_{\mathbf{b}}$ may not be zero, there is a positive integer N such that

$$c_{\mathbf{b}} = 0 \text{ whenever } \|\mathbf{b}\| > N.$$

It suffices to prove that $c_{\mathbf{a}} = c_{\mathbf{a}s_i}$ for each fixed $\mathbf{a} \in \mathbb{Z}^r$ and $1 \leqslant i < r$. Write $\mathbf{a} = (a_1, \ldots, a_r)$. If $a_i = a_{i+1}$, then $\mathbf{a} = \mathbf{a}s_i$ and hence, $c_{\mathbf{a}} = c_{\mathbf{a}s_i}$, as desired. Now let $a_i \neq a_{i+1}$. We may suppose $a_i < a_{i+1}$ (Otherwise, we replace $\mathbf{a}$ by $\mathbf{a}s_i$). We proceed by induction on a_{i+1}. If $|a_{i+1}| > N$, then $\|\mathbf{a}\| = \|\mathbf{a}s_i\| > N$. Thus, $c_{\mathbf{a}} = c_{\mathbf{a}s_i} = 0$. Let $-N \leqslant k \leqslant N$ and suppose that for each $\mathbf{a}$ with $k+1 \leqslant a_{i+1}$, the equality $c_{\mathbf{a}} = c_{\mathbf{a}s_i}$ holds. Now we take $\mathbf{a} \in \mathbb{Z}^r$ with $a_i < a_{i+1} = k$.

By Lemma 2.1, we obtain that

$$\begin{aligned} zT_i &= \sum_{\mathbf{b}\in\mathbb{Z}^r} c_{\mathbf{b}} x_{(r)} X^{\mathbf{b}} T_i \\ &= \sum_{\mathbf{b}\in\mathbb{Z}^r} c_{\mathbf{b}} x_{(r)} T_i X^{\mathbf{b}s_i} + \sum_{\substack{\mathbf{b}\in\mathbb{Z}^r \\ b_i<b_{i+1}}} (q-1) c_{\mathbf{b}} x_{(r)} \sum_{t=1}^{b_{i+1}-b_i} X^{\mathbf{b}s_i - t\alpha_i} \\ &\quad + \sum_{\substack{\mathbf{b}\in\mathbb{Z}^r \\ b_i>b_{i+1}}} (1-q) c_{\mathbf{b}} x_{(r)} \sum_{t=0}^{b_i-b_{i+1}-1} X^{\mathbf{b}s_i + t\alpha_i}. \end{aligned}$$

Since $z \in \mathcal{H}_\vartriangle(r)_{\mathscr{R}} x_{(r)}$ and $x_{(r)}T_i = qx_{(r)}$, it follows that $qz = zT_i$ which gives rise to the equality

$$\begin{aligned} \sum_{\mathbf{b}\in\mathbb{Z}^r} qc_{\mathbf{b}} x_{(r)} X^{\mathbf{b}} &= \sum_{\mathbf{b}\in\mathbb{Z}^r} qc_{\mathbf{b}} x_{(r)} X^{\mathbf{b}s_i} + \sum_{\substack{\mathbf{b}\in\mathbb{Z}^r \\ b_i<b_{i+1}}} (q-1) c_{\mathbf{b}} x_{(r)} \sum_{t=1}^{b_{i+1}-b_i} X^{\mathbf{b}s_i - t\alpha_i} \\ &\quad + \sum_{\substack{\mathbf{b}\in\mathbb{Z}^r \\ b_i>b_{i+1}}} (1-q) c_{\mathbf{b}} x_{(r)} \sum_{t=0}^{b_i-b_{i+1}-1} X^{\mathbf{b}s_i + t\alpha_i}. \end{aligned}$$

For the $\mathbf{a} \in \mathbb{Z}^r$ chosen as above, comparing the coefficients of $x_{(r)}X^{\mathbf{a}}$ on both sides implies that

$$qc_{\mathbf{a}} = qc_{\mathbf{a}s_i} + c' + c'', \tag{B.1.1}$$

where c' (resp., c'') denotes the coefficient of $x_{(r)}X^{\mathbf{a}}$ in the second (resp., third) sum of the right hand side. In the following we calculate c' and c''.

Consider those $\mathbf{b} = (b_1, \ldots, b_r) \in \mathbb{Z}^r$ with $b_i < b_{i+1}$ such that

$$\mathbf{b}s_i - t\alpha_i = \mathbf{a} \text{ for some } 1 \leqslant t \leqslant b_{i+1} - b_i.$$

Then $b_i = a_{i+1} - t$ and $b_{i+1} = a_i + t$, and thus, $t \leqslant b_{i+1} - b_i = a_i - a_{i+1} + 2t$. Hence,

$$a_{i+1} - a_i \leqslant t.$$

Since $\mathbf{a}s_i - t\alpha_i = \mathbf{a} - (t - (a_{i+1} - a_i))\alpha_i$, by substituting t for $t - (a_{i+1} - a_i)$, we obtain that $\mathbf{b} = \mathbf{a} - t\alpha_i$ with $t \geqslant 0$.

Similarly, those $\mathbf{b} = (b_1, \ldots, b_r) \in \mathbb{Z}^r_m$ with $b_i > b_{i+1}$ such that

$$\mathbf{b}s_i + t\alpha_i = \mathbf{a} \text{ for some } 0 \leqslant t \leqslant b_i - b_{i+1} - 1$$

are simply $\mathbf{b} = \mathbf{a}s_i + t\alpha_i$ for all $t \geqslant 0$.

We conclude that

$$c' = (q-1)\sum_{t\geqslant 0} c_{\mathbf{a}-t\alpha_i} \quad\text{and}\quad c'' = (1-q)\sum_{t\geqslant 0} c_{\mathbf{a}s_i+t\alpha_i}.$$

Hence, we obtain from (B.1.1) that

$$\begin{aligned} qc_{\mathbf{a}} &= qc_{\mathbf{a}s_i} + (q-1)\sum_{t\geqslant 0} c_{\mathbf{a}-t\alpha_i} + (1-q)\sum_{t\geqslant 0} c_{\mathbf{a}s_i+t\alpha_i} \\ &= qc_{\mathbf{a}s_i} + (q-1)\big(c_{\mathbf{a}} + \sum_{t\geqslant 1} c_{\mathbf{a}-t\alpha_i}\big) + (1-q)\big(c_{\mathbf{a}s_i} + \sum_{t\geqslant 1} c_{\mathbf{a}s_i+t\alpha_i}\big). \end{aligned}$$

This together with the fact that $\mathbf{a}s_i + t\alpha_i = (\mathbf{a} - t\alpha_i)s_i$ implies that

$$c_{\mathbf{a}} - c_{\mathbf{a}s_i} = \sum_{t\geqslant 1}(q-1)\big(c_{\mathbf{a}-t\alpha_i} - c_{(\mathbf{a}-t\alpha_i)s_i}\big). \tag{B.1.2}$$

Since for each $t \geqslant 1$, $\mathbf{a} - t\alpha_i =: \mathbf{a}^{(t)} = (a_1^{(t)}, \dots, a_r^{(t)})$ satisfies

$$a_i^{(t)} = a_i - t < a_{i+1} + t = a_{i+1}^{(t)} \quad\text{and}\quad a_{i+1}^{(t)} = a_{i+1} + t \geqslant k+1,$$

we have by the induction hypothesis that $c_{\mathbf{a}-t\alpha_i} = c_{(\mathbf{a}-t\alpha_i)s_i}$ for all $t \geqslant 1$. Consequently, $c_{\mathbf{a}} = c_{\mathbf{a}s_i}$. □

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