THE INTEGRAL QUANTUM LOOP ALGEBRA OF $\mathfrak{gl}_n$

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Abstract. We will construct the Lusztig form for the quantum loop algebra of $\mathfrak{gl}_n$ by proving the conjecture [4, 3.8.6] and establish partially the Schur–Weyl duality at the integral level in this case. We will also investigate the integral form of the modified quantum affine $\mathfrak{gl}_n$ by introducing an affine stabilisation property and will lift the canonical bases from affine quantum Schur algebras to a canonical basis for this integral form. As an application of our theory, we will also discuss the integral form of the modified extended quantum affine $\mathfrak{sl}_n$ and construct its canonical basis to verify a conjecture of Lusztig in this case.

1. Introduction

Let $\mathcal{Z} = \mathbb{Z}[v, v^{-1}]$ be the integral Laurent polynomial ring. It is well known that the Lusztig form $U_\mathcal{Z}$ of a quantum enveloping $\mathbb{Q}(v)$-algebra $\mathbf{U}$ associated with a Cartan matrix of finite or affine type is a $\mathcal{Z}$-free subalgebra generated by divided powers of simple root vectors $E_{\alpha_i}, F_{\alpha_i}$ together with group-like elements $K^{\pm}_{\alpha_i^\vee}$. In particular, there is a triangular decomposition $U_\mathcal{Z} = U^+_\mathcal{Z} \cdot U^0_\mathcal{Z} \cdot U^-_\mathcal{Z}$ where, in the simply-laced case, the 0-part $U^0_\mathcal{Z}$ of this form is generated by $K_{\alpha_i^\vee}$ and $\left[{K_{\alpha_i^\vee};0 \atop t}\right]$.

We now consider the quantum loop algebra $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$. It contains a proper subalgebra $'\mathbf{U} = \mathbf{U}_\vartriangle(n)$ generated by $E_i = E_{\alpha_i}, F_i = F_{\alpha_i}$ and $K_i^{\pm}$, $1 \leqslant i \leqslant n$, where $K_iK_{i+1}^{-1} = K_{\alpha_i^\vee}$ with $K_{n+1} = K_1$. This is called the "extended" quantum affine $\mathfrak{sl}_n$ in [4] which is also investigated in [28] (cf. the definition in [28, 7.7]). Note that the subalgebra generated by E_i, F_i and $K_{\alpha_i^\vee}$ is usually called the quantum enveloping algebra of affine $\mathfrak{sl}_n$ type or the quantum loop algebra of $\mathfrak{sl}_n$ (see, e.g., [28, 9.3] or [4, §1.3]). If $'U^+_\mathcal{Z}$ (resp., $'U^-_\mathcal{Z}$) denotes the $\mathcal{Z}$-subalgebra generated by divided powers $E_i^{(m)}$ (resp., $F_i^{(m)}$) and $U^0_\mathcal{Z}$ denotes the $\mathcal{Z}$-subalgebra generated by K_i and $\left[{K_i;0 \atop t}\right]$ $(t \in \mathbb{N}, 1 \leqslant i \leqslant n)$, then the $\mathcal{Z}$-submodule $'U_\mathcal{Z} = {}'U^+_\mathcal{Z} \cdot U^0_\mathcal{Z} \cdot {}'U^-_\mathcal{Z}$ is a $\mathcal{Z}$-free subalgebra of $'\mathbf{U}$ which is the Lusztig form of $'\mathbf{U}$ mentioned above. Now, naturally, one would ask what is a natural Lusztig form for $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$?

By using Drinfeld's presentation for $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$, a so-called restricted integral form $U_v^{\rm res}(\widehat{\mathfrak{gl}}_n)$ was constructed over $\mathbb{C}[v, v^{-1}]$ by Frenkel–Mukhin in [13, §7.2]. However, it is not clear from the construction whether $U_v^{\rm res}(\widehat{\mathfrak{gl}}_n)$ is a Hopf algebra. Another integral form is constructed in [4,

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Supported by the National Natural Science Foundation of China, Fok Ying Tung Education Foundation and the Australian Research Council DP120101436.

2.4.4] by using a double Ringel–Hall algebra presentation for $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$. This integral form is the tensor product of the Lusztig form $'U_{\mathcal{Z}}$ of $'\mathbf{U}$ with an integral central subalgebra. This is a Hopf subalgebra but not large enough to have integral affine quantum Schur algebras as its quotients; see example [4, 5.3.8].

However, there is a natural candidate constructed in [4, §3.8]. By the double Ringel–Hall algebra presentation, we have a triangular decomposition: $\mathbf{U}(\widehat{\mathfrak{gl}}_n) \cong \boldsymbol{\mathfrak{D}}_\vartriangle(n) = \boldsymbol{\mathfrak{H}}_\vartriangle(n) \cdot \mathbf{U}^0 \cdot \boldsymbol{\mathfrak{H}}_\vartriangle(n)^{\mathrm{op}}$, where $\boldsymbol{\mathfrak{H}}_\vartriangle(n)$ is a Ringel–Hall algebra over $\mathbb{Q}(v)$ associated with a cyclic quiver and $\mathbf{U}^0 = \mathbb{Q}(v)[K_1^{\pm 1}, \dots, K_n^{\pm 1}]$ is the 0-part of $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$. The candidate we proposed is to use the (integral) Ringel–Hall algebra $\mathfrak{H}_\vartriangle(n)_{\mathcal{Z}}$ over $\mathcal{Z}$ and the 0-part $U^0_{\mathcal{Z}}$ defined above to form the $\mathcal{Z}$-free submodule[1] $\mathfrak{D}_\vartriangle(n)_{\mathcal{Z}} := \mathfrak{H}_\vartriangle(n)_{\mathcal{Z}} \cdot U^0_{\mathcal{Z}} \cdot \mathfrak{H}_\vartriangle(n)_{\mathcal{Z}}{}^{\mathrm{op}}$. We conjectured in [4, 3.8.6] that $\mathfrak{D}_\vartriangle(n)_{\mathcal{Z}}$ is a $\mathcal{Z}$-subalgebra of $\boldsymbol{\mathfrak{D}}_\vartriangle(n)$. If the conjecture is true, then $\mathfrak{D}_\vartriangle(n)_{\mathcal{Z}}$ is a Hopf subalgebra having integral affine quantum Schur algebras as its quotients.

In this paper, we will prove this conjecture. The proof is a beautiful application of a recent resolution of another conjecture, a realisation conjecture for quantum affine $\mathfrak{gl}_n$, by the authors [11], together with some successful attempts in the classical case [14, 15] (see also [16]). The realisation conjecture is a natural affine generalisation of a new construction for quantum $\mathfrak{gl}_n$ via quantum Schur algebras by A.A. Beilinson, G. Lusztig and R. MacPherson (BLM) in [1]. This remarkable work has important applications to the investigation of integral quantum Schur–Weyl reciprocity [12]. This reciprocity at non-roots of unity was formulated in [20] and its integral version was given in [8, 12], built on the work [1] and the Kazhdan–Lusztig cell theory.

Attempts to generalise the BLM work have been made by Ginzburg–Vasserot [18], Lusztig [28], etc. These constructions are geometric in nature, following BLM's geometric construction, but cannot resolve a realisation for the entire quantum affine $\mathfrak{gl}_n$. The main obstacle is that $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$ cannot be generated by simple root vectors or simple generators. In [11], we discovered certain key multiplication formulas by semisimple generators via the affine Hecke algebra and affine quantum Schur algebras. This allows, by modifying BLM's approach, to introduce a new algebra $\boldsymbol{\mathcal{V}}_\vartriangle(n)$ by a basis together with explicit multiplication formuas of basis elements by semisimple generators. This algebra is isomorphic to $\boldsymbol{\mathfrak{D}}_\vartriangle(n)$ and hence to $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$.

We now construct an integral $\mathcal{Z}$-subalgebra $\mathcal{V}_\vartriangle(n)_{\mathcal{Z}}$ of $\boldsymbol{\mathcal{V}}_\vartriangle(n)$ and then prove that the image of $\mathcal{V}_\vartriangle(n)_{\mathcal{Z}}$ in $\boldsymbol{\mathfrak{D}}_\vartriangle(n)$ coincides with $\mathfrak{D}_\vartriangle(n)_{\mathcal{Z}}$. In this way we prove that $\mathfrak{D}_\vartriangle(n)_{\mathcal{Z}}$ is a subalgebra. As an immediate application, the $\mathbb{Q}(v)$-algebra epimorphism ζ_r given in [4, Th. 3.8.1] restricts to a $\mathcal{Z}$-algebra epimorphism ζ_r from $\mathfrak{D}_\vartriangle(n)_{\mathcal{Z}}$ to the affine quantum Schur algebra $\mathcal{S}_\vartriangle(n,r)_{\mathcal{Z}}$. This establishes partially the Schur–Weyl duality at the integral level and, hence, at roots of unity.

[1]It is denoted by $\widetilde{\mathfrak{D}}_\vartriangle(n)$ in [4, (3.8.1.1)], while $\mathfrak{D}_\vartriangle(n)$ denote the tensor product of $'U_{\mathcal{Z}}$ with the integral central subalgebra in [4, 2.4.4].

There is another application of the key multiplication formulas mentioned above. In [1], the $\mathbb{Q}(v)$-algebra $\mathbf{K}(n)$ was constructed as a result of a stabilisation property. The algebra $\mathbf{K}(n)$ is in fact isomorphic to the modified quantum group $\dot{\mathbf{U}}(\mathfrak{gl}_n)$. We will prove that a stabilisation property continue to hold in the affine case. Thus, we may also introduce a new $\mathbb{Q}(v)$-algebra $\mathbf{K}_\triangle(n)$, which is isomorphic to the modified quantum group $\dot{\mathbf{U}}(\widehat{\mathfrak{gl}}_n)$, and realise $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$ as a subalgebra of the completion algebra $\hat{\mathbf{K}}_\triangle(n)$. In this way, we obtain an (unmodified!) affine generalisation of BLM's construction. We will further discuss the integral form $\mathsf{K}_\triangle(n)_\mathcal{Z}$ of $\mathbf{K}_\triangle(n)$ which is a realisation of $\dot{\mathfrak{D}}_\triangle(n)_\mathcal{Z}$ (see Theorem 6.6) and construct its canonical basis as a lifting of the canonical bases for affine quantum Schur algebras. Applying our theory to the extended quantum affine $\mathfrak{sl}_n$, we will introduce the canonical basis for the modified quantum group $\dot{\mathbf{U}}_\triangle(n)$ and verify in this case a conjecture of Lusztig [28, 9.3] which has been already proved in [32] (cf. [29, 7.9]).

The sections of the paper are organised as follows:

1. Introduction
2. The double Ringel–Hall algebra presentation
3. A BLM type presentation
4. Some integral multiplication formulas
5. Lusztig form of $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$ and integral affine quantum Schur–Weyl reciprocity
6. The affine BLM algebra $\mathsf{K}_\triangle(n)_\mathcal{Z}$
7. Canonical bases for the integral modified quantum affine $\mathfrak{gl}_n$
8. Application to a conjecture of Lusztig.

Notation 1.1. For a positive integer n, let $\Theta_\triangle(n)$ (resp., $\widetilde{\Theta}_\triangle(n)$) be the set of all matrices $A=(a_{i,j})_{i,j\in\mathbb{Z}}$ with $a_{i,j}\in\mathbb{N}$ (resp. $a_{i,j}\in\mathbb{Z}$, $a_{i,j}\geqslant 0$ for all $i\neq j$) such that

(a) $a_{i,j}=a_{i+n,j+n}$ for $i,j\in\mathbb{Z}$;
(b) for every $i\in\mathbb{Z}$, both sets $\{j\in\mathbb{Z}\mid a_{i,j}\neq 0\}$ and $\{j\in\mathbb{Z}\mid a_{j,i}\neq 0\}$ are finite.

Let $\mathbb{Z}^n_\triangle=\{(\lambda_i)_{i\in\mathbb{Z}}\mid \lambda_i\in\mathbb{Z},\ \lambda_i=\lambda_{i-n}\text{ for } i\in\mathbb{Z}\}$ and $\mathbb{N}^n_\triangle=\{(\lambda_i)_{i\in\mathbb{Z}}\in\mathbb{Z}^n_\triangle\mid \lambda_i\geqslant 0\text{ for } i\in\mathbb{Z}\}$. We will sometimes identify $\mathbb{Z}^n_\triangle$ with $\mathbb{Z}^n$ via the natural bijection $\flat:\mathbb{Z}^n_\triangle\longrightarrow\mathbb{Z}^n$ defined by sending $\mathbf{j}$ to $\flat(\mathbf{j})=(j_1,\cdots,j_n)$. Define an order relation $\leqslant$ and "dot" product on $\mathbb{Z}^n_\triangle$ by

$$\lambda\leqslant\mu \iff \lambda_i\leqslant\mu_i\,(1\leqslant i\leqslant n) \quad\text{and}\quad \lambda\centerdot\mu=\lambda_1\mu_1+\cdot+\lambda_n\mu_n=\flat(\lambda)\centerdot\flat(\mu). \tag{1.1.1}$$

We say that $\lambda<\mu$ if $\lambda\leqslant\mu$ and $\lambda\neq\mu$.

Let $\mathbb{Q}(v)$ be the fraction field of $\mathcal{Z}=\mathbb{Z}[v,v^{-1}]$. For integers N,t with $t\geqslant 0$, define Gaussian polynomials and their symmetric version in $\mathcal{Z}$: $\left[\!\!\left[{N \atop t} \right]\!\!\right]=\prod_{1\leqslant i\leqslant t}\frac{v^{2(N-i+1)}-1}{v^{2i}-1}$ and $\left[{N \atop t} \right]=v^{-t(N-t)}\left[\!\!\left[{N \atop t} \right]\!\!\right]$. For $\mu\in\mathbb{Z}^n_\triangle$ and $\lambda\in\mathbb{N}^n_\triangle$, let $[\![{\mu \atop \lambda}]\!]=\prod_{1\leqslant i\leqslant n}[\![{\mu_i \atop \lambda_i}]\!]$ and let $[{\mu \atop \lambda}]=\prod_{1\leqslant i\leqslant n}[{\mu_i \atop \lambda_i}]$. The following identity holds (see [16, 3.3]): for any $\lambda,\mu\in\mathbb{N}^n_\vartriangle$ and $\alpha,\beta\in\mathbb{Z}^n_\vartriangle$,

$$\begin{aligned}&\begin{bmatrix}\alpha+\beta\\ \lambda\end{bmatrix}=\sum_{\mu\in\mathbb{N}^n_\vartriangle,\mu\leqslant\lambda}v^{\alpha\centerdot(\lambda-\mu)-\mu\centerdot\beta}\begin{bmatrix}\alpha\\ \mu\end{bmatrix}\begin{bmatrix}\beta\\ \lambda-\mu\end{bmatrix};\\ &\begin{bmatrix}\alpha\\ \lambda\end{bmatrix}\begin{bmatrix}\alpha\\ \mu\end{bmatrix}=\sum_{\substack{\gamma\in\mathbb{N}^n_\vartriangle\\ \gamma\leqslant\lambda,\gamma\leqslant\mu}}v^{\lambda\centerdot\mu-\alpha\centerdot\gamma}\begin{bmatrix}\alpha\\ \lambda+\mu-\gamma\end{bmatrix}\begin{bmatrix}\lambda+\mu-\gamma\\ \gamma,\lambda-\gamma,\mu-\gamma\end{bmatrix};\end{aligned}\tag{1.1.2}$$

see [5, Exercises 0.14 and 0.15] for a proof in the case of Gaussian polynomials.

2. The Double Ringel–Hall algebra presentation

Let $\vartriangle(n)$ ($n\geqslant 2$) be the cyclic quiver with vertex set $I=\mathbb{Z}/n\mathbb{Z}=\{1,2,\ldots,n\}$ and arrow set $\{i\to i+1\mid i\in I\}$. Note that we will regard I as an abelian group as well as a subset of $\mathbb{Z}$ depending on context.

Let $\mathbb{F}$ be a field. For $i\in I$ and $j\in\mathbb{Z}$ with $i<j$, let S_i denote the one-dimensional representation of $\vartriangle(n)$ with $(S_i)_i=\mathbb{F}$ and $(S_i)_k=0$ for $i\neq k$ and $M^{i,j}$ the unique indecomposable nilpotent representation of dimension $j-i$ with top S_i. Let

$$\Theta^+_\vartriangle(n)=\{A\in\Theta_\vartriangle(n)\mid a_{i,j}=0\text{ for }i\geqslant j\}.$$

Lemma 2.1. *For any $A=(a_{i,j})\in\Theta^+_\vartriangle(n)$, let*

$$M(A)=M_{\mathbb{F}}(A)=\bigoplus_{1\leqslant i\leqslant n,i<j}a_{i,j}M^{i,j}.\tag{2.1.1}$$

Then $\mathcal{M}=\{[M(A)]\}_{A\in\Theta^+_\vartriangle(n)}$ forms a complete set of isomorphism classes of finite dimensional nilpotent representations of $\vartriangle(n)$.

Let $\mathbf{d}(A)\in\mathbb{N}I=\mathbb{N}^n$ be the dimension vector of $M(A)$. For $\boldsymbol{a}=(a_i)\in\mathbb{Z}^n_\vartriangle$ and $\boldsymbol{b}=(b_i)\in\mathbb{Z}^n_\vartriangle$, the Euler form associated with the cyclic quiver $\vartriangle(n)$ is the bilinear form $\langle -,-\rangle:\mathbb{Z}^n_\vartriangle\times\mathbb{Z}^n_\vartriangle\to\mathbb{Z}$ defined by

$$\langle\boldsymbol{a},\boldsymbol{b}\rangle=\sum_{i\in I}a_ib_i-\sum_{i\in I}a_ib_{i+1}.$$

By [31], for $A,B,C\in\Theta^+_\vartriangle(n)$, the Hall polynomial $\varphi^C_{A,B}\in\mathbb{Z}[v^2]$ is defined such that, for any finite field $\mathbb{F}_q$, $\varphi^C_{A,B}|_{v^2=q}$ is equal to the number of submodules N of $M_{\mathbb{F}_q}(C)$ satisfying $N\cong M_{\mathbb{F}_q}(B)$ and $M_{\mathbb{F}_q}(C)/N\cong M_{\mathbb{F}_q}(A)$.

The (generic) twisted *Ringel–Hall algebra* $\mathfrak{H}_\vartriangle(n)_\mathcal{Z}$ of $\vartriangle(n)$ is, by definition, the $\mathcal{Z}$-algebra spanned by basis $\{u_A=u_{[M(A)]}\mid A\in\Theta^+_\vartriangle(n)\}$ whose multiplication is defined by, for all $A,B\in\Theta^+_\vartriangle(n)$,

$$u_Au_B=v^{\langle\mathbf{d}(A),\mathbf{d}(B)\rangle}\sum_{C\in\Theta^+_\vartriangle(n)}\varphi^C_{A,B}u_C.$$

Base change gives the $\mathbb{Q}(v)$-algebra $\boldsymbol{\mathfrak{H}}_\vartriangle(n)=\mathfrak{H}_\vartriangle(n)_\mathcal{Z}\otimes\mathbb{Q}(v)$.

We now describe the semisimple generators $u_\lambda = u_{[S_\lambda]}$ ($\lambda \in \mathbb{N}_\triangle^n$) of $\mathfrak{H}_\triangle(n)_\mathcal{Z}$, where $S_\lambda := \oplus_{i=1}^n \lambda_i S_i$ is a semisimple representation of $\triangle(n)$.[2]

On the set $\mathcal{M}$ of isoclasses of finite dimensional nilpotent representations of $\triangle(n)$, define a multiplication $*$ by $[M] * [N] = [M * N]$ for any $[M], [N] \in \mathcal{M}$, where $M * N$ is the generic extension of M by N. By [3, 30] $\mathcal{M}$ is a monoid with identity $1 = [0]$.

An element λ in $\mathbb{N}_\triangle^n$ is called *sincere* if $\lambda_i > 0$ for all $i \in \mathbb{Z}$. For $1 \leqslant i \leqslant n$ let $\boldsymbol{e}_i^\triangle \in \mathbb{N}_\triangle^n$ be the element satisfying $(\boldsymbol{e}_i^\triangle)_j = \delta_{\bar{i},\bar{j}}$ for $j \in \mathbb{Z}$. Here $\bar{i}$ is the congruence class of i modulo n. Let

$$\widetilde{I} = \{\boldsymbol{e}_1^\triangle, \boldsymbol{e}_2^\triangle, \cdots, \boldsymbol{e}_n^\triangle\} \cup \{\text{all sincere vectors in } \mathbb{N}_\triangle^n\}.$$

Let $\widetilde{\Sigma}$ be the set of words on the alphabet $\widetilde{I}$.

There is a natural surjective map $\wp^+ : \widetilde{\Sigma} \to \Theta_\triangle^+(n)$ ([6, 3.3]) by taking $w = \boldsymbol{a}_1\boldsymbol{a}_2\cdots\boldsymbol{a}_m$ to $\wp^+(w)$, where $\wp^+(w) \in \Theta_\triangle^+(n)$ is defined by

$$[S_{\boldsymbol{a}_1}] * \cdots * [S_{\boldsymbol{a}_m}] = [M(\wp^+(w))].$$

For $A \in \Theta_\triangle^+(n)$, let

$$\widetilde{u}_A = v^{\dim \mathrm{End}(M(A)) - \dim M(A)} u_A.$$

For $\lambda \in \mathbb{N}_\triangle^n$ let $\widetilde{u}_\lambda = \widetilde{u}_{[S_\lambda]}$. Any word $w = \boldsymbol{a}_1\boldsymbol{a}_2\cdots\boldsymbol{a}_m$ in $\widetilde{\Sigma}$ can be uniquely expressed in the *tight form* $w = \boldsymbol{b}_1^{x_1}\boldsymbol{b}_2^{x_2}\cdots\boldsymbol{b}_t^{x_t}$ where $x_i = 1$ if $\boldsymbol{b}_i$ is sincere, and x_i is the number of consecutive occurrences of $\boldsymbol{b}_i$ if $\boldsymbol{b}_i \in \{\boldsymbol{e}_1^\triangle, \boldsymbol{e}_2^\triangle, \cdots, \boldsymbol{e}_n^\triangle\}$. For $w = \boldsymbol{a}_1\boldsymbol{a}_2\cdots\boldsymbol{a}_m \in \widetilde{\Sigma}$ with the tight form $\boldsymbol{b}_1^{x_1}\boldsymbol{b}_2^{x_2}\cdots\boldsymbol{b}_t^{x_t}$, define the associated monomials:

$$\widetilde{u}_{(w)} = \widetilde{u}_{x_1\boldsymbol{b}_1}\widetilde{u}_{x_2\boldsymbol{b}_2}\cdots\widetilde{u}_{x_t\boldsymbol{b}_t} \in \mathfrak{H}_\triangle(n)_\mathcal{Z}.$$

Following [1, 3.5] and [10] we may define the order relation $\preccurlyeq$ on $M_{\triangle,n}(\mathbb{Z})$ as follows. For $A \in M_{\triangle,n}(\mathbb{Z})$ and $i \neq j \in \mathbb{Z}$, let

$$\sigma_{i,j}(A) = \begin{cases} \sum_{s \leqslant i, t \geqslant j} a_{s,t}, & \text{if } i < j; \\ \sum_{s \geqslant i, t \leqslant j} a_{s,t}, & \text{if } i > j. \end{cases}$$

For $A, B \in M_{\triangle,n}(\mathbb{Z})$, define

$$B \preccurlyeq A \text{ if and only if } \sigma_{i,j}(B) \leqslant \sigma_{i,j}(A) \text{ for all } i \neq j. \tag{2.1.2}$$

Put $B \prec A$ if $B \preccurlyeq A$ and, for some pair (i, j) with $i \neq j$, $\sigma_{i,j}(B) < \sigma_{i,j}(A)$.

Associated each $A \in \Theta_\triangle^+(n)$ to a *distinguished word* w_A (see [6, (9.1)]), the following *triangular relation* relative to $\preccurlyeq$ between the monomial basis $\{\widetilde{u}_{(w_A)}\}_{A \in \Theta_\triangle^+(n)}$ and the defining basis $\{\widetilde{u}_A\}_{A \in \Theta_\triangle^+(n)}$ holds (see [6, (9.2)], [10, 6.2]):

[2] For emphasising on semisimple generators, we will use the same notation to denote the matrix in (2.1.1) defining S_λ; see, e.g., Theorem 3.3.

Proposition 2.2. *For $A\in\Theta_\triangle^+(n)$, there exist $w_A\in\widetilde{\Sigma}$ such that $\wp^+(w_A)=A$ and*

$$\widetilde{u}_{(w_A)}=\widetilde{u}_A+\sum_{\substack{B\in\Theta_\triangle^+(n)\\ B\prec A,\,\mathbf{d}(A)=\mathbf{d}(B)}} f_{B,A}\widetilde{u}_B. \tag{2.2.1}$$

where $f_{B,A^+}\in\mathcal{Z}$. In particular, $\mathfrak{H}_\triangle(n)_\mathcal{Z}$ is generated by u_λ for $\lambda\in\mathbb{N}_\triangle^n$ with a monomial basis $\{\widetilde{u}_{(w_A)}\mid A\in\Theta_\triangle^+(n)\}$.

The Hall algebra and its opposite algebra can be used to describe the $\pm$-part of quantum affine $\mathfrak{gl}_n$. Let $\mathfrak{D}_\triangle(n)$ be the (reduced) double Ringel–Hall algebra of the cyclic quiver $\triangle(n)$ over $\mathbb{Q}(v)$ (cf. [34] and [4, (2.1.3.2)]). Then it has a triangular decomposition:

$$\mathfrak{D}_\triangle(n)\cong\mathfrak{D}_\triangle^+(n)\otimes\mathfrak{D}_\triangle^0(n)\otimes\mathfrak{D}_\triangle^-(n)$$

with $\mathfrak{D}_\triangle^+(n)=\mathfrak{H}_\triangle(n)$, $\mathfrak{D}_\triangle^-(n)=\mathfrak{H}_\triangle(n)^{\mathrm{op}}$, and $\mathfrak{D}_\triangle^0(n)=\mathbb{Q}(v)[K_1^{\pm1},\dots,K_n^{\pm1}]$. We will add superscript $+$ or $-$ to u_A, u_λ, $u_{(w)}$, etc. for the corresponding objects in $\mathfrak{D}_\triangle^+(n)$ or $\mathfrak{D}_\triangle^-(n)$. Thus, $\mathfrak{D}_\triangle(n)^\pm$ has basis $\{\widetilde{u}_A^\pm\}_{A\in\Theta_\triangle^+(n)}$, generators $u_\lambda^\pm$, $\lambda\in\mathbb{N}_\triangle^n$ and monomials $\widetilde{u}_{(w)}^\pm$.

Note that it is also natural to use the notation $\{\widetilde{u}_A:=\widetilde{u}_A^+\}_{A\in\Theta_\triangle^+(n)}$ for a basis for $\mathfrak{D}_\triangle^+(n)$ and the notation $\{\widetilde{u}_B:=\widetilde{u}_{{}^tB}^-\}_{B\in\Theta_\triangle^-(n)}$ for the corresponding basis for $\mathfrak{D}_\triangle^-(n)$, where

$$\Theta_\triangle^-(n)=\{A\in\Theta_\triangle(n)\mid a_{i,j}=0\text{ for }i\leqslant j\}.$$

With such notations, the matrix transpose induces the anti-isomorphism

$$\tau:\mathfrak{D}_\triangle^+(n)\longrightarrow\mathfrak{D}_\triangle^-(n),\quad \widetilde{u}_A\longmapsto\widetilde{u}_{{}^tA}. \tag{2.2.2}$$

For $A\in\widetilde{\Theta}_\triangle(n)$, we write

$$A=A^++A^0+A^-,\qquad A^\pm=A^++A^- \tag{2.2.3}$$

where $A^+\in\Theta_\triangle^+(n)$, $A^-\in\Theta_\triangle^-(n)$ and A^0 is a diagonal matrix.

We have the following (not so elegant) presentation for quantum affine $\mathfrak{gl}_n$ via the double Ringel–Hall algebra; see [4, 2.5.3, 2.6.1, 2.6.7 and 2.3.6(2)].

Theorem 2.3. (1) *The (Hopf) algebra $\mathfrak{D}_\triangle(n)$ is isomorphic to Drinfeld's algebra $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$. It is the algebra over $\mathbb{Q}(v)$ which is spanned by basis*

$$\{u_A^+K^{\mathbf{j}}u_B^-\mid A,B\in\Theta_\triangle^+(n),\mathbf{j}\in\mathbb{Z}_\triangle^n\},\ \text{where } K^{\mathbf{j}}=K_1^{j_1}\cdots K_n^{j_n},$$

and is generated by u_λ^+, $K_i^{\pm1}$, u_μ^- ($\lambda,\mu\in\mathbb{N}_\triangle^n$, $1\leqslant i\leqslant n$), and whose multiplication is given by the following relations:

(a) $K_iK_j=K_jK_i$, $K_iK_i^{-1}=1$;
(b) $K^{\mathbf{j}}u_A^+=v^{\langle\mathbf{d}(A),\mathbf{j}\rangle}u_A^+K^{\mathbf{j}}$, $u_A^-K^{\mathbf{j}}=v^{\langle\mathbf{d}(A),\mathbf{j}\rangle}K^{\mathbf{j}}u_A^-$;
(c) $u_\lambda^+u_A^+=\sum_{C\in\Theta_\triangle^+(n)}v^{\langle\lambda,\mathbf{d}(A)\rangle}\varphi_{S_\lambda,A}^Cu_C^+$;
(d) $u_\mu^-u_A^-=\sum_{C\in\Theta_\triangle^+(n)}v^{\langle\mathbf{d}(A),\mu\rangle}\varphi_{A,S_\mu}^Cu_C^-$;

(e) $u_\mu^- u_\lambda^+ - u_\lambda^+ u_\mu^- = \sum_{\substack{\alpha\neq 0,\,\alpha\in\mathbb{N}_\vartriangle^n\\ \alpha\leqslant\lambda,\,\alpha\leqslant\mu}} \sum_{0\leqslant\gamma\leqslant\alpha} x_{\alpha,\gamma} \widetilde{K}^{2\gamma-\alpha} u_{\lambda-\alpha}^+ u_{\mu-\alpha}^-$, *where the coefficients* $x_{\alpha,\gamma}\in\mathcal{Z}$ *are rather complicated as given in* [4, Cor. 2.6.7].

(2) *There exists a central subalgebra* $\mathbf{Z}_\vartriangle(n) = \mathbb{Q}(v)[\mathsf{z}_m^+, \mathsf{z}_m^-]_{m\geqslant 1}$ *such that* $\mathfrak{D}_\vartriangle(n) \cong \mathbf{U}_\vartriangle(n) \otimes \mathbf{Z}_\vartriangle(n)$, *where* $\mathbf{U}_\vartriangle(n)$ *is the subalgebra generated by* $E_i = u^+_{\boldsymbol{e}_i^\vartriangle}, F_i = u^-_{\boldsymbol{e}_i^\vartriangle}, K_i$ *for all* $i\in I$.

We now define a candidate of the Lusztig form of $\mathfrak{D}_\vartriangle(n)$.

Let $\mathfrak{D}_\vartriangle^+(n)_\mathcal{Z} \cong \mathfrak{H}_\vartriangle(n)_\mathcal{Z}$ (resp., $\mathfrak{D}_\vartriangle^-(n)_\mathcal{Z} \cong \mathfrak{H}_\vartriangle(n)_\mathcal{Z}{}^{\mathrm{op}}$) be the $\mathcal{Z}$-submodule of $\mathfrak{D}_\vartriangle(n)$ spanned by the elements u_A^+ (resp., u_A^-) for $A\in\Theta_\vartriangle^+(n)$, and let $\mathfrak{D}_\vartriangle^0(n)_\mathcal{Z}$ be the $\mathcal{Z}$-subalgebra of $\mathfrak{D}_\vartriangle(n)$ generated by $K_i^{\pm 1}$ and $\left[{K_i;0 \atop t}\right]$, for $i\in I$ and $t\in\mathbb{N}$, where

$$\begin{bmatrix} K_i;0 \\ t \end{bmatrix} = \prod_{s=1}^t \frac{K_i v^{-s+1} - K_i^{-1} v^{s-1}}{v^s - v^{-s}}.$$

Let $\mathfrak{D}_\vartriangle(n)_\mathcal{Z} = \mathfrak{D}_\vartriangle^+(n)_\mathcal{Z} \mathfrak{D}_\vartriangle^0(n)_\mathcal{Z} \mathfrak{D}_\vartriangle^-(n)_\mathcal{Z}$. We will prove in Theorem 5.6 that $\mathfrak{D}_\vartriangle(n)_\mathcal{Z}$ is a $\mathcal{Z}$-subalgebra of $\mathfrak{D}_\vartriangle(n)$ and give a realisation for $\mathfrak{D}_\vartriangle(n)_\mathcal{Z}$.

3. A BLM type presentation

We now describe a better presentation for $\mathfrak{D}_\vartriangle(n)$. Let $\mathfrak{S}_{\vartriangle,r}$ be the group consisting of all permutations $w:\mathbb{Z}\to\mathbb{Z}$ such that $w(i+r)=w(i)+r$ for $i\in\mathbb{Z}$. The extended affine Hecke algebra $\mathcal{H}_\vartriangle(r)_\mathcal{Z}$ over $\mathcal{Z}$ associated to $\mathfrak{S}_{\vartriangle,r}$ is the (unital) $\mathcal{Z}$-algebra with basis $\{T_w\}_{w\in\mathfrak{S}_{\vartriangle,r}}$, and multiplication defined by

$$\begin{cases} T_{s_i}^2 = (v^2-1)T_{s_i} + v^2, & \text{for } 1\leqslant i\leqslant r \\ T_w T_{w'} = T_{ww'}, & \text{if } \ell(ww') = \ell(w)+\ell(w'), \end{cases}$$

where $s_i\in\mathfrak{S}_{\vartriangle,r}$ is defined by setting $s_i(j)=j$ for $j\not\equiv i, i+1 \bmod r$, $s_i(j)=j-1$ for $j\equiv i+1 \bmod r$ and $s_i(j)=j+1$ for $j\equiv i \bmod r$. Let $\boldsymbol{\mathcal{H}}_\vartriangle(r) = \mathcal{H}_\vartriangle(r)_\mathcal{Z}\otimes_\mathcal{Z}\mathbb{Q}(v)$.

For $\lambda=(\lambda_i)_{i\in\mathbb{Z}}\in\mathbb{Z}_\vartriangle^n$ let $\sigma(\lambda)=\sum_{1\leqslant i\leqslant n}\lambda_i$. For $r\geqslant 0$ we set

$$\Lambda_\vartriangle(n,r) = \{\lambda\in\mathbb{N}_\vartriangle^n \mid \sigma(\lambda)=r\}.$$

For $\lambda\in\Lambda_\vartriangle(n,r)$, let $\mathfrak{S}_\lambda := \mathfrak{S}_{(\lambda_1,\ldots,\lambda_n)}$ be the corresponding standard Young subgroup of $\mathfrak{S}_r$. For each $\lambda\in\Lambda_\vartriangle(n,r)$, let $x_\lambda = \sum_{w\in\mathfrak{S}_\lambda} T_w \in \mathcal{H}_\vartriangle(r)_\mathcal{Z}$. The endomorphism algebras

$$\mathcal{S}_\vartriangle(n,r)_\mathcal{Z} := \mathrm{End}_{\mathcal{H}_\vartriangle(r)_\mathcal{Z}}\Big(\bigoplus_{\lambda\in\Lambda_\vartriangle(n,r)} x_\lambda\mathcal{H}_\vartriangle(r)_\mathcal{Z}\Big) \text{ and } \boldsymbol{\mathcal{S}}_\vartriangle(n,r) := \mathrm{End}_{\boldsymbol{\mathcal{H}}_\vartriangle(r)}\Big(\bigoplus_{\lambda\in\Lambda_\vartriangle(n,r)} x_\lambda\boldsymbol{\mathcal{H}}_\vartriangle(r)\Big).$$

are called affine quantum Schur algebras (cf. [18, 19, 28]).

For $A\in\widetilde{\Theta}_\vartriangle(n)$ and $r\geqslant 0$, let

$$\sigma(A) = \sum_{1\leqslant i\leqslant n,\, j\in\mathbb{Z}} a_{i,j} \quad\text{and}\quad \Theta_\vartriangle(n,r) = \{A\in\Theta_\vartriangle(n) \mid \sigma(A)=r\}.$$

For $\lambda \in \Lambda_\triangle(n,r)$, let $\mathscr{D}_\lambda^\triangle = \{d \mid d \in \mathfrak{S}_{\triangle,r}, \ell(wd) = \ell(w)+\ell(d) \text{ for } w \in \mathfrak{S}_\lambda\}$ and $\mathscr{D}_{\lambda,\mu}^\triangle = \mathscr{D}_\lambda^\triangle \cap \mathscr{D}_\mu^{\triangle\,-1}$. By [33, 7.4] (see also [10, 9.2]), there is a bijective map

$$\jmath_\triangle : \{(\lambda, d, \mu) \mid d \in \mathscr{D}_{\lambda,\mu}^\triangle, \lambda, \mu \in \Lambda_\triangle(n,r)\} \longrightarrow \Theta_\triangle(n,r)$$

sending (λ, d, μ) to the matrix $A = (|R_k^\lambda \cap dR_l^\mu|)_{k,l\in\mathbb{Z}}$, where

$$R_{i+kn}^\nu = \{\nu_{k,i-1}+1, \nu_{k,i-1}+2, \ldots, \nu_{k,i-1}+\nu_i = \nu_{k,i}\} \ \text{ with } \ \nu_{k,i-1} = kr + \sum_{1\leqslant t\leqslant i-1} \nu_t,$$

for all $1 \leqslant i \leqslant n$, $k \in \mathbb{Z}$ and $\nu \in \Lambda_\triangle(n,r)$.

For $\lambda, \mu \in \Lambda_\triangle(n,r)$ and $d \in \mathscr{D}_{\lambda,\mu}^\triangle$ satisfying $A = \jmath_\triangle(\lambda, d, \mu) \in \Theta_\triangle(n,r)$, define $e_A \in \mathcal{S}_\triangle(n,r)_\mathcal{Z}$ by

$$e_A(x_\nu h) = \delta_{\mu\nu} \sum_{w\in\mathfrak{S}_\lambda d\mathfrak{S}_\mu} T_w h, \tag{3.0.1}$$

where $\nu \in \Lambda_\triangle(n,r)$ and $h \in \mathcal{H}_\triangle(r)_\mathcal{Z}$, and let

$$[A] = v^{-d_A} e_A, \quad \text{ where } \quad d_A = \sum_{\substack{1\leqslant i\leqslant n\\ i\geqslant k, j<l}} a_{i,j}a_{k,l}. \tag{3.0.2}$$

Note that the sets $\{e_A \mid A \in \Theta_\triangle(n,r)\}$ and $\{[A] \mid A \in \Theta_\triangle(n,r)\}$ form $\mathcal{Z}$-bases for $\mathcal{S}_\triangle(n,r)_\mathcal{Z}$.

Let

$$\Theta_\triangle^\pm(n) = \{A \in \Theta_\triangle(n) \mid a_{i,i} = 0 \text{ for all } i\}.$$

For $A \in \Theta_\triangle^\pm(n)$, $\mathbf{j} \in \mathbb{Z}_\triangle^n$ and $\lambda \in \mathbb{N}_\triangle^n$ let

$$A(\mathbf{j}, r) = \sum_{\mu\in\Lambda_\triangle(n,r-\sigma(A))} v^{\mu\cdot\mathbf{j}} [A + \mathrm{diag}(\mu)] \in \mathcal{S}_\triangle(n,r)_\mathcal{Z}.$$

$$A(\mathbf{j}, \lambda, r) = \sum_{\mu\in\Lambda_\triangle(n,r-\sigma(A))} v^{\mu\cdot\mathbf{j}} \begin{bmatrix} \mu \\ \lambda \end{bmatrix} [A + \mathrm{diag}(\mu)] \in \mathcal{S}_\triangle(n,r)_\mathcal{Z}$$

The relationship between $\boldsymbol{\mathfrak{D}}_\triangle(n)$ and $\boldsymbol{\mathcal{S}}_\triangle(n,r)$ can be seen from the following (cf. [18, 28] and [33, Prop. 7.6]).

Theorem 3.1 ([4, 3.6.3, 3.8.1])**.** *For $r \geqslant 0$, the map $\zeta_r : \boldsymbol{\mathfrak{D}}_\triangle(n) \to \boldsymbol{\mathcal{S}}_\triangle(n,r)$ satisfying*

$$\zeta_r(K^{\mathbf{j}}) = 0(\mathbf{j}, r), \ \zeta_r(\widetilde{u}_A^+) = A(\mathbf{0}, r), \ \text{ and } \ \zeta_r(\widetilde{u}_A^-) = ({}^tA)(\mathbf{0}, r),$$

for all $\mathbf{j} \in \mathbb{Z}_\triangle^n$, $A \in \Theta_\triangle^+(n)$ and the transpose ${}^tA$ of A, is a surjective algebra homomorphism.

The map ζ_r defined in Theorem 3.1 induce an algebra homomorphism

$$\zeta = \prod_{r\geqslant 0} \zeta_r : \boldsymbol{\mathfrak{D}}_\triangle(n) \to \boldsymbol{\mathcal{S}}_\triangle(n). \tag{3.1.1}$$

We now describe the image of ζ.

Let

$$\boldsymbol{\mathcal{S}}_\triangle(n) = \prod_{r\geqslant 0} \boldsymbol{\mathcal{S}}_\triangle(n,r).$$

For $A\in\Theta^{\pm}_{\triangle}(n)$, $\mathbf{j}\in\mathbb{Z}^n_{\triangle}$ and $\lambda\in\mathbb{N}^n_{\triangle}$, define elements in $\boldsymbol{\mathcal{S}}_{\triangle}(n)$

$$A(\mathbf{j})=(A(\mathbf{j},r))_{r\geqslant 0},\quad A(\mathbf{j},\lambda)=(A(\mathbf{j},\lambda,r))_{r\geqslant 0}.$$

We set, for $A\in M_{\triangle n}(\mathbb{Z})$ with $a_{i,j}<0$ for some $i\neq j$, $A(\mathbf{j},\lambda)=A(\mathbf{j})=0$.

Let $\boldsymbol{\mathcal{V}}_{\triangle}(n)$ be the $\mathbb{Q}(v)$-subspace of $\boldsymbol{\mathcal{S}}_{\triangle}(n)$ spanned by $A(\mathbf{j},\lambda)$ for $A\in\Theta^{\pm}_{\triangle}(n)$, $\mathbf{j}\in\mathbb{Z}^n_{\triangle}$ and $\lambda\in\mathbb{N}^n_{\triangle}$. By [11, Lem. 4.1], $\{A(\mathbf{j})\mid A\in\Theta^{\pm}_{\triangle}(n),\mathbf{j}\in\mathbb{N}^n_{\triangle}\}$ forms a basis for $\boldsymbol{\mathcal{V}}_{\triangle}(n)$.

Theorem 3.2 ([11, 4.4])**.** *The $\mathbb{Q}(v)$-space $\boldsymbol{\mathcal{V}}_{\triangle}(n)$ is a subalgebra of $\boldsymbol{\mathcal{S}}_{\triangle}(n)$. Furthermore, the restriction of ζ to $\boldsymbol{\mathfrak{D}}_{\triangle}(n)$ induces a $\mathbb{Q}(v)$-algebra isomorphism $\zeta:\boldsymbol{\mathfrak{D}}_{\triangle}(n)\to\boldsymbol{\mathcal{V}}_{\triangle}(n)$. In particular, we have*

$$\zeta(K^{\mathbf{j}})=0(\mathbf{j}),\ \zeta(\widetilde{u}^+_A)=A(\mathbf{0}),\ \text{ and }\ \zeta(\widetilde{u}^-_A)=({}^tA)(\mathbf{0}),$$

for all $A\in\Theta^{\pm}_{\triangle}(n)$ and $\mathbf{j}\in\mathbb{Z}^n_{\triangle}$.

We shall identify $\boldsymbol{\mathfrak{D}}_{\triangle}(n)$ with $\boldsymbol{\mathcal{V}}_{\triangle}(n)$ via the map ζ and identify $\boldsymbol{\mathfrak{D}}_{\triangle}(n)$ with $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$ under the isomorphism given in Theorem 2.3. The following better presentation for $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$, called a *modified* BLM type realisation of quantum affine $\mathfrak{gl}_n$, is given in [11, Th. 1.1].

For $T=(t_{i,j})\in\widetilde{\Theta}_{\triangle}(n)$ let $\delta_T=(t_{i,i})_{i\in\mathbb{Z}}\in\mathbb{Z}^n_{\triangle}$, the "diagonal" of T and let $\widetilde{T}=(\widetilde{t}_{i,j})$, where $\widetilde{t}_{i,j}=t_{i-1,j}$ for all $i,j\in\mathbb{Z}^n$.

For $A\in\widetilde{\Theta}_{\triangle}(n)$, let $\mathrm{ro}(A)=\big(\sum_{j\in\mathbb{Z}}a_{i,j}\big)_{i\in\mathbb{Z}}$ and $\mathrm{co}(A)=\big(\sum_{i\in\mathbb{Z}}a_{i,j}\big)_{j\in\mathbb{Z}}$.

Theorem 3.3. *The quantum loop algebra $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$ is the $\mathbb{Q}(v)$-algebra which is spanned by the basis $\{A(\mathbf{j})\mid A\in\Theta^{\pm}_{\triangle}(n),\mathbf{j}\in\mathbb{Z}^n_{\triangle}\}$ and generated by $0(\mathbf{j})$, $S_\alpha(\mathbf{0})$ and ${}^tS_\alpha(\mathbf{0})$ for all $\mathbf{j}\in\mathbb{Z}^n_{\triangle}$ and $\alpha\in\mathbb{N}^n_{\triangle}$, where $S_\alpha=\sum_{1\leqslant i\leqslant n}\alpha_iE^{\triangle}_{i,i+1}$ and ${}^tS_\alpha$ is the transpose of S_α, and whose multiplication rules are given by:*

(1) $0(\mathbf{j}')A(\mathbf{j})=v^{\mathbf{j}'\cdot\mathrm{ro}(A)}A(\mathbf{j}'+\mathbf{j})$ *and* $A(\mathbf{j})0(\mathbf{j}')=v^{\mathbf{j}'\cdot\mathrm{co}(A)}A(\mathbf{j}'+\mathbf{j})$;

(2) $$S_\alpha(\mathbf{0})A(\mathbf{j})=\sum_{\substack{T\in\Theta_{\triangle}(n)\\ \mathrm{ro}(T)=\alpha}}v^{f_T}\prod_{\substack{1\leqslant i\leqslant n\\ j\in\mathbb{Z},\,j\neq i}}\overline{\left[\!\!\left[\begin{matrix}a_{i,j}+t_{i,j}-t_{i-1,j}\\ t_{i,j}\end{matrix}\right]\!\!\right]}(A+T^{\pm}-\widetilde{T}^{\pm})(\mathbf{j}_T,\delta_T),$$

where $\mathbf{j}_T=\mathbf{j}+\sum_{1\leqslant i\leqslant n}(\sum_{j<i}(t_{i,j}-t_{i-1,j}))\boldsymbol{e}^{\triangle}_i$ *and*

$$\begin{aligned}f_T=&\sum_{\substack{1\leqslant i\leqslant n\\ j\geqslant l,\,j\neq i}}a_{i,j}t_{i,l}-\sum_{\substack{1\leqslant i\leqslant n\\ j>l,\,j\neq i+1}}a_{i+1,j}t_{i,l}-\sum_{\substack{1\leqslant i\leqslant n\\ j\geqslant l,\,j\neq i}}t_{i-1,j}t_{i,l}+\sum_{\substack{1\leqslant i\leqslant n\\ j>l,\,j\neq i,\,j\neq i+1}}t_{i,j}t_{i,l}\\ &+\sum_{\substack{1\leqslant i\leqslant n\\ j<i+1}}t_{i,j}t_{i+1,i+1}+\sum_{1\leqslant i\leqslant n}j_i(t_{i-1,i}-t_{i,i});\end{aligned}$$

(3) $${}^tS_\alpha(\mathbf{0})A(\mathbf{j})=\sum_{\substack{T\in\Theta_{\triangle}(n)\\ \mathrm{ro}(T)=\alpha}}v^{f'_T}\prod_{\substack{1\leqslant i\leqslant n\\ j\in\mathbb{Z},\,j\neq i}}\overline{\left[\!\!\left[\begin{matrix}a_{i,j}-t_{i,j}+t_{i-1,j}\\ t_{i-1,j}\end{matrix}\right]\!\!\right]}(A-T^{\pm}+\widetilde{T}^{\pm})(\mathbf{j}'_T,\delta_{\widetilde{T}}),$$

where $\mathbf{j}'_T=\mathbf{j}+\sum_{1\leqslant i\leqslant n}(\sum_{j>i}(t_{i-1,j}-t_{i,j}))\boldsymbol{e}^\triangle_i$ *and*

$$f'_T=\sum_{\substack{1\leqslant i\leqslant n\\ l\geqslant j,\, j\neq i}}a_{i,j}t_{i-1,l}-\sum_{\substack{1\leqslant i\leqslant n\\ l>j,\, j\neq i}}a_{i,j}t_{i,l}-\sum_{\substack{1\leqslant i\leqslant n\\ j\geqslant l,\, l\neq i}}t_{i-1,j}t_{i,l}+\sum_{\substack{1\leqslant i\leqslant n\\ j>l,\, l\neq i,\, l\neq i+1}}t_{i,j}t_{i,l}$$
$$+\sum_{\substack{1\leqslant i\leqslant n\\ i<j}}t_{i,j}t_{i-1,i}+\sum_{1\leqslant i\leqslant n}j_i(t_{i,i}-t_{i-1,i}).$$

4. Some integral multiplication formulas

Let $^-:\mathcal{Z}\to\mathcal{Z}$ be the ring homomorphism defined by $\bar{v}=v^{-1}$. The following result is proved in [11, 3.6].

Proposition 4.1. *Let* $A\in\Theta_\triangle(n,r)$ *and* $\alpha,\gamma\in\mathbb{N}^n_\triangle$.

(1) *If* $B\in\Theta_\triangle(n,r)$ *satisfies that* $B-\sum\limits_{1\leqslant i\leqslant n}\alpha_iE^\triangle_{i,i+1}$ *is a diagonal matrix and* $\mathrm{co}(B)=\mathrm{ro}(A)$, *then in* $\mathcal{S}_\triangle(n,r)_\mathcal{Z}$:

$$[B][A]=\sum_{\substack{T\in\Theta_\triangle(n),\,\mathrm{ro}(T)=\alpha\\ a_{i,j}+t_{i,j}-t_{i-1,j}\geqslant 0,\,\forall i,j}}v^{\beta(T,A)}\prod_{\substack{1\leqslant i\leqslant n\\ j\in\mathbb{Z}}}\overline{\left[\!\!\left[\begin{matrix}a_{i,j}+t_{i,j}-t_{i-1,j}\\ t_{i,j}\end{matrix}\right]\!\!\right]}[A+T-\widetilde{T}],$$

where $\beta(T,A)=\sum_{1\leqslant i\leqslant n,\,j\geqslant l}(a_{i,j}-t_{i-1,j})t_{i,l}-\sum_{1\leqslant i\leqslant n,\,j>l}(a_{i+1,j}-t_{i,j})t_{i,l}$.

(2) *If* $C\in\Theta_\triangle(n,r)$ *satisfies that* $C-\sum_{1\leqslant i\leqslant n}\gamma_iE^\triangle_{i+1,i}$ *is a diagonal matrix and* $\mathrm{co}(C)=\mathrm{ro}(A)$, *then in* $\mathcal{S}_\triangle(n,r)_\mathcal{Z}$:

$$[C][A]=\sum_{\substack{T\in\Theta_\triangle(n),\,\mathrm{ro}(T)=\gamma\\ a_{i,j}-t_{i,j}+t_{i-1,j}\geqslant 0,\,\forall i,j}}v^{\beta'(T,A)}\prod_{\substack{1\leqslant i\leqslant n\\ j\in\mathbb{Z}}}\overline{\left[\!\!\left[\begin{matrix}a_{i,j}-t_{i,j}+t_{i-1,j}\\ t_{i-1,j}\end{matrix}\right]\!\!\right]}[A-T+\widetilde{T}],$$

where $\beta'(T,A)=\sum_{1\leqslant i\leqslant n,\,l\geqslant j}(a_{i,j}-t_{i,j})t_{i-1,l}-\sum_{1\leqslant i\leqslant n,\,l>j}(a_{i,j}-t_{i,j})t_{i,l}$.

We now derive some integral version of the multiplication formulas.

Proposition 4.2. *Let* $A\in\Theta^\pm_\triangle(n)$, $S_\alpha=\sum_{1\leqslant i\leqslant n}\alpha_iE^\triangle_{i,i+1}$ *and* ${}^tS_\alpha=\sum_{1\leqslant i\leqslant n}\alpha_iE^\triangle_{i+1,i}$ *with* $\alpha\in\mathbb{N}^n_\triangle$. *Let* $\lambda,\mu\in\mathbb{N}^n_\triangle$, $\mathbf{j},\mathbf{j}'\in\mathbb{Z}^n_\triangle$. *The following identities holds in* $\boldsymbol{\mathcal{S}}_\triangle(n)$:

(1) $$0(\mathbf{j}',\mu)A(\mathbf{j},\lambda)=\sum_{\nu\in\mathbb{N}^n_\triangle,\,\nu\leqslant\mu}a_\nu A(\mathbf{j}'+\mathbf{j}-\nu,\lambda+\mu-\nu);$$

where

$$a_\nu=\sum_{\substack{\mathbf{j}''\in\mathbb{N}^n_\triangle\\ \nu-\lambda\leqslant\mathbf{j}''\leqslant\nu}}v^{\mathrm{ro}(A)\centerdot(\mathbf{j}'+\mu-\mathbf{j}'')+\lambda\centerdot(\mu-\mathbf{j}'')}\begin{bmatrix}\mathrm{ro}(A)\\ \mathbf{j}''\end{bmatrix}\begin{bmatrix}\lambda+\mu-\nu\\ \nu-\mathbf{j}'',\,\lambda-\nu+\mathbf{j}'',\,\mu-\nu\end{bmatrix};$$

(2) $$S_\alpha(\mathbf{0})A(\mathbf{j},\lambda)=\sum_{\substack{T\in\Theta_\triangle(n),\,\mathrm{ro}(T)=\alpha\\ \beta,\eta\in\mathbb{N}^n_\triangle,\,\beta\leqslant\delta_T,\,\beta+\eta\leqslant\lambda}}g_{\beta,\eta,T}\cdot(A+T^\pm-\widetilde{T}^\pm)(\mathbf{j}_T+\lambda-\eta-2\beta,\delta_T+\eta),$$

where

$$g_{\beta,\eta,T} = v^{f_T+(\eta+\beta)\centerdot(2\delta_T-\delta_{\widetilde{T}})} \begin{bmatrix} \delta_{\widetilde{T}} - \delta_T \\ \lambda-\eta-\beta \end{bmatrix} \begin{bmatrix} \delta_T+\eta \\ \beta,\, \delta_T-\beta,\, \eta \end{bmatrix} \prod_{\substack{1\leqslant i\leqslant n\\ j\neq i,\, j\in\mathbb{Z}}} \overline{\left[\!\left[\begin{matrix} a_{i,j}+t_{i,j}-t_{i-1,j} \\ t_{i,j} \end{matrix} \right]\!\right]} \in \mathcal{Z},$$

and $\mathbf{j}_T$*,* f_T *are defined as in Theorem 3.3(2);*

(3) $$ {}^tS_\alpha(\mathbf{0})A(\mathbf{j},\lambda) = \sum_{\substack{T\in\Theta_\Delta(n),\, \mathrm{ro}(T)=\alpha\\ \beta,\eta\in\mathbb{N}_\Delta^n,\, \beta\leqslant\delta_{\widetilde{T}},\, \beta+\eta\leqslant\lambda}} g'_{\beta,\eta,T}\cdot(A-T^{\pm}+\widetilde{T}^{\pm})(\mathbf{j}'_T+\lambda-\eta-2\beta,\delta_{\widetilde{T}}+\eta),$$

where

$$g'_{\beta,\eta,T} = v^{f'_T+(\eta+\beta)\centerdot(2\delta_{\widetilde{T}}-\delta_T)} \begin{bmatrix} \delta_T - \delta_{\widetilde{T}} \\ \lambda-\eta-\beta \end{bmatrix} \begin{bmatrix} \delta_{\widetilde{T}}+\eta \\ \beta,\, \delta_{\widetilde{T}}-\beta,\, \eta \end{bmatrix} \prod_{\substack{1\leqslant i\leqslant n\\ j\neq i,\, j\in\mathbb{Z}}} \overline{\left[\!\left[\begin{matrix} a_{i,j}-t_{i,j}+t_{i-1,j} \\ t_{i-1,j} \end{matrix} \right]\!\right]} \in \mathcal{Z},$$

and $\mathbf{j}'_T$*,* f'_T *are defined as in Theorem 3.3(3). The same formulas hold in* $\mathcal{S}_\vartriangle(n,r)_\mathcal{Z}$ *with* $A(\mathbf{j},\lambda)$ *etc. replaced by* $A(\mathbf{j},\lambda,r)$*, etc.*

Proof. The fact $[A][B]\neq 0 \implies \mathrm{ro}(B)=\mathrm{co}(A)$ gives

$$0(\mathbf{j}',\mu,r)A(\mathbf{j},\lambda,r) = \sum_{\alpha\in\Lambda_\vartriangle(n,r-\sigma(A)} v^{(\mathrm{ro}(A)+\alpha)\centerdot\mathbf{j}'+\alpha\centerdot\mathbf{j}} \begin{bmatrix} \mathrm{ro}(A)+\alpha \\ \mu \end{bmatrix} \begin{bmatrix} \alpha \\ \lambda \end{bmatrix} [A+\mathrm{diag}(\alpha)].$$

Applying (1.1.2) yields the required formula. For more details, see [16, 3.4].

Similarly, by Proposition 4.1, the left hand side of (2) at level r becomes

$$\begin{aligned} S_\alpha(\mathbf{0},r)A(\mathbf{j},\lambda,r) &= \sum_{\gamma\in\Lambda_\vartriangle(n,r-\sigma(A))} v^{\gamma\centerdot\mathbf{j}} \begin{bmatrix} \gamma \\ \lambda \end{bmatrix} \left[S_\alpha + \mathrm{diag}\left(\gamma+\mathrm{ro}(A)-\sum_{1\leqslant i\leqslant n}\alpha_i\boldsymbol{e}^\vartriangle_{i+1}\right)\right][A+\mathrm{diag}(\gamma)] \\ &= \sum_{\substack{T\in\Theta_\Delta(n)\\ \mathrm{ro}(T)=\alpha}} \prod_{\substack{1\leqslant i\leqslant n\\ j\in\mathbb{Z},\, j\neq i}} \overline{\left[\!\left[\begin{matrix} a_{i,j}+t_{i,j}-t_{i-1,j} \\ t_{i,j} \end{matrix} \right]\!\right]} x_T \end{aligned}$$

where

$$x_T = \sum_{\gamma\in\Lambda_\vartriangle(n,r-\sigma(A))} v^{\gamma\centerdot\mathbf{j}+\beta(T,A+\mathrm{diag}(\gamma))} \begin{bmatrix} \gamma \\ \lambda \end{bmatrix} \overline{\left[\!\left[\begin{matrix} \gamma+\delta_T-\delta_{\widetilde{T}} \\ \delta_T \end{matrix} \right]\!\right]} [A+T^{\pm}-\widetilde{T}^{\pm}+\mathrm{diag}(\gamma+\delta_T-\delta_{\widetilde{T}})].$$

Let $\nu=\gamma+\delta_T-\delta_{\widetilde{T}}$. Then $\beta(T,A+\mathrm{diag}(\gamma))=\beta_{A,T}+\beta_{\nu,T}$, where $\beta_{\nu,T}=\sum_{1\leqslant i\leqslant n,\, i\geqslant l}\nu_i t_{i,l}-\sum_{1\leqslant i\leqslant n,\, i+1>l}\nu_{i+1}t_{i,l}$ and

$$\begin{aligned} \beta_{A,T} = &\sum_{\substack{1\leqslant i\leqslant n\\ j\geqslant l,\, j\neq i}} (a_{i,j}-t_{i-1,j})t_{i,l} - \sum_{\substack{1\leqslant i\leqslant n\\ j>l,\, j\neq i+1}} a_{i+1,j}t_{i,l} + \sum_{\substack{1\leqslant i\leqslant n\\ j>l,\, j\neq i,i+1}} t_{i,j}t_{i,l} \\ &- \sum_{1\leqslant i\leqslant n} t_{i,i}^2 + \sum_{\substack{1\leqslant i\leqslant n\\ i+1>l}} t_{i+1,i+1}t_{i,l}. \end{aligned}$$

Furthermore, we have $\overline{\left[\!\left[{\nu \atop \delta_T} \right]\!\right]} = v^{\delta_T\boldsymbol{\cdot}(\delta_T-\nu)}\left[{\nu \atop \delta_T} \right]$, $\beta_{A,T}+\delta_T\boldsymbol{\cdot}\delta_T+\mathbf{j}\boldsymbol{\cdot}(\delta_{\widetilde{T}}-\delta_T)=f_T$ and $\beta_{\nu,T}+\nu\boldsymbol{\cdot}(\mathbf{j}-\delta_T)=\nu\boldsymbol{\cdot}\mathbf{j}_T$. This implies that

$$x_T = \sum_{\nu\in\Lambda_\vartriangle(n,r-\sigma(A+T^\pm-\widetilde{T}^\pm))} v^{f_T+\nu\boldsymbol{\cdot}\delta_{T,\mathbf{j}}}\begin{bmatrix}\nu\\ \delta_T\end{bmatrix}\cdot\begin{bmatrix}\nu-\delta_T+\delta_{\widetilde{T}}\\ \lambda\end{bmatrix}[A+T^\pm-\widetilde{T}^\pm+\mathrm{diag}(\nu)].$$

Applying the identities in (1.1.2) yields

$$\begin{aligned}\begin{bmatrix}\nu\\ \delta_T\end{bmatrix}\cdot\begin{bmatrix}\nu-\delta_T+\delta_{\widetilde{T}}\\ \lambda\end{bmatrix} &= \sum_{\substack{\mathbf{x}\in\mathbb{N}^n_\vartriangle\\ \mathbf{x}\leqslant\lambda}} v^{\nu\boldsymbol{\cdot}(\lambda-\mathbf{x})-\mathbf{x}\boldsymbol{\cdot}(\delta_{\widetilde{T}}-\delta_T)}\begin{bmatrix}\nu\\ \delta_T\end{bmatrix}\cdot\begin{bmatrix}\nu\\ \mathbf{x}\end{bmatrix}\cdot\begin{bmatrix}\delta_{\widetilde{T}}-\delta_T\\ \lambda-\mathbf{x}\end{bmatrix}\\
&= \sum_{\substack{\mathbf{x},\beta\in\mathbb{N}^n_\vartriangle,\,\beta\leqslant\delta_T\\ \beta\leqslant\mathbf{x}\leqslant\lambda}} v^{\nu\boldsymbol{\cdot}(\lambda-\mathbf{x}-\beta)+\mathbf{x}\boldsymbol{\cdot}(2\delta_T-\delta_{\widetilde{T}})}\begin{bmatrix}\delta_{\widetilde{T}}-\delta_T\\ \lambda-\mathbf{x}\end{bmatrix}\begin{bmatrix}\delta_T+\mathbf{x}-\beta\\ \beta,\,\delta_T-\beta,\,\mathbf{x}-\beta\end{bmatrix}\\
&\qquad\times\begin{bmatrix}\nu\\ \delta_T+\mathbf{x}-\beta\end{bmatrix}.\end{aligned}$$

Thus,

$$\begin{aligned}x_T &= \sum_{\substack{\mathbf{x},\beta\in\mathbb{N}^n_\vartriangle,\,\beta\leqslant\delta_T\\ \beta\leqslant\mathbf{x}\leqslant\lambda}} v^{f_T+\mathbf{x}\boldsymbol{\cdot}(2\delta_T-\delta_{\widetilde{T}})}\begin{bmatrix}\delta_{\widetilde{T}}-\delta_T\\ \lambda-\mathbf{x}\end{bmatrix}\begin{bmatrix}\delta_T+\mathbf{x}-\beta\\ \beta,\,\delta_T-\beta,\,\mathbf{x}-\beta\end{bmatrix}\\
&\qquad\times(A+T^\pm-\widetilde{T}^\pm)(\mathbf{j}_T+\lambda-\mathbf{x}-\beta,\delta_T+\mathbf{x}-\beta)\\
&= \sum_{\substack{\eta,\beta\in\mathbb{N}^n_\vartriangle,\,\beta\leqslant\delta_T\\ \beta+\eta\leqslant\lambda}} v^{f_T+(\eta+\beta)\boldsymbol{\cdot}(2\delta_T-\delta_{\widetilde{T}})}\begin{bmatrix}\delta_{\widetilde{T}}-\delta_T\\ \lambda-\eta-\beta\end{bmatrix}\begin{bmatrix}\delta_T+\eta\\ \beta,\,\delta_T-\beta,\,\eta\end{bmatrix}\\
&\qquad\times(A+T^\pm-\widetilde{T}^\pm)(\mathbf{j}_T+\lambda-\eta-2\beta,\delta_T+\eta,r)\end{aligned}$$

Consequently, (2) holds. Formula (3) can be proved similarly. □

5. Lusztig form of $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$ and integral affine quantum Schur–Weyl reciprocity

We are now ready to determine the Lusztig form of $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$ by proving the conjecture [4, 3.8.6].

Let $\mathcal{V}_\vartriangle(n)_\mathcal{Z}$ be the $\mathcal{Z}$-submodule of $\boldsymbol{\mathcal{S}}_\vartriangle(n)$ spanned by $\{A(\mathbf{j},\lambda)\mid A\in\Theta^\pm_\vartriangle(n),\,\mathbf{j}\in\mathbb{Z}^n_\vartriangle,\,\lambda\in\mathbb{N}^n_\vartriangle\}$. As seen above, $\mathcal{V}_\vartriangle(n)_\mathcal{Z}$ is a $\mathcal{Z}$-submodule of $\boldsymbol{\mathcal{V}}_\vartriangle(n)$. Our aim is to show that $\mathcal{V}_\vartriangle(n)_\mathcal{Z}$ is a realisation of $\mathfrak{D}_\vartriangle(n)_\mathcal{Z}$ (see Theorem 5.6 below). The following result is [16, 4.8].

Lemma 5.1. *The set $\{A(\mathbf{j},\lambda)\mid A\in\Theta^\pm_\vartriangle(n),\,\mathbf{j},\lambda\in\mathbb{N}^n_\vartriangle,\,j_i\in\{0,1\},\forall i\}$ forms a $\mathcal{Z}$-basis for $\mathcal{V}_\vartriangle(n)_\mathcal{Z}$.*

Proof. Since the 0-part of $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$ is the same as that of $\mathbf{U}(\mathfrak{gl}_n)$, the proof in the finite case [16, 4.2] carries over. □

Let $\mathcal{V}^+_\vartriangle(n)_\mathcal{Z}=\mathrm{span}_\mathcal{Z}\{A(\mathbf{0})\mid A\in\Theta^+_\vartriangle(n)\}$, $\mathcal{V}^-_\vartriangle(n)_\mathcal{Z}=\mathrm{span}_\mathcal{Z}\{A(\mathbf{0})\mid A\in\Theta^-_\vartriangle(n)\}$ and $\mathcal{V}^0_\vartriangle(n)_\mathcal{Z}=\mathrm{span}_\mathcal{Z}\{0(\mathbf{j},\lambda)\mid \mathbf{j}\in\mathbb{Z}^n_\vartriangle,\,\lambda\in\mathbb{N}^n_\vartriangle\}$. By Proposition 4.2(1), $\mathcal{V}^0_\vartriangle(n)_\mathcal{Z}$ is a $\mathcal{Z}$-subalgebra of $\boldsymbol{\mathcal{S}}_\vartriangle(n)$.

Lemma 5.2. *The $\mathcal{Z}$-module $\mathcal{V}_\triangle^+(n)_\mathcal{Z}$ (resp., $\mathcal{V}_\triangle^-(n)_\mathcal{Z}$) is a subalgebras of $\boldsymbol{\mathcal{S}}_\triangle(n)$ which is generated by $(\sum_{1\leqslant i\leqslant n}\alpha_i E^\triangle_{i,i+1})(\mathbf{0})$ (resp., $(\sum_{1\leqslant i\leqslant n}\alpha_i E^\triangle_{i+1,i})(\mathbf{0})$) for $\alpha\in\mathbb{N}_\triangle^n$ as a $\mathcal{Z}$-algebra.*

Proof. Since $\mathfrak{D}_\triangle^+(n)_\mathcal{Z}=\mathfrak{H}_\triangle(n)_\mathcal{Z}$ is a $\mathcal{Z}$-subalgebra of $\boldsymbol{\mathfrak{D}}_\triangle(n)$ and $\mathcal{V}_\triangle^+(n)=\zeta(\mathfrak{D}_\triangle^+(n)_\mathcal{Z})$ by Theorem 3.2, we conclude the first assertion which together with Proposition 2.2 gives the second assertion. □

We now recall the triangular relation for affine quantum Schur algebras. For $A,B\in\widetilde{\Theta}_\triangle(n)$ define

(5.2.1) $$B\sqsubseteq A \text{ if and only if } B\preccurlyeq A,\ \mathrm{co}(B)=\mathrm{co}(A) \text{ and } \mathrm{ro}(B)=\mathrm{ro}(A).$$

Put $B\sqsubset A$ if $B\sqsubseteq A$ and $B\neq A$. According to [10, 6.1] the order relation $\sqsubseteq$ is a partial order relation on $\widetilde{\Theta}_\triangle(n)$ with finite intervals $(-\infty,A]$ for all A; see Lemma 7.5 below.

For $A\in\widetilde{\Theta}_\triangle(n)$ with $\sigma(A)=r$, we denote $[A]=0\in\mathcal{S}_\triangle(n,r)_\mathcal{Z}$ if $a_{i,i}<0$ for some $i\in\mathbb{Z}$. For $A\in\widetilde{\Theta}_\triangle(n)$ let $\boldsymbol{\sigma}(A)=(\sigma_i(A))_{i\in\mathbb{Z}}\in\mathbb{N}_\triangle^n$ where $\sigma_i(A)=a_{i,i}+\sum_{j<i}(a_{i,j}+a_{j,i})$. The following triangular relation for affine quantum Schur algebras is given in [4, 3.7.7]. The first assertion can be seen easily from the proof of loc. cit.

Proposition 5.3. *For $A\in\Theta_\triangle^\pm(n)$ and $\lambda\in\Lambda_\triangle(n,r)$, we have*

$$A^+(\mathbf{0},r)[\mathrm{diag}(\lambda)]A^-(\mathbf{0},r)=[A+\mathrm{diag}(\lambda-\boldsymbol{\sigma}(A))]+\textit{a } \mathcal{Z}\textit{-linear comb. of } [A'] \textit{ with } A'\sqsubset A.$$

In particular, the set

$$\{A^+(\mathbf{0},r)[\mathrm{diag}(\lambda)]A^-(\mathbf{0},r)\mid A\in\Theta_\triangle^\pm(n),\ \lambda\in\Lambda_\triangle(n,r),\ \lambda\geqslant\boldsymbol{\sigma}(A)\}$$

forms a $\mathcal{Z}$-basis for $\mathcal{S}_\triangle(n,r)_\mathcal{Z}$, where the order relation $\leqslant$ is defined in (1.1.1).

For $w\in\widetilde{\Sigma}$, let

$$\mathtt{m}^+_{(w)}=\zeta(\widetilde{u}^+_{(w)})\in\boldsymbol{\mathcal{S}}_\triangle(n)\quad\text{and}\quad\mathtt{m}^-_{(w)}=\zeta(\widetilde{u}^-_{(w)})\in\boldsymbol{\mathcal{S}}_\triangle(n).$$

The triangular relation for affine quantum Schur algebras can be lifted to the $\boldsymbol{\mathcal{S}}_\triangle(n)$ level as follows.

Lemma 5.4. *Let $A\in\Theta_\triangle^\pm(n)$, $\mathbf{j}\in\mathbb{Z}_\triangle^n$ and $\lambda\in\mathbb{N}_\triangle^n$.*

(1) *We have*

$$A^+(\mathbf{0})0(\mathbf{j},\lambda)A^-(\mathbf{0})=\sum_{\substack{\delta\in\mathbb{N}_\triangle^n\\ \delta\leqslant\lambda}}v^{(\mathbf{j}-\delta)\centerdot\boldsymbol{\sigma}(A)}\begin{bmatrix}\boldsymbol{\sigma}(A)\\ \lambda-\delta\end{bmatrix}A(\mathbf{j}+\lambda-\delta,\delta)+f$$

where f is a $\mathcal{Z}$-linear combination of $B(\mathbf{j}',\delta)$ such that $B\in\Theta_\triangle^\pm(n)$, $B\prec A$, $\delta\in\mathbb{N}_\triangle^n$ and $\mathbf{j}'\in\mathbb{Z}_\triangle^n$. In particular, We have $\mathcal{V}_\triangle(n)_\mathcal{Z}=\mathcal{V}_\triangle^+(n)_\mathcal{Z}\mathcal{V}_\triangle^0(n)_\mathcal{Z}\mathcal{V}_\triangle^-(n)_\mathcal{Z}$.

(2) *There exist* $w_{A^+}, w_{A^-} \in \widetilde{\Sigma}$ *such that* $\wp^+(w_{A^+}) = A^+$, $\wp^-(w_{A^-}) := {}^t\wp^+(w_{{}^tA^-}) = A^-$ *and*

$$\mathtt{m}^+_{(w_{A^+})} 0(\mathbf{j},\lambda)\mathtt{m}^-_{(w_{A^-})} = \sum_{\substack{\delta\in\mathbb{N}^n_\vartriangle\\ \delta\leqslant\lambda}} v^{(\mathbf{j}-\delta)\centerdot\boldsymbol{\sigma}(A)} \begin{bmatrix}\boldsymbol{\sigma}(A)\\ \lambda-\delta\end{bmatrix} A(\mathbf{j}+\lambda-\delta,\delta)+g$$

where g *is a* $\mathcal{Z}$*-linear combination of* $B(\mathbf{j}',\delta)$ *such that* $B\in\Theta^\pm_\vartriangle(n)$, $B\prec A$, $\delta\in\mathbb{N}^n_\vartriangle$ *and* $\mathbf{j}'\in\mathbb{Z}^n_\vartriangle$.

Proof. According to Proposition 5.3, for any $\mu\in\Lambda_\vartriangle(n,r)$, we have

$$A^+(\mathbf{0},r)[\mathrm{diag}(\mu)]A^-(\mathbf{0},r) = [A+\mathrm{diag}(\mu-\boldsymbol{\sigma}(A))]+f_{\mu,r}$$

where $f_{\mu,r}$ is a $\mathcal{Z}$-linear combination of $[B]$ such that $B\in\Theta_\vartriangle(n,r)$ and $B\sqsubset A+\mathrm{diag}(\mu-\boldsymbol{\sigma}(A))$. Thus,

$$\begin{aligned} A^+(\mathbf{0},r)0(\mathbf{j},\lambda,r)A^-(\mathbf{0},r) &= \sum_{\mu\in\Lambda_\vartriangle(n,r)} v^{\mathbf{j}\centerdot\mu}\begin{bmatrix}\mu\\ \lambda\end{bmatrix}([A+\mathrm{diag}(\mu-\boldsymbol{\sigma}(A))]+f_{\mu,r})\\ &= \sum_{\nu\in\Lambda_\vartriangle(n-r-\sigma(A))} v^{\mathbf{j}\centerdot(\nu+\boldsymbol{\sigma}(A))}\begin{bmatrix}\nu+\boldsymbol{\sigma}(A)\\ \lambda\end{bmatrix}[A+\mathrm{diag}(\nu)]+f_r, \end{aligned}$$

where $f_r=\sum_{\mu\in\Lambda_\vartriangle(n,r)} v^{\mathbf{j}\centerdot\mu}\left[\begin{smallmatrix}\mu\\ \lambda\end{smallmatrix}\right] f_{\mu,r}$. By (1.1.2), we have

$$\begin{aligned} A^+(\mathbf{0},r)0(\mathbf{j},\lambda,r)A^-(\mathbf{0},r) &= \sum_{\nu\in\mathbb{Z}^n_\vartriangle} v^{\mathbf{j}\centerdot(\nu+\boldsymbol{\sigma}(A))}\sum_{\substack{\delta\in\mathbb{N}^n_\vartriangle\\ \delta\leqslant\lambda}} v^{\nu\centerdot(\lambda-\delta)-\delta\centerdot\boldsymbol{\sigma}(A)}\begin{bmatrix}\nu\\ \delta\end{bmatrix}\begin{bmatrix}\boldsymbol{\sigma}(A)\\ \lambda-\delta\end{bmatrix}[A+\mathrm{diag}(\nu)]+f_r\\ &= \sum_{\substack{\delta\in\mathbb{N}^n_\vartriangle\\ \delta\leqslant\lambda}} v^{(\mathbf{j}-\delta)\centerdot\boldsymbol{\sigma}(A)}\begin{bmatrix}\boldsymbol{\sigma}(A)\\ \lambda-\delta\end{bmatrix}A(\mathbf{j}+\lambda-\delta,\delta)+f_r. \end{aligned}$$

On the other hand, by Lemma 5.2 and Proposition 4.2, we see that $(f_r)_{r\geqslant0}\in\mathcal{V}_\vartriangle(n)_\mathcal{Z}$. Hence, $(f_r)_{r\geqslant0}$ must be a $\mathcal{Z}$-linear combination of $B(\mathbf{j}',\delta)$ such that $B\in\Theta^\pm_\vartriangle(n)$, $B\prec A$, $\delta\in\mathbb{N}^n_\vartriangle$ and $\mathbf{j}'\in\mathbb{Z}^n_\vartriangle$. This proves (1). The assertion (2) follows from (1), Proposition 2.2 and Theorem 3.1. □

For $A\in\widetilde{\Theta}_\vartriangle(n)$, let

$$\|A\| = \sum_{\substack{i<j\\ 1\leqslant i\leqslant n}}\binom{j-i+1}{2}(a_{i,j}+a_{j,i}).$$

Then, $A\prec B$ implies $\|A\|<\|B\|$. The following result is the affine version of [16, Prop. 4.3] which is conjectured in [16, 4.9].

Proposition 5.5. *The* $\mathcal{Z}$*-module* $\mathcal{V}_\vartriangle(n)_\mathcal{Z}$ *is a subalgebra of* $\boldsymbol{\mathcal{S}}_\vartriangle(n)$ *which is generated by the elements* $(\sum_{1\leqslant i\leqslant n}\alpha_iE^\vartriangle_{i,i+1})(\mathbf{0})$, $(\sum_{1\leqslant i\leqslant n}\alpha_iE^\vartriangle_{i+1,i})(\mathbf{0})$, $0(\boldsymbol{e}^\vartriangle_i)$, $0(\mathbf{0},t\boldsymbol{e}^\vartriangle_i)$ *for all* $\alpha\in\mathbb{N}^n_\vartriangle$, $t\in\mathbb{N}$, $1\leqslant i\leqslant n$.

Proof. Let $\mathcal{V}_\vartriangle(n)'_\mathcal{Z}$ be the $\mathcal{Z}$-subalgebra of $\boldsymbol{\mathcal{S}}_\vartriangle(n)$ generated by the indicated elements. According to Proposition 4.2, we have $\mathcal{V}_\vartriangle(n)'_\mathcal{Z}\subseteq\mathcal{V}_\vartriangle(n)'_\mathcal{Z}\mathcal{V}_\vartriangle(n)_\mathcal{Z}\subseteq\mathcal{V}_\vartriangle(n)_\mathcal{Z}$. We shall show by induction on $\|A\|$ that $A(\mathbf{j},\lambda)\in\mathcal{V}_\vartriangle(n)'_\mathcal{Z}$ for all $A\in\Theta^\pm_\vartriangle(n)$, $\mathbf{j}\in\mathbb{Z}^n_\vartriangle$ and $\lambda\in\mathbb{N}^n_\vartriangle$. If $\|A\|=0$, then $A=0$ and

$0(\mathbf{j},\lambda)=\prod_{1\leqslant i\leqslant n}0(\boldsymbol{e}_i^{\triangle})^{j_i}0(\mathbf{0},\lambda_i\boldsymbol{e}_i^{\triangle})\in\mathcal{V}_{\triangle}(n)'_{\mathcal{Z}}$. Now we assume that $\|A\|>0$ and $A'(\mathbf{j},\lambda)\in\mathcal{V}_{\triangle}(n)'_{\mathcal{Z}}$ for all $A',\mathbf{j},\lambda$ with $\|A'\|<\|A\|$. By Lemma 5.4(2) and [4, 3.7.6], there exist $w_{A^+},w_{A^-}\in\widetilde{\Sigma}$ such that

$$\mathtt{m}^{+}_{(w_{A^+})}\mathtt{m}^{-}_{(w_{A^-})}=A(\mathbf{0})+g$$

where g is a $\mathcal{Z}$-linear combination of $B(\mathbf{j}',\delta)$ with $B\in\Theta^{\pm}_{\triangle}(n)$, $\|B\|<\|A\|$, $\delta\in\mathbb{N}^n_{\triangle}$ and $\mathbf{j}'\in\mathbb{Z}^n_{\triangle}$. By the induction hypothesis we have $g\in\mathcal{V}_{\triangle}(n)'_{\mathcal{Z}}$. It follows that $A(\mathbf{0})\in\mathcal{V}_{\triangle}(n)'_{\mathcal{Z}}$ and so $A(\mathbf{j})\in\mathcal{V}_{\triangle}(n)'_{\mathcal{Z}}$ by Theorem 3.3(1). Furthermore, by Proposition 4.2(1) (setting $\mathbf{j}'=\mu-\nu$ there),

$$\begin{aligned}0(\mathbf{j},\lambda)A(\mathbf{0})&=v^{\mathrm{ro}(A)\centerdot(\mathbf{j}+\lambda)}A(\mathbf{j},\lambda)+\sum_{\substack{\mathbf{j}'\in\mathbb{N}^n_{\triangle}\\ \mathbf{j}'<\lambda}}v^{\mathrm{ro}(A)\centerdot(\mathbf{j}+\mathbf{j}')}\begin{bmatrix}\mathrm{ro}(A)\\ \lambda-\mathbf{j}'\end{bmatrix}A(\mathbf{j}+\mathbf{j}'-\lambda,\mathbf{j}')\\ &=v^{\mathrm{ro}(A)\centerdot(\mathbf{j}+\lambda)}A(\mathbf{j},\lambda)+\sum_{\substack{\mathbf{j}'\in\mathbb{N}^n_{\triangle}\\ \sigma(\mathbf{j}')<\sigma(\lambda)}}v^{\mathrm{ro}(A)\centerdot(\mathbf{j}+\mathbf{j}')}\begin{bmatrix}\mathrm{ro}(A)\\ \lambda-\mathbf{j}'\end{bmatrix}A(\mathbf{j}+\mathbf{j}'-\lambda,\mathbf{j}').\end{aligned}\tag{5.5.1}$$

Thus, by induction on $\sigma(\lambda)$, we conclude that $A(\mathbf{j},\lambda)\in\mathcal{V}_{\triangle}(n)'_{\mathcal{Z}}$ for all $\mathbf{j}\in\mathbb{Z}^n_{\triangle}$ and $\lambda\in\mathbb{N}^n_{\triangle}$. □

As indicated in [16, Rem. 4.10(3)], we now use Proposition 5.5 to prove the conjecture formulated in [4, 3.8.6]. Recall from Theorem 3.2 that the homomorphism ζ in (3.1.1) induces an isomorphism $\zeta:\boldsymbol{\mathfrak{D}}_{\triangle}(n)\to\boldsymbol{\mathcal{V}}_{\triangle}(n)$.

Theorem 5.6. *We have $\zeta^{-1}(\mathcal{V}_{\triangle}(n)_{\mathcal{Z}})=\mathfrak{D}_{\triangle}(n)_{\mathcal{Z}}$. In particular, $\mathfrak{D}_{\triangle}(n)_{\mathcal{Z}}$ is a subalgebra of $\boldsymbol{\mathfrak{D}}_{\triangle}(n)$ isomorphic to $\mathcal{V}_{\triangle}(n)_{\mathcal{Z}}$. Moreover, $\mathfrak{D}_{\triangle}(n)_{\mathcal{Z}}$ is a Hopf subalgebra of $\boldsymbol{\mathfrak{D}}_{\triangle}(n)$.*

Proof. Since $\zeta(\mathfrak{D}_{\triangle}(n)_{\mathcal{Z}})=\zeta(\mathfrak{D}^+_{\triangle}(n)_{\mathcal{Z}})\zeta(\mathfrak{D}^0_{\triangle}(n)_{\mathcal{Z}})\zeta(\mathfrak{D}^-_{\triangle}(n)_{\mathcal{Z}})=\mathcal{V}^+_{\triangle}(n)_{\mathcal{Z}}\mathcal{V}^0_{\triangle}(n)_{\mathcal{Z}}\mathcal{V}^-_{\triangle}(n)_{\mathcal{Z}}$, it follows from Lemma 5.4(1) that $\zeta(\mathfrak{D}_{\triangle}(n)_{\mathcal{Z}})=\mathcal{V}_{\triangle}(n)_{\mathcal{Z}}$. Hence, by Proposition 5.5 and Theorem 3.2, $\mathfrak{D}_{\triangle}(n)_{\mathcal{Z}}$ is a subalgebra. By using the semisimple generators for $\mathfrak{D}_{\triangle}(n)_{\mathcal{Z}}$, the last assertion follows from [4, 3.5.7]. □

Remark 5.7. (1) A different integral form $U_v^{\mathrm{res}}(\widehat{\mathfrak{gl}}_n)$ of $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$ was constructed in [13, 7.2]. As pointed out in [13], it is not known if $U_v^{\mathrm{res}}(\widehat{\mathfrak{gl}}_n)$ is a Hopf subalgebra. It would be interstring to find a relation between $\mathfrak{D}_{\triangle}(n)_{\mathcal{Z}}$ and $U_v^{\mathrm{res}}(\widehat{\mathfrak{gl}}_n)$.

(2) There is another form using the Lusztig form of $\mathbf{U}(\widehat{\mathfrak{sl}}_n)$ tensoring with an integral central algebra; see [4, 2.4.4]. However, this form does not map onto the integral affine quantum Schur algebras; see Example 5.3.8 in [4].

We end this section with an application to the affine quantum Schur–Weyl reciprocity at the integerl level. The proof of the following result is the same as that of [4, Th. 3.8.1(1)].

Theorem 5.8. *The restriction of ζ_r to $\mathfrak{D}_{\triangle}(n)_{\mathcal{Z}}$ gives a surjective $\mathcal{Z}$-algebra homomorphism*

$$\zeta_r:\mathfrak{D}_{\triangle}(n)_{\mathcal{Z}}\twoheadrightarrow\mathcal{S}_{\triangle}(n,r)_{\mathcal{Z}}.$$

Let $\mathcal{k}$ be a commutative ring containing an invertible element ε. We will regard $\mathcal{k}$ as a $\mathcal{Z}$-module by specializing v to ε. Let $\mathfrak{D}_\triangle(n)_{\mathcal{k}} = \mathfrak{D}_\triangle(n)_{\mathcal{Z}} \otimes_{\mathcal{Z}} \mathcal{k}$, $\mathcal{S}_\triangle(n,r)_{\mathcal{k}} = \mathcal{S}_\triangle(n,r)_{\mathcal{Z}} \otimes_{\mathcal{Z}} \mathcal{k}$. Then we have $\mathcal{S}_\triangle(n,r)_{\mathcal{k}} \cong \mathrm{End}_{\mathcal{H}_\triangle(r)_{\mathcal{k}}}(\mathcal{T}_\triangle(n,r)_{\mathcal{k}})$, where $\mathcal{T}_\triangle(n,r)_{\mathcal{k}} = \oplus_{\lambda\in\Lambda_\triangle(n,r)}(x_\lambda\mathcal{H}_\triangle(r)_{\mathcal{k}})$ with $\mathcal{H}_\triangle(r)_{\mathcal{k}} = \mathcal{H}_\triangle(r)_{\mathcal{Z}} \otimes_{\mathcal{Z}} \mathcal{k}$.

Corollary 5.9. *For any commutative ring $\mathcal{k}$, there is an algebra epimorphism*

$$\zeta_r \otimes 1 : \mathfrak{D}_\triangle(n)_{\mathcal{k}} \twoheadrightarrow \mathcal{S}_\triangle(n,r)_{\mathcal{k}}.$$

6. The affine BLM algebra $\mathsf{K}_\triangle(n)_{\mathcal{Z}}$

We first derive in Proposition 6.3 the affine stabilisation property for affine quantum Schur algebras, which is the affine analogue of [1, 4.2]. We then construct the affine BLM algebra $\mathsf{K}_\triangle(n)$ and prove that it is isomorphic to the modified quantum group $\dot{\mathfrak{D}}_\triangle(n)$.

Observe the structure constants in Proposition 4.1 and separate the Gaussian polynomial $\left[\!\left[{a_{i,i}+t_{i,i}-t_{i-1,i} \atop t_{i,i}}\right]\!\right]$ from the product. We now introduce, for a second indeterminate v', $T\in\Theta_\triangle(n)$ and $A\in\widetilde{\Theta}_\triangle(n)$, the polynomials

$$P_{T,A}(v,v') = v^{\beta(T,A)} \prod_{\substack{1\leqslant i\leqslant n\\ j\in\mathbb{Z},\, j\neq i}} \overline{\left[\!\!\left[{a_{i,j}+t_{i,j}-t_{i-1,j} \atop t_{i,j}}\right]\!\!\right]} \prod_{\substack{1\leqslant i\leqslant n\\ 1\leqslant s\leqslant t_{i,i}}} \frac{v^{-2(a_{i,i}+t_{i,i}-t_{i-1,i}-s+1)}v'^2-1}{v^{-2s}-1}$$

and

$$Q_{T,A}(v,v') = v^{\beta'(T,A)} \prod_{\substack{1\leqslant i\leqslant n\\ j\in\mathbb{Z},\, j\neq i}} \overline{\left[\!\!\left[{a_{i,j}-t_{i,j}+t_{i-1,j} \atop t_{i,j}}\right]\!\!\right]} \prod_{\substack{1\leqslant i\leqslant n\\ 1\leqslant s\leqslant t_{i-1,i}}} \frac{v^{-2(a_{i,i}-t_{i,i}+t_{i-1,i}-s+1)}v'^2-1}{v^{-2s}-1}$$

in the subring $\mathcal{Z}_1$ of $\mathbb{Q}(v)[v',v'^{-1}]$, where

(6.0.1) $\qquad \mathcal{Z}_1$ is generated (over $\mathbb{Z}$!) by $\displaystyle\prod_{1\leqslant i\leqslant t}\frac{v^{-2(a-i)}v'^2-1}{v^{-2i}-1}$, $\displaystyle\prod_{1\leqslant i\leqslant t}\frac{v^{2(a-i)}v'^{-2}-1}{v^{2i}-1}$, and v^j

for all $a\in\mathbb{Z}$, $t\geqslant 1$ and $j\in\mathbb{Z}$. Note that $\mathcal{Z}_1|_{v'=1}=\mathcal{Z}$.

For $A\in\widetilde{\Theta}_\triangle(n)$ and $p\in\mathbb{Z}$, let

$$_pA = A + pI$$

where $I\in\Theta_\triangle(n)$ is the identity matrix. Then it is clear that $\beta(T,A)=\beta(T,{}_pA)$ and $\beta'(T,A)=\beta'(T,{}_pA)$. Thus, Proposition 4.1 can be generalised as follows.

Lemma 6.1. *Let $A,B\in\widetilde{\Theta}_\triangle(n)$ and assume* $\mathrm{co}(B)=\mathrm{ro}(A)$ *and* $b=\sigma(A)=\sigma(B)$.

(1) *If $B-\sum_{1\leqslant i\leqslant n}\alpha_i E^\triangle_{i,i+1}$ is diagonal for some $\alpha\in\mathbb{N}^n_\triangle$ then, for large p and $r=pn+b$, we have in $\mathcal{S}_\triangle(n,r)_{\mathcal{Z}}$:*

$$[{}_pB][{}_pA] = \sum_{\substack{T\in\Theta_\triangle(n),\,\mathrm{ro}(T)=\alpha\\ a_{i,j}+t_{i,j}-t_{i-1,j}\geqslant 0,\,\forall i\neq j}} P_{T,A}(v,v^{-p})[{}_p(A+T-\widetilde{T})].$$

(2) *If $B - \sum_{1\leqslant i\leqslant n} \alpha_i E^\triangle_{i+1,i}$ is diagonal for some $\alpha \in \mathbb{N}^n_\triangle$ then, for large p and $r = pn + b$, we have in $\mathcal{S}_\triangle(n, r)_\mathcal{Z}$:*

$$[{}_pB][{}_pA] = \sum_{\substack{T\in\Theta_\triangle(n),\,\mathrm{ro}(T)=\alpha\\ a_{i,j}-t_{i,j}+t_{i-1,j}\geqslant 0,\,\forall i\neq j}} Q_{T,A}(v, v^{-p})[{}_p(A - T + \widetilde{T})].$$

Let $\widetilde{\Theta}_\triangle(n)^{ss}$ be the set of $X \in \widetilde{\Theta}_\triangle(n)$ such that either $X - \sum_{1\leqslant i\leqslant n} \alpha_i E^\triangle_{i,i+1}$ or $X - \sum_{1\leqslant i\leqslant n} \alpha_i E^\triangle_{i+1,i}$ is diagonal for some $\alpha \in \mathbb{N}^n_\triangle$. We have the following affine version of [1, 3.9] (see [15, 4.5] for a slightly different version). For completeness, we include a proof.

Proposition 6.2. *Let $A \in \Theta_\triangle(n, r)$. Then there exist upper triangular matrices $A_1, A_2, \cdots, A_s$ and lower triangular matrices $A_{s+1}, A_{s+2}, \cdots, A_t$ in $\widetilde{\Theta}_\triangle(n)^{ss} \cap \Theta_\triangle(n, r)$ such that $\mathrm{co}(A_i) = \mathrm{ro}(A_{i+1})$ $(1 \leqslant i \leqslant t - 1)$ and the following identity holds in $\mathcal{S}_\triangle(n, pn + r)_\mathcal{Z}$: for $p \geqslant 0$,*

$$[{}_p(A_1)]\cdots[{}_p(A_s)]\cdot[{}_p(A_{s+1})]\cdots[{}_p(A_t)] = [{}_pA] + \textit{lower terms relative to } \sqsubset .$$

Proof. By Proposition 2.2, there is a distinguished words w_B for every $B \in \Theta^+_\triangle(n)$ satisfying the triangular relation (2.2.1). Let $x = w_{A^+}$ and $y = {}^tw_{{}^tA^-}$. By Theorem 3.1 and Proposition 2.2, we have in $\mathcal{S}_\triangle(n, r)_\mathcal{Z}$

$$\mathtt{m}^+_{(x),r} := \zeta_r(\widetilde{u}^+_{(x)}) = A^+(\mathbf{0}, r) + f \quad \text{and} \quad \mathtt{m}^-_{(y),r} := \zeta_r(\widetilde{u}^-_{(y)}) = A^-(\mathbf{0}, r) + g,$$

where f (resp., g) is a linear combination of $B(\mathbf{0}, r)$ with $B \in \Theta^+_\triangle(n)$ (resp., $B \in \Theta^-_\triangle(n)$) and $B \prec A^+$ (resp., $B \prec A^-$). By Proposition 5.3, we have for $p \geqslant 0$

$$\mathtt{m}^+_{(x),r}[\mathrm{diag}(\boldsymbol{\sigma}({}_pA))]\mathtt{m}^-_{(y),r} = [{}_pA] + \text{lower terms}.$$

Finally, by writing the words x, y in full, it is clear to see that there exist upper triangular matrices $A_1, A_2, \cdots, A_s$ and lower triangular matrices $A_{s+1}, A_{s+2}, \cdots, A_t$ in $\widetilde{\Theta}_\triangle(n)^{ss}$ such that

$$\mathtt{m}^+_{(x),r}[\mathrm{diag}(\boldsymbol{\sigma}({}_pA))] = [{}_p(A_1)]\cdots[{}_p(A_s)] \quad \text{and} \quad [\mathrm{diag}(\boldsymbol{\sigma}({}_pA))]\mathtt{m}^-_{(y),r} = [{}_p(A_{s+1})]\cdots[{}_p(A_t)],$$

as desired. □

We can now prove the following stabilization property for affine quantum Schur algebras.

Proposition 6.3. *Let $A, B \in \widetilde{\Theta}_\triangle(n)$ and assume $\mathrm{co}(B) = \mathrm{ro}(A)$. Then there exist unique $X_1, \cdots, X_m \in \widetilde{\Theta}_\triangle(n)$, unique $P_1(v, v'), \cdots, P_m(v, v') \in \mathcal{Z}_1$ and an integer $p_0 \geqslant 0$ such that, in $\mathcal{S}_\triangle(n, pn + \sigma(A))_\mathcal{Z}$,*

$$[{}_pB][{}_pA] = \sum_{1\leqslant i\leqslant m} P_i(v, v^{-p})[{}_pX_i] \quad \textit{for all } p \geqslant p_0. \tag{6.3.1}$$

Proof. The proof can be conducted by induction on $\|B\|$. With Lemma 6.1 and Proposition 6.2, the proof is entirely similar to that of [1, 3.9] or [5, Prop. 14.1]. □

Let $\widetilde{\mathsf{K}}_\vartriangle(n)_{\mathcal{Z}_1}$ be the free $\mathcal{Z}_1$-module with basis $\{A \mid A \in \widetilde{\Theta}_\vartriangle(n)\}$. Then, by Proposition 6.3, we may make $\widetilde{\mathsf{K}}_\vartriangle(n)_{\mathcal{Z}_1}$ into an associative $\mathcal{Z}_1$-algebra (without unit) by the multiplication:

$$B \cdot A = \begin{cases} \sum_{1\leqslant i\leqslant m} P_i(v,v')X_i, & \text{if } \mathrm{co}(B) = \mathrm{ro}(A); \\ 0, & \text{otherwise.} \end{cases} \tag{6.3.2}$$

Let

$$\mathsf{K}_\vartriangle(n)_{\mathcal{Z}} = \widetilde{\mathsf{K}}_\vartriangle(n)_{\mathcal{Z}_1} \otimes_{\mathcal{Z}_1} \mathcal{Z},$$

where $\mathcal{Z}$ is regarded as a $\mathcal{Z}_1$-module by specializing v' to 1. Then $\mathsf{K}_\vartriangle(n)_{\mathcal{Z}}$ becomes an associative $\mathcal{Z}$-algebra with basis $\{[A\} := A \otimes 1 \mid A \in \widetilde{\Theta}_\vartriangle(n)\}$. Let $\boldsymbol{\mathsf{K}}_\vartriangle(n) = \mathsf{K}_\vartriangle(n)_{\mathcal{Z}} \otimes_{\mathcal{Z}} \mathbb{Q}(v)$.

Following [1, 5.1], let $\hat{\boldsymbol{\mathsf{K}}}_\vartriangle(n)$ be the vector space of all formal (possibly infinite) $\mathbb{Q}(v)$-linear combinations $\sum_{A\in\widetilde{\Theta}_\vartriangle(n)} \beta_A[A\}$ such that, for any $\mathbf{x} \in \mathbb{Z}^n$, the sets $\{A \in \widetilde{\Theta}_\vartriangle(n) \mid \beta_A \neq 0,\ \mathrm{ro}(A) = \mathbf{x}\}$ and $\{A \in \widetilde{\Theta}_\vartriangle(n) \mid \beta_A \neq 0,\ \mathrm{co}(A) = \mathbf{x}\}$ are finite. We can define the product of two elements $\sum_{A\in\widetilde{\Xi}} \beta_A[A\}$, $\sum_{B\in\widetilde{\Xi}} \gamma_B[B\}$ in $\hat{\boldsymbol{\mathsf{K}}}_\vartriangle(n)$ to be $\sum_{A,B} \beta_A\gamma_B[A\}[B\}$. This defines an associative algebra structure on $\hat{\boldsymbol{\mathsf{K}}}_\vartriangle(n)$. The algebra $\boldsymbol{\mathcal{V}}_\vartriangle(n)$ can also be realized as a $\mathbb{Q}(v)$-subalgebra of $\hat{\boldsymbol{\mathsf{K}}}_\vartriangle(n)$, which we now describe.

The following result can be proved in a way similar to the proof of [9, 6.7] (cf. [15, 6.3]).

Lemma 6.4. *The linear map $\dot{\zeta}_r : \mathsf{K}_\vartriangle(n)_{\mathcal{Z}} \to \mathcal{S}_\vartriangle(n,r)_{\mathcal{Z}}$ defined by*

$$\dot{\zeta}_r([A\}) = \begin{cases} [A] & \text{if } A \in \Theta_\vartriangle(n,r); \\ 0 & \text{otherwise} \end{cases} \tag{6.4.1}$$

is an algebra epimorphism.

The map $\dot{\zeta}_r : \mathsf{K}_\vartriangle(n)_{\mathcal{Z}} \to \mathcal{S}_\vartriangle(n,r)_{\mathcal{Z}}$ induces a surjective algebra homomorphism

$$\widehat{\zeta}_r : \hat{\boldsymbol{\mathsf{K}}}_\vartriangle(n) \to \boldsymbol{\mathcal{S}}_\vartriangle(n,r) \tag{6.4.2}$$

sending $\sum_{A\in\widetilde{\Theta}_\vartriangle(n)} \beta_A[A\}$ to $\sum_{A\in\widetilde{\Theta}_\vartriangle(n)} \beta_A\dot{\xi}_r([A\})$. Consequently, we get a surjective algebra homomorphism

$$\widehat{\zeta} : \hat{\boldsymbol{\mathsf{K}}}_\vartriangle(n) \twoheadrightarrow \boldsymbol{\mathcal{S}}_\vartriangle(n). \tag{6.4.3}$$

defined by sending x to $\widehat{\zeta}(x) := (\widehat{\zeta}_r(x))_{r\geqslant 0}$. It is clear that we have $\widehat{\zeta}(\boldsymbol{\mathsf{K}}_\vartriangle(n)) = \boldsymbol{\mathcal{S}}_\vartriangle^\oplus(n)$ where $\boldsymbol{\mathcal{S}}_\vartriangle^\oplus(n) = \bigoplus_{r\geqslant 0} \boldsymbol{\mathcal{S}}_\vartriangle(n,r)$. Thus, by restriction $\widehat{\zeta}$ to $\boldsymbol{\mathsf{K}}_\vartriangle(n)$, we get a surjective algebra homomorphism from $\boldsymbol{\mathsf{K}}_\vartriangle(n)$ to $\boldsymbol{\mathcal{S}}_\vartriangle^\oplus(n)$.

For $A \in \Theta_\vartriangle^\pm(n)$, $\mathbf{j} \in \mathbb{Z}_\vartriangle^n$ and $\lambda \in \mathbb{N}_\vartriangle^n$, let

$$A(\mathbf{j}\} := \sum_{\mu\in\mathbb{Z}_\vartriangle^n} v^{\mu\cdot\mathbf{j}}[A + \mathrm{diag}(\mu)\} \ \text{ and } \ A(\mathbf{j},\lambda\} := \sum_{\mu\in\mathbb{Z}_\vartriangle^n} v^{\mu\cdot\mathbf{j}} \begin{bmatrix} \mu \\ \lambda \end{bmatrix} [A + \mathrm{diag}(\mu)\}.$$

By Proposition 5.3, the stabilisation property Proposition 6.3 implies that for any $A \in \widetilde{\Theta}_\vartriangle(n)$,

$$A^+(\mathbf{0})[\operatorname{diag}(\boldsymbol{\sigma}(A))]A^-(\mathbf{0}) = [A] + \text{a } \mathcal{Z}\text{-linear comb. of } [A'] \text{ with } A' \sqsubset A. \tag{6.4.4}$$

Let $\mathbf{V}_\vartriangle(n)$ be the $\mathbb{Q}(v)$-subspace of $\hat{\mathbf{K}}_\vartriangle(n)$ spanned by all $A(\mathbf{j})$ ($A \in \Theta^\pm_\vartriangle(n)$ and $\mathbf{j} \in \mathbb{Z}^n_\vartriangle$). Let $\mathsf{V}_\vartriangle(n)_\mathcal{Z}$ be the $\mathcal{Z}$-submodule of $\hat{\mathbf{K}}_\vartriangle(n)$ spanned by $A(\mathbf{j}, \lambda)$ for all $A, \mathbf{j}, \lambda$ as above.

Theorem 6.5. *(1) $\mathbf{V}_\vartriangle(n)$ is a subalgebra of $\hat{\mathbf{K}}_\vartriangle(n)$ and the restriction of $\widehat{\zeta}$ to $\mathbf{V}_\vartriangle(n)$ induces an algebra isomorphism $\widehat{\zeta}: \mathbf{V}_\vartriangle(n) \to \boldsymbol{\mathcal{V}}_\vartriangle(n), A(\mathbf{j}) \mapsto A(\mathbf{j})$.*

(2) The $\mathcal{Z}$-module $\mathsf{V}_\vartriangle(n)_\mathcal{Z}$ is a subalgebra of $\hat{\mathbf{K}}_\vartriangle(n)$ and the restriction of $\widehat{\zeta}$ to $\mathsf{V}_\vartriangle(n)_\mathcal{Z}$ induces an algebra isomorphism $\widehat{\zeta}: \mathsf{V}_\vartriangle(n)_\mathcal{Z} \to \mathcal{V}_\vartriangle(n)_\mathcal{Z}, A(\mathbf{j}, \lambda) \mapsto A(\mathbf{j}, \lambda)$.

Proof. By looking at the kernel of $\widehat{\zeta}$ (cf. [10, §8]), it is clear that the restriction of $\widehat{\zeta}$ to $\mathbf{V}_\vartriangle(n)$ is injective. Note that $\widehat{\zeta}(\mathbf{V}_\vartriangle(n)) = \boldsymbol{\mathcal{V}}_\vartriangle(n)$ and $\widehat{\zeta}(\mathsf{V}_\vartriangle(n)_\mathcal{Z}) = \mathcal{V}_\vartriangle(n)_\mathcal{Z}$. Now the assertion follows from Theorem 3.2 and Proposition 5.5. □

This result together with Theorem 3.2 gives another realisation of $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$. This is an *unmodified* affine generalisation of the BLM construction in [1]. *In particular, we will identify $\mathfrak{D}_\vartriangle(n)$ with $\mathbf{V}_\vartriangle(n)$ and $\mathfrak{D}_\vartriangle(n)_\mathcal{Z}$ with $\mathsf{V}_\vartriangle(n)_\mathcal{Z}$ in the sequel.*

We end this section with a discussion on a realisation of the modified quantum group $\dot{\mathfrak{D}}_\vartriangle(n)$. We will prove that $\dot{\mathfrak{D}}_\vartriangle(n)$ and its integral form $\dot{\mathfrak{D}}_\vartriangle(n)_\mathcal{Z}$ is isomorphic the affine BLM algebras $\mathbf{K}_\vartriangle(n)$ and $\mathsf{K}_\vartriangle(n)_\mathcal{Z}$, respectively.

Let $\Pi_\vartriangle(n) = \{\boldsymbol{e}^\vartriangle_j - \boldsymbol{e}^\vartriangle_{j+1} \mid 1 \leqslant j \leqslant n\}$. According to [14, 3.5.2], the algebra $\mathfrak{D}_\vartriangle(n)$ is a $\mathbb{Z}^n_\vartriangle$-graded algebra with $\deg(u^+_A) = \mathrm{ro}(A) - \mathrm{co}(A)$, $\deg(u^-_A) = \mathrm{co}(A) - \mathrm{ro}(A)$ and $\deg(K^{\pm 1}_i) = 0$ for $A \in \Theta^+_\vartriangle(n)$ and $1 \leqslant i \leqslant n$. For $\nu \in \mathbb{Z}^n_\vartriangle$, let $\mathfrak{D}_\vartriangle(n)_\nu$ be the set of homogeneous elements in $\mathfrak{D}_\vartriangle(n)$ of degree ν. Then we have $\mathfrak{D}_\vartriangle(n) = \bigoplus_{\nu \in \mathbb{Z}\Pi_\vartriangle(n)} \mathfrak{D}_\vartriangle(n)_\nu$.

For $\lambda, \mu \in \mathbb{Z}^n_\vartriangle$ we set ${}_\lambda\mathfrak{D}_\vartriangle(n)_\mu = \mathfrak{D}_\vartriangle(n)/{}_\lambda I_\mu$, where

$${}_\lambda I_\mu = \Big(\sum_{\mathbf{j} \in \mathbb{Z}^n_\vartriangle} (K^\mathbf{j} - v^{\lambda \cdot \mathbf{j}})\mathfrak{D}_\vartriangle(n) + \sum_{\mathbf{j} \in \mathbb{Z}^n_\vartriangle} \mathfrak{D}_\vartriangle(n)(K^\mathbf{j} - v^{\mu \cdot \mathbf{j}})\Big). \tag{6.5.1}$$

Let $\pi_{\lambda,\mu}: \mathfrak{D}_\vartriangle(n) \to {}_\lambda\mathfrak{D}_\vartriangle(n)_\mu$ be the canonical projection. Since $\pi_{\lambda,\mu}(\mathfrak{D}_\vartriangle(n)_{\lambda-\mu}) = {}_\lambda\mathfrak{D}_\vartriangle(n)_\mu$ (cf. [9, Lemma 6.2]), it follows that ${}_\lambda\mathfrak{D}_\vartriangle(n)_\mu$ is spanned by the elements $\pi_{\lambda,\mu}(u^+_A u^-_B)$ for all A, B, λ, μ with $\lambda - \mu = \deg(u^+_A u^-_B)$. Let

$$\dot{\mathfrak{D}}_\vartriangle(n) := \bigoplus_{\lambda, \mu \in \mathbb{Z}^n_\vartriangle} {}_\lambda\mathfrak{D}_\vartriangle(n)_\mu.$$

We define the product in $\dot{\mathfrak{D}}_\vartriangle(n)$ as follows. For $\lambda', \mu', \lambda'', \mu'' \in \mathbb{Z}^n_\vartriangle$ with $\lambda' - \mu', \lambda'' - \mu'' \in \mathbb{Z}\Pi_\vartriangle(n)$ and any $t \in \mathfrak{D}_\vartriangle(n)_{\lambda'-\mu'}$, $s \in \mathfrak{D}_\vartriangle(n)_{\lambda''-\mu''}$, the product $\pi_{\lambda',\mu'}(t)\pi_{\lambda'',\mu''}(s)$ is equal to $\pi_{\lambda',\mu''}(ts)$ if $\mu' = \lambda''$, and it is zero, otherwise. Then $\dot{\mathfrak{D}}_\vartriangle(n)$ becomes an associative $\mathbb{Q}(v)$-algebra with this product. The algebra $\dot{\mathfrak{D}}_\vartriangle(n)$ is naturally a $\mathfrak{D}_\vartriangle(n)$-bimodule defined by $t'\pi_{\lambda',\lambda''}(s)t'' = \pi_{\lambda'+\nu',\lambda''-\nu''}(t'st'')$, for $t' \in \mathfrak{D}_\vartriangle(n)_{\nu'}$, $s \in \mathfrak{D}_\vartriangle(n)$, $t'' \in \mathfrak{D}_\vartriangle(n)_{\nu''}$ and $\lambda', \lambda'' \in \mathbb{Z}^n_\vartriangle$ (cf. [27, 14]). In particular, putting $1_\lambda = \pi_{\lambda,\lambda}(1)$, we have $u_A^+ 1_\lambda u_B^- = \pi_{\lambda+\deg(u_A^+),\lambda-\deg(u_B^-)}(u_A^+ u_B^-)$ and $\dot{\mathfrak{D}}_\triangle(n)$ is spanned by the elements $u_A^+ 1_\lambda u_B^-$ for all A, B, λ.

Let $\dot{\mathfrak{D}}_\triangle(n)_{\mathcal{Z}}$ be the $\mathcal{Z}$-submodule of $\dot{\mathfrak{D}}_\triangle(n)$ spanned by the elements $u_A^+ 1_\lambda u_B^-$ for $A, B \in \Theta_\triangle^+(n)$ and $\lambda \in \mathbb{Z}_\triangle^n$. It is proved in [14, Th. 4.2] that $\dot{\mathfrak{D}}_\triangle(n)_{\mathcal{Z}}$ is a $\mathcal{Z}$-subalgebra of $\dot{\mathfrak{D}}_\triangle(n)$. We now can realise $\dot{\mathfrak{D}}_\triangle(n)$ and $\dot{\mathfrak{D}}_\triangle(n)_{\mathcal{Z}}$ as $\mathsf{K}_\triangle(n)$ and $\mathsf{K}_\triangle(n)_{\mathcal{Z}}$, respectively; cf. [9, Th. 6.3].

Theorem 6.6. *The linear map $\Phi : \dot{\mathfrak{D}}_\triangle(n) \to \mathsf{K}_\triangle(n)$ sending $\pi_{\lambda\mu}(u)$ to $[\operatorname{diag}(\lambda)]u[\operatorname{diag}(\mu)]$ for all $u \in \mathfrak{D}_\triangle(n)$ and $\lambda, \mu \in \mathbb{Z}_\triangle^n$, is an algebra isomorphism. Furthermore we have $\Phi(\dot{\mathfrak{D}}_\triangle(n)_{\mathcal{Z}}) = \mathsf{K}_\triangle(n)_{\mathcal{Z}}$.*

Proof. By a proof similar to that of [9, 6.3], it is easy to see that Φ is an algebra homomorphism. In particular, $\Phi(1_\lambda) = [\operatorname{diag}(\lambda)]$. By (6.4.4), the image of the spanning set $\{u_A^+ 1_\lambda u_B^- \mid A, B \in \Theta_\triangle^+(n), \lambda \in \mathbb{Z}_\triangle^n\}$ is in fact a basis for $\mathsf{K}_\triangle(n)$, proving the first assertion which implies the last assertion by definition. $\square$

We will identify $\dot{\mathfrak{D}}_\triangle(n)$ with $\mathsf{K}_\triangle(n)$ and $\dot{\mathfrak{D}}_\triangle(n)_{\mathcal{Z}}$ with $\mathsf{K}_\triangle(n)_{\mathcal{Z}}$ via the map Φ defined in Theorem 6.6 and identify $\mathfrak{D}_\triangle(n)$ with $\mathsf{V}_\triangle(n)$ and $\mathfrak{D}_\triangle(n)_{\mathcal{Z}}$ with $\mathsf{V}_\triangle(n)_{\mathcal{Z}}$ as in Theorem 6.5. Then the $\mathfrak{D}_\triangle(n)$-bimodule structure on $\dot{\mathfrak{D}}_\triangle(n)$ satisfies the following simple formula: for all $A \in \Theta_\triangle^\pm(n), \mathbf{j}, \lambda \in \mathbb{Z}_\triangle^n$,

$$A(\mathbf{j})[\operatorname{diag}(\lambda)] = [A + \operatorname{diag}(\lambda - \operatorname{co}(A))], \quad [\operatorname{diag}(\lambda)]A(\mathbf{j}) = [A + \operatorname{diag}(\lambda - \operatorname{ro}(A))]. \tag{6.6.1}$$

For $A \in \widetilde{\Theta}_\triangle(n)$, choose words $w_{A^+}, w_{A^-} \in \widetilde{\Sigma}$ such that (2.2.1) and its opposite version (obtained by applying (2.2.2) to (2.2.1)) hold. Then, by (6.4.4),

$$\mathcal{M}^{(A)} := \widetilde{u}^+_{(w_{A^+})} 1_{\boldsymbol{\sigma}(A)} \widetilde{u}^-_{(w_{A^-})} = \widetilde{u}^+_{(w_{A^+})}[\operatorname{diag}(\boldsymbol{\sigma}(A))]\widetilde{u}^-_{(w_{A^-})} = [A] + \sum_{\substack{B \sqsubset A \\ B \in \widetilde{\Theta}_\triangle(n)}} h_{A,B}[B], \tag{6.6.2}$$

where $h_{A,B} \in \mathcal{Z}$. Thus, we have immediately:

Corollary 6.7. *The set $\{\mathcal{M}^{(A)} \mid A \in \widetilde{\Theta}_\triangle(n)\}$ forms a $\mathcal{Z}$-basis for $\dot{\mathfrak{D}}_\triangle(n)_{\mathcal{Z}}$.*

7. Canonical bases for the integral modified quantum affine $\mathfrak{gl}_n$

It is well known that the positive part of a quantum enveloping algebra $\mathbf{U}$ has a canonical basis with remarkable properties (see [21], [23], [24]). In contrast, there is no canonical basis for $\mathbf{U}$. However, the modified form $\dot{\mathbf{U}}$ of $\mathbf{U}$ can have a canonical basis (see [22], [26], [27]). We now define the canonical basis relative the basis $\{[A]\}_{A\in\widetilde{\Theta}_\triangle(n)}$ for $\dot{\mathfrak{D}}_\triangle(n)_{\mathcal{Z}} = \mathsf{K}_\triangle(n)_{\mathcal{Z}}$. Our strategy is to use a stabilisation property for the bar involution on $\mathcal{S}_\triangle(n,r)_{\mathcal{Z}}$ to define a bar involution on $\widetilde{\mathsf{K}}_\triangle(n)_{\mathcal{Z}_1}$ (see (6.3.2)) which then induces a bar involution on $\mathsf{K}_\triangle(n)_{\mathcal{Z}}$.

We first define the bar involution on $\mathcal{S}_\vartriangle(n,r)_\mathcal{Z}$ via the one on the Hecke algebra, following [7] (cf. [33]). Let W_r be the subgroup of $\mathfrak{S}_{\vartriangle r}$ generated by s_i for $1 \leqslant i \leqslant r$. Let ρ be the permutation of $\mathbb{Z}$ sending j to $j+1$ for all $j \in \mathbb{Z}$. Let $\mathcal{H}(W_r)$ be the $\mathcal{Z}$-subalgebra of $\mathcal{H}_\vartriangle(r)_\mathcal{Z}$ generated by T_{s_i} for $1 \leqslant i \leqslant r$. Let $\{C'_w \mid w \in W_r\}$ be the canonical basis of $\mathcal{H}(W_r)$ defined in [23, 1.1(c)]. For $w = \rho^a x \in \mathfrak{S}_{\vartriangle r}$ with $a \in \mathbb{Z}$ and $x \in W_r$, let $C'_w = T_\rho^a C'_x$. Then the set $\{C'_w \mid w \in \mathfrak{S}_{\vartriangle r}\}$ forms a $\mathcal{Z}$-basis for $\mathcal{H}_\vartriangle(r)_\mathcal{Z}$. Note that $C'_{w_{0,\mu}} = v^{-\ell(w_{0,\mu})} x_\mu$. Let $^- : \mathcal{H}_\vartriangle(r)_\mathcal{Z} \to \mathcal{H}_\vartriangle(r)_\mathcal{Z}$ be the ring involution defined by $\bar{v} = v^{-1}$ and $\bar{T}_w = T^{-1}_{w^{-1}}$. We define a map $^- : \mathcal{S}_\vartriangle(n,r)_\mathcal{Z} \to \mathcal{S}_\vartriangle(n,r)_\mathcal{Z}$ such that $\bar{v} = v^{-1}$ and $\bar{f}(C'_{w_{0,\mu}} h) = \overline{f(C'_{w_{0,\mu}})} h$ for $f \in \mathrm{Hom}_{\mathcal{H}_\vartriangle(r)_\mathcal{Z}}(x_\mu \mathcal{H}_\vartriangle(r)_\mathcal{Z}, x_\lambda \mathcal{H}_\vartriangle(r)_\mathcal{Z})$ and $h \in \mathcal{H}_\vartriangle(r)_\mathcal{Z}$. Then the map $^- : \mathcal{S}_\vartriangle(n,r)_\mathcal{Z} \to \mathcal{S}_\vartriangle(n,r)_\mathcal{Z}$ is a ring involution.[3] We need to look some first properties of the bar involution in Lemma 7.2 before proving its stabilisation property in Proposition 7.3.

Given $A \in \Theta_\vartriangle(n,r)$, write $y_A = w$ if $A = \jmath_\vartriangle(\lambda, w, \mu)$, and also write y_A^+ for the unique longest element in $\mathfrak{S}_\lambda w \mathfrak{S}_\mu$. For $\lambda \in \Lambda_\vartriangle(n,r)$, let $w_{0,\lambda}$ be the longest element in $\mathfrak{S}_\lambda$.

Lemma 7.1. *For $A \in \Theta_\vartriangle(n,r)$ we have $\ell(y_A^+) = d_A + \ell(w_{0,\mu})$ where $\mu = \mathrm{co}(A)$ and d_A is given in* (3.0.2).

Proof. For $1 \leqslant i \leqslant n$, let $\nu^{(i)}$ be the composition of μ_i obtained by removing all zeros from column i of A. Let $\lambda = \mathrm{ro}(A)$. According to [4, 3.2.3], $y_A^{-1} \mathfrak{S}_\lambda y_A \cap \mathfrak{S}_\mu = \mathfrak{S}_\nu$, where $\nu = (\nu^{(1)}, \cdots, \nu^{(n)})$. Let x be the longest element in $\mathscr{D}_\nu^\vartriangle \cap \mathfrak{S}_\mu$. Then $y_A^+ = w_{0,\lambda} y_A x$ and $\ell(y_A^+) = \ell(w_{0,\lambda}) + \ell(y_A) + \ell(x)$. Since $w_{0,\nu} x$ is the longest element in $\mathfrak{S}_\mu$, it follows that $w_{0,\mu} = w_{0,\nu} x$ and

$$\ell(x) = \ell(w_{0,\mu}) - \ell(w_{0,\nu}) = \sum_{1 \leqslant i \leqslant n} \left(\binom{\mu_i}{2} - \sum_{k \in \mathbb{Z}} \binom{\nu_k^{(i)}}{2} \right) = \sum_{\substack{1 \leqslant i \leqslant n \\ s<t}} \nu_s^{(i)} \nu_t^{(i)}.$$

Hence,

$$\ell(y_A^+) = \ell(w_{0,\lambda}) + \ell(y_A) + \ell(x) = \ell(w_{0,\lambda}) + \ell(y_A) + \sum_{\substack{1 \leqslant i \leqslant n \\ s<t}} a_{s,i} a_{t,i} \tag{7.1.1}$$

By [11, 5.3], $d_A - \ell(y_A) = \sum_{1 \leqslant i \leqslant n;\, j<l} a_{i,j} a_{i,l}$. Furthermore, we have

$$\ell(w_{0,\lambda}) - \ell(w_{0,\mu}) = \sum_{1 \leqslant i \leqslant n} \left(\frac{\lambda_i(\lambda_i - 1)}{2} - \frac{\mu_i(\mu_i - 1)}{2} \right) = \sum_{\substack{1 \leqslant i \leqslant n \\ k<l}} (a_{i,k} a_{i,l} - a_{k,i} a_{l,i}).$$

Thus, by (7.1.1), we conclude that $d_A - \ell(y_A) - (\ell(w_{0,\lambda}) - \ell(w_{0,\mu})) = \ell(y_A^+) - \ell(w_{0,\lambda}) - \ell(y_A)$. Consequently, $\ell(y_A^+) = d_A + \ell(w_{0,\mu})$. $\square$

For $d \in \mathscr{D}_{\lambda,\mu}^\vartriangle$ let

$$\widetilde{T}_{\mathfrak{S}_\lambda d \mathfrak{S}_\mu} = v^{-\ell(d^+)} T_{\mathfrak{S}_\lambda d \mathfrak{S}_\mu},$$

[3] See [7, Prop. 3.2] for a proof.

where d^+ is the unique longest element in $\mathfrak{S}_\lambda d\mathfrak{S}_\mu$. Recall from Theorem 3.3 and Propoition 4.2 that the matrix $S_\alpha = \sum_{1\leqslant i\leqslant n}\alpha_i E^{\triangle}_{i,i+1}$ defines a semisimple representation of the cyclic quiver $\triangle(n)$.

Lemma 7.2. *For $\alpha,\beta\in\mathbb{N}^n_\triangle$, let $A=S_\alpha+\mathrm{diag}(\beta)\in\Theta_\triangle(n,r)$. Then, in $\mathcal{S}_\triangle(n,r)_\mathcal{Z}$, $\overline{[A]}=[A]$ and $\overline{[{}^tA]}=[{}^tA]$. In particular, we have $\overline{S_\alpha(\mathbf{0},r)}=S_\alpha(\mathbf{0},r)$, $\overline{{}^tS_\alpha(\mathbf{0},r)}={}^tS_\alpha(\mathbf{0},r)$ for $\alpha\in\mathbb{N}^n_\triangle$.*

Proof. Let $\lambda=\mathrm{ro}(A)$ and $\mu=\mathrm{co}(A)$. Then, by Lemma 7.1, we have $[A](C'_{w_{0,\mu}})=\widetilde{T}_{\mathfrak{S}_\lambda y_A\mathfrak{S}_\mu}$ and $[{}^tA](C'_{w_{0,\lambda}})=\widetilde{T}_{\mathfrak{S}_\mu y_{{}^tA}\mathfrak{S}_\lambda}$ (note that $y_{{}^tA}=y_A^{-1}$). By [11, (2.0.2)] (cf. the proof of [11, Prop. 3.5]), we have $y_A=\rho^{-\alpha_n}$ and $y_{{}^tA}=\rho^{\alpha_n}$. It follows from [2, (1.10)] that $C'_{y_A^+}=\widetilde{T}_{\mathfrak{S}_\lambda y_A\mathfrak{S}_\mu}$ and $C'_{y_{{}^tA}^+}=\widetilde{T}_{\mathfrak{S}_\mu y_{{}^tA}\mathfrak{S}_\lambda}$. Thus,

$$\overline{[A]}(C'_{w_{0,\mu}})=\overline{[A](C'_{w_{0,\mu}})}=\overline{C'_{y_A^+}}=C'_{y_A^+}=[A](C'_{w_{0,\mu}})$$
$$\overline{[{}^tA]}(C'_{w_{0,\lambda}})=\overline{[{}^tA](C'_{w_{0,\lambda}})}=\overline{C'_{y_{{}^tA}^+}}=C'_{y_{{}^tA}^+}=[{}^tA](C'_{w_{0,\lambda}}).$$

Consequently $\overline{[A]}=[A]$ and $\overline{[{}^tA]}=[{}^tA]$. The last assertion is clear. □

The stabilisation property developed at the beginning of last section gives the following stabilisation property.

Proposition 7.3. *For $A\in\widetilde{\Theta}_\triangle(n)$ there exist $C_1,\cdots,C_m\in\widetilde{\Theta}_\triangle(n)$, elements $H_i(v,v')\in\mathcal{Z}_1$ ($1\leqslant i\leqslant m$) and an integer $p_0\geqslant 0$ such that, in $\mathcal{S}_\triangle(n,pn+\sigma(A))_\mathcal{Z}$,*

$$\overline{[{}_pA]}=\sum_{1\leqslant i\leqslant m}H_i(v,v^{-p})[{}_pC_i]\quad\text{for all } p\geqslant p_0.$$

Proof. We prove the assertion by induction on $\|A\|$. If $\|A\|=0$ then $\overline{[{}_pA]}=[{}_pA]$ for all large enough p. Assume now that $\|A\|\geqslant 1$ and the result is true for all A' with $\|A'\|<\|A\|$. By Lemma 6.1 and Proposition 6.2, there exist $A_i\in\widetilde{\Theta}_\triangle(n)^{ss}$, $Z_j\in\widetilde{\Theta}_\triangle(n)$ and $Q_j(v,v')\in\mathcal{Z}_1$ ($1\leqslant i\leqslant N$, $1\leqslant j\leqslant m$) such that the following identity holds in $\mathcal{S}_\triangle(n,pn+\sigma(A))_\mathcal{Z}$

$$[{}_pA]=[{}_pA_1]\cdots[{}_pA_N]-\sum_{1\leqslant j\leqslant m}Q_j(v,v^{-p})[{}_pZ_j]$$

for all large enough p, where $\|Z_i\|<\|A\|$ for $1\leqslant i\leqslant m$. It follows from Lemma 7.2 that

$$\overline{[{}_pA]}=[{}_pA_1]\cdots[{}_pA_N]-\sum_{1\leqslant j\leqslant m}\overline{Q_j(v,v^{-p})}\cdot\overline{[{}_pZ_j]}.$$

Now the assertion follows from the induction hypothesis. □

Recall the ring $\mathcal{Z}_1$ defined in (6.0.1). It admits a ring involution (i.e., a ring automorphism of order two) $^-$satisfying $\bar v=v^{-1}$ and $\bar{v'}=v'^{-1}$. Extend the bar involution on $\mathcal{Z}_1$ to define a ring involution$^-:\widetilde{\mathsf{K}}_\triangle(n)_{\mathcal{Z}_1}\to\widetilde{\mathsf{K}}_\triangle(n)_{\mathcal{Z}_1}$ by setting $\overline{A}=\sum_{1\leqslant i\leqslant m}H_i(v,v')C_i$ (notation of Proposition 7.3). This involution induces a ring involution

$$\bar{\ }: \mathsf{K}_\vartriangle(n)_{\mathcal{Z}} \to \mathsf{K}_\vartriangle(n)_{\mathcal{Z}} \text{ which satisfies } \overline{v^j[A]} = v^{-j} \sum_{1\leqslant i\leqslant m} H_i(v,1)[C]. \tag{7.3.1}$$

The involution $\bar{\ }$ on $\mathsf{K}_\vartriangle(n)_{\mathcal{Z}}$ induces a $\mathbb{Q}$-algebra involution $\bar{\ }: \hat{\mathsf{K}}_\vartriangle(n) \to \hat{\mathsf{K}}_\vartriangle(n)$ such that $\overline{\sum_{A\in\widetilde{\Theta}_\vartriangle(n)} \beta_A[A]} = \sum_{A\in\widetilde{\Theta}_\vartriangle(n)} \overline{\beta_A}\overline{[A]}$.

Corollary 7.4. (1) *For $\alpha,\beta\in\mathbb{N}_\vartriangle^n$, if $A = S_\alpha + \mathrm{diag}(\beta) \in \Theta_\vartriangle(n,r)$, then $\overline{[A]} = [A]$ and $\overline{[{}^tA]} = [{}^tA]$. In particular, for any $\alpha\in\mathbb{N}_\vartriangle^n$, $\overline{S_\alpha(\mathbf{0})} = S_\alpha(\mathbf{0})$, $\overline{{}^tS_\alpha(\mathbf{0})} = {}^tS_\alpha(\mathbf{0})$.*

(2) *There is a unique $\mathbb{Q}$-algebra involution*[4]

$$\bar{\ }: \mathfrak{D}_\vartriangle(n) \to \mathfrak{D}_\vartriangle(n) \textit{ satisfying } \bar{v} = v^{-1},\ \overline{\widetilde{u}_\lambda^\pm} = \widetilde{u}_\lambda^\pm \textit{ and } \overline{K_i} = K_i^{-1} \textit{ for } \lambda\in\mathbb{N}_\vartriangle^n,\ 1\leqslant i\leqslant n.$$

(3) *The bar involution on $\mathsf{K}_\vartriangle(n)_{\mathcal{Z}}$ preserves the bimodule structure on $\mathsf{K}_\vartriangle(n)_{\mathcal{Z}}$.*

Proof. Clearly, by the definition of the bar involution on $\mathsf{K}_\vartriangle(n)_{\mathcal{Z}}$, (1) follows from Proposition 7.3 and Lemma 7.2. (2) follows from (1), Theorems 5.6 and 6.6. Finally, (3) is clear as the bimodule structure on $\mathsf{K}_\vartriangle(n)_{\mathcal{Z}}$ is induced by the algebra structure of $\hat{\mathsf{K}}_\vartriangle(n)$ on which the bar involution is an ring automorphism. □

We first look at an algebraic construction of the canonical basis for affine quantum Schur algebras (see [28] for a geometric construction). We need the following interval finite condition.

Lemma 7.5. *For $A\in\Theta_\vartriangle^\pm(n)$, the set $\{B\in\Theta_\vartriangle^\pm(n) \mid B\prec A\}$ is finite. Hence, the intervals $(-\infty,A'] := \{B\in\widetilde{\Theta}_\vartriangle(n) \mid B\sqsubseteq A'\}$ for all $A'\in\widetilde{\Theta}_\vartriangle(n)$ are finite.*

Proof. There exist $j_0\geqslant n$ such that $a_{s,j}=0$ for $1\leqslant s\leqslant n$ and $j\in\mathbb{Z}$ with $|j|>j_0$. Let $\mathcal{X}_A = \{B\in\Theta_\vartriangle^\pm(n) \mid b_{s,j}=0 \text{ for } 1\leqslant s\leqslant n \text{ and } |j|>j_0,\ \sigma(B)<\|A\|\}$. Then, $\mathcal{X}_A$ is a finite set. If $B\prec A$, $1\leqslant i\leqslant n$ and $j_0<j$, then

$$b_{i,j} \leqslant \sigma_{i,j}(B) \leqslant \sigma_{i,j}(A) = \sum_{\substack{1\leqslant s\leqslant n\\ s<t,\, j\leqslant t}} a_{s,t}|\{b\in\mathbb{N} \mid s-bn\leqslant i<j\leqslant t-bn\}| = 0.$$

This implies that if $B\prec A$, then $b_{i,j}=0$ for $1\leqslant i\leqslant n$ and $j>j_0$. Similarly, if $B\prec A$, then $b_{i,j}=0$ for $1\leqslant i\leqslant n$ and $j<-j_0$. Furthermore, by [4, 3.7.6], we conclude that $\sigma(B)\leqslant\|B\|<\|A\|$ for $B\in\Theta_\vartriangle^\pm(n)$ with $B\prec A$. Consequently, $\{B\in\Theta_\vartriangle^\pm(n) \mid B\prec A\}\subseteq\mathcal{X}_A$, proving the first assertion. The last assertion is clear from (5.2.1). □

Proposition 7.6. (1) *There is a unique $\mathcal{Z}$-basis $\{\theta_{A,r} \mid A\in\Theta_\vartriangle(n,r)\}$ for $\mathcal{S}_\vartriangle(n,r)_{\mathcal{Z}}$ such that $\overline{\theta_{A,r}} = \theta_{A,r}$ and*

$$\theta_{A,r} - [A] = \sum_{\substack{B\in\Theta_\vartriangle(n,r)\\ B\sqsubset A}} g_{B,A,r}[B] \in \sum_{\substack{B\in\Theta_\vartriangle(n,r)\\ B\sqsubset A}} v^{-1}\mathbb{Z}[v^{-1}][B]. \tag{7.6.1}$$

[4]This bar involution can also be induced from the bar involutions on $\mathcal{S}_\vartriangle(n,r)_{\mathcal{Z}}$ via $\boldsymbol{\mathcal{S}}_\vartriangle(n)$ and $\boldsymbol{\mathcal{V}}_\vartriangle(n)$. Thus, we may avoid using the stabilisation property.

(2) *For the canonical basis* $\{A\}$, $A\in\Theta_\vartriangle(n,r)$, *of* $\mathcal{S}_\vartriangle(n,r)_\mathcal{Z}$ *defined in* [28, 4.1(d)], *we have* $\{A\}=\theta_{A,r}$. *In particular,* $g_{B,A,r}$ *can be described in terms of Kazhdan–Lusztig polynomials.*

Proof. By Proposition 2.2, for each $A\in\Theta_\vartriangle(n,r)$, we may choose words $w_{A^+}\in\widetilde{\Sigma}$ such that (2.2.1) hold. Let $w_{A^-}={}^tw_{{}^tA^-}$. By (2.2.1) and its opposite version for $\widetilde{u}^-_{w_{A^-}}=\tau(\widetilde{u}^+_{{}^t(A^-)})$ (see (2.2.2)) and Proposition 5.3, we have

$$m^{(A)}:=\zeta_r(\widetilde{u}^+_{w_{A^+}})[\mathrm{diag}(\boldsymbol{\sigma}(A))]\zeta_r(\widetilde{u}^-_{w_{A^-}})=[A]+\sum_{\substack{B\sqsubset A\\ B\in\Theta_\vartriangle(n,r)}}h_{A,B}[B]\quad(h_{A,B}\in\mathcal{Z}).\tag{7.6.2}$$

Now the interval finite condition in Lemma 7.5 implies that there exist $h'_{A,B}\in\mathcal{Z}$ such that

$$[A]=m^{(A)}+\sum_{\substack{B\in\Theta_\vartriangle(n,r)\\ B\sqsubset A}}h'_{A,B}m^{(B)}.$$

Furthermore, by Lemma 7.2, we have $\overline{m^{(A)}}=m^{(A)}$ for $A\in\Theta_\vartriangle(n,r)$. Thus, (7.6.2) implies

$$\overline{[A]}=m^{(A)}+\sum_{\substack{B\in\Theta_\vartriangle(n,r)\\ B\sqsubset A}}\overline{h'_{A,B}}m^{(B)}=[A]+\sum_{\substack{B\in\Theta_\vartriangle(n,r)\\ B\sqsubset A}}k_{A,B}[B],$$

where $k_{A,B}\in\mathcal{Z}$. Now (1) follows from a standard argument; see, e.g., [25, 7.10]. Let $\leqslant$ be the partial order on $\Theta_\vartriangle(n,r)$ defined in [28, 4.1]. According to [29, §7], if $A,B\in\Theta_\vartriangle(n,r)$ and $B<A$ then $B\sqsubset A$. Thus, by [28, 4.1(e)] and [33, Remark 7.6], we conclude (2). □

We now construct the canonical basis for $\mathsf{K}_\vartriangle(n)_\mathcal{Z}$ as follows. See [17] for a construction in the non-affine case.

Theorem 7.7. (1) *There exists a unique* $\mathcal{Z}$*-basis* $\{\theta_A\mid A\in\widetilde{\Theta}_\vartriangle(n)\}$ *for* $\mathsf{K}_\vartriangle(n)_\mathcal{Z}=\dot{\mathfrak{D}}_\vartriangle(n)_\mathcal{Z}$ *such that* $\overline{\theta_A}=\theta_A$ *and* $\theta_A-[A]\in\sum_{B\in\widetilde{\Theta}_\vartriangle(n),B\sqsubset A}v^{-1}\mathbb{Z}[v^{-1}][B]$.

(2) *The algebra homomorphism* $\dot{\zeta}_r:\mathsf{K}_\vartriangle(n)_\mathcal{Z}\to\mathcal{S}_\vartriangle(n,r)_\mathcal{Z}$ *given in* (6.4.1) *preserves the bar involution and the canonical bases:*

(a) $\dot{\zeta}_r(\bar{u})=\overline{\dot{\zeta}_r(u)}$ *for all* $u\in\mathsf{K}_\vartriangle(n)_\mathcal{Z}$; (b) $\dot{\zeta}_r(\theta_A)=\begin{cases}\theta_{A,r}, & \text{if } A\in\Theta_\vartriangle(n,r);\\ 0, & \text{otherwise.}\end{cases}$

(3) *There is an anti-automorphism* $\dot{\tau}$ *on* $\mathsf{K}_\vartriangle(n)_\mathcal{Z}$ *such that* $\dot{\tau}([A])=[{}^tA]$ *and* $\dot{\tau}(\theta_A)=\theta_{{}^tA}$.

Proof. Consider the monomial basis $\{\mathcal{M}^{(A)}\mid A\in\widetilde{\Theta}_\vartriangle(n)\}$ given in Corollary 6.7. Then Lemma 7.2 implies $\overline{\mathcal{M}^{(A)}}=\mathcal{M}^{(A)}$ and (6.6.2) together with the interval finite property Lemma 7.5 implies $[A]=\mathcal{M}^{(A)}+h$, where h is a $\mathcal{Z}$-linear combination of $\mathcal{M}^{(C)}$ with $C\in\widetilde{\Theta}_\vartriangle(n)$ and $C\sqsubset A$. Thus, we conclude that $\overline{[A]}-[A]\in\sum_{\substack{C\in\widetilde{\Theta}_\vartriangle(n)\\ C\sqsubset A}}\mathcal{Z}[C]$. Hence, like the proof of Proposition 7.6, a standard argument proves (1).

According to (6.4.1) and Lemma 7.2 we see that $\dot{\zeta}_r(\overline{\mathcal{M}^{(A)}})=\overline{\dot{\zeta}_r(\mathcal{M}^{(A)})}$ for $A\in\widetilde{\Theta}_\vartriangle(n)$. Furthermore, by Corollary 6.7, the set $\{\mathcal{M}^{(A)}\mid A\in\widetilde{\Theta}_\vartriangle(n)\}$ forms a $\mathcal{Z}$-basis for $\mathsf{K}_\vartriangle(n)_\mathcal{Z}$. Thus,

$\dot\zeta_r(\bar u)=\overline{\dot\zeta_r(u)}$ for $u\in \mathsf{K}_\vartriangle(n)_{\mathcal Z}$. The second assertion in (2) follows from the argument for the uniqueness of canonical basis.

By [28, 1.11], the $\mathcal Z$-linear map $\tau_r:\boldsymbol{\mathcal S}_\vartriangle(n,r)\to\boldsymbol{\mathcal S}_\vartriangle(n,r)$, $[A]\mapsto[{}^tA]$ is an algebra anti-automorphism, where ${}^tA$ is the transpose of A. By Proposition 6.3, the maps τ_r induce an algebra anti-automorphism $\dot\tau:\mathsf{K}_\vartriangle(n)_{\mathcal Z}\to\mathsf{K}_\vartriangle(n)_{\mathcal Z}$ such that $\dot\tau([A])=[{}^tA]$ for $A\in\widetilde\Theta_\vartriangle(n)$. Finally, applying $\dot\tau$ to $\theta_A-[A]$ yields $\dot\tau(\theta_A)=\theta_{{}^tA}$ by the uniqueness of canonical bases. □

Remark 7.8. The basis constructed in Theorem 7.7(1) is the canonical basis for the integral modified quantum affine $\mathfrak{gl}_n$. Theorem 7.7(2b) shows that this basis is the lifting of the canonical bases for affine quantum Schur algebras. A similar basis with a similar property for the modified quantum affine $\mathfrak{sl}_n$ was conjectured by Lusztig in [28, 9.3]. This conjecture (rather its slight modified version) was proved by Vasserot and Schiffmann in [32]. Thus, Theorems 6.5, 6.6 and 7.7 can be regarded as of a generalisation of the conjecture of Lusztig to the quantum loop algebra $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$. We will address an extension of our approach to the extended quantum affine $\mathfrak{sl}_n$ case in the last section.

We end this section with a comparison of this canonical basis and the canonical basis for the Ringel–Hall algebra of a cyclic quiver. According to [33, Prop 7.5] (see also [24]), there is a unique $\mathcal Z$-basis $\{\theta^+_A\mid A\in\Theta^+_\vartriangle(n)\}$ for the Ringel–Hall algebra $\mathfrak{H}_\vartriangle(n)_{\mathcal Z}=\mathfrak{D}^+_\vartriangle(n)_{\mathcal Z}$ such that $\overline{\theta^+_A}=\theta^+_A$ and

$$\theta^+_A-\widetilde u^+_A\in\sum_{\substack{B\prec A,\,B\in\Theta^+_\vartriangle(n)\\ \mathbf{d}(B)=\mathbf{d}(A)}}v^{-1}\mathbb Z[v^{-1}]\widetilde u^+_B. \tag{7.8.1}$$

Proposition 7.9. *Assume $A\in\Theta^+_\vartriangle(n)$ and $\lambda\in\mathbb Z^n_\vartriangle$. Then we have $\theta^+_A[\mathrm{diag}(\lambda)]=\theta_{A+\mathrm{diag}(\lambda-co(A))}$. In particular, we have $\theta^+_A=\sum_{\mu\in\mathbb Z^n_\vartriangle}\theta_{A+\mathrm{diag}(\mu)}$.*

Proof. By (6.6.1) and (7.8.1),

$$\theta^+_A[\mathrm{diag}(\lambda)]-[A+\mathrm{diag}(\lambda-co(A))]\in\sum_{\substack{B\in\Theta^+_\vartriangle(n),B\prec A\\ \mathbf{d}(B)=\mathbf{d}(A)}}v^{-1}\mathbb Z[v^{-1}][B+\mathrm{diag}(\lambda-co(B))].$$

It is direct to check that, for $\mathbf{d}(B)=\mathbf{d}(A)$ and $B\in\Theta^+_\vartriangle(n)$, $\mathrm{ro}(B)-\mathrm{co}(B)=\mathrm{ro}(A)-\mathrm{co}(A)$. Hence,

$$\theta^+_A[\mathrm{diag}(\lambda)]-[A+\mathrm{diag}(\lambda-co(A))]\in\sum_{\substack{C\in\widetilde\Theta_\vartriangle(n)\\ C\sqsubset A+\mathrm{diag}(\lambda-\mathrm{co}(A))}}v^{-1}\mathbb Z[v^{-1}][C].$$

Also, by Corollary 7.4(3), $\overline{\theta^+_A[\mathrm{diag}(\lambda)]}=\overline{\theta^+_A}\,\overline{[\mathrm{diag}(\lambda)]}=\theta^+_A[\mathrm{diag}(\lambda)]$. Hence, the first assertion follows from the uniqueness of the canonical basis. Now, the identity element $1=\sum_{\lambda\in\mathbb Z^n_\vartriangle}[\mathrm{diag}(\lambda)]$ gives the last assertion. □

8. Application to a conjecture of Lusztig

Let $\mathbf{U}_\triangle(n)$ be the extended affine $\mathfrak{sl}_n$ as defined in Theorem 2.3(2) and let $\dot{\mathbf{U}}_\triangle(n) = \oplus_{\lambda,\mu\in\mathbb{Z}^n_\triangle}\mathbf{U}_\triangle(n)/{}_\lambda{}'I_\mu$, where ${}_\lambda{}'I_\mu := \sum_{\mathbf{j}\in\mathbb{Z}^n_\triangle}(K^{\mathbf{j}} - v^{\lambda\cdot\mathbf{j}})\mathbf{U}_\triangle(n) + \sum_{\mathbf{j}\in\mathbb{Z}^n_\triangle}\mathbf{U}_\triangle(n)(K^{\mathbf{j}} - v^{\mu\cdot\mathbf{j}})$. Since ${}_\lambda{}'I_\mu = {}_\lambda I_\mu \cap \mathbf{U}_\triangle(n)$ (see Theorem 2.3(2)), it follows that $\dot{\mathbf{U}}_\triangle(n) \cong \oplus_{\lambda,\mu\in\mathbb{Z}^n_\triangle}{}_\lambda\mathbf{U}_\triangle(n)_\mu$, where ${}_\lambda\mathbf{U}_\triangle(n)_\mu = \pi_{\lambda,\mu}(\mathbf{U}_\triangle(n))$. Thus, we will regard $\dot{\mathbf{U}}_\triangle(n)$ as this subalgebra of $\dot{\boldsymbol{\mathfrak{D}}}_\triangle(n) = \mathbf{K}_\triangle(n)$. We now look at an application to the conjecture given in [28, 9.3] which is proved in [32].

Let $\dot{U}_\triangle(n)_\mathcal{Z}$ be the $\mathcal{Z}$-subalgebra of $\dot{\boldsymbol{\mathfrak{D}}}_\triangle(n)$ generated by

$$\widetilde{u}^+_{m\boldsymbol{e}^\triangle_i}[\mathrm{diag}(\lambda)] = E_i^{(m)}[\mathrm{diag}(\lambda)],\quad \widetilde{u}^-_{m\boldsymbol{e}^\triangle_i}[\mathrm{diag}(\lambda)] = F_i^{(m)}[\mathrm{diag}(\lambda)]$$

for all $1\leqslant i\leqslant n$, $m\in\mathbb{N}$ and $\lambda\in\mathbb{Z}^n_\triangle$. Then $\dot{U}_\triangle(n)_\mathcal{Z}$ is a subalgebra of $\dot{\mathfrak{D}}_\triangle(n)_\mathcal{Z} = \mathsf{K}_\triangle(n)_\mathcal{Z}$.

Call a matrix $A=(a_{i,j})\in\widetilde{\Theta}_\triangle(n)$ to be *aperiodic* if for every integer $l\neq 0$ there exists $1\leqslant i\leqslant n$ such that $a_{i,i+l}=0$. Let $\widetilde{\Theta}^{\mathrm{ap}}_\triangle(n)$ be the set of all aperiodic matrices in $\widetilde{\Theta}_\triangle(n)$.

Recall the monomial basis for $\dot{\mathfrak{D}}_\triangle(n)_\mathcal{Z}$ given in Corollary 6.7.

Lemma 8.1. *The set $\{\mathcal{M}^{(A)} \mid A\in\widetilde{\Theta}^{\mathrm{ap}}_\triangle(n)\}$ forms a $\mathcal{Z}$-basis for $\dot{U}_\triangle(n)_\mathcal{Z}$.*

Proof. By [6, Th. 7.5(1)], the elements $\widetilde{u}^+_{(w_A)}$, $A\in\Theta^+_\triangle(n)\cap\widetilde{\Theta}^{\mathrm{ap}}_\triangle(n)$, form a basis for the +-part $U^+_\triangle(n)_\mathcal{Z}$ generated by all $E_i^{(m)}$. Hence, the set $\{\mathcal{M}^{(A)} \mid A\in\widetilde{\Theta}^{\mathrm{ap}}_\triangle(n)\}$ spans $\dot{U}_\triangle(n)_\mathcal{Z}$. By (6.6.2), the set is linearly independent. □

For each $A\in\widetilde{\Theta}^{\mathrm{ap}}_\triangle(n)$, use the coefficients $h_{A,B}$ given in (6.6.2) and the order $\sqsubseteq$ given in (5.2.1) to define (cf. [6, Def. 7.2]) recursively the elements $\mathcal{E}_A\in\dot{U}_\triangle(n)_\mathcal{Z}$ by

$$\mathcal{E}_A = \begin{cases}\mathcal{M}^{(A)}, & \text{if } A \text{ is minimal relative to } \sqsubseteq;\\ \mathcal{M}^{(A)} - \sum_{\substack{B\sqsubset A\\ B\in\widetilde{\Theta}^{\mathrm{ap}}_\triangle(n)}} h_{A,B}\mathcal{E}_B. & \text{otherwise.}\end{cases}\tag{8.1.1}$$

Lemma 8.2. (1) *The set $\{\mathcal{E}_A \mid A\in\widetilde{\Theta}^{\mathrm{ap}}_\triangle(n)\}$ forms a $\mathcal{Z}$-basis for $\dot{U}_\triangle(n)_\mathcal{Z}$.*

(2) *For $A\in\widetilde{\Theta}^{\mathrm{ap}}_\triangle(n)$ we have $\mathcal{E}_A - [A] \in \sum_{\substack{B\in\widetilde{\Theta}_\triangle(n)\setminus\widetilde{\Theta}^{\mathrm{ap}}_\triangle(n)\\ B\sqsubset A}}\mathcal{Z}[B]$.*

Proof. Statement (1) follows from Lemma 8.1 and the definition $\mathcal{E}_A$ (8.1.1). We prove (2) by induction on $\|A\|$. The assertion is clear for by $\|A\|=0$. Assume now $\|A\|\geqslant 1$ By (6.6.2) and (8.1.1), we have

$$\mathcal{E}_A - [A] + \sum_{\substack{B\in\widetilde{\Theta}^{\mathrm{ap}}_\triangle(n)\\ B\sqsubset A}} h_{A,B}(\mathcal{E}_B - [B]) = \sum_{\substack{B\in\widetilde{\Theta}_\triangle(n)\setminus\widetilde{\Theta}^{\mathrm{ap}}_\triangle(n)\\ B\sqsubset A}} h_{A,B}[B].$$

Now the assertion follows from induction since $B\sqsubset A$ implies $\|B\|<\|A\|$. □

Note that the restriction of the bar involution (7.3.1) gives a bar involution on $\dot{U}_\triangle(n)_\mathcal{Z}$.

Proposition 8.3. *There exists a unique $\mathcal{Z}$-basis $\{\theta'_A \mid A \in \widetilde{\Theta}^{\rm ap}_\triangle(n)\}$ for $\dot{U}_\triangle(n)_\mathcal{Z}$ such that $\overline{\theta'_A} = \theta'_A$ and*

$$\theta'_A - \mathcal{E}_A \in \sum_{B\in\widetilde{\Theta}^{\rm ap}_\triangle(n),\, B\sqsubset A} v^{-1}\mathbb{Z}[v^{-1}]\mathcal{E}_B.$$

Proof. Since, by (8.1.1),

$$\mathcal{E}_A = \mathcal{M}^{(A)} + \text{a } \mathcal{Z}\text{-linear combination of } \mathcal{M}^{(C)} \text{ with } C \in \widetilde{\Theta}^{\rm ap}_\triangle(n) \text{ and } C \sqsubset A,$$

it follows that $\overline{\mathcal{E}_A} - \mathcal{E}_A \in \sum_{\substack{C\in\widetilde{\Theta}^{\rm ap}_\triangle(n)\\ C\sqsubset A}} \mathcal{Z}\mathcal{E}_C$. Now the assertion follows from a standard argument. □

Remark 8.4. Motivated by [28, Th. 8.2], it would be natural to conjecture that $\theta_A \in \dot{U}_\triangle(n)_\mathcal{Z}$ for all $A \in \widetilde{\Theta}^{\rm ap}_\triangle(n)$. Equivalently, $\theta'_A = \theta_A$ if $A \in \widetilde{\Theta}^{\rm ap}_\triangle(n)$ (cf. [6, Th. 8.5]). In the rest of the section, we show some strong evidence for the truth of this conjecture.

Let $\mathcal{L}_r = \sum_{A\in\Theta_\triangle(n,r)} \mathbb{Z}[v^{-1}][A] \in \mathcal{S}_\triangle(n,r)_\mathcal{Z}$ and let $\mathscr{P}$ be the $\mathcal{Z}$-submodule of $\dot{\mathfrak{D}}_\triangle(n)_\mathcal{Z}$ spanned by the *periodic* elements $[B]$ with $B \in \widetilde{\Theta}_\triangle(n)\backslash\widetilde{\Theta}^{\rm ap}_\triangle(n)$. Recall the algebra homomorphisms ζ in Theorem 3.2 and $\dot{\zeta}_r$ in (6.4.1) and note that $\dot{\zeta}_r(\mathscr{P}) \cap \mathbf{U}_\triangle(n,r) = 0$, where $\zeta_r(\mathbf{U}_\triangle(n)) = \mathbf{U}_\triangle(n,r)$.

Let $\Theta^{\rm ap}_\triangle(n,r) = \widetilde{\Theta}^{\rm ap}_\triangle(n) \cap \Theta_\triangle(n,r)$.

Lemma 8.5. *Assume $A \in \widetilde{\Theta}^{\rm ap}_\triangle(n)$.*

(1) *If $A \notin \Theta_\triangle(n,r)$ then we have $\dot{\zeta}_r(\mathcal{E}_A) = 0$.*
(2) *If $A \in \Theta_\triangle(n,r)$ then we have $\dot{\zeta}_r(\mathcal{E}_A) - [A] \in v^{-1}\mathcal{L}_r$.*

Proof. If $A \notin \Theta_\triangle(n,r)$, Lemma 8.2(2) implies $\dot{\zeta}_r(\mathcal{E}_A) = \dot{\zeta}_r(\mathcal{E}_A) - \dot{\zeta}_r([A]) \in \dot{\zeta}_r(\mathscr{P}) \cap \mathbf{U}_\triangle(n,r) = 0$, proving (1).

Now we assume $A \in \Theta_\triangle(n,r)$. If $\|A\| = 0$ then $\mathcal{E}_A = [A]$ and $\dot{\zeta}_r(\mathcal{E}_A) - [A] = 0$. Now we assume $\|A\| > 0$. We write $\theta_{A,r}$ as in (7.6.1). By Lemma 8.2 and [28, 8.2], we see that

$$\begin{aligned}\theta_{A,r} - \Big(\dot{\zeta}_r(\mathcal{E}_A) + \sum_{\substack{B\in\Theta^{\rm ap}_\triangle(n,r)\\ B\sqsubset A}} g_{B,A,r}\dot{\zeta}_r(\mathcal{E}_B)\Big) &= ([A] - \dot{\zeta}_r(\mathcal{E}_A)) + \sum_{\substack{B\in\Theta^{\rm ap}_\triangle(n,r)\\ B\sqsubset A}} g_{B,A,r}([B] - \dot{\zeta}_r(\mathcal{E}_B))\\ &\quad + \sum_{\substack{B\in\Theta_\triangle(n,r)\backslash\Theta^{\rm ap}_\triangle(n,r)\\ B\sqsubset A}} g_{B,A,r}[B],\end{aligned}$$

which belongs to $\dot{\zeta}_r(\mathscr{P}) \cap \mathbf{U}_\triangle(n,r) = 0$. Thus, by the induction hypothesis,

$$\dot{\zeta}_r(\mathcal{E}_A) - [A] = \sum_{\substack{B\in\Theta^{\rm ap}_\triangle(n,r)\\ B\sqsubset A}} g_{B,A,r}([B] - \dot{\zeta}_r(\mathcal{E}_B)) + \sum_{\substack{B\in\Theta_\triangle(n,r)\backslash\Theta^{\rm ap}_\triangle(n,r)\\ B\sqsubset A}} g_{B,A,r}[B] \in v^{-1}\mathcal{L}_r$$

as required. □

We now show that the basis θ'_A satisfies a property similar to Theorem 7.7(2b).

Theorem 8.6. *Let $A \in \widetilde{\Theta}^{\mathrm{ap}}_\vartriangle(n)$. Then we have*

$$\dot{\zeta}_r(\theta'_A) = \begin{cases} \theta_{A,r} & \text{if } A \in \Theta_\vartriangle(n,r); \\ 0 & \text{if } A \notin \Theta_\vartriangle(n,r). \end{cases}$$

Hence, we have $\dot{\zeta}_r(\theta'_A) = \dot{\zeta}_r(\theta_A)$ for $A \in \widetilde{\Theta}^{\mathrm{ap}}_\vartriangle(n)$.

Proof. If $A \notin \Theta_\vartriangle(n,r)$ then, by Proposition 8.3 and Lemma 8.5, we see that

$$\dot{\zeta}_r(\theta'_A) = \dot{\zeta}_r(\theta'_A - \mathcal{E}_A) \in \sum_{\substack{B\in\Theta^{\mathrm{ap}}_\vartriangle(n,r)\\ B\sqsubset A}} v^{-1}\mathbb{Z}[v^{-1}]\dot{\zeta}_r(\mathcal{E}_B) \subseteq v^{-1}\mathcal{L}_r.$$

If $A \in \Theta_\vartriangle(n,r)$ then, by loc. cit., we have

$$\dot{\zeta}_r(\theta'_A) \in \dot{\zeta}_r(\mathcal{E}_A) + \sum_{\substack{B\in\Theta^{\mathrm{ap}}_\vartriangle(n,r)\\ B\sqsubset A}} v^{-1}\mathbb{Z}[v^{-1}]\dot{\zeta}_r(\mathcal{E}_B) \subseteq [A] + v^{-1}\mathcal{L}_r.$$

Furthermore, we have $\overline{\dot{\zeta}_r(\theta'_A)} = \dot{\zeta}_r(\theta'_A)$ for all $A \in \widetilde{\Theta}^{\mathrm{ap}}_\vartriangle(n)$. The assertion follows the uniqueness of the canonical basis. □

Theorem 8.6 gives an algebraic construction of the conjecture of Lusztig stated at the end of [28, §9.3][5] for the modified extended quantum affine $\mathfrak{sl}_n$, $\dot{U}_\vartriangle(n)_\mathcal{Z}$, idempotented on $\mathbb{Z}^n$; see [32] for a proof for the (polynomial weighted) modified quantum affine $\mathfrak{sl}_n$ which is idempotented on $\mathbb{N}^n$ (compare the construction in [29, §7] for the modified quantum affine $\mathfrak{sl}_n$ idempotented on $\mathbb{Z}^{n-1}$). Note that, by the presentation for $\dot{U}_\vartriangle(n)_\mathcal{Z}$ given in [27, 31.1.3], this modified algebra of Schiffmann–Vasserot is a homomorphic image of $\dot{U}_\vartriangle(n)_\mathcal{Z}$.

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